EXTENSIONS FOR SYSTEMS OF CONSERVATION LAWS

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Abstract. Entropies (convex extensions) play a central role in the theory of hyperbolic conservation laws by providing intrinsic selection criteria for weak solutions and local well-posedness for the Cauchy problem. While many systems occurring in physical models are equipped with extensions, it is well-known that existence of a non-trivial (i.e. non-linear) extension requires the solution of an over-determined system of equations. On the other hand, so-called rich systems are equipped with large sets of entropies. Beyond these general facts little seems to be known about "how many" extensions a particular system of conservation laws has.

For a given hyperbolic system $u_t + f(u)_x = 0$, a standard approach is to analyze directly the second order PDE system for the extensions. Instead we find it advantageous to consider the equations satisfied by the lengths β^i of the right eigenvectors r_i of Df, as measured with respect to the inner product defined by an extension. For a given eigen-frame $\{r_i\}$ the extensions are determined uniquely, up to trivial affine parts, by these lengths.

This geometric formulation provides a natural and systematic approach to existence of extensions. By considering the eigen-fields r_i as prescribed our results automatically apply to all systems with the same eigen-frame. As a computational benefit we note that the equations for the lengths β^i form a first order algebraic-differential system (the β-system) to which standard integrability theorems can be applied. The size of the set of extensions follows by determining the number of free constants and functions present in the general solution to the β-system. We provide a complete breakdown of the various possibilities for 3×3-systems, as well as for rich frames in any dimension provided the β-system has trivial algebraic part. The latter case covers 2×2-systems, strictly hyperbolic rich systems of any size, and any rich system with an orthogonal eigen-frame.

Our analysis is relevant whenever there exists a non-trivial conservative system whose eigen-frame coincides with the given frame. This issue was analyzed by the authors in [25], where the problem was formulated in terms of another algebraic-differential system, the "λ-system," whose solutions provide the characteristic speeds (eigenvalues) of the resulting conservative systems. We investigate the relationships between the λ- and β-systems and recover standard results for symmetric systems (orthogonal frames). It turns out that despite close structural connections between the λ- and the β-system, there is no general relationship between the sizes of their solution sets.

We provide a list of examples that illustrate our results.

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1. Introduction, background, and discussion

1.1. **Notation and conventions.** Unless otherwise stated the following will be in force:

- $u = (u^1, \ldots, u^n)$ denotes a fixed coordinate system on an open domain $\Omega \subset \mathbb{R}^n$. The domain Ω is assumed to be smoothly contractible to a point.
- We denote the (i, j)-entry (i.e., the element in the ith row and the jth column) of a matrix A by A^i_j. Superscript ^{T} denotes transpose.
- All vectors and vector functions are assumed to be column vectors, except gradients of scalar maps $\phi : \mathbb{R}^n \to \mathbb{R}$ which are row vectors: $\nabla\phi = \big(\frac{\partial\phi}{\partial u^1}, \ldots, \frac{\partial\phi}{\partial u^n}\big) \in \mathbb{R}^{1\times n}$.
- The Hessian of ϕ is $D^2\phi = \big(\frac{\partial^2\phi}{\partial u^i \partial u^j}\big) \in \mathbb{R}^{n\times n}$. For a map $f : \Omega \to \mathbb{R}^n$ the Jacobian matrix of f is denoted by $Df = \big(\frac{\partial f^i}{\partial u^j}\big) \in \mathbb{R}^{n\times n}$. An $n \times n$-matrix $A(u)$ is a u-Hessian (u-Jacobian) if there is a map $\phi : \mathbb{R}^n \to \mathbb{R}$ ($f : \mathbb{R}^n \to \mathbb{R}^n$) such that $A(u) = D^2\phi(u)$ ($A(u) = Df(u)$). For emphasis we sometimes use a subscript to indicate the coordinates in which Hessian or Jacobian are computed: $A = D^2_u\phi$ ($A = D_u f$).
- An "inner product" is a symmetric, but not necessarily positive definite, 2-tensor. We also refer to symmetric $n \times n$-matrices as inner products on $\mathbb{R}^n$.
- Summation convention is *not* used.
- We write $\epsilon(i, j, k) = 1$ to mean $i \neq j \neq k \neq i$.
- For convenience, by "smooth" we mean C^k-smooth for some sufficiently large $k \geq 2$.

1.2. **Hyperbolic conservation laws and entropies.** Consider a system of n conservation laws in one spatial dimension

$$u_t + f(u)_x = 0\,, \qquad t,\, x \in \mathbb{R}\,, \tag{1.1}$$

where the unknown $u = u(t, x) = (u^1(t, x), \ldots, u^n(t, x))^T \in \mathbb{R}^n$ is the column vector of conserved quantities. The map $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is referred to as the flux function. We consider the case of *hyperbolic* systems for which the Jacobian $Df(u)$ is diagonalizable over $\mathbb{R}$ and with a basis of eigenvectors at each $u \in \Omega$. Many 1-dimensional systems (1.1) are derived from multi-dimensional models in continuum mechanics by assuming that $u(t, \cdot)$ varies along only one fixed spatial direction, [12]. The example par excellence is the 1-d compressible Euler equations that model uni-directional inviscid gas flow, [12, 18, 40]

It is well-known that solutions of the initial value problem for (1.1) generically develop discontinuities in finite time [12]. In the context of the Euler system this provides a model for gas-dynamical shocks, i.e. narrow transition regions where the flow suffers steep gradients. To progress the solution beyond shock formation it is necessary to admit weak (distributional) solutions. However, extending the solution space to include discontinuous functions introduces non-uniqueness: solutions are not unique within the full class of all weak solutions. A central issue is to regain uniqueness by imposing appropriate selection criteria.

Motivated by physical models it makes good sense to consider (1.1) as an idealization which discards higher order dissipative terms. One approach is to admit a solution $u(t,x)$ of (1.1) only if it can be realized as a vanishing viscosity limit of solutions $u^\varepsilon(t,x)$ to

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon\,, \qquad \text{as } \varepsilon \downarrow 0. \tag{1.2}$$

However, the construction and convergence of solutions to (1.2) is a notoriously hard problem in itself [5]. A more intrinsic approach, formalized by Kružkov [27] and Lax [28], is to impose so-called *entropy inequalities*. These are derived from (1.2) by considering the equation satisfied by $\eta(u^\varepsilon)$, where η is a given scalar field intended to generalize the physical entropy in gas-dynamics. Since the classical entropy is a convex function of the state variables (local thermodynamic equilibrium, [18]) we insist that the Hessian matrix $D^2\eta$ is positive semidefinite. Assuming for now that there is an associated entropy flux q, i.e. $\nabla q = \nabla\eta Df$, we obtain from (1.2) that

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \le \varepsilon\eta(u^\varepsilon)_{xx} \qquad (\varepsilon > 0)\,.$$

Assuming further that the viscous solutions u^ε do converge in a sufficiently strong sense to $u(t,x)$, we obtain an intrinsic selection criterion for (1.1): a weak solution $u(t,x)$ of (1.1) is said to be *admissible* provided it satisfies the entropy inequality

$$\eta(u)_t + q(u)_x \le 0 \qquad \text{(distributional sense)} \tag{1.3}$$

whenever (η, q) is a convex entropy pair [28]. While the issue of admissibility criteria is far from settled for multi-dimensional problems (see [12, 13, 15]), entropies are of central importance in the theory of hyperbolic conservation laws.

1.3. **Extensions and entropies for systems of conservation laws.** We start by making the following (not entirely standard) definition:

Definition 1.1. *Let (1.1) have a smooth flux function $f : \Omega \to \mathbb{R}^n$. A smooth scalar field $\eta : \Omega \to \mathbb{R}$ is an* extension *for (1.1) provided the map $u \mapsto \nabla\eta(u)Df(u)$ is the u-gradient of a scalar field $q(u)$. If so, $q(u)$ is the* flux *of the extension η. An extension $\eta(u)$ is a (strict)* entropy *for (1.1) provided it is a (strictly) convex function of the conserved quantities, i.e. the Hessian $D^2\eta(u)$ is (strictly) positive semi-definite on Ω.*

The following facts about extensions and entropies are well-known, [2, 6, 11, 12, 18–20, 28, 29, 33, 35, 42]:

(1) Any system (1.1) is equipped with *trivial extensions*, i.e. affine maps

$$\eta(u) = a \cdot u + b\,, \qquad a \in \mathbb{R}^n,\, b \in \mathbb{R} \tag{1.4}$$

with corresponding fluxes $q(u) = a^T f(u) + c$, $c \in \mathbb{R}$.

(2) For a single equation ($n = 1$) any scalar field is an extension.

(3) Strictly hyperbolic rich systems (Section 3.1), and in particular strictly hyperbolic systems of two conservation laws ($n = 2$), possess large (i.e. "rich") families of extensions. More precisely, given a strictly hyperbolic, rich system (1.1), a base point $\bar u \in \Omega$, and a choice of n functions of one variable each; then there is an extension η that reduces to each of the given functions along each Riemann coordinate curves through $\bar u$. See [11, 36, 42] and Theorem 3.1 below.

(4) An extension for (1.1) with $n \ge 3$ must satisfy an overdetermined system of differential constraints. Consequently one would expect that a "randomly" chosen system (1.1) would not possess many, if any, non-trivial extensions. On the other hand, many systems appearing in physical applications *are* equipped with extensions/entropies. The prime example is given by the Euler system for a medium with a convex equation of state (see Example 6.1 below).

(5) (Friedrichs-Lax [16]) If (1.1) is equipped with a strict entropy η, then it is Friedrichs symmetrizable: pre-multiplication of (1.1) by $D^2\eta$ yields a quasi-linear system with symmetric coefficient matrices. This implies energy estimates and the Cauchy problem for such systems is well-posed in appropriate Sobolev spaces [2].

(6) (Mock [33], Godunov [19]) If (1.1) is equipped with a strict entropy η, then the change of variables $u \mapsto v := \nabla_u\eta$ transforms (1.1) into a symmetric, conservative system in gradient form: $[\nabla\phi(v)]_t +$

$[\nabla\psi(v)]_x = 0$. Conversely, if (1.1) is ("conservatively") symmetrizable in this sense, then it possesses a strict entropy.

(7) The classes of extensions and entropies for the Euler system of compressible gas dynamics, have been determined (for general equations of state). See Example 6.1 and [20, 21, 34, 35].

(8) The entropy inequality (1.3) is related to other types of selection criteria such as the Lax-, Liu-, and entropy-rate criterion; see [12] for an overview.

(9) For strictly hyperbolic systems there is a general relationship between interaction coefficients, eigenvalues and eigenvectors of the system; see [37] and Observation 2.12 below.

The main goal of the present work is to gain a better understanding of "how many" extensions a system of conservation laws has. Except for the case of strictly hyperbolic rich systems, we are not aware of general results that provide detailed information about the size of the set of extensions. In this article we provide a geometric approach to this problem and consider the lengths of eigen-vectors as measured with respect to the inner product determined by the extensions, rather than the extensions themselves, as the primary unknowns. This leads to a natural formulation in terms of flat connections and opens up for the application of standard integrability theorems.

To give a precise formulation of the problem we first recall how the requirement of possessing an extension places restrictions on (1.1). By definition (1.1) is equipped with an extension η if and only if

$$\partial_i\big[\big(\nabla\eta Df\big)_j\big] = \partial_j\big[\big(\nabla\eta Df\big)_i\big] \qquad \forall i<j \qquad\qquad (\partial_i = \frac{\partial}{\partial u^i})\,. \tag{1.5}$$

Performing the differentiations we get the equivalent conditions that

$$\partial_i\big(\nabla\eta\big)\cdot\partial_j f = \partial_j\big(\nabla\eta\big)\cdot\partial_i f \qquad \forall i<j\,. \tag{1.6}$$

Thus, an extension must satisfy $\frac{n(n-1)}{2}$ second order (linear, variable-coefficients) differential equations. In general these cannot be satisfied in a nontrivial manner (i.e. by a nonlinear field η) unless $n=1$ or $n=2$.

We assume that the system (1.1) is hyperbolic with a basis $\{R_i(u)\}_{i=1}^n$ of right eigenvectors of Df with corresponding eigenvalues $\{\lambda^i(u)\}_{i=1}^n$. The requirements (1.6) are then equivalent to the requirement that the matrix $D^2\eta\, Df$ is symmetric and therefore,

$$\big(R_i^T D^2\eta\big)\big(DfR_j\big) = \big(R_j^T D^2\eta\big)\big(DfR_i\big) \qquad \forall i<j\,,$$

Since $DfR_i = \lambda^i\, R_i$ for $i=1,\dots,n$, we thus require

$$\text{for each pair } 1\le i\neq j\le n, \text{ either} \quad \lambda^j=\lambda^i \quad \text{or} \quad R_i^T\big(D^2\eta\big)R_j = 0\,. \tag{1.7}$$

Also, by hyperbolicity, convexity of η is equivalent to

$$R_i^T\big(D^2\eta\big)R_i \ge 0 \qquad \forall i=1,\dots,n\,. \tag{1.8}$$

Summing up we have:

Proposition 1.2. *Given a hyperbolic system (1.1) such that Df has right eigenvectors $\{R_i(u)\}_{i=1}^n$ and eigenvalues $\{\lambda^i(u)\}_{i=1}^n$. Then η is an extension for (1.1) if and only if (1.7) holds, and it is an entropy if and only if (1.7) and (1.8) hold. If (1.1) is* strictly *hyperbolic (i.e., $\lambda^i(u)\neq\lambda^j(u)$ for all $u\in\Omega$), then η is an extension for (1.1) if and only if $\{R_i(u)\}_{i=1}^n$ is orthogonal with respect to the inner product $D^2\eta$.*

1.4. **Discussion and outline.** It is clear from Proposition 1.2 that the eigen-frame $\mathfrak{R} := \{R_1,\dots,R_n\}$ plays a central role in admitting or preventing non-trivial extensions for (1.1). Our main goal is to analyze, *in terms of the frame* $\mathfrak{R}$, how large the class of extensions is. We will therefore prescribe the frame $\mathfrak{R}$, and then determine how many scalar fields η have the property that $\mathfrak{R}$ is orthogonal with respect to $D^2\eta$. As detailed below, this problem leads to an over-determined algebraic-differential system, which we call the "β-system."

Remark 1.3. *The β-system seems to be derived for the first time by Conlon and Liu (*[11]*, Section 2). These authors work in the setting of a given, strictly hyperbolic system, and record the fact that systems with a coordinate system of Riemann invariants have rich families of extensions and entropies. The latter fact was observed independently by Tsarev* [42]*; see also* [36] *and Theorem 3.1 below. We are not aware of further results on how many extensions there are in the absence of a coordinate system of Riemann invariants.*

The unknowns in the β-system are the lengths of the frame-vectors R_i as measured with respect to the inner-product $D^2\eta$. The Hessian matrix $D^2\eta$ is determined from these lengths and the given frame $\mathfrak{R}$. In turn, the actual extensions η are determined from $D^2\eta$ by solving a system of $\frac{n(n+1)}{2}$ linear, second order PDEs, which amounts to successive integration of $n(n+1)$ first order linear ODEs (see Remark 2.4).

Our primary concern in the present work is to analyze the size of the solution set of the β-system. (The issue of existence of *entropies*, i.e. whether the β-system admits solutions with all $\beta^i > 0$, will be pursued elsewhere.) After introducing an appropriate geometric setup we can employ standard integrability theorems (Frobenius, Darboux, Cartan-Kähler) to analyze the size of the solution set. The answer will specify how many free constants and functions appear in a general solution, and thus provides an answer to how "rich" the class of extensions is. This geometric approach appears more convenient than a direct analysis of the second order system (1.6) for the extensions η themselves.

Of course, having prescribed the frame $\mathfrak{R}$, an immediate issue is whether there *are* systems (1.1) with $\mathfrak{R}$ as their eigen-frame. This question was studied by the authors in [25] where it was formulated in terms of another over-determined algebraic-differential system, the "λ-system", whose unknowns are the eigenvalues-to-be in a corresponding system (1.1). It is a non-obvious fact that there are frames $\mathfrak{R}$ with the property that the only systems (1.1) with eigen-frame $\mathfrak{R}$, are trivial systems of the form $u_t + \bar{\lambda}u_x = 0$, $\bar{\lambda} \in \mathbb{R}$. See Examples 6.5 and 6.14.

It is important to note that a given frame $\mathfrak{R}$ can give rise to a family of systems (1.1) which may include both strictly-hyperbolic and non strictly-hyperbolic systems. From (1.7) we see that for a strictly hyperbolic system of conservation laws with eigenframe $\mathfrak{R}$, the solutions of the β-system give rise to all possible extensions. On the other hand, for a non-strictly hyperbolic system (1.1) the solutions of the β-system may produce only a proper subset of all extensions, since the β-system does not account for extensions that are due to coalescing eigenvalues. (See Example 6.3 and Remark 2.8 for more details.)

The paper is organized as follows. In Section 2.1, we formulate the problem of finding extensions η such that a given frame is orthogonal with repsect to the inner product $D^2\eta$ (see the second part of (1.7)). This leads to the β-system in Section 2.2. In Section 2.3, we review briefly the problem of constructing conservative systems (1.1) with a prescribed eigen-frame (analyzed in [25]), and record the corresponding λ-system. Section 2.4 details the relationships between the two systems. Not surprisingly there are close connections, but also important differences, between the λ- and the β-systems.

In Section 2.5 we review relevant facts about connections on frame bundles and provide coordinate-free formulations of the λ- and the β-systems. This general framework reveals the geometric structure that underlies the problem. This geometric structure immediately provides us with important identities on the coefficients of the λ- and the β-systems that play important part in deriving compatibility conditions for these systems. It also readily shows how the systems behave under changes of coordinates. This is useful for example when formulating the β-system for rich systems in the associated Riemann coordinates (see Section 3.1).

