RICHNESS OR SEMI-HAMILTONICITY OF QUASI-LINEAR SYSTEMS WHICH ARE NOT IN EVOLUTION FORM

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Abstract. The aim of this paper is to consider quasi-linear systems which are not in the form of evolution equations. We propose new condition of Richness or Semi-Hamiltonicity for such a system and prove that the blow up analysis along characteristic curves can be performed for it in an analogous manner. This opens a possibility to use this ansatz also for geometric problems. We apply the results to the problem of Polynomial integral for geodesic flows on the 2-torus.

1. Motivation and the result

Consider a quasi-linear system for vector function $u(x,y) = (u_1, ..., u_n)$ which has the following form

$$A(u)u_x + B(u)u_y = 0 \tag{1}$$

It may happen in practice that one of the matrices $A(u)$ and $B(u)$ can degenerate somewhere (and even both of them can degenerate somewhere).

Throughout this paper our main assumption on these matrices is that the homogeneous polynomial P in α, β is not a zero polynomial at any point (x,y) :

$$P = \det(\alpha B - \beta A),\ deg(P) = n. \tag{P}$$

This assumption is obviously satisfied if one of the matrices $A(u)$ and $B(u)$ is non-degenerate, however we shall assume everywhere the weaker version-(P). We shall see in the example of the last section that (P) is in fact the correct assumption.

Moreover, we shall assume in the following that the system is strictly hyperbolic that is the polynomial P has n distinct roots $[\beta_i : \alpha_i]$. We define unite characteristic vector fields on the plane $\mathbf{R}^2(x,y)$ by

$$v_i = \cos\phi_i \partial_x + \sin\phi_i \partial_y,$$

where the angles ϕ_i, we shall call them characteristic angles, are such that $[\sin\phi_i : \cos\phi_i] = [\beta_i : \alpha_i], \phi_i \neq \phi_j\ (mod\ \pi)$.

One can use a regular change of variables and multiplication from the left on a invertible matrix in order to transform system (1) to an equivalent

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one. Namely, if $u = \Phi(w)$ is a regular change of variables and $C(w)$ is an invertible matrix then system (1) takes the form

$$C(w)A(\Phi(w))D\Phi(w)w_x + C(w)B(\Phi(w))D\Phi(w)w_y = 0,$$

where D is the differential. Obviously such a transformation preserves the roots of P.

Notice that if one of the matrices, say A is non-degenerate then the system is equivalent to one in the evolution form. Recall the notion of the evolution system to be Rich or Semi-Hamiltonian (see [10],[11] and [12],[5]), we shall call them Rich for the sake of brevity. Strictly Hyperbolic evolution system is called Rich if it can be written in Riemann invariants (diagonal form)

$$(r_i)_x + \lambda_i(r_1, ..., r_n)(r_i)_y = 0, i = 1, ..., n,$$

and moreover the eigenvalues $\lambda_i = \beta_i/\alpha_i$ of $A^{-1}B$ satisfy the following identities:

$$\partial_{r_k}\left(\frac{\partial_{r_i}\lambda_j}{\lambda_i - \lambda_j}\right) = \partial_{r_i}\left(\frac{\partial_{r_k}\lambda_j}{\lambda_k - \lambda_j}\right). \tag{R}$$

This condition allows one to perform blow-up analysis along characteristics as it is shown in [10] and applied for a mechanical example in [1]. It was proved by B. Sevennec [11] and later understood in differential-geometric terms [7] that strictly Hyperbolic system in evolution form which is written in Riemann invariants is Rich, if and only if there are local coordinates in which the system takes the form of conservation laws.

The unsatisfactory thing, however,with the condition (R) for the system (1) is the fact that characteristic curves can pass from the chart where A is non-degenerate to the chart where B is non-degenerate or even reach those points where both matrices degenerate. This does not allow one to use the analysis of corresponding Riccati equations for all times.

