On Multi-Dimensional Sonic-Subsonic Flow

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Abstract

In this paper, a compensated compactness framework is established for sonic-subsonic approximate solutions to the n-dimensional($n \geq 2$) Euler equations for steady irrotational flow that may contain stagnation points. This compactness framework holds provided that the approximate solutions are uniformly bounded and satisfy $H^{-1}_{loc}(\Omega)$ compactness conditions. As illustration, we show the existence of sonic-subsonic weak solution to n-dimensional($n \geq 2$) Euler equations for steady irrotational flow past obstacles or through an infinitely long nozzle. This is the first result concerning the sonic-subsonic limit for n-dimension($n \geq 3$).

1 Introduction

The n-dimensional($n \geq 2$) Euler equations for the steady irrotational flow reads

$$\begin{cases} \text{curl } u = 0, \\ \text{div}(\rho u) = 0, \\ \text{div}(\rho u \otimes u + pI) = 0, \end{cases} \tag{1.1}$$

where $u = (u_1, \cdots, u_n)$ are the flow velocities, and ρ and p represent the density and pressure-density function, respectively, and $(\text{curl } u)_{ij} = \partial_{x_j} u_i - \partial_{x_i} u_j, \ i, j = 1, \cdots, n$, is a $n \times n$ matrix and I is a $n \times n$ unit matrix. Usually, we require $p'(\rho) > 0$ for $\rho > 0$.

The famous Bernoulli's law can be easily derived from $(1.1)_1$ and $(1.1)_3$:

$$h(\rho) + \frac{1}{2} q^2 = const, \tag{1.2}$$

where $q^2 = |u|^2 = \sum_{i=1}^{n} u_i^2$ is the flow speed and $h(\rho) = \int_1^{\rho} \frac{p'(s)}{s} ds$ is the enthalpy.

In this paper, we are interested in the polytropic gas, that is $p = p(\rho) = \frac{\rho^\gamma}{\gamma}$, for $\gamma > 1$. Then (1.2) is converted to the normalized formula

$$\rho = \rho(q) = (1 - \frac{\gamma - 1}{2} q^2)^{\frac{1}{\gamma - 1}}. \tag{1.3}$$

The local sound speed is defined by

$$c^2 = p'(\rho) = 1 - \frac{\gamma - 1}{2} q^2. \tag{1.4}$$

At the sonic point $q = c$, (1.4) implies $q^2 = \frac{2}{\gamma+1}$. The critical speed q_{cr} is defined as

$$q_{cr} = \sqrt{\frac{2}{\gamma + 1}},$$

and the Bernoulli's law is rewritten as

$$q^2 - q_{cr}^2 = \frac{2}{\gamma+1}(q^2 - c^2).$$

Thus the flow is subsonic when $q < q_{cr}$, sonic when $q = q_{cr}$, and supersonic when $q > q_{cr}$.

For the isothermal flow, $p = \bar{c}^2\rho$, where $\bar{c} > 0$ is the constant sound speed, Bernoulli's law takes:

$$\rho = \rho(q) = \rho_0 \exp\left(-\frac{q^2}{2\bar{c}^2}\right) \tag{1.5}$$

for some constant $\rho_0 > 0$.

It is well known that the steady irrotational Euler equations (1.1) is of mixed type of partial differential equations, which is elliptic if $q < q_{cr}$, parabolic if $q = q_{cr}$, hyperbolic if $q > q_{cr}$. Since the equations of uniform subsonic flow possess ellipticity, its solutions have extra-smoothness to those related to transonic flow or supersonic flow. There are a large of literatures on the smooth uniform subsonic solutions, for instance, see [5],[26],[29],[44],[45] for two dimensional flow and [21],[22],[23],[27],[41],[48],[49] for three dimensional flow. Among them, Frankl and Keldysh [29] obtained the first result about the subsonic flow past a two dimensional finite body (or airfoil). Shiffman in [44], [45] proved there exists a unique subsonic potential flow around a given profile with finite energy provided that the infinite free stream flow speed q_∞ is less than some critical speed, which was improved by Bers [5]. Finn and Gilbarg [26] proved the uniqueness of the two dimensional potential subsonic flow past a bounded obstacle with given circulation and velocity at infinity. The first result for three dimensional subsonic flow past an obstacle was given by Finn and Gilbarg [27] in which they studied the existence, uniqueness and the asymptotic behavior with some restriction on Mach number. Dong [21] extended the results of Finn and Gilbarg [27] to maximum Mach number $M < 1$ and to arbitrary dimensions. Recently, Du, Xin and Yan [23] obtained the smooth uniform subsonic for n-dimensional($n \geq 2$) flow in an infinitely long nozzle. For other related results, we refer to [2, 3, 4, 6, 25, 28, 30, 31, 32] and references therein.

