EXISTENCE OF SOLUTIONS FOR DEGENERATE PARABOLIC EQUATIONS WITH ROUGH COEFFICIENTS

D. MITROVIC

Abstract. We prove that a sequence of quasi-solutions to non-degenerate degenerate parabolic equations with rough coefficients is strongly L^1_{loc}-precompact. The result is obtained using the H-measures and a new concept of quasi-homogeneity. A consequence of the precompactness is existence of a weak solution to the equation under consideration.

1. Introduction

In the current contribution, we consider the following equation

$$\partial_t u + \mathrm{div}_x f(t,x,u) = D^2 \cdot A(t,x,u) + s(t,x,u), \tag{1}$$

where

- The convective term $f(t,x,\lambda)$ is the Caratheodory vector, i.e. continuously differentiable in $\lambda \in \mathbf{R}$ and in $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ it belongs to $L^s(\mathbf{R}^+ \times \mathbf{R}^d) \cap BV(\mathbf{R}^+ \times \mathbf{R}^d)$, $s > 2$.
- The matrix $A(t,x,\lambda) = (a_{ij}(t,x,\lambda))_{i,j=1,\dots,d} \in \left(C^1(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}^d))\right)^{d\times d}$, is strictly increasing with respect to $\lambda \in \mathbf{R}$:
$$\begin{gathered}\forall (t,x) \in \mathbf{R}^{1+d},\ \lambda_1, \lambda_2 \in \mathbf{R},\ \lambda_1 > \lambda_2,\ \forall \xi \in \mathbf{R}^{1+d},\\ \left(A(t,x,\lambda_1) - A(t,x,\lambda_2)\right)\xi \cdot \xi \geq 0.\end{gathered} \tag{2}$$
- For any $p \in \mathbf{R}$ the distribution
$$\mathrm{div}_x f(t,x,p) + D^2 \cdot A(t,x,p) = \gamma_p \in M_{loc}(\mathbf{R}^+ \times \mathbf{R}^d). \tag{3}$$
We assume that $\gamma_p = \omega_p + \gamma_p^s$, where $\omega_p \in L^1(\mathbf{R}^+ \times \mathbf{R}^d)$ is the regular part of the measure, while γ_p^s is a singular measure (supported on the set of measure zero).
- The function $s(t,x,\lambda)$ is continuous in $\lambda \in \mathbf{R}$ and in $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ it belongs to the space of locally bounded measures, i.e.
$$s(t,x,\lambda) \in C(\mathbf{R}; \mathcal{M}(\mathbf{R}^+ \times \mathbf{R}^d)).$$

We shall prove that a Cauchy problem for equation (1) admits a weak solution by proving that a sequence of approximate solutions to (1) (e.g. the ones generated by the vanishing viscosity method) is strongly L^1_{loc}-precompact.

The latter equation is very important and it describes phenomena containing the combined effects of nonlinear convection, degenerate diffusion, and nonlinear reaction. Thus, it attracted significant amount of attention recently. It appears in a broad spectrum of applications, such as flow in porous media and sedimentation-consolidation processes which very often occur in highly heterogeneous media causing rather rough coefficients in (1) (e.g. during CO_2 sequestration process, the gas is disposed around 800m under the earth which is highly stratified surounding). However, due to obvious technical obstacles, most of the previous literature was dedicated either to homogeneous degenerate parabolic equations either to equations where the flux and diffusion are regular functions (e.g. [24, 5, 6, 7]).

Recently, several existence results for (1) in the case when the coefficients are irregular were obtained. In [19, 18] the authors considered ultra-parabolic equations satisfying (3) while in [12], a degenerate parabolic equation was considered, but with homogeneous diffusion matrix (i.e. the matrix A was independent on $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$).

We are extending the previous results on the case of fully heterogenous equation (1) with rough coefficients. The basic tools that we are using is the same as in [19, 18, 12] – H-measures [21, 11] and the kinetic approach [15, 7].

What makes the difference is the new concept of quasi-homogeneity. We shall say that the diffusion matrix is quasi-homogeneous if it can be locally estimated by a homogenous matrix.

For instance, assume that the matrix $A = A(x,y,\lambda) = diag(x^2\lambda, y^2\lambda)$. The matrix is clearly non-negative but it reaches zero at $Q = \{0\} \times \mathbf{R} \cup \mathbf{R} \times \{0\}$. However, in a neighborhood U of any point $(x,y) \in \mathbf{R}^2 \setminus Q$ we can estimate it by

$$\inf_{(x,y)\in U} \min\{|x|,|y|\}(\xi_1^2 + \xi_2^2) \leq x^2\xi_1^2 + y^2\xi_2^2,$$

i.e the matrix $A = A(x,y,\lambda) = diag(x^2\lambda, y^2\lambda)$ can be locally almost everywhere estimated by the unit matrix. If this is a case, we can adapt the techniques from the homogeneous case in order to obtain wanted results.

In general, if the matrix A'_λ is non-negative definite in the classical sense, then we can find a function $P(t,x,\lambda)$ and a constant $c > 0$ such that, at least locally in $\mathbf{R}_\lambda \times \mathbf{R}_t^+ \times \mathbf{R}^d$, it holds

$$cP(t,x,\lambda)|\xi|^2 \leq \langle A'_\lambda(t,x,\lambda)\xi, \xi\rangle.$$

Then, if the function P regular enough (and this is the case when P is continuous, and thus, according to [3], even when it is of bounded variation), we can almost everywhere estimate it by a function depending only on λ. Thus, we see that the quasi-homogeneity concept covers wide class of diffusion matrices. Here, we shall not get into this issue deeper.