It turns out that the analysis of the β-system is more involved than for the λ-system. We first consider the case when $\beta(\mathfrak{R})$ contains no algebraic part (Section 3). The corresponding systems (1.1) are then necessarily rich. (This covers all 2×2-systems as well as rich systems with orthogonal eigen-frames.) We then treat systems of three equations (Sections 4 and 5), and give a complete analysis for both rich and non-rich systems through a breakdown similar to what was done for the λ-system in [25]. Finally, in Section 6, we consider several examples that illustrate our approach and results. In particular, we treat the case when the frame $\mathfrak{R}$ is that of the Euler system.

We have developed MAPLE code[1] to calculate solutions of the λ- and β-systems.

2. Problem formulation; β-system and λ-system

2.1. **Problem formulation.** By definition, a *frame* $\mathfrak{R}$ on Ω is a collection of smooth vector fields $\{r_1, \dots, r_n\}$ which are linearly independent at each point of Ω. Its dual *co-frame* is $\mathfrak{L} := \{\ell^1, \dots, \ell^n\}$ where the differential 1-forms ℓ^i satisfy $\ell^i(r_j) = \delta^i_j$.

[1] Posted on `http://www.math.ncsu.edu/~iakogan/symbolic/geometry_of_conservation_laws.html`

Let $u^1, \ldots, u^n$ be a fixed coordinate system on Ω. By the *u-representation* of $\mathfrak{R}$ we mean the collection $\{R_1(u), \ldots, R_n(u)\}$ of column vectors $R_i = [R_i^1, \ldots, R_i^n]^T$ given by

$$r_i\big|_u = \sum_{k=1}^n R_i^k(u) \frac{\partial}{\partial u^k}\Big|_u \qquad\qquad i = 1, \ldots, n\,.$$

We let $R(u)$, $L(u) \in \mathbb{R}^{n\times n}$ be given by

$$R(u) := [R_1(u) \,|\, \cdots \,|\, R_n(u)] \qquad \text{and} \qquad L(u) := R^{-1}(u) = \begin{bmatrix} L^1(u) \\ \hline \vdots \\ \hline L^n(u) \end{bmatrix}. \tag{2.1}$$

Then

$$\ell^i\big|_u = \sum_{m=1}^n L_m^i(u) du^m\big|_u\,, \tag{2.2}$$

and we define the $(n \times 1)$-vector of 1-forms ℓ by

$$\ell := \begin{bmatrix} \ell^1 \\ \vdots \\ \ell^n \end{bmatrix} = L du\,. \tag{2.3}$$

We refer to $L(u)$, or $\{L^1(u), \ldots, L^n(u)\}$, as the *u-representation* of the dual co-frame $\mathfrak{L}$.

Given $\mathfrak{R}$ we would like to find all scalar fields $\eta(u)$ that satisfy the second condition in (1.7):

$$R_i(u)^T \big(D_u^2 \eta(u)\big) R_j(u) = 0 \quad \text{for each pair } 1 \le i \ne j \le n. \tag{2.4}$$

If $\mathfrak{R}$ is an eigen-frame for the flux f of a conservative system (1.1), then such an η is an extension of (1.1).

Problem 1. *Let $\mathfrak{R}$ be a given frame on Ω. Find all scalar fields $\eta(u)$ defined on a neighborhood of a point $\bar{u} \in \Omega$, such that the frame $\mathfrak{R}$ is orthogonal with respect to the inner product defined by the Hessian matrix $D_u^2\eta$.*

Remark 2.1. *Note that if $D_u^2\eta$ is strictly positive definite, then it defines a positive definite (Riemannian) metric on an open subset of $\mathbb{R}^n$, that can be locally expressed as the Hessian of a smooth function. Such metrics are called* Hessian metrics *(see Remark 2.14 for a coordinate free definition of a Hessian metric). They were introduced as real analogs of Kählerian metrics on complex manifolds and have been extensively studied, see* [39] *and references therein. The problem stated above, however, does not seem to appear in the literature.*

We proceed to show that an extension η is completely determined by the values

$$\beta^i(u) := R_i(u)^T \big(D_u^2 \eta(u)\big) R_i(u) \qquad i = 1, \ldots, n, \tag{2.5}$$

and reformulate our problem accordingly. From (2.4) and (2.5) we have that

$$R^T \big(D_u^2 \eta\big) R = \operatorname{diag}[\beta^1, \ldots, \beta^n]\,.$$

Recalling (2.1), we thus obtain an equivalent condition that

$$D_u^2 \eta = L^T \operatorname{diag}[\beta^1, \ldots, \beta^n]\, L\,. \tag{2.6}$$

We now recall two well-known consequences of Poincaré's Lemma [41]. The first result will be used immediately, while the second is used in the following sections.

Proposition 2.2. *A u-Jacobian on Ω is symmetric if and only if it is a u-Hessian.*

Proposition 2.3. *An $n \times n$-matrix $A(u)$ defined on Ω is a u-Jacobian if and only if*

$$dA(u) \wedge du = 0\,. \tag{2.7}$$

Combining (2.6) with Proposition 2.2 we conclude that η is an extension for a conservative system with eigen-frame $\mathfrak{R}$ provided

$$L^T \operatorname{diag}[\beta^1, \dots, \beta^n] L \qquad \text{is a } u\text{-Jacobian.} \tag{2.8}$$

Considering the frame $\mathfrak{R}$ as given, we view (2.8) as a condition on $\beta^1, \dots, \beta^n$, and obtain the following reformulation of Problem 1.

Problem 2. *Let $\mathfrak{R}$ be a given frame defined near $\bar{u} \in \Omega$, and let the matrix $L(u)$ be the u-representation of the dual co-frame $\mathfrak{L}$, as in (2.1). Find n scalar fields $\beta^1(u), \dots, \beta^n(u)$ defined on a neighborhood $\mathcal{U} \subset \Omega$ of $\bar{u}$ such that with*

$$\mathcal{B} := \operatorname{diag}[\beta^1, \dots, \beta^n]\,, \tag{2.9}$$

the symmetric matrix

$$L^T(u)\mathcal{B}(u)L(u) \qquad \text{is the u-Jacobian of some map } \Psi : \mathcal{U} \to \mathbb{R}^n. \tag{2.10}$$

We are further interested in how large the set of solutions is, i.e. how many arbitrary constants and functions appear in a general solution $\beta^1(u), \dots, \beta^n(u)$.

In Section 2.2 we derive a system of differential and algebraic equations that is equivalent to condition (2.10). The following remark outlines how one can recover the solution of Problem 1 from the solution of Problem 2.

Remark 2.4. *Condition (2.6) provides the link between Problem 1 and Problem 2. The right-hand side of (2.6) can be computed from a given frame $\mathfrak{R}$ and $\beta^1, \dots, \beta^n$. By symmetry of the matrices involved, (2.6) provides $\frac{n(n-1)}{2}$ linear second order PDEs for the scalar field η.*

Vice versa, given the solution of Problem 2, we can recover η by successively solving $n^2 + n$ ODEs. Indeed, $L^T(u)\mathcal{B}(u)L(u)$ is the Jacobian of a map $\Psi = (\Psi^1, \dots, \Psi^n) : \mathcal{U} \to \mathbb{R}^n$. This means that the i-th row of $L^T(u)\mathcal{B}(u)L(u)$ is the gradient of the function Ψ^i, $i = 1, \dots, n$. Recovering a function of n variables from its gradient requires successive integration of n ODEs. Therefore, we can recover $\Psi = (\Psi^1, \dots, \Psi^n)$ in (2.10) by solving n^2 ODEs. Also, by Proposition 2.2 we have that

$$(\Psi^1, \dots, \Psi^n) = \nabla\eta \qquad \text{for some scalar field } \eta.$$

From this we recover η by successively solving n ODEs. Thus, all in all we can recover η from $L^T(u)\mathcal{B}(u)L(u)$ by integrating of $n^2 + n$ ODEs. Finally, a function is recovered from its gradient uniquely up to an additive constant. Therefore, the above steps allow us to uniquely recover η from $\beta^1, \dots, \beta^n$, up to addition of a trivial extension (1.4). Examples are given in Section 6.

2.2. **The β-system.** Applying Proposition 2.3, we conclude that $L^T\mathcal{B}L$ is a Jacobian (condition (2.10)) if and only if $\beta = (\beta^1, \dots, \beta^n)$ satisfies the following *β-system*:

$$\beta(\mathfrak{R}) : \qquad d[L^T\mathcal{B}L] \wedge du = 0 \qquad (\mathcal{B} \text{ given by (2.9)}). \tag{2.11}$$

We proceed to rewrite (2.11) as an algebraic-differential system. Define

$$\Gamma^k_{ij} := L^k(DR_j)R_i \qquad \text{and} \qquad c^k_{ij} := \Gamma^k_{ij} - \Gamma^k_{ji} \tag{2.12}$$

(see Section 2.5 for the geometric meaning of these functions). By applying the product rule to (2.11), multiplying by R^T from the left, and using that $(dL)R = -L(dR)$ we obtain that (2.11) is equivalent to

$$d\mathcal{B} \wedge Ldu = \Big[\mathcal{B}L(dR) + \big(\mathcal{B}L(dR)\big)^T\Big] \wedge L\,du\,, \tag{2.13}$$

which may be re-written as

$$d\mathcal{B} \wedge \ell = \Big[\mathcal{B}\mu + \big(\mathcal{B}\mu\big)^T\Big] \wedge \ell, \tag{2.14}$$

where ℓ is given by (2.3) and the matrix μ is given by

$$\mu^k_j := \big(LdR\big)^k_j = \sum_{i=1}^n \Gamma^k_{ij}\ell^i\,.$$

Applying (2.14) to pairs of frame vectors (r_i, r_j) we obtain an explicit formulation of $\beta(\mathfrak{R})$:

$$r_i(\beta^j) = \beta^j\,(\Gamma^j_{ij} + c^j_{ij}) - \beta^i\,\Gamma^i_{jj} \qquad \text{for } i \neq j, \tag{2.15}$$

$$\beta^k\, c^k_{ij} + \beta^j\Gamma^j_{ik} - \beta^i\,\Gamma^i_{jk} = 0 \qquad \text{for } i < j,\ \epsilon(i,j,k) = 1, \tag{2.16}$$

where there are no summations. Note that (2.15) gives $n(n-1)$ linear, homogeneous PDEs, while (2.16) gives $\frac{n(n-1)(n-2)}{2}$ algebraic relations. We observe that the left-hand side of (2.16) is skew-symmetric in i and j, and that all coefficients Γ^k_{ij} with $\epsilon(i,j,k) = 1$ appear in (2.16). We proceed with some simple but important properties of the β-system.

Observation 2.5. *(Trivial solutions) We observe that $\beta(\mathfrak{R})$ always has the trivial solution $\beta^1 = \cdots = \beta^n = 0$, which corresponds to the trivial, affine extensions in (1.4).*

Definition 2.6. *A solution $\beta = (\beta^1, \ldots, \beta^n)$ of the β-system (2.11) is called* non-degenerate *in Ω provided $\beta^i(u) \neq 0$ for all $i = 1, \ldots, n$, $u \in \Omega$.*

Observation 2.7. *(The effect of scaling) Given smooth functions $\alpha^j : \Omega \to \mathbb{R} \setminus \{0\}$, $1 \leq j \leq n$, we scale the given frame and its dual frame according to:*

$$\tilde{R}_j(u) = \alpha^j(u) R_j(u) \qquad \textit{and} \qquad \tilde{L}^j(u) := \alpha^j(u)^{-1} L^j(u)\,.$$

Letting $\tilde{R} := R\alpha$, $\tilde{L} := \alpha^{-1}L$, where $\alpha(u) = \mathrm{diag}[\alpha^1(u) \ldots \alpha^n(u)]$, we may consider the β-system $\beta(\tilde{\mathfrak{R}})$. Clearly,

$$L^T(u)\mathcal{B}(u)L(u) \quad \textit{is a Jacobian if and only if} \quad [\alpha(u)\tilde{L}(u)]^T\mathcal{B}(u)\alpha(u)\tilde{L}(u) \quad \textit{is a Jacobian.}$$

Therefore, $\beta = (\beta^1, \ldots, \beta^n)$ solves $\beta(\mathfrak{R})$ if and only if $\tilde{\beta} = ((\alpha^1)^2\beta^1, \ldots, (\alpha^n)^2\beta^n)$ solves $\beta(\tilde{\mathfrak{R}})$. However, while the solution set of the β-system is affected by scalings in this way, the two systems $\beta(\mathfrak{R})$ and $\beta(\tilde{\mathfrak{R}})$ generate the same class of extensions. This is clear from (2.6) since $\tilde{L} := \alpha^{-1}L$.

2.3. **The λ-system.** A solution of the $\beta(\mathfrak{R})$-system provides an extension for any conservative systems (1.1) whose Jacobian $D_u f$ has eigen-frame $\mathfrak{R}$. This raises a natural question: *What is the class of flux functions $f(u)$ with the property that the set of eigenvectors of Df coincide with $\mathfrak{R}$?* This problem was treated by the authors in [25], and we briefly review some results from that paper. (A complete breakdown for the case $n = 3$ is recorded in Proposition 4.1.)

Given n scalar fields $\lambda^1(u), \ldots, \lambda^n(u) : \Omega \to \mathbb{R}$, we define the diagonal matrix

$$\Lambda(u) := \mathrm{diag}[\lambda^1(u), \ldots, \lambda^n(u)]\,.$$

The λ-*system* $\lambda(\mathfrak{R})$ associated with a frame $\mathfrak{R}$ is the system of equations which encodes that the matrix $A(u) := R(u)\Lambda(u)L(u)$ is the Jacobian matrix of some map $f : \Omega \to \mathbb{R}^n$. The set of such maps provide all possible flux functions f in conservative systems (1.1) with $\mathfrak{R}$ as their eigen-frame. The unknowns of the λ-system are the eigenvalues $\lambda^1(u), \ldots, \lambda^n(u)$ of the resulting Jacobian $D_u f$. Proposition 2.3 gives the following formulation of the λ-system:

$$\lambda(\mathfrak{R}) : \qquad d[R\Lambda L] \wedge du = 0\,. \tag{2.17}$$

Evaluating (2.17) on pairs (r_i, r_j) of frame vector fields we obtain the following algebraic-differential system, whose coefficients are given by (2.12):

$$r_i(\lambda^j) \;=\; \Gamma^j_{ji}(\lambda^i - \lambda^j) \qquad \text{for } i \neq j, \tag{2.18}$$

$$(\lambda^i - \lambda^k)\Gamma^k_{ji} \;=\; (\lambda^j - \lambda^k)\Gamma^k_{ij} \qquad \text{for } i < j,\ \epsilon(i,j,k) = 1, \tag{2.19}$$

where there are no summations. Note that (2.18) gives $n(n-1)$ linear, homogeneous PDEs, while (2.19) gives $\frac{n(n-1)(n-2)}{2}$ linear algebraic relations, with coefficients Concerning scaling of $\mathfrak{R}$ (cf. Observation 2.7) we have that the solution set of the λ-system is unaffected by scaling. Indeed, since $\tilde{R}\Lambda\tilde{L} \equiv R\Lambda L$, λ solves $\lambda(\tilde{\mathfrak{R}})$ if and only if λ solves $\lambda(\mathfrak{R})$.

As is clear from (2.17), $\lambda(\mathfrak{R})$ always has a one-parameter family of *trivial solutions* given by $\Lambda(u) \equiv \bar{\lambda}I_{n\times n}$, $\bar{\lambda} \in \mathbb{R}$, corresponding to diagonal affine fluxes

$$f(u) = \bar{\lambda}u + \bar{u}. \tag{2.20}$$

There are frames for which the λ-system has only trivial solutions, and there are frames for which the λ-system has a large family of non-trivial solutions; see [25].

Remark 2.8. *Note that even when a frame $\mathfrak{R}$ admits strictly hyperbolic fluxes (i.e. there is a solution of $\lambda(\mathfrak{R})$ such that $\lambda^1(u), \ldots, \lambda^n(u)$ are distinct $\forall u \in \Omega$), it also gives rise to non-strictly hyperbolic systems - in particular the trivial fluxes (2.20). The β-system (2.11) produces all extensions for strictly hyperbolic fluxes that correspond to $\mathfrak{R}$, but, in general, it does not produce all extensions for corresponding non-strictly hyperbolic fluxes. The extensions for fluxes with coalescing eigenvalues, which satisfy the first part of (1.7), are not necessarily covered by the β-system.*

2.4. **Relation between the λ- and the β-systems.** By restricting ourselves to searching for those extensions η that satisfy (1.7) due to orthogonality, we have separated the problem of finding fluxes that correspond to the frame $\mathfrak{R}$, from the problem of finding extensions that correspond to the same frame. Indeed, no unkonwn of $\beta(\mathfrak{R})$ appears in $\lambda(\mathfrak{R})$ or vice versa. The two systems can therefore be solved independently of each other.