However, few result is known until now for the sonic-subsonic flow and transonic flow, because the uniform ellipticity is lost and shocks may present. That is, smooth solutions may not exist, and weak solution is necessarily considered. Morawetz [37, 38] firstly introduced the compensated compactness method to study steady flow of irrotational Euler equations. Indeed, Morawetz established a compactness framework under assumption that the stagnation points and cavitation points are excluded. Morawetz's result was improved by Chen, Slemrod and Wang [14] in which the approximate solutions are constructed by a viscous perturbation. On the other hand, the first compactness framework on sonic-subsonic irrotational flow in two dimension was recently due to Chen, Dafermos, Slemrod and Wang [13] by combining the mass conservation, momentum, and irrotational equations. The key point of [13] is based on the fact that the two dimensional steady flow can be regarded as the one dimensional system of conservation laws, that is, x is regarded as time t, so that the div-curl lemma can be applied to the two momentum equations. In fact, the authors [13] first applied the momentum equations to reduce the support of the corresponding Young measure to two points, then again used the irrotational equation and the mass equation to deduce the Young measure to a Dirac measure. As application, the two dimensional sonic-subsonic flow past an airfoil was obtained in [13]. Soon after, Xie and Xin [49] investigated the sonic-subsonic limit for the three-dimensional axis-symmetric flow(it is similar to the two dimensional case) through an axis-symmetric nozzle.

However, the compactness framework established in [13] is no longer effective for n-d ($n \geq 3$) steady irrotational Euler equations, which can not be reduced to one dimensional system of conservation laws, and the famous div-curl lemma is no longer valid for the momentum equations. In this paper, we find that it is enough, by only using the mass conservation equation and irrotational equations, to reduce the Young measure to a Dirac measure for arbitrary dimension. Thus we establish a compactness framework of approximate solutions for steady irrotational flow in n-dimension($n \geq 2$). It is worthy to point out that the famous Bernoulli's law plays a key role in our proof. As application, we show the sonic-subsonic limit for steady irrotational flows past obstacles or through an infinitely long nozzle in n-dimension($n \geq 2$).

The rest of this paper is organized as follows. In section 2, we establish the compactness framework of sonic-subsonic approximate solutions for the system of steady irrotational equations in n-dimension($n \geq 2$). In section 3, we give two applications of the compactness framework to show the existence of sonic-subsonic flow over obstacles or through an infinitely long nozzle.

2 Compensated Compactness Framework for Steady Irrotaitional Flow in n-Dimension

Let a sequence of function $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$, defined on open subset $\Omega \subset \mathbb{R}^n$, satisfy the following Conditions:

(A.1) $q^\varepsilon(x) = |u^\varepsilon(x)| \leq q_{cr}$ a.e. in Ω;

(A.2) curl u^ε, and $\operatorname{div}(\rho(q^\varepsilon)u^\varepsilon)$ are confined in a compact set in $H_{loc}^{-1}(\Omega)$.

Based on the above conditions, the famous div-curl lemma [24] and the Young measure representation theorem for a uniformly bounded sequence of functions imply:

$$< \rho(q)q^2, \nu(u) >= \sum_{i=1}^{n} < \rho(q)u_i, \nu(u) >< u_i, \nu(u) >, \tag{2.1}$$

where $\nu = \nu_x(u)$ is the associated Young measure (a probability measure) for the sequence $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$. Now, the main effort is to establish a compensated compactness framework, namely, to prove that ν is a Diac measure by using the identity (2.1). This in turn implies the compactness of the sequence $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$ in $L_{loc}^1(\Omega)$.