The paper is organized as follows.

In Section 2, we introduce the variants of H-measures that we are going to use and prove number of auxiliary lemmas. Also, we rigorously introduce the quasi-homogeneity concept. In Section 3, we basically prove that under a genuine nonlinearity conditions, a sequence of approximate solutions to (1) is strongly precompact in $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^d)$. As a consequence, we prove existence of solutions to the Cauchy problem corresponding to (1).

2. H-measures, quasi-homogeneity, and auxiliary statements

In this section, we shall recall the notion of the H-measures – the basic tools that we are going to use. Also, we shall introduce the notion of quasi-homogeneity which is the new concept providing an improvement with respect to the previous result on the subject. Finally, we shall prove numerous auxiliary lemmas which will provide easier exposition of main results.

We start with the H-measures. The following theorem lies in the basis of the H-measures. It is given by L.Tartar [21] (and independently by P.Gerard [11] in a more general framework; see below).

Theorem 1. [21, Theorem 1.1] *If* $(\mathsf{u}_n)=((u_n^1,\dots,u_n^r))$ *is a sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$ *such that* $u_n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there exists a subsequence* $(\mathsf{u}_{n'})$ *and a complex, positive semi-definite matrix Radon measure* $\boldsymbol{\mu}=\{\mu^{ij}\}_{i,j=1,\dots,d}$ *on* $\mathbf{R}^d\times S^{d-1}$*, such that for all* $\varphi_1,\varphi_2\in C_c(\mathbf{R}^d)$ *and* $\psi\in C(S^{d-1})$*:*

$$\begin{aligned}\lim_{n'}\int\limits_{\mathbf{R}^d}\mathcal{F}(\varphi_1 u^i_{n'})(\xi)\overline{\mathcal{F}(\varphi_2 u^j_{n'})(\xi)}\psi(\frac{\xi}{|\xi|})d\xi &= \int\limits_{\mathbf{R}^d\times S^{d-1}}\varphi_1(x)\overline{\varphi_2(x)}\psi(\xi)d\mu^{ij}(x,\xi)\\ &= \langle\mu^{ij},\varphi_1\bar\varphi_2\otimes\psi\rangle,\quad i,j=1,\dots,d,\end{aligned}\tag{4}$$

where $\otimes$ *stands for the tensor product of functions in different variables.*

The measure $\boldsymbol{\mu}$ from the above theorem is called the H-measure associated to (a sub)sequence (of) (u_n).

By a complex Radon measure on a locally compact Hausdorff space X we denote an element from the dual space $(\mathrm{C}_0(X))'$. Complex Radon measures form a Banach space denoted by $\mathcal{M}_b(X)$.

The above theorem remains valid if a sequence u_n is taken from $\mathrm{L}^2_{\rm loc}$, but in that case the corresponding variant H-measure does not have to be a (complex) Radon measure, but a distribution of order 0.

By using multiplier operators associated to functions defined on S^{d-1}, we can conveniently rewrite (4). More precisely, for a function $\psi\in\mathrm{C}(\mathrm{P}^d)$ we define an operator $\mathcal{A}_\psi$ on $L^2(\mathbf{R}^d)$ by $\mathcal{A}_\psi u := ((\psi\circ\pi_P)\hat u)^\vee$, i.e.

$$(\mathcal{A}_\psi u)(\mathbf{x})=\int_{\mathbf{R}^d}e^{2\pi i x\cdot\xi}\psi\Big(\frac{\xi}{|\xi|}\Big)\hat u(\xi)\,d\xi\,.\tag{5}$$

Clearly, $\mathcal{A}_\psi$ is a bounded operator, called *(the Fourier) multiplier operator*, with norm equal to $\|\psi\|_{\mathrm{L}^\infty}$.

By applying Plancherel's theorem to (4), we can rewrite it in the form:

$$\lim_{n'}\int\limits_{\mathbf{R}^d}\Big(\mathcal{A}_\psi\,\varphi_1 u^i_{n'}\Big)(x)\,\overline{\varphi_2 u^j_{n'}(x)}dx=\int\limits_{\mathbf{R}^d\times S^{d-1}}\varphi_1(x)\overline{\varphi_2(x)}\psi(\xi)d\mu^{ij}(x,\xi).$$

The proof of Theorem 1 is based on the following commutation lemma which we will also need in the sequel.

Lemma 2. [23, Lemma 28.2] *Let* $b\in C_c(\mathbf{R}^d)$ *and* $a\in L^\infty(\mathbf{R}^d)$ *such that*

$$\begin{aligned}&\forall M>0\ \epsilon>0\ \ \exists\kappa>0\ \ \text{such that}\\ &|\xi_1-\xi_2|\le M\quad\Longrightarrow\quad|a(\xi_1)-a(\xi_2)|\le\epsilon.\end{aligned}$$

Let $\mathcal{A}_a$ *be the multiplier operator with the symbol* a*, and* B *the operator of multiplication by* b*. Then the commutator*

$$\mathcal{A}_a\circ B-B\circ\mathcal{A}_a$$

is a compact $L^2\to L^2$ *mapping.*

Notice that Theorem 1 is formulated for sequences of functions taking values in a finite dimensional Hilbert space, $\mathbf{C}^r$. In [13], for the parabolic variant of the H-measures [2], we introduced an extension of the object corresponding to sequences of functions indexed in an uncountable set. The same result holds for the classical H-measures given by Theorem 4 (it is simply enough to replace the manifold P from [13] by S^{d-1}).