We note that both systems $\beta(\mathfrak{R})$ and $\lambda(\mathfrak{R})$ encode the property that a matrix is a u-Jacobian. Also, the coefficients of both systems are computed from the u-representation of the given frame $\mathfrak{R}$. Nonetheless, we will show in Section 4 that there is in general no relationship between the sizes of the solution sets of $\beta(\mathfrak{R})$ and $\lambda(\mathfrak{R})$. We observe that *orthogonal* frames provide a notable exception:

Observation 2.9. *(Orthonormal frames) Comparing the β-system (2.11) and the λ-system (2.17) we see that if $\mathfrak{R}$ is orthonormal relative to the standard inner product, then $R = L^T$ and the two systems coincide. By Proposition 2.2 we deduce that the corresponding conservative systems (1.1) are* gradient systems*: their flux function is the gradient of an extension, $f = (\nabla\eta)^T$.*

When the frame $\mathfrak{R}$ is orthogonal the solutions to the λ- and β-systems are related by scaling, and the β-system necessarily have non-trivial solutions:

Observation 2.10. *(Orthogonal frames) Assume $\mathfrak{R}$ is an orthogonal frame relative to the standard inner product, and let $\alpha^i = |R_i|^{-1}$, such that $\tilde{\mathfrak{R}} = \{\alpha^1 R_1, \ldots, \alpha^n R_n\}$ is the orthonormal scaling of $\mathfrak{R}$. From Observations 2.7 and 2.9, and the fact that the solution set of $\lambda(\tilde{\mathfrak{R}})$ coincides with that of $\lambda(\mathfrak{R})$, we obtain:*

$$\beta^1, \ldots, \beta^n \text{ solves } \beta(\mathfrak{R}) \text{ if and only if } (\alpha^1)^2\,\beta^1, \ldots, (\alpha^n)^2\,\beta^n \text{ solves } \lambda(\mathfrak{R}). \tag{2.21}$$

It follows from this that $\beta(\mathfrak{R})$ has non-trivial solutions. Indeed, $\lambda(\mathfrak{R})$ has a one-parameter family of trivial solutions $\lambda^1 = \cdots = \lambda^n \equiv \bar{\lambda} \in \mathbb{R}$. However, as solutions to $\beta(\tilde{\mathfrak{R}}) \equiv \lambda(\tilde{\mathfrak{R}})$, these are non-trivial solutions. They provide the extensions $\eta(u) = \frac{1}{2}\bar{\lambda}\,|u|^2$, which (according to the last part of Observation 2.7) are also extensions corresponding to the original, un-scaled frame $\mathfrak{R}$. In particular, we obtain the well-known fact that if (1.1) is a gradient system with flux $f = (\nabla\eta)^T$, then it has η as an extension and $|u|^2$ as a strict entropy ([12,16,19]*).*

We finally note that any *other solution of $\lambda(\mathfrak{R})$ also provides an extension of the original gradient system $u_t + f(x)_x = 0$ (see Example 6.4).*

If $n = 2$ neither the β-system nor the λ-system has an algebraic part. For $n = 3$ we shall see that the algebraic parts (2.19) and (2.16) have the same rank (Proposition 4.2). However, Example 6.10 illustrates that equality of ranks of the algebraic parts does *not* generalize to $n \geq 4$. The rank zero case provides an exception in this respect:

Observation 2.11. *The following three statements are equivalent*

1. $\Gamma^i_{jk} = 0$ *whenever* $\epsilon(i,j,k) = 1$.
2. *There are no algebraic conditions in the λ-system (2.18)-(2.19).*
3. *There are no algebraic conditions in the β-system (2.15)-(2.16).*

When the algebraic parts (2.16) and (2.19) of the β-system and the λ-system are non-trivial, we can easily derive, *under the assumption of strict hyperbolicity*, the following relationship between solutions of the β-system and the λ-system (see Proposition 8 in [37]):

Observation 2.12. *Under the assumption of strict hyperbolicity:*

$$\frac{c^i_{jk}\beta^i}{\lambda^j-\lambda^k}+\frac{c^j_{ki}\beta^j}{\lambda^k-\lambda^i}+\frac{c^k_{ij}\beta^k}{\lambda^i-\lambda^j}=0 \qquad \textit{whenever } i<j<k. \tag{2.22}$$

In the present setup this may be deduced directly from the algebraic parts of the λ- and β-systems: replace Γ^k_{ji} with $\Gamma^k_{ij}-c^k_{ij}$ in the algebraic part of λ system (2.19), and solve for Γ^k_{ij}

$$\Gamma^k_{ij}=\frac{\lambda^i-\lambda^k}{\lambda^i-\lambda^j}c^k_{ij}\,, \qquad \epsilon(i,j,k)=1.$$

Substituting

$$\Gamma^j_{ik}=\frac{\lambda^i-\lambda^j}{\lambda^i-\lambda^k}c^j_{ik} \qquad \textit{and} \qquad \Gamma^i_{jk}=\frac{\lambda^j-\lambda^i}{\lambda^j-\lambda^k}c^i_{jk}$$

in the algebraic part (2.16) of the β-system yields (2.22).

From the point of view of constructing relevant systems (1.1) from a given frame $\mathfrak{R}$, one would first solve $\lambda(\mathfrak{R})$ and then proceed to solve $\beta(\mathfrak{R})$, provided the former possesses non-trivial solutions. However, to highlight the fact that the solution sets of $\lambda(\mathfrak{R})$ and $\beta(\mathfrak{R})$ may be of different size, we have also included Example 6.14. The two frames in that example show that $\lambda(\mathfrak{R})$ may have only trivial solutions while $\beta(\mathfrak{R})$ possesses non-trivial solutions.

2.5. **Connections on frame bundles and a coordinate free formulation.** The coefficients Γ^k_{ij} that appear in both the β- and the λ-systems have a natural geometric interpretation as connection components (Christoffel symbols) of a flat and symmetric connection relative to the frame $\mathfrak{R}$. In this section we review the basic facts about connections on a frame bundle, which we will use to analyze the β-system. In Remark 2.14 we give a coordinate-free definition of a Hessian metric and indicate how the $\beta(\mathfrak{R})$-system can be obtained in a coordinate-free manner. Similarly, Remark 2.15 provides a coordinate-free definition of a Jacobian tensor and describes a coordinate-free approach to the $\lambda(\mathfrak{R})$-system. A comprehensive account of the differential geometry material used in this section can be found in [10], [30], and [39].

Given an n-dimensional smooth manifold M we let $\mathcal{X}(M)$ and $\mathcal{X}^*(M)$ denote the set of smooth vector fields and differential 1-forms on M, respectively. A *frame* $\mathfrak{R}=\{r_1,\dots,r_n\}$ is a set of vector fields which span the tangent space T_pM at each point $p\in M$. A *coframe* $\{\ell^1,\dots,\ell^n\}$ is a set of n differential 1-forms which span the cotangent space T^*_pM at each point $p\in M$. The coframe and frame are *dual* if $\ell^i(r_j)=\delta^i_j$ (Kronecker delta). If $u^1,\dots,u^n$ are local coordinate functions on M, then $\{\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^n}\}$ is the corresponding local *coordinate frame*, while $\{du^1,\dots,du^n\}$ is the dual local *coordinate coframe*. The *structure coefficients* c^k_{ij} of $\mathfrak{R}$ are defined by

$$[r_i,r_j]=\sum_{k=1}^n c^k_{ij}\,r_k\,, \tag{2.23}$$

and the dual coframe has related structure equations given by

$$d\ell^k=-\sum_{i<j}c^k_{ij}\,\ell^i\wedge\ell^j\,. \tag{2.24}$$

It can be shown (see Proposition 5.14 in [41]) that there exist coordinate functions $w^1,\dots,w^n$ on an open subset of Ω such that $r_i=\frac{\partial}{\partial w^i}$, $i=1\dots,n$, if and only if $r_1,\dots,r_n$ commute, i.e. all structure coefficients are zero. We recall the slightly weaker requirement of richness:

Definition 2.13. *The frame $\{r_i\}_{i=1}^n$ is* rich *provided its structure coefficients satisfy*

$$c^k_{ij}=0 \qquad \textit{whenever } \epsilon(i,j,k)=1. \tag{2.25}$$

We note that a rich frame may be scaled so as to yield a commutative frame [12].

An *affine connection* ∇ on M is an $\mathbb{R}$-bilinear map

$$\mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M) \qquad\qquad (X,Y)\mapsto\nabla_XY$$

such that for any smooth function f on M

$$\nabla_{fX}Y = f\nabla_X Y, \qquad \nabla_X(fY) = (Xf)Y + f\nabla_X Y\,. \tag{2.26}$$

By $\mathbb{R}$-bilinearity and (2.26) the connection is uniquely defined by prescribing it on a frame:

$$\nabla_{r_i} r_j = \sum_{k=1}^{n} \Gamma_{ij}^k r_k,$$

where the smooth coefficients Γ_{ij}^k are called *connection components, or Christoffel symbols,* relative to the frame $\{r_1, \ldots, r_n\}$. Any choice of a frame and n^3 functions Γ_{ij}^k, $i, j, k = 1, \ldots, n$, defines an affine connection on M. A change of frame induces a change of the connection components, and this change is not tensorial. E.g., a connection with zero components relative to a coordinate frame, may have non-zero components relative to a non-coordinate frame.

A connection uniquely defines *a covariant derivative* $\nabla_X T$ of any tensor field T on M in the direction of a vector field X (see for instance Section 1.2 of [39]).

Given a frame $\{r_1, \ldots, r_n\}$ with associated Christoffel symbols Γ_{ij}^k and the dual frame $\{\ell^1, \ldots, \ell^n\}$, we define the *connection 1-forms* μ_i^j by

$$\mu_i^j := \sum_{k=1}^{n} \Gamma_{ki}^j \ell^k\,. \tag{2.27}$$

In turn, these are used to define the *torsion* 2-forms

$$\mathbf{T}^i := d\ell^i + \sum_{k=1}^{n} \mu_k^i \wedge \ell^k = \sum_{k<m} T_{km}^i \ell^k \wedge \ell^m, \qquad i = 1, \ldots, n\,, \tag{2.28}$$

and the *curvature* 2-forms

$$\mathbf{R}_i^j := d\mu_i^j + \sum_{k=1}^{n} \mu_k^j \wedge \mu_i^k = \sum_{k<m} R_{i\,km}^j \ell^k \wedge \ell^m\,. \tag{2.29}$$

The second equalities of (2.28) and (2.29) define the components of torsion and curvature tensors, respectively:

$$T_{km}^i = \Gamma_{km}^i - \Gamma_{mk}^i - c_{km}^i \tag{2.30}$$

$$R_{i\,km}^j = r_k\big(\Gamma_{mi}^j\big) - r_m\big(\Gamma_{ki}^j\big) + \sum_{s=1}^{n} \big(\Gamma_{ks}^j \Gamma_{mi}^s - \Gamma_{ms}^j \Gamma_{ki}^s - c_{km}^s \Gamma_{si}^j\big). \tag{2.31}$$

We can write equations (2.28) and (2.29) in the compact matrix form

$$\mathbf{T} = d\ell + \mu \wedge \ell, \quad \mathbf{R} = d\mu + \mu \wedge \mu \tag{2.32}$$

where $\mathbf{T} = (\mathbf{T}^1, \ldots, \mathbf{T}^n)^T$, and $\mathbf{R}$ and μ are the matrices with components $\mathbf{R}_i^j$ and μ_i^j, respectively. The connection is called *symmetric* if the torsion form is identically zero and it is called *flat* if the curvature form is identically zero. Equivalently:

$$d\ell = -\mu \wedge \ell \qquad \text{(Symmetry)}, \qquad d\mu = -\mu \wedge \mu \qquad \text{(Flatness)}. \tag{2.33}$$

In terms of Christoffel symbols and structure coefficients this is equivalent to

$$c_{km}^i = \Gamma_{km}^i - \Gamma_{mk}^i \qquad \text{(Symmetry)} \tag{2.34}$$

and

$$r_m\big(\Gamma_{ki}^j\big) - r_k\big(\Gamma_{mi}^j\big) = \sum_{s=1}^{n} \big(\Gamma_{ks}^j \Gamma_{mi}^s - \Gamma_{ms}^j \Gamma_{ki}^s - c_{km}^s \Gamma_{si}^j\big) \qquad \text{(Flatness)}. \tag{2.35}$$

One can also show that a connection ∇ is symmetric and flat if and only if in a neighborhood of each point there exist coordinate functions $u^1, \ldots, u^n$ with the property that the Christoffel symbols relative to the coordinate frame are zero:

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0 \qquad \text{for all } i, j = 1, \ldots, n. \tag{2.36}$$

Such a coordinate system is called an *affine coordinate system* with respect to ∇. Manifolds with a symmetric and flat connection are called *flat*.

Remark 2.14. *(*COORDINATE-FREE DEFINITION OF A HESSIAN AND THE β-SYSTEM*) In* [39]*, g is called a Hessian metric on a manifold M with a* flat, symmetric *connection ∇, if there exists a function $\eta\colon M \to \mathbb{R}$ such that*

$$g = \nabla d\eta, \tag{2.37}$$

Explicitly (2.37) says that for any two vector fields $X, Y \in \mathcal{X}(M)$:

$$g(X,Y) = (\nabla_X d\eta)(Y) := X(d\eta(Y)) - d\,\eta(\nabla_X Y). \tag{2.38}$$

The advantage of the condition (2.37) is that it provides us with a coordinate free definition of Hessian metrics, whereas a definition based on Hessian matrices requires a choice of coordinates. The pair (∇, g) is called a Hessian structure *on M. Applying (2.38) to an affine coordinate frame $\frac{\partial}{\partial u^i}$, $i = 1, \dots, n$, we can verify that $g(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}) = \frac{\partial^2 \eta}{\partial u^i\,\partial u^j}$. If $\mathfrak{R} = \{r_1, \dots, r_n\}$ is a frame, whose Christoffel symbols relative to ∇ are Γ^k_{ij}, then a simple computation shows that*

$$g(r_i, r_j) = r_i(r_j(\eta)) - \sum_{k=1}^{n} \Gamma^k_{i,j}\, r_k(\eta). \tag{2.39}$$

We observe from (2.39) that the symmetry of the metric, $g(r_i, r_j) = g(r_j, r_i)$, is equivalent to the commutator identity $[r_i, r_j]\eta = \sum_{k=1}^{n} c^k_{ij}\, r_k(\eta)$.

It is shown in [39] *that a pair (∇, g) of a flat, symmetric connection ∇ and a metric g, is a Hessian structure if and only if it satisfies the Codazzi equations*

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z), \quad \forall X, Y, Z \in \mathcal{X}(M). \tag{2.40}$$

Let us further assume that $\mathfrak{R}$ is an orthogonal frame relative to g, i.e. $g(r_i, r_j) = \delta^i_j \beta^i$. Recalling that $(\nabla_X g)(Y, Z) := X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$, and substituting $X = r_i$, $Y = r_j$ with $i \neq j$, and $Z = r_k$ in (2.40), we obtain the $\beta(\mathfrak{R})$-system (2.15)-(2.16).

Remark 2.15. *(*COORDINATE-FREE DEFINITION OF A JACOBIAN AND THE λ-SYSTEM*) We can similarly provide a coordinate-free definition of a Jacobian. For a manifold M with a* flat, symmetric *connection ∇, a linear map $J\colon \mathcal{X}(M) \to \mathcal{X}(M)$ is called a* Jacobian map *if there exists a vector field $V \in \mathcal{X}(M)$ such that*

$$J = \nabla V \quad \Leftrightarrow \quad J(X) = \nabla_X V, \quad \forall X \in \mathcal{X}(M). \tag{2.41}$$

If (2.41) holds we say that J is the Jacobian of V *and use the notation J_V. The flatness and symmetry of ∇ implies that for $\forall X, Y \in \mathcal{X}(M)$*

$$\nabla_X J_V(Y) - \nabla_Y J_V(X) := \nabla_X(\nabla_Y(V)) - \nabla_Y(\nabla_X(V)) = \nabla_{[X,Y]} V =: J_V([X, Y]). \tag{2.42}$$

Let $(u^1, \dots, u^n)$ be an affine system of coordinates (see (2.36)) and $V = \sum_{i=1}^{n} f^i(u) \frac{\partial}{\partial u^i}$, then according to (2.41)

$$J_V(\frac{\partial}{\partial u^j}) = \sum_{i}^{n} \frac{\partial f^i}{\partial u^j}\,\frac{\partial}{\partial u^i},$$

which is exactly j-th column vector of the usual Jacobian matrix of a vector valued function $f(u) = (f^1, \dots, f^n)$. If $\mathfrak{R} = \{r_1, \dots, r_n\}$ is a frame whose Christoffel symbols relative to ∇ are Γ^k_{ij} and $V = \sum_{i=1}^{n} \tilde{f}^i\, r_i$ then (2.41) implies:

$$J_V(r_j) = \sum_{i=1}^{n} \left[r_j(\tilde{f}^i) + \sum_{k=1}^{n} \Gamma^i_{jk}\, \tilde{f}^k \right] r_i. \tag{2.43}$$

Let us further assume that $\mathfrak{R}$ is a set of eigenvector-fields of J with real-valued eigenvalues λ^i:

$$J(r_i) = \nabla_{r_i} V = \lambda^i r_i\,.$$

Substitution of $X = r_i$ and $Y = r_j$, with $i \neq j$, into (2.42) produces the $\lambda(\mathfrak{R})$-system (2.18)-(2.19).

Before proceeding with the analysis of the β-system we return to the setting of Sections 2.1-2.3 where the coordinate system $(u^1, \ldots, u^n)$ and the frame $\mathfrak{R} = \{r_1, \ldots, r_n\}$ on Ω are fixed. From now on ∇ will denote the unique flat and symmetric connection satisfying (2.36). As indicated at the beginning of Section 2.5 the coefficients Γ_{ij}^k defined in (2.12), and appearing in the λ- and β-systems, are then the Christoffel symbols of the connection ∇ relative to the frame $\mathfrak{R}$. Also, the notation for the coefficients c_{ij}^k in (2.12) is consistent with symmetry (2.34) of ∇.

3. Analysis of $\beta(\mathfrak{R})$ with no algebraic part

In this section we analyze β-systems with no algebraic part. We show that the frames corresponding to such systems are necessarily rich. Examples in Section 6.2.2 illustrate that the converse is not true: there are rich systems whose corresponding β-system imposes non-trivial algebraic constraints. It is true, however, that the β-system for a rich frame that admits a *strictly hyperbolic* flux has no algebraic constraints.

In [25] we analyzed $\lambda(\mathfrak{R})$ for general rich frames $\mathfrak{R}$. When the algebraic part is non-trivial this analysis is complicated due to possible additional algebraic constraints imposed by the differential part of $\lambda(\mathfrak{R})$. A similar breakdown does not seem feasible for $\beta(\mathfrak{R})$ unless $n \leq 3$. In this section we therefore treat rich systems of any size, but with trivial algebraic part. Rich systems of three equations with algebraic constraints are covered in Section 4.2.