We propose the following generalization of the Richness condition whose naturality we shall justify below:

Definition 1.1. We call strictly Hyperbolic system (1) Rich if it can be written in the diagonal form

$$L_{v_i}r_i = \cos\phi_i(r_i)_x + \sin\phi_i(r_i)_y = 0,\ i = 1, ..., n,\ \phi_i \neq \phi_j\ (mod\ \pi) \tag{2}$$

for a regular change of variables $(u_1, ..., u_n) \to (r_1, ..., r_n)$ and the following conditions on the characteristic angles $\phi_i(r_1, ..., r_n)$ holds true

$$\partial_{r_k}\left(\frac{\partial_{r_i}\phi_j}{\tan(\phi_i - \phi_j)}\right) = \partial_{r_i}\left(\frac{\partial_{r_k}\phi_j}{\tan(\phi_k - \phi_j)}\right) \tag{Φ}$$

It is important fact that this definition is invariant with respect to rotations of the plane. We shall continue to call r_i in (2) by Riemann invariants. Our first result is the following

Theorem 1.2. *If a strictly Hyperbolic system (1) satisfying (P) is Rich according Definition 1.1, so that the conditions (2),(Φ) hold true, then the derivatives of i-th Riemann invariant $w_i = L_{v_i^\perp}r_i$ in the orthogonal direction to characteristics satisfy the following Riccati equation:*

$$L_{v_i}(\exp(-G_i)w_i) + \exp(G_i)\partial_{r_i}(\phi_i)(\exp(-G_i)w_i)^2 = 0,$$

for any $i = 1, ..., n$, where G_j is a function of Riemann invariants satisfying

$$\partial_{r_i} G_j = \frac{\partial_{r_i}\phi_j}{\tan(\phi_i - \phi_j)}.$$

Here $v^\perp$ stands for the vector field rotated from v by 90° counterclockwise.

We shall see in lemma below that the conditions (R) and (Φ) are almost equivalent. This lemma enables us to prove the following theorem which is a generalization to our case of the result of [11].

Theorem 1.3. *Given any strictly Hyperbolic diagonal system*

$$\cos\phi_i (r_i)_x + \sin\phi_i (r_i)_y = 0, \; i = 1, ..., n.$$

Then the condition (Φ) is satisfied if and only if the system can be written in the form of n conservation laws

$$(g_i)_x + (h_i)_y = 0, i = 1, ..., n.$$

We prove in sections 2,3 the main theorems. The last section contains a geometric example originated from Classical Mechanics.

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2. Derivation along characteristics. Proof of Theorem 1.2.

Differentiate the j-th equation of (2) with respect to the field $v_j^\perp$. We have

$$0 = L_{v_j^\perp} L_{v_j} r_j = L_{v_j} L_{v_j^\perp} r_j - L_{[v_j, v_j^\perp]} r_j. \tag{3}$$

Compute now the derivative along the commutator:

$$\begin{gathered} L_{[v_j, v_j^\perp]} r_j = L_{v_j} L_{v_j^\perp} r_j - L_{v_j^\perp} L_{v_j} r_j = \\ = L_{v_j}(-\sin\phi_j (r_j)_x + \cos\phi_j (r_j)_y) - L_{v_j^\perp}(\cos\phi_j (r_j)_x + \sin\phi_j (r_j)_y) = \\ = (r_j)_x(-\cos^2\phi_j (\phi_j)_x - \cos\phi_j \sin\phi_j (\phi_j)_y) + \\ + (r_j)_y(-\sin\phi_j \cos\phi_j (\phi_j)_x - \sin^2\phi_j (\phi_j)_y) + \\ + (r_j)_x(-\sin^2\phi_j (\phi_j)_x + \sin\phi_j \cos\phi_j (\phi_j)_y) + \\ + (r_j)_y(\sin\phi_j \cos\phi_j (\phi_j)_x - \cos^2\phi_j (\phi_j)_y) = \\ = -(r_j)_x(\phi_j)_x - (r_j)_y(\phi_j)_y \end{gathered} \tag{4}$$

Notice that the derivatives $(r_j)_x, (r_j)_y$ can be expressed by the following two identities:

$$\cos\phi_j (r_j)_x + \sin\phi_j (r_j)_y = 0,$$
$$-\sin\phi_j (r_j)_x + \cos\phi_j (r_j)_y = L_{v_j^\perp} r_j.$$

Therefore

$$(r_j)_x = -\sin\phi_j L_{v_j^\perp} r_j,$$
$$(r_j)_x = \cos\phi_j L_{v_j^\perp} r_j. \tag{5}$$

Substituting back to (4) we get

$$L_{[v_j,v_j^\perp]}r_j=(L_{v_j^\perp}r_j)(\sin\phi_j(\phi_j)_x-\cos\phi_j(\phi_j)_y)=-(L_{v_j^\perp}r_j)(L_{v_j^\perp}\phi_j) \tag{6}$$