Theorem 2.1 *(Compensated compactness framework) Let a sequence of functions $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$ satisfy conditions (A.1) and (A.2). Then the associated Young measure ν is a Dirac mass and the sequence $u^\varepsilon(x)$ is compact in $L_{loc}^1(\Omega)$; that is, there is a subsequence (still labeled) $u^\varepsilon(x) \to u(x) = (u_1, \cdots, u_n)(x)$ a.e. as $\varepsilon \to 0$ and satisfying $q(x) = |u(x)| \leq q_{cr}$, a.e. $x \in \Omega$.*

Proof. Let

$$I(u^{(1)}, u^{(2)}) = \sum_{i=1}^{n} (u_i^{(1)} - u_i^{(2)})(\rho(q^{(1)})u_i^{(1)} - \rho(q^{(2)})u_i^{(2)}), \tag{2.2}$$

where $u^{(i)} = (u_1^{(i)}, \cdots, u_n^{(i)})$ and $q^{(i)} = |u^{(i)}|$ for $i = 1, 2$ be two independent vector variables.

After some basic calculations on $I(u^{(1)}, u^{(2)})$, we have

$$I(u^{(1)}, u^{(2)}) = \rho(q^{(1)})[(q^{(1)})^2 - \sum_{i=1}^{n} u_i^{(1)}u_i^{(2)}] + \rho(q^{(2)})[(q^{(2)})^2 - \sum_{i=1}^{n} u_i^{(1)}u_i^{(2)}]. \tag{2.3}$$

Then the Cauchy inequality implies

$$\begin{aligned} I(u^{(1)}, u^{(2)}) &\geq \rho(q^{(1)})[(q^{(1)})^2 - q^{(1)}q^{(2)}] + \rho(q^{(2)})[(q^{(2)})^2 - q^{(1)}q^{(2)}] \\ &= (q^{(1)} - q^{(2)})(\rho(q^{(1)})q^{(1)} - \rho(q^{(2)})q^{(2)}) \\ &= (q^{(1)} - q^{(2)})^2 \frac{d(\rho q)}{dq}(\tilde{q}). \end{aligned} \tag{2.4}$$

where $\tilde{q}$ lies between $q^{(1)}$ and $q^{(2)}$ due to the mean-value theorem. The famous Bernoulli's law (1.3) gives that for $\gamma > 1$,

$$\rho(q) = (1 - \frac{\gamma - 1}{2} q^2)^{\frac{1}{\gamma - 1}},$$

which immediately implies

$$\frac{d(\rho q)}{dq} = (1 - \frac{\gamma - 1}{2} q^2)^{\frac{1}{\gamma-1} - 1}(1 - \frac{\gamma + 1}{2} q^2) = (1 - \frac{\gamma - 1}{2} q^2)^{\frac{1}{\gamma-1} - 1}(1 - \frac{q^2}{q_{cr}^2}). \tag{2.5}$$

For $\gamma = 1$, the Bernoulli's law is

$$\rho(q) = \rho_0 \exp\Big(- \frac{q^2}{2q_{cr}^2} \Big),$$

which gives

$$\frac{d(\rho q)}{dq} = \rho_0 (1 - \frac{q^2}{q_{cr}^2}) \exp\Big(- \frac{q^2}{2q_{cr}^2} \Big). \tag{2.6}$$

Thus , for sonic-subsonic flows, namely, $q^{(1)}, q^{(2)} \leq q_{cr}$, (2.4) – (2.6) imply

$$I(u^{(1)}, u^{(2)}) = (q^{(1)} - q^{(2)})^2 \frac{d(\rho q)}{dq}(\tilde{q}) \geq 0 \tag{2.7}$$

and

$$(q^{(1)} - q^{(2)})^2 \frac{d(\rho q)}{dq}(\tilde{q}) = 0, \text{ if and only if } q^{(1)} = q^{(2)}. \tag{2.8}$$

From the identity (2.1), noticing that the Young measure ν is a probability measure, we have

$$< I(u^{(1)}, u^{(2)}), \nu(u^{(1)}) \otimes \nu(u^{(2)}) >= 0, \tag{2.9}$$

which together with (2.7) and (2.8) implies $q^{(1)} = q^{(2)}$, where $\nu(u^{(1)}) \otimes \nu(u^{(2)})$ is understood as a product measure of $\nu(u^{(1)})$ and $\nu(u^{(2)})$. With the property $q^{(1)} = q^{(2)}$ at hand, we have from (2.2)

$$\begin{aligned} 0 &= < I(u^{(1)}, u^{(2)}), \nu(u^{(1)}) \otimes \nu(u^{(2)}) > \\ &= < \rho(q^{(1)}) \sum_{i=1}^{n} (u_i^{(1)} - u_i^{(2)})^2, \nu(u^{(1)}) \otimes \nu(u^{(2)}) >, \end{aligned} \tag{2.10}$$

which immediately implies $u^{(1)} = u^{(2)}$, i.e, the Young measure is a Dirac measure. This completes Theorem 2.1.