3.1. **Absence of the algebraic part implies richness.** Given a frame $\mathfrak{R}$ such that $\beta(\mathfrak{R})$ does not impose any algebraic constraints. According to Observation 2.11 this is the case if and only if the same is true for the λ-system, and occurs if and only if

$$\Gamma_{ij}^k = 0 \qquad \text{whenever } \epsilon(i,j,k) = 1. \tag{3.1}$$

The symmetry conditions (2.34) then imply that

$$c_{ij}^k = 0 \qquad \text{whenever } \epsilon(i,j,k) = 1, \tag{3.2}$$

whence, according to Definition 2.13, the frame $\mathfrak{R}$ is rich, and we have

$$[r_i, r_j] \in \operatorname{span}\{r_i, r_j\} \qquad \text{for all } 1 \leq i,\, j \leq n. \tag{3.3}$$

It follows from (3.3) that for every $i = 1, \ldots, n$, the vector fields $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n$ are in involution. By Frobenius' theorem (Chapter 6 in [41], Section 7.3 in [12]) there exist scalar fields w^i, $i = 1, \ldots, n$, on an open neighborhood (again denoted by Ω) of an arbitrary point in Ω, such that

$$r_j(w^i) \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases} \qquad \forall u \in \Omega\,.$$

By setting $\tilde{r}_i := r_i / r_i(w^i)$ we achieve the normalization

$$\tilde{r}_j(w^i) \equiv \delta_j^i \iff \tilde{r}_i = \frac{\partial}{\partial w^i}. \tag{3.4}$$

By Observation 2.7 we may assume, without loss of generality, that the given rich frame $\mathfrak{R}$ satisfies (3.4) and therefore is commutative. We denote the change of coordinates map by ρ:

$$u \mapsto \rho(u) = (w^1(u), \ldots, w^n(u)).$$

The w-coordinates are referred to as *Riemann coordinates.*

3.2. $\beta(\mathfrak{R})$ **in Riemann coordinates.** In Riemann coordinates the β-system (2.15)-(2.16) becomes

$$\partial_i \gamma^j = Z_{ji}^j \gamma^j - Z_{jj}^i \gamma^i \qquad \text{for } 1 \leq i \neq j \leq n, \qquad \left(\partial_i = \tfrac{\partial}{\partial w^i}\right) \tag{3.5}$$

$$Z_{ik}^j \gamma^j = Z_{jk}^i \gamma^i \qquad \text{for } 1 \leq k \neq i < j \neq k \leq n, \tag{3.6}$$

where

$$\gamma^i(w) := \beta^i \circ \rho^{-1}(w) \quad \text{and} \quad Z_{ij}^k(w) := \Gamma_{ij}^k \circ \rho^{-1}(w)\,. \tag{3.7}$$

Symmetry and flatness of the connection ∇ imply the following relations among the $Z_{ij}^k(w)$:

$$Z_{ij}^k = Z_{ji}^k\,, \qquad \forall\, i,\, j,\, k, \ \text{(symmetry)} \tag{3.8}$$

$$\partial_m\big(Z^j_{ik}\big) - \partial_k\big(Z^j_{im}\big) = \sum_{t=1}^n \big(Z^j_{tk}Z^t_{im} - Z^j_{tm}Z^t_{ik}\big)\,, \qquad \forall\, i,\ j,\ k,\ m \ \text{(flatness)}. \tag{3.9}$$

3.3. **Solution of $\beta(\mathfrak{R})$ for systems with no algebraic part.** We now assume that

$$Z^k_{ij} = 0 \qquad \text{whenever } \epsilon(i,j,k) = 1. \tag{3.10}$$

In this case $\beta(\mathfrak{R})$ reduces to the pure PDE system (3.5). Darboux's theorem (see Theorem 4.1 in [25]) applies and shows that the solutions to $\beta(\mathfrak{R})$ depend on n functions of 1 variable. More precisely:

Theorem 3.1. *Given a C^2-smooth, rich frame $\{r_1, \dots, r_n\}$ in a neighborhood of $\overline{w} \in \mathbb{R}^n$. Let $(w^1, \dots, w^n)$ be associated Riemann invariants and assume the normalization (3.4). Let the connection coefficients Z^k_{ij} be defined by (3.7) and assume that $Z^k_{ij} = 0$ whenever $\epsilon(i,j,k) = 1$.*

Then, for given functions φ_i, $i = 1, \dots, n$, of one variable, there is a unique local solution $\gamma^1(w), \dots, \gamma^n(w)$ to the β-system (3.5) with

$$\gamma^i(\overline{w}^1, \dots, \overline{w}^{i-1}, w^i, \overline{w}^{i+1}, \dots \overline{w}^n) = \varphi_i(w^i).$$

Proof. The compatibility condition required by Darboux's theorem is that the mixed second order partial derivatives calculated from (3.5) should agree. That is, we need to verify that

$$\partial_k\big(Z^j_{mj}\gamma^j - Z^m_{jj}\gamma^m\big) = \partial_m\big(Z^j_{kj}\gamma^j - Z^k_{jj}\gamma^k\big) \tag{3.11}$$

holds whenever $\epsilon(j,k,m) = 1$, when the derivatives are calculated according to (3.5). Performing the differentiations, applying (3.5), and collecting terms, we obtain the conditions

$$\begin{aligned} &\big(\partial_m Z^j_{kj} - \partial_k Z^j_{mj}\big)\gamma^j - \big(\partial_m Z^k_{jj} + Z^k_{jj}Z^k_{mk} + Z^m_{jj}Z^k_{mm} - Z^j_{mj}Z^k_{jj}\big)\gamma^k \\ &\qquad + \big(\partial_k Z^m_{jj} + Z^m_{jj}Z^m_{km} + Z^k_{jj}Z^m_{kk} - Z^j_{kj}Z^m_{jj}\big)\gamma^m = 0\,, \qquad \text{whenever } \epsilon(j,k,m) = 1. \end{aligned} \tag{3.12}$$

We shall show that the coefficients of γ^j, γ^k, and γ^m vanish identically due to flatness and symmetry. First, due to symmetry and to (3.9) with $i = j$, we have

$$\partial_m Z^j_{kj} - \partial_k Z^j_{mj} = \partial_m Z^j_{jk} - \partial_k Z^j_{jm} = \sum_{t=1}^n \big(Z^j_{tk}Z^t_{jm} - Z^j_{tm}Z^t_{jk}\big)\,.$$

Recalling (3.10) and that $\epsilon(j,k,m) = 1$ we conclude that the latter sum is zero. This shows that the coefficient of γ^j in (3.12) vanishes identically.

Next, using (3.9) with $i = k$ and then interchanging j and k yields the general identity

$$\partial_m Z^k_{jj} - \partial_j Z^k_{jm} = \sum_{t=1}^n \big(Z^k_{tj}Z^t_{jm} - Z^k_{tm}Z^t_{jj}\big)\,.$$

Then apply this to the case when $\epsilon(j,k,m) = 1$ to get that

$$0 = \partial_m Z^k_{jj} - \sum_{t=1}^n \big(Z^k_{tj}Z^t_{jm} - Z^k_{tm}Z^t_{jj}\big) = \partial_m Z^k_{jj} - Z^k_{jj}Z^j_{jm} + Z^k_{km}Z^k_{jj} + Z^k_{mm}Z^m_{jj}\,. \tag{3.13}$$

Applying symmetry (3.8) in the last expression shows that the coefficient of γ^k in (3.12) vanishes identically.

Finally, since we need to verify that the coefficient of γ^m vanishes identically for all triples j,k,m with $\epsilon(j,k,m) = 1$, we may as well interchange m and k in the coefficient of γ^m in (3.12). The result is the right-hand side in (3.13), which vanishes. □

In Example 6.4 we consider the rich, orthogonal frame whose Riemann coordinates are cylindrical coordinates in $\mathbb{R}^3$. According to the analysis in Section 4.2.1 of [25], the algebraic part of the λ-system corresponding to any rich, orthogonal frame is necessarily trivial. By Observation 2.11 above the same is true for the algebraic part of the β-system. This example illustrates Theorem 3.1, as well as Observations 2.10 and 2.7.

Remark 3.2. *We note that this result applies to any frame when $n = 2$ since in that case there are no algebraic constraints. Theorem 3.1 is well-known in the setting of a given, strictly hyperbolic system (1.1), see* [11, 36, 42].

4. Analysis of $\beta(\mathfrak{R})$ for $n = 3$

We now restrict attention to frames on open subsets of $\mathbb{R}^3$. Before considering the associated β-systems we recall the breakdown of possible cases for the λ-systems. In [25], it was demonstrated that there are essentially only four possibilities in this case, as described in the following proposition:

Proposition 4.1 ([25])**.** *Given a smooth frame $\mathfrak{R}$ on $\Omega \subset \mathbb{R}^3$, the solution set of $\lambda(\mathfrak{R})$ is described by one of the following cases:*

- I: *if* rank[(2.19)] $= 0$ *(no algebraic constraints) then $\mathfrak{R}$ is necessarily rich and a general solution of $\lambda(\mathfrak{R})$ depends on 3 functions of 1 variable. There are strictly hyperbolic solutions in this class.*
- II: *if* rank[(2.19)] $= 1$ *(a single algebraic constraint) then there are two possibilities:*
  - IIa. *All three λ^i appear in the algebraic constraint. The solution of the λ-system is either trivial or depends on 2 arbitrary constants. In the latter case, there are strictly hyperbolic systems in the family. There are no rich systems in class IIa.*
  - IIb. *Exactly two λ^i appear in the algebraic constraint. Two λ^i coincide and the general solution is either trivial or depends on 1 arbitrary function of 1 variables and 1 constant. There are no strictly hyperbolic systems, but there are rich systems, in class IIb.*
- III: *if* rank[(2.19)] $= 2$ *then there are only trivial solutions $\lambda^1 = \lambda^2 = \lambda^3 \equiv$ constant.*

Furthermore, each of the above possibilities can occur.

In our analysis of the β-system, Case I in Proposition 4.1 is covered by the analysis in Section 3.3. We consider Case II in sections 4.2 and 4.3 below; in particular this covers the case of rich frames with one algebraic constraint. The various sub-cases in this latter category indicate that the analysis of rich systems of size $n \geq 4$ with algebraic constraints (i.e. the rich systems *not* treated in Section 3.3), is quite involved. Finally, Case III is largely irrelevant as it admits only trivial linear systems $u_t + \bar{\lambda} u_x = 0$, for which any scalar field is an extension. However, it does highlight the fact that our approach of characterizing extensions that satisfy (1.7) due to orthogonality alone, may not provide *all* extensions for *all* systems with a given eigen-frame $\mathfrak{R}$; see Example 6.3.

4.1. **The algebraic parts of $\lambda(\mathfrak{R})$ and $\beta(\mathfrak{R})$ for $n = 3$.** Before analyzing $\beta(\mathfrak{R})$ we establish a useful relationship between the algebraic parts of $\lambda(\mathfrak{R})$ and $\beta(\mathfrak{R})$. The former is given by (2.19):

$$A_\lambda\,(\lambda^1, \lambda^2, \lambda^3)^T = 0\,, \qquad \text{where} \qquad A_\lambda = \begin{bmatrix} c^1_{23} & \Gamma^1_{32} & -\Gamma^1_{23} \\ \Gamma^2_{31} & c^2_{13} & -\Gamma^2_{13} \\ \Gamma^3_{21} & -\Gamma^3_{12} & c^3_{12} \end{bmatrix}. \tag{4.1}$$

For $n = 3$ the algebraic part (2.16) of the β-system is

$$A_\beta\,(\beta^1, \beta^2, \beta^3)^T = 0 \qquad \text{where} \qquad A_\beta = \begin{bmatrix} c^1_{23} & -\Gamma^2_{31} & \Gamma^3_{21} \\ -\Gamma^1_{32} & c^2_{13} & \Gamma^3_{12} \\ -\Gamma^1_{23} & \Gamma^2_{13} & c^3_{12} \end{bmatrix}. \tag{4.2}$$

A calculation shows that

$$A_\lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_\beta^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.3}$$

Observing that $\operatorname{rank}(A_\lambda) \leq 2$, we have:

Proposition 4.2. *For $n = 3$ the ranks of the algebraic parts of $\lambda(\mathfrak{R})$ and $\beta(\mathfrak{R})$ are equal and ≤ 2.*

Remark 4.3. *Example 6.10 shows that this result does not generalize to $n \geq 4$.*

4.2. **Rich frames with algebraic part of $\beta(\mathfrak{R})$ of rank 1.** Given a rich frame $\mathfrak{R} = \{r_1, r_2, r_3\}$, we make a choice $w = (w^1, w^2, w^3)$ of Riemann coordinates and scale the frame according to (3.4). The matrices A_λ and A_β are then

$$A_\lambda = \begin{bmatrix} 0 & Z^1_{23} & -Z^1_{23} \\ Z^2_{13} & 0 & -Z^2_{13} \\ Z^3_{12} & -Z^3_{12} & 0 \end{bmatrix} \qquad \text{and} \qquad A_\beta = \begin{bmatrix} 0 & -Z^2_{13} & Z^3_{12} \\ -Z^1_{23} & 0 & Z^3_{12} \\ -Z^1_{23} & Z^2_{13} & 0 \end{bmatrix}, \tag{4.4}$$

where Z^k_{ij} are given by $(3.7)_2$. Assume now that rank(3.6) = 1, i.e. $\text{rank}(A_\lambda) = \text{rank}(A_\beta) = 1$, and observe that this is the case if and only if exactly two of the functions Z^3_{12}, Z^2_{13} and Z^1_{23} vanish identically. Without loss of generality we assume that

$$Z^1_{23} \neq 0, \qquad \text{while} \qquad Z^3_{12} = Z^2_{13} \equiv 0\,. \tag{4.5}$$

Remark 4.4. *In this case the algebraic part (2.19) of the λ-system requires $\lambda^1 \equiv \lambda^2$ and, according to Proposition 4.1 there are only two possibilities for the solutions of the λ-system:*

- *only the trivial solution: $\lambda^1(u) = \lambda^2(u) = \lambda^3(u) \equiv \hat{\lambda} \in \mathbb{R}$ (Example 6.6);*
- *$\lambda^1(u) = \lambda^2(u) \neq \lambda^3(u)$ and the general solution depends on 1 arbitrary function of one variable and one arbitrary constant, which in some examples may be absorbed in the arbitrary function. (See Examples 6.6).*

We proceed to show that for "rich, rank 1" frames, there are exactly three possibilities for the solution set of the β-system. In all cases the corresponding extensions are degenerate.

Theorem 4.5. *Given a rich frame $\mathfrak{R} = \{r_1, r_2, r_3\}$ on $\Omega \subset \mathbb{R}^3$, consider the β-system (3.5)-(3.6) expressed in Riemann coordinates. Assume that the algebraic part (3.6) of $\beta(\mathfrak{R})$ has rank 1. Then there are three possibilities for the solution set of the β-system:*

1. *Only the trivial solution: $\beta^1 = \beta^2 = \beta^3 \equiv 0$*
2. *Exactly two β^i are zero and the third depends on 1 arbitrary function of 1 variable.*
3. *Exactly one β^i is zero and the other two β^i depend on 2 arbitrary functions of 1 variable.*

Proof. As in Section 3.1, we let $\gamma^i(w(u)) = \beta^i(u)$ and $Z^k_{ij}(w(u)) = \Gamma^k_{ij}(u)$. Then the algebraic part (3.6) of the β-system is equivalent to

$$\gamma^1 \equiv 0\,, \tag{4.6}$$

showing that there are no non-degenerate extensions in this case. The differential part (3.5) of the β-system reduces to:

$$0 = \gamma^2\, Z^2_{11} \tag{4.7}$$
$$0 = \gamma^3\, Z^3_{11} \tag{4.8}$$
$$\partial_1(\gamma^2) = \gamma^2\, Z^2_{12} \tag{4.9}$$
$$\partial_3(\gamma^2) = \gamma^2\, Z^2_{32} - \gamma^3\, Z^3_{22} \tag{4.10}$$
$$\partial_1(\gamma^3) = \gamma^3\, Z^3_{13} \tag{4.11}$$
$$\partial_2(\gamma^3) = \gamma^3\, Z^3_{23} - \gamma^2\, Z^2_{33}\,. \tag{4.12}$$

We proceed to consider the various possibilities:

- **a.** If $Z^2_{11} \neq 0$ and $Z^3_{11} \neq 0$ then $\gamma^2 \equiv 0$ and $\gamma^3 \equiv 0$. Together with (4.6) this shows that the β-system has only the trivial solution $\beta^1 = \beta^2 = \beta^3 \equiv 0$ in this case.
- **b.** If $Z^2_{11} \neq 0$ and $Z^3_{11} \equiv 0$ then from (4.7) it follows that $\gamma^2 \equiv 0$, and the system (4.7)-(4.12) reduces to:

$$0 = \gamma^3\, Z^3_{22} \tag{4.13}$$
$$\partial_1(\gamma^3) = \gamma^3\, Z^3_{13} \tag{4.14}$$
$$\partial_2(\gamma^3) = \gamma^3\, Z^3_{23}. \tag{4.15}$$

If $Z^3_{22} \neq 0$ then (4.13) implies $\gamma^3 = 0$ and the β-system has only the trivial solution $\beta^1 = \beta^2 = \beta^3 \equiv 0$. Otherwise the system reduces to (4.14)-(4.15), which we claim is a compatible system. The only integrability condition is $\partial_1(\partial_2\gamma^3) = \partial_2(\partial_1\gamma^3)$ which, upon using (4.14)-(4.15), amounts to the condition that $\partial_2 Z^3_{13} = \partial_1 Z^3_{23}$. This is indeed the case by flatness (3.9), symmetry (3.8), and the assumptions (4.5) and $Z^3_{11} \equiv 0$. We may therefore apply Darboux's theorem (Theorem 4.1 in [25]) and conclude that γ^3 depends on 1 arbitrary function of 1 variable.

c. If $Z^2_{11} = 0$ and $Z^3_{11} \neq 0$ then from (4.8) it follows that $\gamma^3 \equiv 0$, while γ^2 depends on 1 arbitrary function of 1 variable. This case is equivalent to Case **b** by permutation of the third and the second eigenvectors (recall the symmetry (3.8) for rich systems).

d. If $Z^2_{11} \equiv Z^3_{11} \equiv 0$, then the differential system reduces to (4.9)-(4.12) for the unknown functions γ^2 and γ^3. The two integrability conditions are $\partial_3(\partial_1\gamma^2) = \partial_1(\partial_3\gamma^2)$ and $\partial_2(\partial_1\gamma^3) = \partial_1(\partial_2\gamma^3)$, and these are satisfied as identities thanks symmetry, flatness, and the current assumptions. We apply Darboux's theorem and conclude that γ^2 depends on 1 function of 1 variable, as does γ^3. Example 6.6 belongs to this category.