Such type of extensions was given by Gerard [11] (we repeat, simultaneously and independently of L.Tartar) and Panov [17, 18]. Gerard's extension is rather general, but also rather abstract and, from our point of view, therefore hard to use. On the other hand, Panov's extension concerns H-measures corresponding to L^∞-sequences of the form $(u_n(\mathbf{x},y))_{y\in\mathbf{R}}$ which are uniformly continuous with respect to y outside a zero-measure set. In [13], we proposed the extension which is somewhere in between Panov's and Gerard's level of generality, and the representation of our object (Proposition 5) is relatively simple and thus easy to use.

Here, we shall provide the statement that we need. It is slightly modified with respect to the published version from [13]. Thus, let (u_n) and (v_n) be arbitrary bounded sequences of functions in variables $\mathbf{x}\in\mathbf{R}^d$ and $\mathbf{y}\in\mathbf{R}^m$, weakly converging to zero in $\mathrm{L}^2(\mathbf{R}^d\times\mathbf{R}^m)$. The following theorem holds (see also [21, Corollary 1.5] and [22, Remark 2, a)]).

Theorem 3. *There exists subsequences $(u_{n'})$ and $(v_{n'})$ of the sequences (u_n) and (v_n) such that there exists a measure $\mu\in\mathrm{L}^2_{\mathrm{w}*}(\mathbf{R}^{2m};\mathcal{M}_b(\mathbf{R}^d\times\S^{d-1}))$ such that for all $\rho\in\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^{2m})$, $\varphi_1\in L^s(\mathbf{R}^d)$, $s>2$, $\varphi_2\in C_c(\mathbf{R}^d)$ and $\psi\in C(S^{d-1})$:*

$$\begin{aligned}\lim_{n'}\int\limits_{\mathbf{R}^{2m}}\int\limits_{\mathbf{R}^{1+d}}\rho(\mathbf{p},\mathbf{q})\Big(P_\psi\,\varphi_1u_{n'}(\cdot,\mathbf{p})\Big)(t,\mathbf{x})\,\overline{\varphi_2(t,\mathbf{x})v_{n'}(t,\mathbf{x},\mathbf{q})}dtd\mathbf{x}d\mathbf{p}d\mathbf{q}\\ =\int\limits_{\mathbf{R}^{2m}}\rho(\mathbf{p},\mathbf{q})\,\langle\mu(\mathbf{p},\mathbf{q}),\varphi_1\bar\varphi_2\otimes\psi\rangle d\mathbf{p}d\mathbf{q}.\end{aligned}\tag{6}$$

Proof: The theorem is proved in [13, Theorem 3] in the case when $(u_n)=(v_n)$, and the sequence (u_n) converges weakly to zero in L^2 for the test functions $\varphi_1,\varphi_2\in C_c(\mathbf{R}^d)$.

The proof in the case when (u_n) and (v_n) are different sequences weakly converging to zero in $L^2(\mathbf{R}^d)$ is the same as when $(u_n)=(v_n)$.

In order to prove that we can use $\varphi_1\in L^s(\mathbf{R}^d)$, $s>2$, $\varphi_2\in C_c(\mathbf{R}^d)$, it is enough to approximate φ_1 by a family of $C_c(\mathbf{R}^d)$ functions $(\varphi_1^\varepsilon)_\varepsilon$ and to define:

$$\int\limits_{\mathbf{R}^{2m}}\rho(\mathbf{p},\mathbf{q})\,\langle\mu(\mathbf{p},\mathbf{q}),\varphi_1\bar\varphi_2\otimes\psi\rangle d\mathbf{p}d\mathbf{q}=\lim_{\varepsilon\to0}\int\limits_{\mathbf{R}^{2m}}\rho(\mathbf{p},\mathbf{q})\,\langle\mu(\mathbf{p},\mathbf{q}),\varphi_1^\varepsilon\bar\varphi_2\otimes\psi\rangle d\mathbf{p}d\mathbf{q}$$

Since (u_n) and (v_n) are bounded sequences, the latter limes is well-defined. Indeed, for any $\varepsilon_1,\varepsilon_2$, it holds

$$\begin{aligned}&\Big|\int\limits_{\mathbf{R}^{2m}}\rho(\mathbf{p},\mathbf{q})\,\langle\mu(\mathbf{p},\mathbf{q}),\varphi_1^{\varepsilon_1}\bar\varphi_2\otimes\psi\rangle d\mathbf{p}d\mathbf{q}-\int\limits_{\mathbf{R}^{2m}}\rho(\mathbf{p},\mathbf{q})\,\langle\mu(\mathbf{p},\mathbf{q}),\varphi_1^{\varepsilon_2}\bar\varphi_2\otimes\psi\rangle d\mathbf{p}d\mathbf{q}\Big|\\ &\le C_0\int_q\|\mathcal{F}(\varphi_1^{\varepsilon_1}-\varphi_1^{\varepsilon_2})(\cdot,\mathbf{q})\|_{L^2(\mathbf{R}^d)}d\mathbf{q}=\mathcal{O}(\varepsilon_1-\varepsilon_2),\end{aligned}$$

and conclusion now follows from the Cauchy criterion for convergence. ❑

Remark 4. The last theorem remains valid in the case when the test functions φ_1,φ_2 depend on the velocity variable ($\mathbf{p}$ or $\mathbf{q}$) as well, i.e. when they are taken from the space $\mathrm{C}_0(\mathbf{R}^{1+d}\times\mathbf{R}^m)$. Analogically, we can assume that $\psi\in C_c(S^d\times\mathbf{R}^m)$ as well.