Remark 2.2 *Theorem 2.1 is valid for any $n \geq 2$. Namely, a compactness framework in Theorem 2.1 is established for sonic-subsonic limit for steady irrotational flow in arbitrary dimension. From the Bernoulli's law* (1.2)*, it is straightforward to extend Theorem 2.1 to the general pressure-density function p satisfying $p'(\rho) > 0$ for $\rho > 0$.*

We now consider a sequence of approximate solutions u^ε to the Euler equations $(1.1)_1$, $(1.1)_2$ and the Bernoulli's law (1.3) or (1.5). That is, besides Conditions $(A.1)$ and $(A.2)$, the approximate solutions u^ε further satisfy

$$\begin{cases} \text{curl } u^\varepsilon = o_1(\varepsilon), \\ \text{div}(\rho(q^\varepsilon) u^\varepsilon) = o_2(\varepsilon), \end{cases} \tag{2.11}$$

where $o_1(\varepsilon), o_2(\varepsilon) \to 0$ in the sense of distributions as $\varepsilon \to 0$. Then, as a corollary of Theorem 2.1, we have

Theorem 2.3 *(Convergence of approximate solutions) Let* $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$ *be a sequence of approximate solutions satisfying* (2.11) *and the Bernoulli's law* (1.3) *or* (1.5)*. Then, there is a subsequence (still labeled)* $u^\varepsilon(x)$ *that converges a.e. as* $\varepsilon \to 0$ *to a weak solution* $u(x) = (u_1, \cdots, u_n)(x)$ *to the Euler equations of* $(1.1)_1$, $(1.1)_2$ *and the Bernoulli's law* (1.3) *or* (1.5) *satisfying* $q(x) = |u(x)| \leq q_{cr}$*, a.e.* $x \in \Omega$*.*

Remark 2.4 *For any functions* $Q(u) = (Q_1(u), \cdots, Q_n(u))$ *satisfying*

$$div(Q(u^\varepsilon)) = o(\varepsilon), \tag{2.12}$$

where $o(\varepsilon) \to 0$ *in the sense of distributions as* $\varepsilon \to 0$*, from the strong convergence of* u^ε*,* $Q(u) = 0$ *holds in the sense of distributions. So if we have*

$$div(\rho(q^\varepsilon)u^\varepsilon \otimes u^\varepsilon + p^\varepsilon I) = o(\varepsilon) \to 0, \text{ in the sense of distributions}, \tag{2.13}$$

the weak solution in Theorem 2.3 *also satisfies the momentum equations* $(1.1)_3$ *in the sense of distributions.*

There are various ways to construct approximate solutions by either numerical methods or analytical methods such as vanishing viscosity methods. In the next section, we will show two examples to apply the compactness framework built in Theorem 2.1.

3 Sonic Limit of Irrotational Subsonic Flows in n-Dimension

In this section, we wish to apply the compactness framework established in Theorem 2.1 to obtain the sonic limit of n-dimensional($n \geq 2$) steady irrotational subsonic flows.

Firstly, we give an example of subsonic-sonic limit past obstacles. Let the obstacle Γ be one or several closed $n-1(n \geq 2)$ dimensional hypersurfaces. We shall always assume $\Gamma \in C^{2,\tau_0}$. Denote by $\mathcal{D}(\Gamma)$ the domain exterior to Γ, see Fig 3.1,

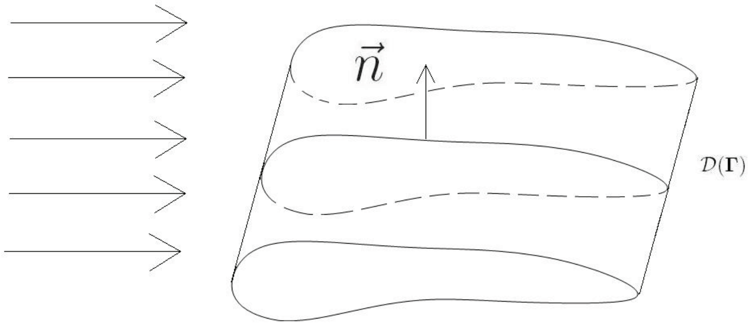

Figure 3.1: General high dimensional case.