□

4.3. **Non-rich frames with algebraic part of $\beta(\mathfrak{R})$ of rank 1.** The following theorem lists all possible degrees of freedom that the solutions of the β-system may enjoy in this case.

Theorem 4.6. *Given a non-rich frame $\mathfrak{R} = \{r_1, r_2, r_3\}$ on $\Omega \subset \mathbb{R}^3$ for which the algebraic part (2.16) of $\beta(\mathfrak{R})$ has rank 1. Then the solution set of the β-system (2.15)-(2.16) is covered by one of the following possibilities, all of which can occur:*

(1) *Only the trivial solution: $\beta^1 = \beta^2 = \beta^3 \equiv 0$*
(2) *Exactly two β^i are zero and the third depends on 1 arbitrary function of one variable.*
(3) *Exactly one β^i is zero and the other two β^i depend on*
  (3a) *2 arbitrary functions of one variable.*
  (3b) *1 common arbitrary constant.*
(4) *There is a non-degenrate solution (all β^i are non-zero) which depends on*
  (4a) *1 arbitrary function of one variable and 1 arbitrary constant.*
  (4b) *2 arbitrary constants.*
  (4c) *1 arbitrary constant.*

Proof. In Section 5 we show that there are no other possibilities for the solutions set of the β-system. On the other hand, the examples in Section 6 confirm that every scenario from the above list is realizable: (Cases with Roman numerals refer to the cases listed in Proposition 4.1)

- **Case (1)** is realized by Example 6.11. The corresponding λ-system belongs to Case IIb of Proposition 4.1 and has non-trivial solutions.
- **Case (2)** is realized by Examples 6.7 and 6.12. In the former example, the corresponding λ-system belongs to Case IIa and has non-trivial solutions depending on 2 constants. There are strictly hyperbolic conservative systems that correspond to this frame. In the latter example, the corresponding λ-system belongs to Case IIb and has non-trivial solutions depending on 1 function of one variable and 1 constant. This frame does not admit strictly hyperbolic conservative systems.
- **Case (3a)** is realized by Examples 6.8 and 6.13, Example 6.15 when $g \equiv 0$, as well as Example 6.14. In the first four examples the λ-system is of Case IIb, whereas in the last example the λ-system is of the Case IIa. The λ-systems in Examples 6.8, 6.13 and 6.15 have non-trivial solutions, while in the other two examples the corresponding λ-system has only the trivial solution.
- **Case (3b)** is realized by Example 6.15 with $g \not\equiv 0$, and the corresponding λ-system belongs to Case IIb and has non-trivial solutions.
- **Case (4a)** is realized by Examples 6.1 (non-rich gas dynamics) and 6.16. In the former example the corresponding λ-system belongs to Case IIa and admits strictly hyperbolic conservative systems, whereas in the latter example the corresponding λ-system belong to Case IIb and does not admits strictly hyperbolic conservative systems.
- **Case (4b)** is realized by Example 6.17. The corresponding λ-system belongs to Case IIb and has non-trivial solutions.
- **Case (4c)** is realized by Example 6.18. The corresponding λ-system belongs to Case IIb and has non-trivial solutions.

□

Remark 4.7. *We note that cases (1)-(3a) listed in Theorem 4.6 include all cases from Theorem 4.5. Thus* non-richness *of the frame opens for more possibilities (Cases 3b and 4a-c).*

By combining Theorem 4.6 and Theorem 4.5 we see that if a frame with a non-trivial algebraic constraint admits a non-degenerate extension, then the frame must necessarily be non-rich.

On the other hand, by Theorem 4.6 and Theorem 3.1, the largest set of extensions occurs in the algebraically unconstrained (and hence) rich case described in Section 3. Indeed, in this case (i.e. $\Gamma_{ij}^k \equiv 0$ whenever $\epsilon(i,j,k) = 1$) the general solution of the β-system depends on 3 arbitrary functions of one variable, while for any algebraically constrained case the maximal degree of freedom is 2 arbitrary functions of one variable (Case (3a)).

5. Completion of the proof of Theorem 4.6

In this section, we show that the cases listed in Theorem 4.6 exhaust all possibilities for solutions of $\beta(\mathfrak{R})$, whenever $\mathfrak{R}$ is a non-rich frame on $\Omega \subset \mathbb{R}^3$ whose β-system has algebraic rank 1.

5.1. **Setup.** By assumption the matrix A_β in (4.2) has rank 1. Before showing how each of the possibilities arise we make a choice of indices. By assumption $\mathfrak{R}$ is non-rich such that there a triple (i,j,k) with $\epsilon(i,j,k) = 1$ and $c_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i \neq 0$. If necessary we permute the prescribed eigenfields r_1, r_2, r_3 and assume

$$c_{32}^1 \neq 0 \qquad \text{and} \qquad \Gamma_{32}^1 \neq 0\,. \tag{5.1}$$

We define

$$a(u) = \frac{\Gamma_{32}^1(u)}{c_{32}^1(u)} \neq 0\,.$$

Since $\operatorname{rank}(A_\beta) = 1$ it follows that

$$c_{31}^2 = a\,\Gamma_{31}^2 \qquad \text{and} \qquad \Gamma_{12}^3 = a\,\Gamma_{21}^3. \tag{5.2}$$

Using (5.1) we solve the first algebraic constraint in (4.2) for β^1 to get

$$\beta^1 = \alpha^2\,\beta^2 + \alpha^3\,\beta^3 \qquad \text{where} \qquad \alpha^2 = -\frac{\Gamma_{31}^2}{c_{32}^1} \text{ and } \alpha^3 = \frac{\Gamma_{21}^3}{c_{32}^1}. \tag{5.3}$$

For $n = 3$ the differential part of the β-system consists of the following 6 PDEs:

$$r_2(\beta^1) = \beta^1\,(\Gamma_{21}^1 + c_{21}^1) - \beta^2\,\Gamma_{11}^2 \tag{5.4}$$

$$r_3(\beta^1) = \beta^1\,(\Gamma_{31}^1 + c_{31}^1) - \beta^3\,\Gamma_{11}^3 \tag{5.5}$$

$$r_1(\beta^2) = \beta^2\,(\Gamma_{12}^2 + c_{12}^2) - \beta^1\,\Gamma_{22}^1 \tag{5.6}$$

$$r_3(\beta^2) = \beta^2\,(\Gamma_{32}^2 + c_{32}^2) - \beta^3\,\Gamma_{22}^3 \tag{5.7}$$

$$r_1(\beta^3) = \beta^3\,(\Gamma_{13}^3 + c_{13}^3) - \beta^1\,\Gamma_{33}^1 \tag{5.8}$$

$$r_2(\beta^3) = \beta^3\,(\Gamma_{23}^3 + c_{23}^3) - \beta^2\,\Gamma_{33}^2\,. \tag{5.9}$$

Substituting (5.3) into (5.4)-(5.9), and using (5.7) and (5.9), yield the PDE system

$$\begin{aligned} \alpha^2\,r_2(\beta^2) = \beta^2\,&\left[\alpha^2\,(\Gamma_{21}^1 + c_{21}^1) - r_2\,(\alpha^2) + \alpha^3\,\Gamma_{33}^2 - \Gamma_{11}^2\right] \\ &+ \beta^3\left[\alpha^3\,(\Gamma_{21}^1 + c_{21}^1 - \Gamma_{23}^3 - c_{23}^3) - r_2(\alpha^3)\right] \end{aligned} \tag{5.10}$$

$$\begin{aligned} \alpha^3\,r_3(\beta^3) = \beta^2\,&\left[\alpha^2\,(\Gamma_{31}^1 + c_{31}^1 - \Gamma_{32}^2 - c_{32}^2) - r_3\,(\alpha^2)\right] \\ &+ \beta^3\left[\alpha^3\,(\Gamma_{31}^1 + c_{31}^1) - r_3(\alpha^3) + \alpha^2\,\Gamma_{22}^3 - \Gamma_{11}^3\right] \end{aligned} \tag{5.11}$$

$$r_1(\beta^2) = \beta^2\,(\Gamma_{12}^2 + c_{12}^2 - \alpha^2\,\Gamma_{22}^1) - \alpha^3\,\beta^3\,\Gamma_{22}^1 \tag{5.12}$$

$$r_3(\beta^2) = \beta^2\,(\Gamma_{32}^2 + c_{32}^2) - \beta^3\,\Gamma_{22}^3 \tag{5.13}$$

$$r_1(\beta^3) = \beta^3\,(\Gamma_{13}^3 + c_{13}^3 - \alpha^3\,\Gamma_{33}^1) - \alpha^2\,\beta^2\Gamma_{33}^1 \tag{5.14}$$

$$r_2(\beta^3) = \beta^3\,(\Gamma_{23}^3 + c_{23}^3) - \beta^2\,\Gamma_{33}^2\,. \tag{5.15}$$

We proceed to analyze this system according to a number of β's involved in the algebraic constraint (5.3).

5.2. **All three β^i appear in the unique algebraic constraint.** In this case we have $\alpha^2 \neq 0$ and $\alpha^3 \neq 0$ (equivalently $\Gamma^2_{31} \neq 0$ and $\Gamma^3_{21} \neq 0$), and we can rewrite (5.10)-(5.15) as

$$r_i(\beta^s) = \phi_i^s(u)\beta^2 + \psi_i^s(u)\beta^3\,, \qquad i = 1,\,2,\,3, \quad s = 2,\,3. \tag{5.16}$$

This is a "Frobenius type" system which prescribes the derivatives of 2 unknowns along three independent vector-fields in $\mathbb{R}^3$. According to the Frobenius Theorem [41] the associated integrability conditions require that

$$[r_i, r_j](\beta^s) = \sum_{k=1}^{n} c_{ij}^k\, r_k(\beta^s) \qquad 1 \leq i < j \leq 3, \quad s = 2,\,3, \tag{5.17}$$

should hold as identities in u, β^2, β^3, when both sides are evaluated by using (5.16). A calculation shows that these requirements amounts to 6 relations of the form

$$\mathcal{A}_{ij}^s(u)\beta^2 + \mathcal{B}_{ij}^s(u)\beta^3 = 0\,, \qquad 1 \leq i < j \leq 3, \quad s = 2,\,3, \tag{5.18}$$

where the $\mathcal{A}_{ij}^s$ and $\mathcal{B}_{ij}^s$ are given in terms of ϕ_i^s, ψ_i^s, their derivatives, and the c_{ij}^k. We have the following possibilities

- If all $\mathcal{A}_{ij}^s$ and $\mathcal{B}_{ij}^s$ vanish identically, then the integrability conditions are satisfied for all β^2 and β^3, and by the Frobenius Theorem the general solution of (5.16) depends on two arbitrary constants. The remaining β^1 is expressed in terms of β^2 and β^3 using (5.3). The two arbitrary constants can be chosen so that all β^i are non-zero on an open subset of Ω. This scenario is listed in ***Case 4b*** of Theorem 4.6.

- If the system (5.18) is of rank one then there are two possibilities
  - Conditions (5.18) are satisfied when $\beta^2 \equiv 0$ and β^3 is an arbitrary function. Substituting into (5.10)-(5.15) yields an over-determined algebraic-differential system for β^3 with 3 algebraic constraints and 3 PDEs prescribing $r_i(\beta^3)$, $i = 1,\,2,\,3$. Provided the algebraic relations are satisfied and the appropriate integrability conditions are met for all values of β^3, Frobenius' Theorem again applies and shows that the general solution of the β^3-system depends on 1 arbitrary constant. From (5.3) it follows that $\beta^1 = \alpha^3\,\beta^3$ depends on the same constant, and we obtain ***Case 3b*** of the Theorem. Otherwise, $\beta^3 \equiv 0$ is the only solution and therefore from (5.3) it follows that $\beta^1 = 0$. This is ***Case 1*** (trivial solution).
  - Conditions (5.18) are satisfied when $\beta^3 \equiv 0$ and β^2 is an arbitrary function. By the same argument as above we are either in ***Case 3b*** or ***Case 1*** of the Theorem.
  - Otherwise, from (5.18) one can write $\beta^2 = \Phi(u)\,\beta^3$, where $\Phi(u)$ is a non-zero function. Substituting into (5.10)-(5.15) yields an over-determined algebraic-differential system for β^3 with 3 algebraic constraints and 3 PDEs prescribing $r_i(\beta^3)$, $i = 1,\,2,\,3$. Provided the algebraic relations are satisfied and the appropriate integrability conditions are met for all values of β^3, Frobenius' Theorem implies that the general solution of the β^3-system depends on 1 arbitrary constant. Then $\beta^2 = \Phi(u)\,\beta^3$ depends on the same constant, as does β^1 due to (5.3). This is ***Case 4c*** of the Theorem. Otherwise the PDEs for β^3 imply that $\beta^3 \equiv 0$ and hence $\beta^2 = \Phi(u)\,\beta^3 \equiv 0$. From (5.3) it follows that $\beta^1 \equiv 0$ as well and we are in ***Case 1*** (trivial solution).

- Finally, if the system (5.18) is of rank 2, then $\beta^2 = \beta^3 \equiv 0$ is the only solution for (5.16) which, together with (5.3), yield the trivial solution of $\beta(\mathfrak{R})$ (***Case 1***).

5.3. **Exactly two β^i appear in the algebraic constraint.** In this case exactly one of α^2 and α^3 is non-zero; without loss of generality we assume that

$$\alpha^2 \equiv 0 \qquad \text{and} \qquad \alpha^3 \neq 0 \qquad (\text{equivalently } \Gamma^2_{31} \equiv 0 \text{ and } \Gamma^3_{21} \neq 0). \tag{5.19}$$

From (5.3) it then follows that

$$\beta^1 = \alpha^3\beta^3, \tag{5.20}$$

and the differential part (5.10)-(5.15) of the β-system reduces to

$$0 = \beta^2 \left(\alpha^3 \Gamma^2_{33} - \Gamma^2_{11}\right) + \beta^3 \left[\alpha^3 (\Gamma^1_{21} + c^1_{21} - \Gamma^3_{23} - c^3_{23}) - r_2(\alpha^3)\right] \tag{5.21}$$

$$\alpha^3 r_3(\beta^3) = \beta^3 \left[\alpha^3 (\Gamma^1_{31} + c^1_{31}) - r_3(\alpha^3) - \Gamma^3_{11}\right] \tag{5.22}$$

$$r_1(\beta^2) = \beta^2 (\Gamma^2_{12} + c^2_{12}) - \alpha^3 \beta^3 \Gamma^1_{22} \tag{5.23}$$

$$r_3(\beta^2) = \beta^2 (\Gamma^2_{32} + c^2_{32}) - \beta^3 \Gamma^3_{22} \tag{5.24}$$

$$r_1(\beta^3) = \beta^3 (\Gamma^3_{13} + c^3_{13} - \alpha^3 \Gamma^1_{33}) \tag{5.25}$$

$$r_2(\beta^3) = \beta^3 (\Gamma^3_{23} + c^3_{23}) - \beta^2 \Gamma^2_{33}\,. \tag{5.26}$$

From $(5.2)_1$ and $(5.19)_1$ we conclude that $c^2_{13} \equiv 0$. It follows that r_1 and r_3 may be scaled to obtain *commuting* vector fields $\tilde{r}_1$ and $\tilde{r}_3$. We can then choose a coordinate system $(v^1, v^2, v^3) = \rho(u^1, u^2, u^3)$ in $\mathbb{R}^3$ such that $\tilde{r}_1$ and $\tilde{r}_3$ are *coordinate* vectors. (Since $\tilde{r}_1$ and $\tilde{r}_3$ are in involution, Frobenius' Theorem gives the existence of a coordinate system $(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)$ in which the level sets of $\tilde{u}^2$, say, are integral manifolds of the 2-distribution span$\{\tilde{r}_1, \tilde{r}_3\}$. Since $\tilde{r}_1$ and $\tilde{r}_3$ commute there is a change of the two coordinate functions $v^1 = \tau^1(\tilde{u}^1, \tilde{u}^3)$, $v^3 = \tau^3(\tilde{u}^1, \tilde{u}^3)$ such that $\tilde{r}_1 = \frac{\partial}{\partial v^1}$ and $\tilde{r}_3 = \frac{\partial}{\partial v^3}$. Let $v^2 = \tilde{u}^2$.)

In v-coordinates we then have

$$r_1 = \xi^1(v)\,\frac{\partial}{\partial v^1}\,, \quad r_2 = \xi^2(v)\,\frac{\partial}{\partial v^1} + \xi^3(v)\,\frac{\partial}{\partial v^2} + \xi^4(v)\,\frac{\partial}{\partial v^3}\,, \quad r_3 = \xi^5(v)\,\frac{\partial}{\partial v^3}\,,$$

where ξ^1, ξ^3 and ξ^5 are non zero functions. For $i = 1, 2, 3$, let γ^i denote the pull-back of β^i under the change of coordinates ρ: $\gamma^i(\rho(u)) = \beta^i(u)$. We rewrite the system (5.21) -(5.26) in terms of v-coordinates and reorder the equations to group derivations of γ^2 and γ^3 together. With $\partial_i = \frac{\partial}{\partial v^i}$ we then have the system

$$0 = \mathcal{A}_1\,\gamma^2 + \mathcal{B}_1\,\gamma^3 \tag{5.27}$$

$$\xi^1\,\partial_1(\gamma^2) = \Phi^2_1\,\gamma^2 + \Psi^2_1\,\gamma^3 \tag{5.28}$$

$$\xi^5\,\partial_3(\gamma^2) = \Phi^2_3\,\gamma^2 + \Psi^2_3\,\gamma^3 \tag{5.29}$$

$$\xi^1\,\partial_1(\gamma^3) = \Psi^3_1\,\gamma^3 \tag{5.30}$$

$$[\xi^2\,\partial_1 + \xi^3\,\partial_2 + \xi^4\,\partial_3](\gamma^3) = \Phi^3_2\,\gamma^2 + \Psi^3_2\,\gamma^3 \tag{5.31}$$

$$\xi^5\,\partial_3(\gamma^3) = \Psi^3_3\,\gamma^3\,, \tag{5.32}$$

where $\mathcal{A}$, $\mathcal{B}$, Φ^i_j and Ψ^i_j are functions of (v^1, v^2, v^3) that can be expressed in terms of functions Γ^i_{jk}, α^3 and the change of coordinates ρ.