The theorem also remains valid if instead of the bounded sequence (v_n) we put the sequence $(\mathcal{A}_\psi v_n)$ where $\mathcal{A}_\psi$ is the multiplier operator with the bounded symbol ψ.

Remark finally that the H-measure defined in the last theorem is absolutely continuous with respect to the Lebesgue measure (for more detailed explanation see [22, Remark 2, a)])).

If in Theorem 3, it holds $(u_n) = (v_n)$, we proved in [14], that we can describe the object μ defined in Theorem 3 more precisely by showing that it can be represented as $\mu(p,q,x,\xi) = f(p,q,x,\xi)\nu(x,\xi)$, where $\nu \in \mathcal{M}(\mathbf{R}^d \times S^{d-1})$ is a positive Radon measure, and $f \in L^1_{loc}(\mathbf{R}^d \times S^{d-1}; L^2(\mathbf{R}^{2m}) : \nu)$. More precisely, the following proposition holds.

Proposition 5. *The object $\mu \in \mathcal{M}_+(\mathbf{R}^d \times S^{d-1}; L^2(\mathbf{R}^{2m})$ defined in Theorem 3 has the form*

$$\mu(p,q,x,\xi) = f(p,q,x,\xi)\nu(x,\xi), \tag{7}$$

where $\nu \in \mathcal{M}(\mathbf{R}^d \times P)$ is a non-negative Radon measure and $f \in L^1_{loc}(\mathbf{R}^d; L^2(\mathbf{R}^{2m} \times S^{d-1}) : \nu)$.

Proof: First, notice that from (6), the Cauchy-Schwartz inequality, and the density arguments, it follows that for almost every $p, q \in \mathbf{R}$:

$$|\mu(p,q,\cdot,\cdot)(\varphi)| \leq C|\mu(p,p,\cdot,\cdot)(\varphi)|^{1/2}|\mu(q,q,\cdot,\cdot)(\varphi)|^{1/2} \tag{8}$$

for any nonnegative $\varphi \in C_0(\mathbf{R}^d \times S^{d-1})$ and an appropriate constant C (see also the proof of [17, Lemma 1]). Note that here positivity of the measures $\mu(p,p,\cdot,\cdot)$ should also be taken into account.

Then, for $\varphi \in C_0(\mathbf{R}^d \times P)$, define by

$$\nu(x,\xi)(\varphi) := \int_{\mathbf{R}^m} \mu(p,p,x,\xi)\varphi dp$$

the nonnegative Radon measure ν on $\mathbf{R}^d \times P$. Clearly, from (8), it follows that the measures $\mu(p,q,\cdot,\cdot)$ are absolutely continuous with respect to ν for almost every $p, q \in \mathbf{R}$:

$$\mu(p,q,\cdot,\cdot) << \nu. \tag{9}$$

Indeed, if $\nu(\varphi) = 0$ for some nonnegative function $\varphi \in C_0(\mathbf{R}^d \times P)$, then from the definition of $\nu(\varphi)$ and positivity of $\mu(p,p,\cdot,\cdot)$, it follows that $\mu(p,p,\cdot,\cdot)(\varphi) = 0$ for almost every $p \in \mathbf{R}$. From this fact and (8), we immediately obtain (9).

Now, the conclusion (7) follows from the Radon-Nikodym theorem. ❑

In the sequel, we shall work in the space $\mathbf{R}^d_+ = \mathbf{R}^+ \times \mathbf{R}^d$, and we shall write

$$f'_\lambda(t,x,\lambda) = F(t,x,\lambda), \quad A'_\lambda(t,x,\lambda) = a(t,x,\lambda), \quad t \in \mathbf{R}^+, \ x \in \mathbf{R}^d.$$

Also, we shall omit the conjugation when applying (6) if it does not affect the final conclusions.

If not stated otherwise, we shall assume that $\varphi \in C_c(\mathbf{R}^d_+)$, $\psi \in C(S^{d-1})$, $\rho \in C_c(\mathbf{R}^2)$.

Let us now rigorously introduce the quasi-homogeneity concept.

Definition 6. We say that the diffusion matrix $A \in \left(C^1(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}^d)))\right)^{d\times d}$ is *quasi-homogeneous on* $(a,b) \times \Omega \subset \mathbf{R} \times \mathbf{R}^{d+1}$ if there exists a matrix $q \in \left(C^1(\mathbf{R})\right)^{d\times d}$ and a constant $c > 0$ such that

$$0 \leq c\langle q(\lambda)\xi,\xi\rangle \leq \langle a(t,x,\lambda)\xi,\xi\rangle, \quad \lambda \in (a,b), \ (t,x) \in \Omega. \tag{10}$$

We shall say that the matrix a is quasi-homogeneous if it is quasi-homogeneous in a neighborhood of almost every point $(t,x,\lambda) \in \mathbf{R}^d_+ \times \mathbf{R}$.