By the chain rule for $L_{v_j^\perp}\phi_j$ the last equation can be rewritten as follows

$$L_{[v_j,v_j^\perp]}r_j=-L_{v_j^\perp}r_j\sum_{i=1}^{n}(\partial_{r_i}\phi_j)(L_{v_j^\perp}r_i)=$$

$$-(L_{v_j^\perp}r_j)^2(\partial_{r_j}\phi_j)-(L_{v_j^\perp}r_j)\sum_{i\neq j}(\partial_{r_i}\phi_j)(L_{v_j^\perp}r_i). \tag{7}$$

Let me express now the derivative

$$L_{v_j^\perp}r_i=-\sin\phi_j(r_i)_x+\cos\phi_j(r_i)_y, \tag{8}$$

via $L_{v_j}r_i$ as follows. Write

$$\cos\phi_i(r_i)_x+\sin\phi_i(r_i)_y=0,$$

$$\cos\phi_j(r_i)_x+\sin\phi_j(r_i)_y=L_{v_j}r_i. \tag{9}$$

From these two identities we have

$$(r_i)_x=\frac{\sin\phi_i}{\sin(\phi_i-\phi_j)}L_{v_j}r_i,\quad (r_j)_y=-\frac{\cos\phi_i}{\sin(\phi_i-\phi_j)}L_{v_j}r_i. \tag{10}$$

Substitute expressions (10) into (8) to get:

$$L_{v_j^\perp}r_i=-\frac{L_{v_j}r_i}{tan(\phi_i-\phi_j)}. \tag{11}$$

Denote by

$$w_i:=L_{v_i^\perp}r_i.$$

Plug this together with (11) into equation (7) and then to (3):

$$L_{v_j}(w_j)+(\partial_{r_j}\phi_j)(w_j)^2-w_j\sum_{i\neq j}(\partial_{r_i}\phi_j)\frac{1}{tan(\phi_i-\phi_j)}L_{v_j}r_i=0. \tag{12}$$

By Richness (Φ) we have that for all $j=1,...,n$ there exist functions

$$G_j(r_1,...,r_n):\quad \partial_{r_i}G_j=\frac{(\partial_{r_i}\phi_j)}{tan(\phi_i-\phi_j)},\ i\neq j. \tag{13}$$

By (13) we can rewrite (12) as

$$L_{v_j}(w_j)+(\partial_{r_j}\phi_j)(w_j)^2-w_j\sum_{i\neq j}(\partial_{r_i}G_j)L_{v_j}r_i=0,$$

which is the same as

$$L_{v_j}(w_j)+(\partial_{r_j}\phi_j)(w_j)^2-w_jL_{v_j}G_j=0. \tag{14}$$

Multiplying (14) by $exp(-G_j)$ we get the Riccati equation of the first theorem for j instead of i. This completes the proof of theorem 1.2.

3. Conservation laws. Proof of theorem 1.3.

We shall need the following key observation.

Lemma 3.1. *Given two sets of functions* $\lambda_i(r_1, ..., r_n)$; $\phi_i(r_1, ...r_n)$, $i = 1, ..., n)$ *such that*

$$\lambda_i \neq \lambda_j, \ \phi_i \neq \pi/2 \ (mod\ \pi), \ \lambda_i = \tan\phi_i.$$

Then the conditions (R) and (Φ) are equivalent.

The proof which I know is computational. It would be interesting to find more conceptual proof.

Proof. Let us prove first that (R) implies (Φ).

Denote by

$$a_{ij} := \frac{\partial_{r_i}\lambda_j}{\lambda_i - \lambda_j}.$$

Then a_{ij} satisfy the following identities ([10]):

$$\partial_{r_i} a_{kj} = \partial_{r_k} a_{ij} = a_{ki}a_{ij} + a_{ik}a_{kj} - a_{kj}a_{ij}. \tag{15}$$

In order to prove them differentiate with respect to r_k the identity $\partial_{r_i}\lambda_j = a_{ij}(\lambda_i - \lambda_j)$ then interchange the order of i, k, subtract one from the other and divide by $\lambda_i - \lambda_k$.