We observe that we can solve equations (5.28)-(5.32) for the derivatives of γ^2 and γ^3 and obtain a system of Darboux type, see [25]. Derivatives of γ^2 are prescribed along two coordinate directions, while derivatives of γ^3 are prescribed along all three coordinate directions:

$$\partial_1(\gamma^2) = \phi^2_1\,\gamma^2 + \psi^2_1\,\gamma^3 \tag{5.33}$$

$$\partial_3(\gamma^2) = \phi^2_3\,\gamma^2 + \psi^2_3\,\gamma^3 \tag{5.34}$$

$$\partial_1(\gamma^3) = \psi^3_1\,\gamma^3 \tag{5.35}$$

$$\partial_2(\gamma^3) = \phi^3_2\,\gamma^2 + \psi^3_2\,\gamma^3 \tag{5.36}$$

$$\partial_3(\gamma^3) = \psi^3_3\,\gamma^3\,, \tag{5.37}$$

where ϕ^i_j and ψ^i_j are known functions of (v^1, v^2, v^3). According to Darboux's theorem (Theorem 4.1 in [25]) the relevant integrability conditions are obtained by evaluating the identities $\partial_i\partial_j\gamma^s - \partial_j\partial_i\gamma^s = 0$ by using equations (5.33)-(5.37). A calculation shows that this gives rise to 5 linear homogeneous algebraic equations on γ^2 and γ^3:

$$\mathcal{A}_i\,\gamma^2 + \mathcal{B}_i\,\gamma^3 = 0, \quad i = 1, \ldots, 5, \tag{5.38}$$

where $\mathcal{A}_i$ and $\mathcal{B}_i$ are known functions of (v^1, v^2, v^3). Together with (5.27) we therefore obtain a system $\mathcal{L}$ of 6 linear homogeneous algebraic equations on γ^2 and γ^3. We break down all possibilities according to the rank of $\mathcal{L}$.

- If $\operatorname{rank}\mathcal{L} = 0$, i.e. the integrability conditions (5.38) together with (5.27) are satisfied for all γ^2 and γ^3, then according to Darboux's theorem the general solution of (5.33)-(5.37) depends on 1 arbitrary function of one variable and 1 constant. This is ***Case 4a*** of the theorem.

- We split the case of $\operatorname{rank}\mathcal{L} = 1$ into three sub-cases:
  - $\mathcal{L}$ is satisfied for $\gamma^2 \equiv 0$ (and so $\beta^2 \equiv 0$) and arbitrary γ^3. In this case system (5.33)-(5.37) reduces to

$$0 = \psi_1^2\,\gamma^3 \tag{5.39}$$
$$0 = \psi_3^2\,\gamma^3 \tag{5.40}$$
$$\partial_1(\gamma^3) = \psi_1^3\,\gamma^3 \tag{5.41}$$
$$\partial_2(\gamma^3) = \psi_2^3\,\gamma^3 \tag{5.42}$$
$$\partial_3(\gamma^3) = \psi_3^3\,\gamma^3\,, \tag{5.43}$$

  If either $\psi_1^2 \neq 0$ (equivalently $\Gamma_{22}^1 \neq 0$) or $\psi_3^2 \neq 0$ (equivalently $\Gamma_{22}^3 \neq 0$) then (5.39), respectively (5.40), shows that $\gamma^3 = 0$. Thus $\beta^3 = 0$ and $\beta^1 = 0$ by (5.20). In this case the β-system has only the trivial solution (***Case 1***). Otherwise, (5.41), (5.42), (5.43) is a compatible system of Frobenius type for γ^3. (One shows this by verifying that compatibility is a consequence of the assumption that $\mathcal{L}$ is satisfied for $\gamma^2 = 0$ and arbitrary γ^3. We drop details of this routine calculation.) Therefore γ^3, and hence also β^3, depends on 1 constant; $\beta^1 = \alpha^3\beta^3$ (recall (5.20)) depends on the same constant, while $\beta^2 = 0$. This is ***Case 3b*** of the theorem.
  - $\mathcal{L}$ is satisfied for $\gamma^3 \equiv 0$ (and so $\beta^3 \equiv 0$) and arbitrary γ^2. Then the system (5.33)-(5.37) reduces to

$$\partial_1(\gamma^2) = \phi_1^2\,\gamma^2 \tag{5.44}$$
$$\partial_3(\gamma^2) = \phi_3^2\,\gamma^2 \tag{5.45}$$
$$0 = \phi_2^3\,\gamma^2 \tag{5.46}$$

  If $\phi_2^3 \not\equiv 0$ (equivalently $\Gamma_{33}^2 \not\equiv 0$) then (5.46) shows that $\gamma^2 \equiv 0$, such that $\beta^2 \equiv 0$. Since $\beta^3 \equiv 0$ also $\beta^1 \equiv 0$ by (5.20), and we are in ***Case 1***. Otherwise, (5.44)-(5.45) is a compatible system of Darboux type for γ^2. (Again, compatibility is a consequence of our assumption that $\mathcal{L}$ is satisfied for $\gamma^3 \equiv 0$ and arbitrary γ^2.) Therefore γ^2 (and hence β^2) depends on 1 arbitrary function of one variable and $\beta^1 = \alpha^3\beta^3 \equiv 0$. This is ***Case 2*** of the theorem.
  - $\mathcal{L}$ is satisfied for $\gamma^3 = \mathcal{A}(v)\gamma^2$, where $\mathcal{A}$ is a non-zero function. Substituting $\gamma^3 = \mathcal{A}\gamma^2$ into (5.33)-(5.37), we obtain an overdetermined system of equations of Frobenious type (all partial derivatives of γ^2 are specified). If the system is compatible for all γ^2, then by Frobenious theorem its general solution for γ^2 (and hence β^2) depends on 1 constant. Then $\gamma^3 = \mathcal{A}\gamma^2$, and thus β^3, depends on the same constant. Combining this with (5.20) we conclude that the the general solution of the β-system depends on 1 arbitrary constant. This is ***Case 4c*** of the theorem. Otherwise, the system is compatible only for $\gamma^2 \equiv 0$, and $\gamma^3 = \mathcal{A}\gamma^2 \equiv 0$. Hence $\beta^2 = \beta^3 \equiv 0$ in this case. By (5.20), also $\beta^1 \equiv 0$, and the β-system has only the trivial solution (***Case 1***).
- Finally if $\operatorname{rank}\mathcal{L} = 2$ then $\mathcal{L}$ is satisfied only for $\gamma^2 = \gamma^3 \equiv 0$. Hence $\beta^2 = \beta^3 \equiv 0$. Again $\beta^1 \equiv 0$ and the β-system has only the trivial solution (***Case 1***).

5.4. **Exactly one β^i appears in the algebraic constraint.** In this case $\alpha^2 = \alpha^3 \equiv 0$ (equivalently $\Gamma^2_{31} = \Gamma^3_{21} \equiv 0$) and $\beta^1 \equiv 0$ by (5.3). The system (5.10)-(5.15) thus reduces to

$$0 = \beta^2 \, \Gamma^2_{11} \tag{5.47}$$

$$0 = \beta^3 \, \Gamma^3_{11} \tag{5.48}$$

$$r_1(\beta^2) = \beta^2 \, (\Gamma^2_{12} + c^2_{12}) \tag{5.49}$$

$$r_3(\beta^2) = \beta^2 \, (\Gamma^2_{32} + c^2_{32}) - \beta^3 \, \Gamma^3_{22} \tag{5.50}$$

$$r_1(\beta^3) = \beta^3 \, (\Gamma^3_{13} + c^3_{13}) \tag{5.51}$$

$$r_2(\beta^3) = \beta^3 \, (\Gamma^3_{23} + c^3_{23}) - \beta^2 \, \Gamma^2_{33} \, . \tag{5.52}$$

From the assumptions of this case, together with (5.2) and (5.3) we have

$$\Gamma^2_{31} = \Gamma^2_{13} = c^2_{13} = \Gamma^3_{12} = \Gamma^3_{21} = c^3_{12} \equiv 0 \tag{5.53}$$

We consider the possible sub-cases:

- $\Gamma^2_{11} \not\equiv 0$ and $\Gamma^3_{11} \not\equiv 0$. In this case $\beta^2 \equiv 0$ and $\beta^3 \equiv 0$, such that the β-system has only the trivial solution, i.e. ***Case 1*** of the theorem.
- $\Gamma^2_{11} \equiv 0$ and $\Gamma^3_{11} \not\equiv 0$. In this case $\beta^3 \equiv 0$ and the system (5.47)-(5.52) reduces to

$$0 = \beta^2 \, \Gamma^2_{33} \tag{5.54}$$

$$r_1(\beta^2) = \beta^2 \, (\Gamma^2_{12} + c^2_{12}) \tag{5.55}$$

$$r_3(\beta^2) = \beta^2 \, (\Gamma^2_{32} + c^2_{32}). \tag{5.56}$$

  If $\Gamma^2_{33} \not\equiv 0$ then the β-system has only the trivial solution (***Case 1***). Otherwise a direct calculation (using flatness and symmetry conditions (2.34), (2.35), as well as assumptions (5.53) and $\Gamma^2_{11} = \Gamma^2_{33} \equiv 0$ that are now in force), shows that the compatibility condition

$$[r_1, r_3](\beta^2) = \sum_{k=1}^{3} c^k_{13} r_k(\beta^2)$$

  holds as an identity in u, β^3, when calculated according to (5.55) and (5.56). Since $c^2_{13} \equiv 0$ we can introduce the same coordinates as in Section 5.3. Rewritten in these new coordinates, the system (5.55), (5.56) specifies the derivatives of one unknown function in two coordinate directions and therefore is of Darboux type. Applying Darboux's theorem (see [25]) we conclude that β^2 depends on 1 function of one variable. In this case $\beta^1 = \beta^3 \equiv 0$ and we obtain ***Case 2*** of the theorem.
- $\Gamma^3_{11} \equiv 0$ and $\Gamma^2_{11} \not\equiv 0$. This case reduces to the previous one upon permuting the second and third eigenvectors. We obtain either a trivial solution, or $\beta^1 = \beta^2 \equiv 0$ while β^3 depends on 1 arbitrary function of one variable. This is again ***Case 2*** of the theorem.
- $\Gamma^2_{11} \equiv \Gamma^3_{11} \equiv 0$. In this case (5.47) and (5.48) are satisfied for all β^2 and β^3. A direct computation (using flatness and symmetry conditions (2.34), (2.35), as well as assumptions (5.53) and $\Gamma^2_{11} = \Gamma^3_{11} \equiv 0$ that are now in force), shows that the compatibility conditions for the remaining PDE system (5.49)-(5.52), viz.

$$[r_1, r_3](\beta^2) = \sum_{k=1}^{3} c^k_{13} r_k(\beta^2), \qquad [r_1, r_2](\beta^3) = \sum_{k=1}^{3} c^k_{12} r_k(\beta^3) \, ,$$

  hold as identities in u, β^2 and β^3, when calculated according to (5.49)-(5.52).

  Remark 5.1. *Explicitly, the integrability conditions are given by*

$$r_3(\Gamma^2_{12} + c^2_{12}) - r_1(\Gamma^2_{32} + c^2_{32}) + c^1_{13} \, (\Gamma^2_{12} + c^2_{12}) + c^3_{13} \, (\Gamma^2_{32} + c^2_{32}) = 0 \tag{5.57}$$

$$r_1(\Gamma^3_{22}) - \Gamma^3_{22} \, (\Gamma^2_{12} + c^2_{12} - \Gamma^3_{13}) = 0 \tag{5.58}$$

$$r_2(\Gamma^3_{13} + c^3_{13}) - r_1(\Gamma^3_{23} + c^3_{23}) + c^1_{12} \, (\Gamma^3_{13} + c^3_{13}) + c^2_{12} \, (\Gamma^3_{23} + c^3_{23}) = 0 \tag{5.59}$$

$$r_1(\Gamma^2_{33}) - \Gamma^2_{33} \, (\Gamma^3_{13} + c^3_{13} - \Gamma^2_{12}) = 0 \, . \tag{5.60}$$

Unfortunately no change of variables seems to bring (5.49)-(5.52) into Darboux form (i.e. we do not obtain an equivalent system in which each equation contains only one partial derivative of one unknown function). We therefore need apply the more general Cartan-Kähler theorem. We omit the lengthy calculations that lead to the conclusion that the general solution of (5.49)-(5.52) depends on 2 arbitrary functions of one variable and two constants. Two arbitrary constants specify the values of β^2 and β^3 at an initial point $\bar{u}$ and 2 arbitrary functions of one variable prescribe the directional derivatives $s^2 = r_2(\beta^2)$ and $s^3 = r_3(\beta^3)$ along a curve. These arbitrary functions absorb the arbitrary constants. Thus, the general solution depends on 2 arbitrary functions of one variable. Recalling that $\beta^1 \equiv 0$ we conclude that this provides ***Case 3a*** of the theorem.

With this we have verified that Theorem 4.6 describes all possible degrees of freedom for the general solution of β-system, when its algebraic part has rank 1.

6. Examples

6.1. **The Euler system for 1-dimensional compressible flow.**

Example 6.1. *Extensions and entropies for the 1-d compressible Euler system have been considered by several authors, (see Remark 6.2). We now treat this particular case within our setup of prescribed eigen-frames. I.e., we first determine the eigen-frame $\mathfrak{R}$ of the Euler system and then analyze the associated λ- and β-systems. In Lagrangian variables the system is:*

$$v_t - u_x = 0 \tag{6.1}$$

$$u_t + p_x = 0 \tag{6.2}$$

$$E_t + (up)_x = 0\,, \tag{6.3}$$

where v, u, p are the specific volume, velocity, and pressure, respectively, and $E = \epsilon + \frac{u^2}{2}$ is the total specific energy. Here ϵ denotes the specific internal energy and we assume that it is given in the form of a so-called complete equation of state (EOS) $\epsilon = \epsilon(v, S)$, [32]*. The thermodynamic variables are related through Gibbs' relation $d\epsilon = TdS - pdv$, where T is absolute temperature and S is specific entropy. We make the standard sign assumptions:*

$$T = T(v, S) = \epsilon_S(v, S) > 0\,, \qquad \text{and} \qquad p = p(v, S) = -\epsilon_v(v, S) > 0\,.$$

Thermodynamic stability requires that ϵ is convex at each state (v, S) [32]*:*

$$\epsilon_{vv} > 0\,, \qquad \epsilon_{vv}\epsilon_{SS} > \epsilon_{vS}^2\,, \tag{6.4}$$

which implies that

$$p_v(v, S) = -\epsilon_{vv}(v, S) < 0\,. \tag{6.5}$$

For smooth solutions (6.1)-(6.3) may be rewritten as

$$v_t - u_x = 0 \tag{6.6}$$

$$u_t + p_x = 0 \tag{6.7}$$

$$S_t = 0\,. \tag{6.8}$$

The Jacobian of the flux $f(v, u, S) = (-u,\, p,\, 0)^T$ has eigenvalues

$$\lambda^1 = -\sqrt{-p_v}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_v}\,,$$

with corresponding right and left eigenvectors (normalized according to $R_i \cdot L^j \equiv \delta_i^j$)

$$R_1 = \left[1,\, \sqrt{-p_v},\, 0\right]^T, \qquad R_2 = \left[-p_S,\, 0,\, p_v\right]^T, \qquad R_3 = \left[1,\, -\sqrt{-p_v},\, 0\right]^T, \tag{6.9}$$

and

$$L^1 = \tfrac{1}{2}\left[1\,,\, \tfrac{1}{\sqrt{-p_v}}\,,\, \tfrac{p_S}{p_v}\right], \qquad L^2 = \left[0\,,\, 0\,,\, \tfrac{1}{p_v}\right], \qquad L^3 = \tfrac{1}{2}\left[1\,,\, -\tfrac{1}{\sqrt{-p_v}}\,,\, \tfrac{p_S}{p_v}\right]. \tag{6.10}$$

The λ-system associated with the frame $\mathfrak{R} = \{R_1,\, R_2,\, R_3\}$ was analyzed in [25]*, and there are two distinct cases:*

(a) $\left(\frac{p_S}{p_v}\right)_v \equiv 0$*: The Euler system is rich with no algebraic constraints, and the general solution of the λ-system depends on three functions of one variable.*

(b) $\left(\frac{p_S}{p_v}\right)_v \neq 0$*: There is a single algebraic relation among the eigenvalues:*

$$\lambda^1 + \lambda^3 = 2\lambda^2 \,.$$

The general solution of the λ-system is described by ***Case IIa*** *of Proposition 4.1 and depends on 2 constants $\bar{\lambda}$, C according to:*

$$\lambda^1 = \bar{\lambda} - C\sqrt{-p_v}\,, \qquad \lambda^2 \equiv \bar{\lambda}\,, \qquad \lambda^3 = \bar{\lambda} + C\sqrt{-p_v}\,.$$

We note that Case (a) occurs if and only if the pressure has the particular form

$$p(v,S) = \mathcal{P}(v + \phi(S)) \tag{6.11}$$

for functions $\mathcal{P}(\cdot)$, $\phi(\cdot)$ of 1 variable.