In order to make use of the latter concept we need several auxiliary results.

Lemma 7. *For every fixed $\lambda \in \mathbf{R}$, the multiplier operators $\mathcal{A}_{\frac{1}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}$ and $\left(\mathcal{A}_{\frac{1}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}\right)'_\lambda = \mathcal{A}_{\frac{-\langle p'(\lambda)\xi,\xi\rangle}{(|\xi|+\langle q(\lambda)\xi,\xi\rangle)^2}}$ with the symbols $\frac{1}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}$ and $\frac{-\langle q(\lambda)\xi,\xi\rangle}{(|\xi|+\langle q(\lambda)\xi,\xi\rangle)^2}$, respectively, are bounded mappings from $L^s(\mathbf{R}^d_+) \to W^{1,s}(\mathbf{R}^d_+)$, i.e. they are compact operators from $L^s(\mathbf{R}^d_+)$ to $L^s_{loc}(\mathbf{R}^d_+)$, $s > 1$.*

Proof: We shall prove the statement for the multiplier operator $\mathcal{A}_{\frac{1}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}$. The proof is the same for the multiplier operator $\mathcal{A}_{\frac{-\langle p'(\lambda)\xi,\xi\rangle}{(|\xi|+\langle q(\lambda)\xi,\xi\rangle)^2}}$.

Accordingly, take an arbitrary $u \in L^2(\mathbf{R}^d)$ and notice that for a fixed $k \in \{1,\dots,d\}$, according to the Hörmander-Mikhlin theorem [16]

$$\|\partial_{x_k}\mathcal{A}_{\frac{1}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}} u\|_{L^s(\mathbf{R}^d)} \leq K\|u\|_{L^s(\mathbf{R}^d)},$$

for a constant $K > 0$ (independent on λ). Thus, the multiplier operator $\mathcal{A}_{\frac{1}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}$ is bounded as a mapping from $L^s(\mathbf{R}^d_+) \to H^s(\mathbf{R}^d_+)$, and thus, according to the Relich theorem, compact mapping from $L^s(\mathbf{R}^d_+)$ to $L^s_{loc}(\mathbf{R}^d_+)$.

The nontrivial moment here is to prove that the symbols $\frac{-\xi_k\langle p'(\lambda)\xi,\xi\rangle}{(|\xi|+\langle q(\lambda)\xi,\xi\rangle)^2}$ and $\frac{\xi_k}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}$ satisfy the conditions of the Hörmander-Mikhlin theorem. The proof is cumbersome but standard and we shall omit it. However, we address readers on the proof of [12, Corollary 23 and Corollary 24] and [20, Sect. 3.2, Example 2]. ❑

To proceed, assume that we have a sequence (v_n) bounded in $L^\infty(\mathbf{R}^d_+ \times \mathbf{R})$, and such that $v_n \rightharpoonup 0$ weak-$\star$. Denote by

$$A_0 = \{(\xi,\lambda) \in \mathbf{R}^d_+ \times \mathbf{R} : \ \langle q(\lambda)\xi,\xi\rangle = 0\}$$

$$\chi_0(\xi,\lambda) = \begin{cases} 1, & (\xi,\lambda) \in A_0 \\ 0, & (\xi,\lambda) \notin A_0 \end{cases},$$

where p is the matrix given by the homogenization conditions (6).

For a fixed $\lambda \in \mathbf{R}$, let $\mathcal{A}_{\chi_0}$ and $\mathcal{A}_{1-\chi_0}$ be multiplier operators with symbols χ_0 and $1-\chi_0$, respectively.

Denote

- by μ_a the H-measure corresponding to the sequences (v_n) and (v_n);
- by μ_{0a} the H-measure corresponding to the sequences (v_n) and $(\mathcal{A}_{\chi_0}v_n)$;
- by μ_{1a} the H-measure corresponding to the sequences (v_n) and $(\mathcal{A}_{1-\chi_0}v_n)$;
- by μ_{11} the H-measure corresponding to the sequences $(\mathcal{A}_{1-\chi_0}v_n)$ and $(\mathcal{A}_{1-\chi_0}v_n)$;
- by μ_{00} the H-measure corresponding to the sequences $(\mathcal{A}_{\chi_0}v_n)$ and $(\mathcal{A}_{\chi_0}v_n)$.

We shall need the following consequence of Lemma 2. The notations are taken from Lemma 2.

Lemma 8. *For any $k = 1, \dots, d$, the symbol $\frac{\xi_k}{|\xi| + \langle q(\lambda)\xi, \xi\rangle}$ satisfies conditions from Lemma 2, i.e. the commutator*

$$\mathcal{A}_{\frac{\xi_k}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}} \circ B - B \circ \mathcal{A}_{\frac{\xi_k}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}$$

is a compact operator from $L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$.