Problem $\mathbf{P}(u_\infty)$: Let $n \geq 2$. Find functions $u = (u_1, \cdots, u_n)$ satisfy $(1.1)_1, (1.1)_2$ with the Bernoulli's law (1.3) or (1.5), and the slip boundary condition

$$u \cdot \vec{n} = 0 \ \text{ on } \ \Gamma, \tag{3.1}$$

where $\vec{n}$ denotes the unit inward normal of domain $\mathcal{D}(\Gamma)$, and the limit

$$u_\infty = \lim_{|x|\to\infty} u(x), \tag{3.2}$$

exists and is finite.

The main result of [21] is described as follows:

Theorem 3.1 (Uniform Subsonic Flows Past An Obstacle for n-D Case [21]) *Let $q_\infty := |u_\infty|$. There exists a positive number $\hat{q} < q_{cr}$, so that $\mathbf{P}(u_\infty)$ has a uniform subsonic solutions if $0 \le q_\infty < \hat{q}$. Furthermore, let $q_m(q_\infty) = \sup_{x \in \mathcal{D}(\Gamma)} |u(x)|$, then the function $q_m(q_\infty) \in C[0, \hat{q})$ and $q_m(q_\infty) \to q_{cr}$ as $q_\infty \to \hat{q}$.*

Theorem 3.2 (Sonic Limit Past An Obstacle) *Let $u^\varepsilon_\infty \to \hat{u}_\infty$ as $\varepsilon \to 0$ be a sequence of speeds at ∞ with $q^\varepsilon_\infty < \hat{q} = |\hat{u}_\infty|$, and $u^\varepsilon = (u^\varepsilon_1, \cdots, u^\varepsilon_n)$ be the corresponding solutions to Problem $\mathbf{P}(u^\varepsilon_\infty)$. Then, as $u^\varepsilon_\infty \to \hat{u}_\infty$, the solution sequence $u^\varepsilon(x)$ possess a subsequence (still denoted by) $u^\varepsilon(x)$ that converge a.e. in $\mathcal{D}(\Gamma)$ to a vector function $u(x) = (u_1, \cdots, u_n)(x)$ which is a weak solution of Problem $\mathbf{P}(u_\infty)$ with $q_\infty = \hat{q}$. Furthermore the limit velocity $u = (u_1, \cdots, u_n)$ also satisfies $(1.1)_3$ in the sense of distributions and the boundary conditions (3.1) as the normal trace of the divergence-measure field $(u_1, \cdots, u_n)$ on the boundary (see [11]).*

Proof. The strong solutions u^ε satisfy (1.1), and the Bernoulli's law and are uniform subsonic solutions of Problem $\mathbf{P}(u^\varepsilon_\infty)$. Hence Theorem 2.1 immediately implies that the Young measure is a Dirac mass and the convergence is strong a.e. in $\mathcal{D}(\Gamma)$. The boundary conditions are satisfied for u in the sense of Chen-Frid [11]. On the other hand, Since $(1.1)_3$ holds for the sequence of subsonic solutions $u^\varepsilon(x)$, it is straightforward to see that u also satisfies $(1.1)_3$ in the sense of distributions. This completes the proof of Theorem 3.2.

Now we give another example of subsonic-sonic limit through an infinite long nozzle. As in [23], denote the multi-dimensional nozzle domain by Ω which satisfies the following regularity assumption: there exists an invertible $C^{2,\alpha}$ map $T : \bar{\Omega} \to \bar{C} : x \to y$ satisfying

$$\begin{cases} T(\partial\Omega) = \partial C, \\ \text{For any } k \in \mathbb{R},\ T(\Omega \cap \{x_n = k\}) = B(0,1) \times \{y_n = k\}, \\ \|T\|_{C^{2,\alpha}}, \|T^{-1}\|_{C^{2,\alpha}} \le K < \infty, \\ \text{The nozzle approaches to a cylinder in the far fields}, i.e, \\ \Omega \cap \{x_n = k\} \to S_\pm \text{ as } k \to \pm\infty, \text{respectively}, \end{cases} \tag{3.3}$$

where K is a uniform constant, $C = B(0,1) \times (-\infty, +\infty)$ is a unit cylinder in $\mathbb{R}^n$, $B(0,1)$ is unit ball in $\mathbb{R}^{n-1}$ centered at the origin, $S_\pm$ are $n-1$ dimensional simply connected $C^{2,\alpha}$, x_n is the longitudinal coordinate and $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$, see Fig 3.2,