Denote by

$$b_{ij} := \frac{\partial_{r_i}\phi_j}{\tan(\phi_i - \phi_j)} = \frac{\partial_{r_i}\lambda_j}{\lambda_i - \lambda_j}\frac{1 + \lambda_i\lambda_j}{1 + \lambda_j^2} = a_{ij}\frac{1 + \lambda_i\lambda_j}{1 + \lambda_j^2}. \tag{16}$$

To prove (Φ) we have to verify that the difference

$$d = \partial_{r_k} b_{ij} - \partial_{r_i} b_{kj}$$

vanishes. Let us compute d explicitly:

$$d = (\partial_{r_k} a_{ij})\frac{1 + \lambda_i\lambda_j}{1 + \lambda_j^2} - (\partial_{r_i} a_{kj})\frac{1 + \lambda_k\lambda_j}{1 + \lambda_j^2} +$$

$$+ a_{ij}\partial_{r_k}\left(\frac{1 + \lambda_i\lambda_j}{1 + \lambda_j^2}\right) - a_{kj}\partial_{r_i}\left(\frac{1 + \lambda_k\lambda_j}{1 + \lambda_j^2}\right).$$

By the identities (15) and the condition (R) we have

$$d = (a_{ki}a_{ij} + a_{ik}a_{kj} - a_{kj}a_{ij})\frac{\lambda_j(\lambda_i - \lambda_k)}{1 + \lambda_j^2} +$$

$$+ a_{ij}\frac{\partial_{r_k}(\lambda_i\lambda_j)(1 + \lambda_j^2) - (1 + \lambda_i\lambda_j)2\lambda_j\partial_{r_k}(\lambda_j)}{(1 + \lambda_j^2)^2} -$$

$$- a_{kj}\frac{\partial_{r_i}(\lambda_k\lambda_j)(1 + \lambda_j^2) - (1 + \lambda_k\lambda_j)2\lambda_j\partial_{r_i}(\lambda_j)}{(1 + \lambda_j^2)^2}. \tag{17}$$

Substitute now into the nominators of (17) the following expressions for the derivatives of λ_j from the definition of a_{ij}:

$$\partial_{r_i}\lambda_j = a_{ij}(\lambda_i - \lambda_j).$$

Then one has

$$d=(a_{ki}a_{ij}+a_{ik}a_{kj}-a_{kj}a_{ij})\frac{\lambda_j(\lambda_i-\lambda_k)}{1+\lambda_j^2}+$$

$$+a_{ij}\frac{(a_{kj}(\lambda_k-\lambda_j)\lambda_i+\lambda_j a_{ki}(\lambda_k-\lambda_i)}{1+\lambda_j^2}-a_{kj}\frac{(a_{ij}(\lambda_i-\lambda_j)\lambda_k+\lambda_j a_{ik}(\lambda_i-\lambda_k)}{1+\lambda_j^2}-$$

$$-2a_{ij}\frac{(1+\lambda_i\lambda_j)\lambda_j a_{kj}(\lambda_k-\lambda_j)}{(1+\lambda_j^2)^2}+2a_{kj}\frac{(1+\lambda_k\lambda_j)\lambda_j a_{ij}(\lambda_i-\lambda_j)}{(1+\lambda_j^2)^2}. \qquad (18)$$

Notice that the identity (18) is a quadratic expression in $a_{ij}s$. Collecting the coefficients of $a_{ij}a_{kj},\ a_{ik}a_{kj},\ a_{ki}a_{ij}$ one comes to $d=0$. This proves lemma in one direction.

Proof of the converse statement is very much analogous but with even harder computations. I shall reproduce them sketchy. So we assume the identities (Φ) are satisfied. First one can obtain the identity analogous to (15) for the derivatives $\partial_{r_k}b_{ij}$ in the following way. Write

$$\partial_{r_i}\phi_j=b_{ij}\tan(\phi_i-\phi_j)=b_{ij}\frac{\lambda_i-\lambda_j}{1+\lambda_i\lambda_j},\ \partial_{r_i}\lambda_j=b_{ij}\frac{(1+\lambda_j^2)(\lambda_i-\lambda_j)}{1+\lambda_i\lambda_j}. \ (19).$$

Differentiating the first equality of (19) with respect to r_k, using the identities (19) again and taking into account (16) one has

$$\partial_{r_k}\partial_{r_i}\phi_j=\partial_{r_k}(b_{ij})\frac{(\lambda_i-\lambda_j)}{1+\lambda_i\lambda_j}+$$

$$+b_{ij}\left(1+\frac{(\lambda_i-\lambda_j)^2}{(1+\lambda_i\lambda_j)^2}\right)\left(b_{ki}\frac{\lambda_k-\lambda_i}{1+\lambda_k\lambda_i}-b_{kj}\frac{\lambda_k-\lambda_j}{1+\lambda_k\lambda_j}\right).$$