Proof: Take arbitrary $\xi^1, \xi^2 \in \mathbf{R}^{d+1}$ (since we have the time variable we are in $(d+1)$-dimensional space) and put $\xi^2 = \xi^1 + \zeta(\xi^2)$. If we assume $|\xi^1 - \xi^2| \le M$, we also have $|\zeta(\xi^2)| < M$. Having this in mind, we have

$$\begin{aligned}&\Big|\frac{\xi_k^2}{|\xi^2| + \langle q(\lambda)\xi^2, \xi^2\rangle} - \frac{\xi_k^1}{|\xi^1| + \langle q(\lambda)\xi^1, \xi^1\rangle}\Big| \\ &= \Big|\frac{|\xi_1| \cdot \mathcal{O}(\zeta(\xi_2))}{(|\xi^2| + \langle q(\lambda)\xi^2, \xi^2\rangle)(|\xi^1| + \langle q(\lambda)\xi^1, \xi^1\rangle)}\Big| \le \frac{M}{|\xi^1|}.\end{aligned}$$

For $|\xi^1|$ large enough, we immediately obtain the conditions of Lemma 2 fulfilled.

❑

Before we continue, we shall make a simplifying assumption. Namely, we shall assume that the sequence (v_n) is uniformly compactly supported. This is not any lost of generality since the H-measures have a local character, i.e. they are defined via compactly supported test functions. Thus, if (v_n) is not compactly supported, we can partition $\mathbf{R}^d_+$ on the sequence of balls $B(0,k)$ and on each ball define the H-measure μ_k through (v_n). Since $B(0,k)$, $k \in \mathbf{N}$, is the countable number of sets, we can assume that $\mu_i \equiv \mu_j$ on $B(0,i) \subset B(0,j)$, $i,j \in \mathbf{N}$ which implies that we can choose an H-measures defined by (v_n) on entire $\mathbf{R}^d_+$ as the inductive limit of the H-measures μ_k, $k \in \mathbf{N}$. Thus, in order to prove certain property of μ it is enough to prove it for μ_k, $k \in \mathbf{N}$.

Having this in mind, we can prove the following statement.

Lemma 9. *The following relation hold for the H-measures defined above:*

$$\mu_a = \mu_{0a} + \mu_{1a} \tag{11}$$

$$\mu_{11} = \mu_{0a} \ \text{ and } \ \mu_{00} = \mu_{1a}. \tag{12}$$

Proof: First, notice that if $v_n \rightharpoonup 0$ in $L^2(\mathbf{R}^d)$ then the sequences $(\mathcal{A}_{\chi_0} v_n)$ and $(\mathcal{A}_{1-\chi_0} v_n)$ converge weakly to zero in $L^2(\mathbf{R}^d)$ as well. Indeed, it is enough to use the Plancherel theorem.

Thus, according to Remark 4, all the mentioned H-measures are well defined. Now, it is enough to notice

$$v_n = \mathcal{A}_{\chi_0} v_n + \mathcal{A}_{1-\chi_0} v_n \tag{13}$$

to prove (11).

As it comes to (12), consider the sets

$$A_0^C = \{(\lambda, \xi) : \ \langle q(\lambda)\xi, \xi\rangle > 0\} \ \text{ and } \ A_0 = \{(\lambda, \xi) : \ \langle q(\lambda)\xi, \xi\rangle = 0\}.$$

Denote by μ_{01} the H-measure defined by the sequences $(\mathcal{A}_{\chi_0} v_n)$ and $(\mathcal{A}_{1-\chi_0} v_n)$. The restriction of the H-measure μ_{01} on the sets $Proj_{S^d}(A_0^C) = \{(\lambda, \xi) \in \mathbf{R} \times S^d :$ $\langle q(\lambda)\xi, \xi\rangle > 0\}$, and $Proj_{S^d}(A_0) = \{(\lambda, \xi) \in \mathbf{R} \times S^d : \ \langle q(\lambda)\xi, \xi\rangle = 0\}$, is zero.

Indeed, take an arbitrary $\psi \in C_c(Proj_{S^d}(A_0^C))$, and consider the limit defining the H-measure:

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\varphi\mathcal{A}_{1-\chi_0}v_n(\cdot,\lambda)\mathcal{A}_{\psi(\xi/|\xi|,\lambda)}(\varphi\mathcal{A}_{\chi_0}v_n(\cdot,p))$$
$$\overset{Lemma\ 8}{=}\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\mathcal{A}_{1-\chi_0}v_n(\cdot,\lambda)\mathcal{A}_{\psi(\xi/|\xi|,\lambda)}(\varphi^2\mathcal{A}_{\chi_0}v_n(\cdot,p))$$
$$\overset{Plancherel}{=}\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}(1-\chi_0)\psi(\xi/|\xi|,\lambda)\mathcal{F}(v_n(\cdot,\lambda))\mathcal{F}(\varphi^2\mathcal{A}_{\chi_0}v_n(\cdot,p))=0,$$

since $(1-\chi_0)\psi=0$. The statement for the set A_0 is analogical. Since the H-measure is zero on both A_0^C and on A_0 it is zero everywhere. From here and (13), we immediately conclude that (12) is correct.