We proceed to analyze the β-system corresponding to $\mathfrak{R}$. A calculation (using the expressions for the coefficients Γ^k_{ij} from [25]*) shows that the three algebraic relations of $\beta(\mathfrak{R})$ are identical and express the single constraint that*

$$\left(\frac{p_S}{p_v}\right)_v \left(\beta^1 - \beta^3\right) = 0\,. \tag{6.12}$$

The two cases (a) and (b) thus yield different answers:

- *(a)* $\left(\frac{p_S}{p_v}\right)_v \equiv 0$*: There is no algebraic constraint in the β-system, and the analysis in Section 3.3 applies. The general solution of the β-system depends on 3 functions of 1 variable.*
- *(b)* $\left(\frac{p_S}{p_v}\right)_v \neq 0$*: The unique algebraic constraint is $\beta^1 \equiv \beta^3$. We claim that the β-system in this case is described by* ***Case 4a*** *of Theorem 4.6: the general solution of the β-system depends on 1 constant and 1 function of 1 variable. Furthermore, this variable is the physical entropy S. Indeed, the solution of the β-system provided by our MAPLE code is*

$$\beta^1 = \beta^3 = K_1\, p_v \qquad \text{and} \qquad \beta^2 = \frac{K_1 p_v^2}{2}\left(\int_{K_2}^{v} p_{SS}(\tau,S)\,d\tau - \frac{p_S^2}{p_v}(v,S) + F(S)\right), \tag{6.13}$$

where K_1 and K_2 are arbitrary constants and F is an arbitrary function of one variable. Since $p = -\epsilon_v$ we may re-write this as

$$\beta^1 = \beta^3 = K\,\epsilon_{vv} \qquad \text{and} \qquad \beta^2 = \frac{K\epsilon_{vv}^2}{2}\left[\frac{\epsilon_{vv}\epsilon_{SS} - \epsilon_{vS}^2}{\epsilon_{vv}} + G(S)\right], \tag{6.14}$$

where we have set $K := -K_1$ and $G(S) := -\epsilon_{SS}(K_2, S) - F(S)$.

Let's focus on the non-rich case and determine the extensions $\eta(v,u,S)$ for the Euler system. Using the expressions in (6.9) and (6.14) we need to determine η from the six relations:

$$\begin{aligned}
R_1^T(D^2\eta)R_1 &= Ke_{vv}\\
R_3^T(D^2\eta)R_3 &= Ke_{vv}\\
R_2^T(D^2\eta)R_2 &= \frac{K\epsilon_{vv}^2}{2}\left[\frac{\epsilon_{vv}\epsilon_{SS} - \epsilon_{vS}^2}{\epsilon_{vv}} + G(S)\right]\\
R_i^T(D^2\eta)R_j &= 0 \qquad \text{for } 1 \le i < j \le 3\,.
\end{aligned}$$

A straightforward calculation shows that these imply that $\eta(v,u,S)$ has the form:

$$\eta(v,u,S) = C_1 v + C_2 u + C_3\Big[\epsilon(v,S) + \frac{u^2}{2}\Big] + H(S)\,, \tag{6.15}$$

From (6.14) we can immediately determine when the extension is convex: the scalar field $\eta(v,u,S)$ given by (6.15) is strictly convex if and only if

$$C_3\epsilon_{vv} > 0 \qquad \text{and} \qquad \frac{C_3}{\epsilon_{vv}}\left(\epsilon_{vv}\epsilon_{SS} - \epsilon_{vS}^2\right) + H''(S) > 0 \qquad \text{for all } (v,S) \in \mathbb{R}_+ \times \mathbb{R}.$$

Under the assumption of thermodynamic stability (6.4), η is convex if and only if

$$C_3 > 0 \qquad \text{and} \qquad H''(S) > -\frac{C_3}{\epsilon_{vv}}\left(\epsilon_{vv}\epsilon_{SS} - \epsilon_{vS}^2\right).$$

Remark 6.2. *Extensions and entropies for the Euler system with general equations of state have been analyzed in* [21, 34, 35]. *In* [34] *the general form of an extension was derived for Case (b). In* [35] *the general form of extensions (for both Case (a) and Case (b)) was determined (*[35] *Exercise 3.19, p. 86), and the convexity of extensions of the form $g(S)$ (g a scalar map, S the physical entropy) was characterized (*[35] *Exercise 3.18, p. 85). The latter issue was also treated in* [21], *extending the analysis in* [20] *for ideal, polytropic gases.*

6.2. **Examples with rich frames.**

6.2.1. **Rich frames admitting strictly hyperbolic conservative systems.** According to the analysis in Section 3.1, any n-frame with the property that Γ_{ij}^k vanishes identically whenever $\epsilon(i,j,k) = 1$, is necessarily rich. Frames of this type admit a large family of corresponding conservative systems (1.1), parameterized by n arbitrary functions of one variable. The family contains both strictly hyperbolic and non-strictly hyperbolic systems. The solutions of the β-system enjoy the same degree of freedom (Theorem 3.1). For a strictly hyperbolic systems the solutions to the β-system provides *all* possible extensions, while non-strictly hyperbolic system may have additional extensions arising from the first part of the condition (1.7).

Example 6.3. *This example shows that $\beta(\mathfrak{R})$ may not provide all extensions for all systems (1.1) corresponding to a given frame, even when the latter admits strictly hyperbolic systems (1.1). Let $\mathfrak{R} = \{\partial_{u^1}, \partial_{u^2}\}$, the standard coordinate frame on $\mathbb{R}^2$. In this case any pair of functions of the form $(\lambda^1(u), \lambda^2(u)) = (\phi(u^1), \psi(u^2))$ solves $\lambda(\mathfrak{R})$, which thus admits strictly hyperbolic solutions. From (1.7), and the fact that the λ- and β-systems are "separated" (no unknown of one occurs in the other), one might expect that the β-system in this case would provide* all *extensions for* all *systems (1.1) with the same eigen-frame $\mathfrak{R}$. However, a simple example shows that this is incorrect. E.g., among the systems with eigen-frame $\mathfrak{R}$ there are the trivial systems $u_t + \bar{\lambda} u_x = 0$ ($\bar{\lambda} \in \mathbb{R}$), which admits any scalar function $\tilde{\eta}(u)$ as an extension. Consider the choice $\tilde{\eta}(u) = (u^1 u^2)^2/2$. The lengths of the given eigenvectors, measured with respect to the inner-product $D^2\tilde{\eta}(u)$, are $\beta^1 = (u^2)^2$ and $\beta^2 = (u^1)^2$. However, these do* not *solve $\beta(\mathfrak{R})$, which in this case consists of the two PDEs $\partial_{u^i}\beta^j = 0$, $1 \le i \ne j \le 2$.*

Example 6.4. *(*RICH ORTHOGONAL FRAME*)*

$$R_1 := (u^1,\, u^2,\, 0)^T, \qquad R_2 = (-u^2,\, u^1,\, 0)^T, \qquad R_3 = (0,\, 0,\, 1)^T.$$

This is a rich, orthogonal, and commutative frame on $\mathbb{R}^3 - \{(0,0,u_3)\}$. The λ- and β-systems impose no algebraic constraints and its general solution depends on 3 arbitrary functions of one variable (Theorem 3.1). Introducing $v = (u^1)^2 + (u^2)^2$, we have

$$\lambda^1 = F_1(v), \qquad \frac{1}{\sqrt{v}} \int_*^{\sqrt{v}} F_1(\tau^2)\, d\tau + \frac{1}{u^1} F_2\left(\frac{u^2}{u^1}\right), \qquad \lambda^3 = F_3(u^3);$$

$$\beta^1 = v\, G_1(v), \qquad \beta^2 = \sqrt{v} \int_*^{\sqrt{v}} G_1(\tau^2)\, d\tau + u^1\, G_2\left(\frac{u^2}{u^1}\right), \qquad \beta^3 = G_3(u^3).$$

The frame is orthogonal, but not orthonormal; in accordance with Observation 2.10 the solutions of the β-system may be obtained by scaling the solutions of the λ-system. Indeed, choosing

$$G_1 \equiv F_1\,, \qquad G_2(\xi) = (1+\xi^2)F_2(\xi)\,, \qquad \textit{and} \qquad G_3 \equiv F_3\,,$$

we obtain

$$\beta^1 = v\lambda^1, \qquad \beta^2 = v\lambda^2, \qquad \beta^3 = \lambda^3\,.$$

We also note any solution of the λ-system can be combined with any solution of the β-system. I.e., by choosing a particular set of functions F_1, F_2, F_3 we specify $\lambda^1, \lambda^2, \lambda^3$, and hence a conservative system (1.1) (unique up to adding a trivial flux). For this fixed conservative system any *choice of functions G_1, G_2, G_3 will provide us with an extension.*

The Riemann coordinates for this frame (in the first octant, say) are

$$w^1 = \tfrac{1}{2} \ln\left[(u^1)^2 + (u^2)^2\right], \qquad w^2 = \arctan\left(\frac{u^2}{u^1}\right), \qquad w^3 = u^3\,.$$

In terms of the Riemann coordinates we have

$$\beta^1 = e^{2w^1}\, G_1(e^{2w^1}) = \psi_1(w^1)$$

$$\beta^2 = e^{w^1} \int_*^{e^{w^1}} G_1(\tau^2)\, d\tau + e^{w^1} \cos(w^2)\, G_2\left(\tan w^2\right) = e^{w^1} \int_*^{e^{w^1}} \frac{\psi_1(\ln\tau)}{\tau^2}\, d\tau + e^{w^1}\, \psi_2(w^2)$$

$$\beta^3 = G_3(w^3) = \psi_3(w^3)\,,$$

where ψ_1, ψ_2 and ψ_3 are arbitrary functions of 1 variable. In accordance with Theorem 3.1, for a fixed point $(\bar w^1, \bar w^2, \bar w^3)$ and three arbitrary functions $\varphi_1(w^1), \varphi_2(w^2)$ and $\varphi_3(w^3)$, there is a unique solution of the β-system such that

$$\beta^1(w^1, \bar w^2, \bar w^3) = \varphi_1(w^1), \qquad \beta^2(\bar w^1, w^2, \bar w^3) = \varphi_2(w^2) \qquad \textit{and} \qquad \beta^3(\bar w^1, \bar w^2, w^3) = \varphi_3(w^3),$$

Indeed, ψ_1, ψ_2 and ψ_3 are uniquely determined from the equations:

$$\psi_1(w^1) = \varphi_1(w^1), \quad e^{\bar w^1} \int_*^{e^{\bar w^1}} \frac{\psi_1(\ln\tau))}{\tau^2}\, d\tau + e^{\bar w^1}\, \psi_2(w^2) = \varphi_2(w^2) \ \textit{and}\ \psi_3(w^3) = \varphi_3(w^3).$$

Note that dependence of the general solution on the constant $\bar w^1$ can be "hidden" in the arbitrary functions by replacing arbitrary function φ_2 with $e^{-\bar w^1}\, \varphi_2 - \int_*^{e^{\bar w^1}} \frac{\varphi_1(\ln\tau))}{\tau^2}\, d\tau$.

6.2.2. **Rich frames with no corresponding strictly hyperbolic conservative systems.** There are rich frames for which there exists a triple i, j, k, such that $\epsilon(i,j,k) = 1$ and $\Gamma_{ij}^k \neq 0$. Such frames do not admit strictly hyperbolic conservative systems and in some cases admit only trivial systems. Even in the latter case the β-system may have non-trivial solutions. This is another indication of the lack of general relationships between the number of solutions to $\lambda(\mathfrak{R})$ and $\beta(\mathfrak{R})$.

Example 6.5. Rich frame with only trivial solutions of the λ-system and non-trivial, but degenerate solutions of the β-system:

$$R_1 = (u^1,\, u^2,\, 0)^T, \qquad R_2 = (-u^2,\, u^1,\, 0)^T, \qquad R_3 = (-u^2,\, u^1,\, 1)^T.$$

This is a rich frame on $\mathbb{R}^3 - \{(0,0,u_3)\}$ with rank(2.16) = rank(2.19) = 2. *Hence the λ-system has only the trivial solution $\lambda^1 = \lambda^2 = \lambda^3 = C$. The algebraic constraints of the β-system are $\beta^1 = 0 = \beta^1 + \beta^2$, and the general solution of the β-system depends on 1 arbitrary function of one variable:*

$$\beta^1 = \beta^2 = 0, \qquad \beta^3 = F(u^3).$$

The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1, u^2, u^3) = G(u^3)\,, \qquad \textit{where } G'' = F.$$

Example 6.6. Rich frame with no strictly-hyperbolic solutions of the λ-system and non-trivial, but degenerate solutions of the β-system:

$$R_1 := (u^1,\, u^2,\, u^3)^T, \qquad R_2 = (u^1,\, u^2,\, 0)^T, \qquad R_3 = (u^1,\, 0,\, u^3)^T.$$

This is a rich commutative *frame on $\mathbb{R}^3 \setminus \{(0,0,u_3)\}$. The β-system contains 1 algebraic constraint and its solution set is described by part (3) of Theorem 4.5: the general solution depends on 2 arbitrary functions G_1 and G_2 of one variable,*

$$\beta^1 = 0, \qquad \beta^2 = u^1\, G_1\Big(\frac{u^3}{u^1}\Big), \qquad \beta^3 = u^1\, G_2\Big(\frac{u^2}{u^1}\Big).$$

A set of Riemann coordinates for this frame (in the first octant, say) are

$$w^1 = \ln u^2 + \ln u^3 - \ln u^1\,, \qquad w^2 = \ln u^1 - \ln u^3\,, \qquad w^3 = \ln u^1 - \ln u^2\,.$$

In terms of the Riemann coordinates we have

$$\beta^1(w) = 0\,, \qquad \beta^2(w) = e^{w^1+w^3}\, \psi_2(w^2)\,, \qquad \beta^3(w) = e^{w^1+w^2}\, \psi_3(w^3)\,,$$

where ψ_2, ψ_3 are arbitrary functions. In accordance with Theorem 3.1, for a fixed point $(\bar w^1, \bar w^2, \bar w^3)$ and two arbitrary functions $\varphi_2(w^2)$ and $\varphi_3(w^3)$, there is a unique solution of the β-system with

$$\beta^2(\bar w^1, w^2, \bar w^3) = \varphi_2(w^2) \qquad \textit{and} \qquad \beta^3(\bar w^1, \bar w^2, w^3) = \varphi_3(w^3)\,.$$

Indeed, ψ_2 and ψ_3 are uniquely determined by

$$\varphi_2(w^2) = e^{\bar{w}^1+\bar{w}^3}\,\psi_2(w^2) \qquad \text{and} \qquad \varphi_3(w^3) = e^{\bar{w}^1+\bar{w}^2}\,\psi_3(w^3).$$

This example shows how two arbitrary constants $\bar{w}^1 + \bar{w}^2$ and $\bar{w}^1 + \bar{w}^3$ may be "hidden" in the arbitrary functions by replacing φ_2 with $e^{-(\bar{w}^1+\bar{w}^3)}\varphi_2$ and ψ_3 with $e^{-(\bar{w}^1+\bar{w}^3)}\varphi_3$.

The solution of the λ-system is

$$\lambda^1 = F\left(\frac{u^3\,u^2}{u^1}\right) + \frac{u^3\,u^2}{u^1}F'\left(\frac{u^3\,u^2}{u^1}\right), \quad \lambda^2 = \lambda^3 = F\left(\frac{u^3\,u^2}{u^1}\right), \tag{6.16}$$

where F is an arbitrary function. In terms of the Riemann coordinates we have

$$\lambda^1(w) = h(w^1) + e^{w^1}h'(w^1), \quad \lambda^2(w) = \lambda^3(w) = h(w^1),$$

where h is an arbitrary function. In accordance with Theorem 4.3 in [25]*, for a fixed point $(\bar{w}^1, \bar{w}^2, \bar{w}^3) \in \Omega$, an arbitrary function $\phi(w^1)$, and a constant $\hat{h}$ there is a unique solution such that:*

$$\lambda^1(w^1, \bar{w}^2, \bar{w}^3) = \phi(w^1) \qquad \text{and} \qquad \lambda^2(\bar{w}^1, \bar{w}^2, \bar{w}^3) = \lambda^3(\bar{w}^1, \bar{w}^2, \bar{w}^3) = \hat{h}.$$

Indeed, the solution (6.16) is obtained by solving the ODE $\phi(w^1) = h(w^1) + e^{w^1}h'(w^1)$ for h with initial value $\hat{h} = h(\bar{w}^1)$.

6.3. **Examples with non-rich frames.**

6.3.1. **Non-rich frames admitting strictly hyperbolic conservative systems.** Any frame corresponding to a non-rich Euler system (Example 6.1) is of this type. While non-rich Euler frames admit nondegenerate extensions, Examples 6.7 and 6.8 below show that this is not always the case.