Problem $\tilde{\mathbf{P}}(m_0)$: Let $n \ge 2$. Find functions $u = (u_1, \cdots, u_n)$ satisfy $(1.1)_1, (1.1)_2$ with the Bernoulli's law (1.3) or (1.5), and the slip boundary condition

$$u \cdot \vec{n} = 0 \ \text{ on } \ \partial\Omega, \tag{3.4}$$

where $\vec{n}$ denotes the unit outward normal of domain Ω; and the mass flux condition

$$\int_{S_0} \rho(|u|^2) u \cdot \vec{l} dS = m_0 > 0 \tag{3.5}$$

The main result of [23] is stated as follows:

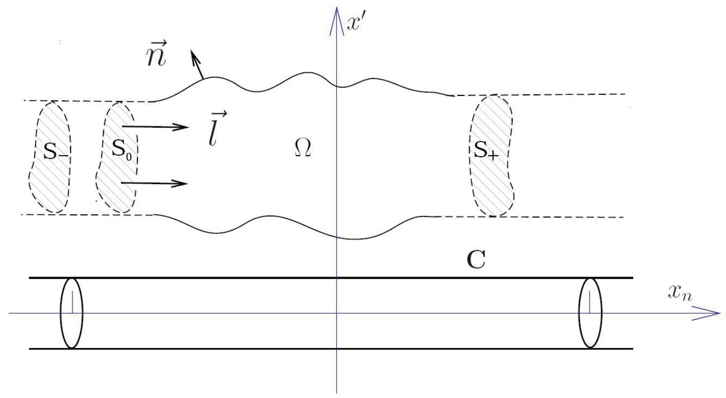

Figure 3.2: n-dimensional nozzle

Theorem 3.3 (Uniform Subsonic Flows in n-D Nozzle) *There is a critical mass flux $M_c > 0$, which depends only on Ω, such that if $0 < m_0 < M_c$, then $\tilde{\mathbf{P}}(m_0)$ has a unique uniformly subsonic flow through the nozzle, i.e, $q_m(m_0) := \sup_{x\in\Omega} |u(x)| < q_{cr}$. The velocity $(u_1, \cdots, u_n)$ is Holder continuous. Moreover, $q_m(m_0) \to q_{cr}$ as $m_0 \to M_c$.*

Similar to Theorem 3.2, we have

Theorem 3.4 (Sonic Limit Through A Nozzle) *Let $0 < m_0^\varepsilon < M_c$ be a sequence of mass flux, and let $u^\varepsilon(x)$ be the corresponding solution to $\tilde{\mathbf{P}}(m_0^\varepsilon)$. Then as $m_0^\varepsilon \to M_c$, the solution sequence $u^\varepsilon(x)$ possess a subsequence (still denoted by) $u^\varepsilon(x)$ that converge strongly a.e. in Ω to a vector function $u(x) = (u_1, \cdots, u_n)(x)$ which is a weak solution of $\tilde{\mathbf{P}}(M_c)$ with Bernoulli's law. Furthermore the limit velocity $u = (u_1, \cdots, u_n)$ also satisfies $(1.1)_3$ in the sense of distributions and the boundary conditions (3.4) as the normal trace of the divergence-measure field $(u_1, \cdots, u_n)$ on the boundary (see [11]).*

Remark 3.5 *In this section we only give two examples as applications of compactness framework established in Theorem 2.1. Certainly it can be used to other cases as long as conditions $(A.1)$ and $(A.2)$ are satisfied.*

Acknowledgments: The first author would like to thank Prof. Cathleen S.Morawetz for her kind hospitality and helpful discussion when he visited Courant Institute in 2006. The research of FMH was supported in part by NSFC Grant No. 10825102 for distinguished youth scholar, and National Basic Research Program of China (973 Program) under Grant No. 2011CB808002.

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