❑

Lemma 10. *It holds for any $\varphi \in C_c(\mathbf{R}^d_+)$, $\rho \in C^1_c(\mathbf{R}^2)$, and every $\psi \in C(S^{d-1})$*

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\varphi f_k(t,x,\lambda)\mathcal{A}_{\chi_0}v_n\cdot\mathcal{A}_{\frac{\xi_k\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi v_n(\cdot,p)) \tag{14}$$
$$=\int_{p,\lambda}\int_{\mathbf{R}^{d+1}\times S^d}f_k(t,x,\lambda)\xi_k d\mu_{0a}.$$

Proof: We have (we use Lemma 8 on the first, the Plancherel theorem on the second, and $\langle q(\lambda)\xi,\xi\rangle=0$ on the third step):

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\varphi f_k(t,x,\lambda)\mathcal{A}_{\chi_0}v_n\cdot\mathcal{A}_{\frac{\xi_k\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi v_n(\cdot,p)) \tag{15}$$
$$=\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\mathcal{A}_{\chi_0}v_n\cdot\mathcal{A}_{\frac{\xi_k\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi^2 f_k(t,x,\lambda)v_n(\cdot,p))$$
$$=\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\frac{\xi_k\psi(\xi/|\xi|)\chi_0}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\mathcal{F}(v_n)\mathcal{F}(\varphi^2 f_k(t,x,\lambda)v_n(\cdot,p))$$
$$=\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\frac{\xi_k\psi(\xi/|\xi|)}{|\xi|}\mathcal{F}(\mathcal{A}_{\chi_0}v_n)\mathcal{F}(\varphi^2 f_k(t,x,\lambda)v_n(\cdot,p))$$
$$=\int_{p,\lambda}\int_{\mathbf{R}^{d+1}\times S^d}f_k(t,x,\lambda)\xi_k\varphi^2(t,x)\psi(\xi)\rho(p,\lambda)d\mu_{0a}dpd\lambda,$$

where on the last step we used Theorem 3. This concludes the proof of the lemma.
❑

Lemma 11. *It holds for any $\varphi \in C_c(\mathbf{R}^d_+)$, $\rho \in C^1_c(\mathbf{R}^2)$, and every $\psi \in C_c(\mathbf{R}\times S^{d-1})$*

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\rho(p,\lambda)\varphi f_k(t,x,\lambda)\mathcal{A}_{1-\chi_0}v_n\cdot\mathcal{A}_{\frac{\xi_k\psi(\lambda,\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi v_n(\cdot,p))=0. \tag{16}$$

Proof: According to Lemma 8 and the Plancherel theorem, we have:

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\rho(p,\lambda)\varphi f_k(t,x,\lambda)\mathcal{A}_{1-\chi_0}v_n\cdot\mathcal{A}_{\frac{\xi_k\psi(\xi/|\xi|,\lambda)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi v_n(\cdot,p)) \qquad (17)\\
&=\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\rho(p,\lambda)\frac{\xi_k\psi(\xi/|\xi|,\lambda)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\mathcal{F}(v_n)(\varphi^2 f_k(t,x,\lambda)v_n(\cdot,p))\\
&=\lim_{n\to\infty}\int_{p,\lambda}\int_{A_0^C}\rho(p,\lambda)\frac{\xi_k\psi(\xi/|\xi|,\lambda)\psi(\lambda,\xi)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\mathcal{F}(v_n)(\varphi^2 f_k(t,x,\lambda)v_n(\cdot,p))\\
&=I(\psi),
\end{aligned}
$$

where the limit exists (at least) along a subsequence. Then, notice that the mapping

$$
\psi\mapsto I(\psi),\quad \psi\in C_0(Proj_{S^d}(A_0^C))
$$

where $Proj_{S^d}(A_0^C)=\{(\lambda,\xi)\in\mathbf{R}\times S^d:\ \langle q(\lambda)\xi,\xi\rangle>0\}$, represents the Radon measure on the set $Proj_{S^d}(A_0^C)$. We shall prove that the operator I is actually equal to zero. Indeed, take an arbitrary $\psi\in C_0(Proj_{S^d}(A_0^C))$. Since it is supported out of the boundary of $Proj_{S^d}(A_0^C)$, there exists $j\in\mathbf{N}$ such that

$$
\langle q(\lambda)\xi,\xi\rangle\geq\frac{1}{j},\quad (\lambda,\xi)\in\operatorname{supp}(\psi).
$$

Having this in mind, we get from (17)

$$
\begin{aligned}
&\Big|\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\rho(p,\lambda)\frac{\xi_k\psi(\xi/|\xi|,\lambda)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\mathcal{F}(v_n)(\varphi^2 f_k(t,x,\lambda)v_n(\cdot,p))\Big| \qquad (18)\\
&\leq\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\rho(p,\lambda)\frac{j\psi(\xi/|\xi|,\lambda)}{j+|\xi|}\mathcal{F}(v_n)(\varphi^2 f_k(t,x,\lambda)v_n(\cdot,p))\Big|=0,
\end{aligned}
$$

since $\mathcal{F}(v_n)\to 0$ pointwisely (which provides convergence of the integral over any finite ball $B(0,M)$), and since $\frac{j\psi(\xi/|\xi|,\lambda)}{j+|\xi|}\to 0$ as $|\xi|\to\infty$ (which provide boundedness by $\mathcal{O}(1/M)$ of the integral over the complement of $B(0,M)$). Due to arbitrariness of M, we conclude that (18) is correct. The same reasoning is used e.g. in the proof of Tartar's first commutation lemma [21, Lemma 1.7].

From (18) we conclude that the variation of the measure I equals zero $|I|=0$ (see e.g. [7, p. 89]). Applying this on (17), we conclude the lemma.