Example 6.7. Non-rich frame with strictly hyperbolic solutions of the λ-system and exactly two vanishing β^i*:*

$$R_1 = (-1,\, 0,\, u^2+1)^T\,, \qquad R_2 = \Big(\frac{u^3}{(u^2)^2-1},\, -1,\, u^1\Big)^T\,, \qquad R_3 = (1,\, 0,\, 1-u^2)^T\,.$$

This is a non-rich frame with rank(2.19) = rank(2.16) = 1*. The unique algebraic λ-constraint is $2\lambda^2 = (1-u^2)\lambda^1 + (1+u^2)\lambda^3$, and the λ-system belongs to* ***Case IIa*** *of Proposition 4.1. Its general solution depends on two constants C_1, C_2 and is given by*

$$\lambda^1 = C_1 - 2\,C_2, \qquad \lambda^2 = C_1 + (u^2-1)\,C_2, \qquad \lambda^3 = C_1\,.$$

The flux in the corresponding conservative system (1.1) is given by

$$f(u) = \begin{pmatrix} (C_1 + C_2\,(u^2-1))\,u^1 \; + C_2\,u^3, \\ u^2\,(C_1 - C_2 + \tfrac{1}{2}C_2\,u^2), \\ C_2\,u^1\,(1-(u^2)^2) - C_2\,u^2\,u^3 + (C_1 - C_2)\,u^3 \end{pmatrix}. \tag{6.17}$$

The unique algebraic β-constraint is $(u^2-1)\beta^1 = (u^2+1)\beta^3$. The general solution of the β-system belongs to ***Case 2*** *in Theorem 4.6 and is given by*

$$\beta^1 \equiv 0, \qquad \beta^2 = F(u^2), \qquad \beta^3 \equiv 0,$$

where F is an arbitrary function. The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1, u^2, u^3) = G(u^2)\,, \qquad \text{where } G'' = F.$$

It is known that there are (uniformly) strictly hyperbolic systems of the type (6.17) whose weak solutions may exhibit finite-time blowup in either BV or L^∞; see Section 9.10 in [12]*, and* [23, 24]*. However, it not known whether this can occur for systems equipped with a strictly convex entropy. For further examples of blowup phenomena see* [38, 43]*.*

Example 6.8. Non-rich frame with strictly hyperbolic solutions of the λ-system and exactly one vanishing β^i:

$$R_1 = (u^1,\, u^2,\, u^3)^T\,, \qquad R_2 = (u^2,\, u^1,\, u^3)^T\,, \qquad R_3 = (u^1,\, u^3,\, u^2)^T\,.$$

This is a non-rich frame with rank(2.19) = rank(2.16) = 1. *The unique algebraic λ-constraint is* $(u^1 - 2\lambda^2 + u^3)\lambda^1 + (u^2 - u^3)\lambda^2 + (u^2 - u^1)\lambda^3 = 0$, *and the λ-system belongs to **Case IIa** of Proposition 4.1. Its general solution depends on two constants C_1, C_2 and is given by*

$$\lambda^1 = C_1, \qquad \lambda^2 = C_1 + \frac{C_2(u^1 - u^2)}{(u^1 + u^2 + u^3)^2}, \qquad \lambda^3 = C_1 + \frac{C_2(u^2 - u^3)}{(u^1 + u^2 + u^3)^2}\,.$$

*The unique algebraic β-constraint reduces to $\beta^1 \equiv 0$. In accordance with **Case 3a** of Theorem 4.6, the general solution of the β-system depends on two arbitrary functions of 1 variable:*

$$\beta^1 \equiv 0, \qquad \beta^2 = \frac{(u^2 - u^1)^2}{u^1} F\Big(\frac{u^2 + u^3}{u^1}\Big), \qquad \beta^3 = \frac{(u^3 - u^2)^2}{u^1 + u^2} G\Big(\frac{u^1 + u^2}{u^3}\Big).$$

By exchanging R_3 in this example with the vector field $(u^1,\, u^3,\, u^3)^T$ we obtain a frame with the same type of λ- and $\beta-$ dependencies.

Remark 6.9. *The case of non-rich Euler frames together with the two examples above show that non-rich frames on $\Omega \subset \mathbb{R}^3$, admitting strictly hyperbolic conservative systems (1.1), may have none, one, or two associated β^i vanishing. We have not been able to find examples of this type for which all three β^i vanish identically.*

6.3.2. **Non-rich frames with no corresponding strictly hyperbolic conservative systems.** There are many non-rich frames which do not admit strictly hyperbolic conservative systems, and some of these admit only trivial conservative systems. The size of the solution set of the corresponding β-system does not, in general, correspond to the size of the solution set of the λ-system. In this category of frames there are examples where the β-system has only trivial solutions (Example 6.11), non-trivial but degenerate solutions (Examples 6.12, 6.13, 6.14, 6.15), as well as examples with non-degenerate solution (Examples 6.16, 6.17, 6.18).

Example 6.10. $n = 4$ frame with different algebraic ranks of the λ- and the β-systems. *This example shows that Proposition 4.2 does not generalize to systems with more than 3 equations: the algebraic parts of $\lambda(\mathfrak{R})$ and $\beta(\mathfrak{R})$ may have different ranks when $n \geq 4$. Consider the frame*

$$R_1 = (1, 0, u^2, u^4)^T, \qquad R_2 = (0, 1, u^1, 0)^T, \qquad R_3 = (u^3, 0, 1, 0)^T, \qquad R_4 = (1, 0, 0, 0)^T.$$

In this case the rank of the algebraic part of the λ-system is 3. On the other hand the algebraic part of the β-system is given by

$$\beta^3 = 0\,, \qquad \beta^4 = 0\,, \qquad -\beta^3 + u^1\beta^4 = 0, \qquad \beta^3 + (u^3)^2\beta^4 = 0\,,$$

which is of rank 2. The solution to the β system depends on two arbitrary functions F_1 and F_2 of one variable:

$$\beta^1(u) = F_1(u^4)\,, \quad \beta^2(u) = F_2(u^2)\,, \quad \beta^3(u) = \beta^4(u) \equiv 0,$$

while the λ-system has only trivial solutions.

Example 6.11. a frame with non-trivial solutions of the λ-system and only trivial solutions of the β-system:

$$R_1 = (u^1,\, -u^2,\, 0)^T, \qquad R_2 = (-u^1,\, u^2,\, 1)^T, \qquad R_3 = (1,\, 1,\, 1)^T.$$

This is a non-rich frame with rank(2.19) = 1 *on the subset of $\mathbb{R}^3$ where $u_1 \neq -u_2$. The λ-system belongs to **Case IIb** of Proposition 4.1 and has a non-trivial solution, depending on 1 constant and 1 function of 1 variable:*

$$\lambda^1 = \lambda^2 \equiv C, \quad \lambda^3 = (u^1 + u^2)\, F(u^1\, u^2) + C.$$

The only solution of the β-system is the trivial solution

$$\beta^1 = \beta^2 = \beta^3 \equiv 0\,.$$

*There are only trivial (affine) extensions in this case. This is an example of **Case 1** in Theorem 4.6.*

Example 6.12. A FRAME WITH NON-TRIVIAL SOLUTIONS OF THE λ-SYSTEM AND NON-TRIVIAL, BUT DEGENERATE SOLUTIONS OF THE β-SYSTEM:

$$R_1 = (1,\, u^2,\, 0)^T, \qquad R_2 = (u^3,\, 1,\, 0)^T, \qquad R_3 = (0,\, 0,\, 1)^T.$$

This is a non-rich frame on $\mathbb{R}^3$ *with* $\operatorname{rank}(2.19) = 1$. *The* λ*-system belongs to* ***Case IIb*** *of Proposition 4.1 and its general solution depends on 1 constant and 1 function of 1 variable:*

$$\lambda^1 = \lambda^2 \equiv C, \quad \lambda^3 = H(u^3).$$

Two of the β*'s vanish identically, while the third depends on 1 function of 1 variable:*

$$\beta^1 = \beta^2 \equiv 0\,, \qquad \beta^3 = F(u^3)\,.$$

The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1, u^2, u^3) = G(u^3)\,, \qquad\qquad \textit{where } G'' = F.$$

This is an example of ***Case 2*** *in Theorem 4.6.*

Example 6.13. A FRAME WITH NON-TRIVIAL SOLUTIONS OF THE λ-SYSTEM AND NON-TRIVIAL, BUT DEGENERATE SOLUTIONS OF THE β-SYSTEM:

$$R_1 = (1,\, 0,\, 0)^T, \qquad R_2 = (u^2,\, u^3,\, 1)^T, \qquad R_3 = (0,\, 1,\, 0)^T.$$

This is a non-rich frame on $\mathbb{R}^3$ *with* $\operatorname{rank}(2.19) = \operatorname{rank}(2.16) = 1$. *The* λ*-system belongs to* ***Case IIb*** *of Proposition 4.1 and its general solution depends on 1 constant and 1 function of 1 variable:*

$$\lambda^1 = \lambda^2 \equiv C, \quad \lambda^3 = H\big((u^3)^2 - 2\,u^2\big).$$

The general solution of β*-system depends on 2 arbitrary functions of one variable:*

$$\beta^1 = 0, \qquad \beta^2 = \tfrac{1}{2} F_1\big((u^3)^2 - 2u^2\big) + F_2(u^3), \qquad \beta^3 = F_1'\big((u^3)^2 - 2\,u^2\big)\,.$$

The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1, u^2, u^3) = \tfrac{1}{4} G_1\big((u^3)^2 - 2\,u^2\big) + G_2(u^3)\,, \qquad \textit{where } G_1' = F_1 \textit{ and } G_2'' = F_2.$$

This is an example of ***Case 3a*** *in Theorem 4.6.*

Example 6.14. TWO FRAMES WITH ONLY TRIVIAL SOLUTIONS OF THE λ-SYSTEM AND NON-TRIVIAL, BUT DEGENERATE SOLUTIONS OF THE β-SYSTEM:

$$R_1 = (1,\, 0,\, 0)^T, \qquad R_2 = (u^2,\, u^3,\, -u^2)^T, \qquad R_3 = (u^1 + u^3,\, 1,\, 0)^T.$$

This is a non-rich frame on the subset of $\mathbb{R}^3$*, where* $u^2 \neq 0$ *with* $\operatorname{rank}(2.19) = 1$. *The* λ*-system belongs to* ***Case IIb*** *of Proposition 4.1, and admits only trivial solutions:*

$$\lambda^1 = \lambda^2 = \lambda^3 \equiv C \in \mathbb{R}.$$

The general solution of β*-system depends on 2 arbitrary functions of one variable:*

$$\beta^1 = 0, \quad \beta^2 = (u^2)^2 \left[\int_*^{u^2} F_1(\tau^2 + (u^3)^2)\Big[1 + \frac{(u^3)^2}{\tau^2}\Big]\, d\tau + F_2(u^3) \right], \quad \beta^3 = u^2\, F_1\big((u^2)^2 + (u^3)^2\big).$$

The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1, u^2, u^3) = \tfrac{1}{2} \int_*^{u^2} G_1\big(\tau^2 + (u^3)^2\big)\, d\tau + G_2(u^3)\,, \qquad \textit{where } G_1' = F_1 \textit{ and } G_2'' = F_2.$$

This is an example of ***Case 3a*** *in Theorem 4.6.*

Another example of the same is provided by

$$R_1 := (0,\, 0,\, 1)^T, \qquad R_2 = (0,\, 1,\, u^1)^T, \qquad R_3 = (u^3,\, 0,\, 1)^T.$$

This frame belongs to ***Case IIa*** *of Proposition 4.1 and to* ***Case 3a*** *of Theorem 4.6. Its* λ*-system admits only trivial solutions* $\lambda^1 = \lambda^2 = \lambda^3 \equiv C \in \mathbb{R}$*, while the general solution of the* β*-system depends on 2 arbitrary functions of one variable:*

$$\beta^1 = 0, \qquad \beta^2 = F_2(u^2), \qquad \beta^3 = F_1(u^1)\,(u^3)^2\,.$$

Example 6.15. A FAMILY OF FRAMES WITH NON-TRIVIAL SOLUTIONS OF THE λ-SYSTEM AND NON-TRIVIAL, BUT DEGENERATE SOLUTIONS OF THE β-SYSTEM:

$$R_1 = (0,\, u^2,\, u^3)^T\,, \qquad R_2 = (g(u^1),\, 0,\, u^3)^T\,, \qquad R_3 = (1,\, 1,\, 0)^T\,.$$

These vector-fields form a frame on an open subset of $\mathbb{R}^3$ *where* $u^3\,(u^2 + g(u^1)) \neq 0$*. On this set the algebraic part (2.16) of* $\beta(\mathfrak{R})$ *for this frame is*

$$g'(u^1)\;\beta_1 + \beta^2 - g(u^1)\,u^2\,\beta^3 = 0\,. \tag{6.18}$$

For a generic function $g(u^1)$*, all three* β*'s are involved in the algebraic relation and we are in the situation described in Section 5.2. The general solution of the* β*-system depends on 1 arbitrary constant* C*:*

$$\beta^1 = 0 \quad \beta^2 = C(g(u^1) + u^2), \quad \beta^3 = C\,\frac{g(u^1) + u_2}{g(u^1)\,u^2}.$$

This is an Example of ***Case 3b*** *in Theorem 4.6.*

The number of β*'s involved in the algebraic relations drops in the following specific cases:*

(i) *If* $g(u^1) \equiv k \neq 0 \in \mathbb{R}$ *then (6.18) reduces to* $\beta^2 - k\,u^2\,\beta^3 = 0$ *and involves only two of* β^i*. We are then in the situation descried in Section 5.3, but the general solution of the* β*-system still depends on 1 arbitrary constant:*

$$\beta^1 = 0 \quad \beta^2 = C(k + u^2), \quad \beta^3 = C\,\frac{k + u^2}{u^2}\,.$$

Thus, this particular case also falls into ***Case 3b*** *of Theorem 4.6.*

(ii) *If* $g \equiv 0$ *then the algebraic* β*-relation (6.18) reduces to* $\beta^2 = 0$*. This situation is described in Section 5.4, and the general solution of the* β*-system depends on 2 arbitrary functions of 1 variables:*

$$\beta^1 = (u^2)^2\,G_1(u^2 - u^1) \quad \beta^2 = 0, \quad \beta^3 = G_2(u^1)\,.$$

This particular case falls into ***Case 3a*** *of Theorem 4.6.*

Interestingly, the type of the λ*-system does not depend on* g*. The algebraic part of* $\lambda(\mathfrak{R})$ *is always equivalent to* $\lambda^1 - \lambda^2 = 0$*, such that we are in* ***Case IIb*** *of Proposition 4.1. Its general solution depends on 1 arbitrary function of 1 variable and 1 constant:*

$$\text{if } g \neq 0, \text{ then } \lambda^1 = \lambda^2 \equiv K \text{ and } \lambda^3 = \frac{u^2 + g(u^1)}{g(u^1)\,u^2}\,F\left(\frac{u^3}{u^2}\,e^{-\int \frac{d\,u^1}{g(u^1)}}\right) + K; \tag{6.19}$$

$$\text{if } g \equiv 0, \text{ then } \lambda^1 = \lambda^2 \equiv K \text{ and } \lambda^3 = F(u^1). \tag{6.20}$$

Example 6.16. A FAMILY OF FRAMES WITH NON-TRIVIAL SOLUTIONS OF THE λ-SYSTEM AND NON-DEGENERATE SOLUTIONS OF THE β-SYSTEM:

$$R_1 = (1,\, g(u),\, 0)^T, \qquad R_2 = (-g(u),\, 1,\, 0)^T, \qquad R_3 = (0,\, 1,\, 1)^T.$$

The algebraic part (2.16) of $\beta(\mathfrak{R})$ *for this frame is*

$$\frac{\partial_2 g + \partial_3 g}{1 + g^2}\,(\beta^2 - \beta^1) = 0.$$

If $\partial_2 g + \partial_3 g = 0 \Leftrightarrow g(u) = h(u_1, u_2 - u_3)$ *for some function* h *of two variables, then the frame is rich of rank 0, and the general solution of the* β *(as well as for the* λ *system) depends on 3 arbitrary functions of 1 variable.*

Otherwise this is a non-rich frame on $\mathbb{R}^3$ *with* $\mathrm{rank}(2.19) = \mathrm{rank}(2.16) = 1$*. The* λ*-system belongs to* ***Case IIb*** *of Proposition 4.1 and its general solution depends on 1 constant and 1 function of 1 variable:*

$$\lambda^1 = \lambda^2 \equiv C, \quad \lambda^3 = F(u^3)$$

The general solution of the β*-system depends on 1 constant and 1 function of 1 variable:*

$$\beta^1 \equiv \beta^2 = K\,(1 + g(u)^2)\,, \qquad \beta^3 = F(u^3)\,.$$

The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1, u^2, u^3) = \tfrac{K}{2}\left[(u^1)^2 + (u^2 - u^3)^2\right] + G(u^3)\,, \qquad \text{where } G'' = F.$$

This is an example of ***Case 4a*** *in Theorem 4.6.*

Example 6.17. A FRAME WITH NON-TRIVIAL SOLUTIONS OF THE λ-SYSTEM AND NON-DEGENERATE SOLUTIONS OF THE β-SYSTEM:

$$R_1 = (0,\, u^2,\, u^3)^T, \qquad R_2 = (u^1,\, 0,\, u^3)^T, \qquad R_3 = (1,\, 1,\, 0)^T.$$

This is a non-rich frame on the open subset of $\mathbb{R}^3$ *where* $u^1 + u^2 \neq 0$ *with* $\mathrm{rank}(2.19) = \mathrm{rank}(2.16) = 1$*. The* λ*-system belongs to* **Case IIb** *of Proposition 4.1, and its general solution depends on 1 constant and 1 function of 1 variable:*

$$\lambda^1 = \lambda^2 \equiv C, \quad \lambda^3 = \frac{u^1+u^2}{u^1\,u^2} F\left(\frac{u^3}{u^1\,u^2}\right) + C$$

The general solution of the β*-system depends on 2 arbitrary constants:*

$$\beta^1 = (K_1 - K_2)(u^1+u^2)\,, \qquad \beta^2 = K_2(u^1+u^2)\,, \qquad \beta^3 = K_1 \frac{(u^1+u^2)}{u^1u^2}\,.$$

The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1,u^2,u^3) = K_1\big[u^1\ln u^1 + u^2 \ln u^2 - u^3 \ln u^3\big] + K_2(u^1-u^2)\ln u^3\,.$$

This is an example of ***Case 4b*** *in Theorem 4.6.*

Example 6.18. A FRAME WITH NON-TRIVIAL SOLUTIONS OF THE λ-SYSTEM AND NON-DEGENERATE SOLUTIONS OF THE β-SYSTEM:

$$R_1 = (1,\, u^2,\, u^3)^T\,, \qquad R_2 = (1,\, 0,\, u^3)^T\,, \qquad R_3 = (1,\, 1,\, 0)^T\,.$$

This is a non-rich frame on an open subset of $\mathbb{R}^3$ *where* $u^2 \neq 0$ *with* $\mathrm{rank}(2.19) = 1$*. The* λ*-system belongs to* **Case IIb** *of Proposition 4.1 and its general solution depends on 1 constant and 1 function of 1 variable:*

$$\lambda^1 = \lambda^2 \equiv C, \quad \lambda^3 = F(u^3\,e^{-u^1})$$

The β*-system admits non-degenerate solutions and the general solution depends on 1 constant:*

$$\beta^1 = -K\,u^2\,, \qquad \beta^2 = K\,u^2\,, \qquad \beta^3 \equiv K\,.$$

The corresponding extensions (modulo affine parts) are given by

$$\eta(u^1,u^2,u^3) = K\,\left[\tfrac{1}{2}(u^1)^2 + (1-u^2)\ln u^3\right]\,.$$

This is an example of ***Case 4c*** *in Theorem 4.6.*

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