Interchanging in this identity the order of indexes i and k and using $\partial_{r_k}b_{ij}=\partial_{r_i}b_{kj}$ one has the identity:

$$\partial_{r_k}(b_{ij})\frac{(\lambda_i-\lambda_k)(1+\lambda_j^2)}{(1+\lambda_i\lambda_j)(1+\lambda_k\lambda_j)}=$$

$$=b_{kj}\left(1+\frac{(\lambda_k-\lambda_j)^2}{(1+\lambda_k\lambda_j)^2}\right)\left(b_{ik}\frac{\lambda_i-\lambda_k}{1+\lambda_i\lambda_k}-b_{ij}\frac{\lambda_i-\lambda_j}{1+\lambda_i\lambda_j}\right)-$$

$$-b_{ij}\left(1+\frac{(\lambda_i-\lambda_j)^2}{(1+\lambda_i\lambda_j)^2}\right)\left(b_{ki}\frac{\lambda_k-\lambda_i}{1+\lambda_i\lambda_k}-b_{kj}\frac{\lambda_k-\lambda_j}{1+\lambda_k\lambda_j}\right). \qquad (20)$$

In order to verify (R) one computes

$$\partial_{r_k}a_{ij}-\partial_{r_i}a_{kj}=\partial_{r_k}b_{ij}\frac{\lambda_j(\lambda_k-\lambda_i)(1+\lambda_j^2)}{(1+\lambda_i\lambda_j)(1+\lambda_k\lambda_j)}+$$

$$+b_{ij}\partial_{r_k}\left(\frac{1+\lambda_j^2}{1+\lambda_i\lambda_j}\right)-b_{kj}\partial_{r_i}\left(\frac{1+\lambda_j^2}{1+\lambda_k\lambda_j}\right). \qquad (21)$$

The last step is to plug into (21) the expression (20) and also to differentiate the last two brackets of (21) using the expression for the derivatives (19). Then one finally gets a quadratic expression in $b_{ij}s$. Collecting similar terms one verifies that the right hand side of (21) vanishes. Therefore (R) holds true. This proves the lemma. □

It is easy now to prove Theorem 1.3.

Proof. Notice first of all that the statement of the second theorem is local. Given a system which is strictly Hyperbolic and is written in the diagonal form

$$\cos\phi_i(r_i)_x + \sin\phi_i(r_i)_y = 0,\ i = 1, ..., n,$$

Let us give a proof first in one direction, namely assume that the system can be written in the form of conservation laws

$$(g_i)_x + (h_i)_y = 0, i = 1, ..., n.$$

Let me explain that then it must satisfy condition (Φ). If among ϕ_i there is one with $\cos\phi_i = 0$ then one can apply a small rotation of the plane $\mathbf{R}^2(x, y)$ and to get a new system which has all angles different from $\pm\pi/2$, $\phi_i \neq \phi_j\ (mod\ \pi)$. Notice that the rotated system remains in the form of conservation laws and in addition the differential Dg becomes a non singular matrix, since otherwise $\alpha = 1, \beta = 0$ would be the root of (P) but this is impossible by $\phi_i \neq \pm\pi/2$. Denote

$$\lambda_i := \tan\phi_i.$$

Use now Sevennec' theorem saying that the diagonal system

$$(r_i)_x + \lambda_i(r_i)_y = 0$$

which can be written in the form of conservation laws

$$(g_i)_x + (h_i)_y = 0, i = 1, ..., n$$

with the non-singular Jacobi matrix $\left(\partial_{r_j} g_i\right)$ must satisfy (R). But by lemma in this case (R) and (Φ) are equivalent. So we get condition (Φ) for rotated system. But this condition is obviously rotationally invariant. Thus it holds also for the original system.

The proof in the opposite direction is analogous. First rotate the plane exactly as above. Condition (Φ) remains valid since it is rotationally invariant. Then by the lemma (R) is valid as well and then by Sevennec' theorem the rotated system can written in the form of conservation laws. But then obviously the original one as well. This completes the proof.

□

4. Geometric example.

In this section we give a geometric example originated from Classical Mechanics where the results of the previous sections apply.

Let ρ be a Riemannian metric on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$, ρ^t denotes the geodesic flow. Assume that ρ is written in conformal way:

$$ds^2 = \Lambda(x, y)(dx^2 + dy^2).$$

Let $F : T^*\mathbb{T}^2$ be a function on the cotangent bundle which is homogeneous polynomial of degree n with respect to the fibre:

$$F = \sum_{k=0}^{n} a_k(x, y) p_1^{n-k} p_2^k.$$

We are looking for such an F which is an integral of motion for the geodesic flow ρ^t, i.e. $F \circ \rho^t = F$. We shall also assume that this F is irreducible, i.e of minimal possible degree. Let us mention that this problem is classical there are very well studied examples of the geodesic flows on the 2-torus which have integrals F of degree one and two. We refer to books [4] and [9] for the history and discussion of this classical question with references therein. In our recent papers with A.E.Mironov we used the so called semi-geodesic coordinates. In these coordinates one arrives to a remarkable Rich quasi-linear system of equations in evolution form on the coefficients of the integral F ([2], [3]). However it is very natural to be able to work in conformal coordinates as well. In this case the quasi-linear system on the coefficients has no evolution form any more but looks like:

$$A(U)U_x + B(U)U_y = 0.$$

Let me write down explicitly the matrices for the case $n = 3$ (this case is already very interesting and not trivial see for example [6]).

$$A(U) = \begin{pmatrix} 1 & 0 & 3a \\ 0 & 1 & 3b \\ \Lambda & 0 & u \end{pmatrix}, \ B(U) = \begin{pmatrix} 0 & -1 & 3b \\ 1 & 0 & -3a \\ 0 & \Lambda & v \end{pmatrix}, \ U = \begin{pmatrix} u \\ v \\ \Lambda \end{pmatrix}. \quad (22)$$

Here a, b, u, v are related to the coefficients of the integral a_i by the following

$$a_0 = a + \frac{u}{\Lambda},\ a_1 = 3b + \frac{v}{\Lambda},\ a_2 = -3a + \frac{u}{\Lambda},\ a_3 = -b + \frac{v}{\Lambda}.$$

It was noticed in [8] that a, b are in fact constants. Computing polynomial $P = \det(\alpha B - \beta A)$ one has:

$$P = \alpha^3(v + 3b\Lambda) + \alpha^2\beta(-u - 9a\Lambda) + \alpha\beta^2(v - 9b\Lambda) + \beta^3(-u + 3a\Lambda).$$

Let remark that it may happen at some points that both matrices A, B are degenerate, however polynomial P for any point can not vanish identically. This is because otherwise both constants a, b vanish, but then one checks that in such a case the integral F is a product of the Hamiltonian with an integral of degree one in momenta, therefore reducible.

Notice that quasi-linear system (22) is written in the form of conservation laws

$$(g_i)_x + (h_i)_y = 0,$$
$$g_1 = u + 3a\Lambda,\ g_2 = v + 3b\Lambda,\ g_3 = u\Lambda,$$
$$h_1 = -v + 3b\Lambda,\ h_2 = u - 3a\lambda,\ h_3 = v\Lambda.$$

Moreover by a very general argument in the Hyperbolic region this system can be written in the diagonal form (2). Indeed introduce angular coordinate ϕ on the fibres of the energy level

$$\{\frac{1}{\Lambda}(p_1^2 + p_2^2) = 1\}:\ p_1 = \sqrt{\Lambda}\cos\phi, p_2 = \sqrt{\Lambda}\sin\phi,$$

then one can verify that the condition on a function F to be an integral of the flow reads

$$F_x \cos\phi + F_y \sin\phi + F_\phi \left(\frac{\Lambda_y}{2\Lambda}\cos\phi - \frac{\Lambda_x}{2\Lambda}\sin\phi\right) = 0$$

At the points where F_ϕ vanishes this equation takes particularly nice form:

$$F_x \cos\phi + F_y \sin\phi = 0.$$

Therefore critical values of F on the fibre are Riemann invariants. One can check also that the polynomial P is proportional in fact to the derivative of F in the direction of the fibre. Moreover one can check, as we did in [2], that in the Hyperbolic region Riemann invariants form a regular change of variables. As a consequence of Theorem 1.3 one concludes that in the Hyperbolic region the system of this example is Rich in our generalized sense. And therefore Theorem 1.2 tells us that the Riccati equation along characteristics applies. This result is in fact general and is not restricted to the case $n = 3$. For any n the quasi-linear system (1) on the coefficients of the polynomial integral of motion is Rich in the generalized sense. The details will appear elsewhere.

5. Questions

Several questions are very natural:

1. It would be interesting to find more conceptual proof of the lemma in the framework differential-geometric approach by Dubrovin-Novikov [5].

2. How does generalized Hodograph method by Tsarev [12] work in our case?

3. How to analyze the behavior of the Riccati equation for the example of previous section? It seems that genuine non-linearity condition can not be expected for all eigenvalues.

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