❑

Lemma 12. *Assume that the matrix A is quasi-homogeneous on $(a,b)\times\Omega\subset\mathbf{R}\times\mathbf{R}^d_+$, i.e. that (10) holds in the set $(a,b)\times\Omega$. Then, for any $\varphi\in C^1_c(\Omega)$, $\rho\in C^1_0((a,b)^2)$,*

$$
\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\varphi\rho(p,\lambda)\sum_{k,j=1}^{d}a_{kj}(t,x,\lambda)\mathcal{A}_{\chi_0}v_n\mathcal{A}_{\frac{\xi_i\xi_j\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi v_n)=0. \qquad (19)
$$

Proof: The proof of the lemma is relatively simple and it is similar to the proof of Lemma 10. We use Lemma 8 and Plancherel's theorem:

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\varphi\rho(p,\lambda)\sum_{k,j=1}^{d}a_{kj}(t,x,\lambda)\mathcal{A}_{\chi_0}v_n\mathcal{A}_{\frac{\xi_i\xi_j\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi v_n) \tag{20}$$

$$=\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\rho(p,\lambda)\sum_{k,j=1}^{d}\mathcal{A}_{\chi_0}v_n\mathcal{A}_{\frac{\xi_i\xi_j\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi^2 a_{kj}(\cdot,\lambda)v_n(\cdot,\lambda))$$

$$=\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\rho(p,\lambda)\mathcal{F}(v_n)\mathcal{F}\left(\chi_0\sum_{k,j=1}^{d}\frac{a_{kj}(\cdot,\lambda)\xi_i\xi_j\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}(\varphi^2 v_n(\cdot,\lambda))\right)=0,$$

according to (10) and properties of the function χ_0. ❑

The final lemma in this section is the following.

Lemma 13. *Assume that the matrix A is quasi-homogeneous on $(a,b)\times\Omega\subset\mathbf{R}\times\mathbf{R}^{d+1}$, i.e. that (10) holds in the set $(a,b)\times\Omega$. Then, for any $\varphi\in C^1_0(\Omega)$, $\rho\in C^1_0((a,b)^2)$,*

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{t,x}\varphi(t,x)\rho(p,\lambda)\sum_{k,j=1^d}a_{kj}(t,x,\lambda)\mathcal{A}_{1-\chi_0}v_n\mathcal{A}_{\frac{\xi_i\xi_j\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}(\varphi v_n) \tag{21}$$

$$=\int_{p,\lambda}\int_{\mathbf{R}^d_+\times S^d}\rho(p,\lambda)\frac{\langle a(t,x,\lambda)\xi,\xi\rangle}{\langle q(\lambda)\xi,\xi\rangle}\psi(\xi)\varphi^2 d\mu_{1a}(t,x,\xi,p,\lambda)d\lambda dp.$$

Proof: Initial steps of the proof are the same as in the proof of the previous lemma. In order to prove the lemma, similarly as in (20), we need to show:

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\rho(\lambda)\rho(p)\mathcal{F}(v_n)\mathcal{F}\left((1-\chi_0(\lambda,\xi))\sum_{k,j=1^d}\frac{a_{kj}(\cdot,\lambda)\xi_i\xi_j\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\varphi^2 v_n(\cdot,\lambda)\right) \tag{22}$$

$$=\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\rho(\lambda)\rho(p)\mathcal{F}(v_n)\mathcal{F}\left((1-\chi_0)\frac{\langle a(\cdot,\lambda)\xi,\xi\rangle\psi(\xi/|\xi|)}{\langle q(\lambda)\xi,\xi\rangle}\varphi^2 v_n(\cdot,\lambda)\right).$$

To prove the latter, it is enough to consider the difference between the symbols on the right and left-hand side of (22) and to notice:

$$(1-\chi_0)\Big|\frac{\langle a(\cdot,\lambda)\xi,\xi\rangle}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}-\frac{\langle a(\cdot,\lambda)\xi,\xi\rangle}{\langle q(\lambda)\xi,\xi\rangle}\Big|\leq(1-\chi_0)\frac{C_1|\xi|}{|\xi|+\langle q(\lambda)\xi,\xi\rangle},$$

implying that it is enough to prove

$$\lim_{n\to\infty}\int_{p,\lambda}\int_{\xi}\rho(\lambda)\rho(p)\mathcal{F}(v_n)\mathcal{F}\left(\psi(\xi/|\xi|)(1-\chi_0(\lambda,\xi))\frac{|\xi|}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}(\varphi^2 v_n(\cdot,\lambda))\right)=0$$

and this is done in the same way as in Lemma 11.

❑

3. Quasi-solutions and kinetic formulation

In this section, we shall introduce the notion of quasi-solution to (1). In a special situation, the quasi-solution is an entropy admissible solution that singles out a physically relevant solutions to the equation (1). The notion of quasi-solution will lead to an appropriate kinetic formulation of the equation under consideration which will enable us to use the H-measures.

Definition 14. A measurable function u defined on $\mathbf{R}^+ \times \mathbf{R}$ is called a quasi-solution to (1) if $f_i(t,x,u), A_{ij}(t,x,u), s(t,x,u) \in L^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^d)$, $i,j = 1,\dots,d$, and a.e. $p \in \mathbf{R}$ the Kruzhkov type entropy equality holds

$$\begin{aligned}&\partial_t |u-p|^{\pm} + \operatorname{div}\left[\operatorname{sgn}_{\pm}(u-p)(f(t,x,u)-f(t,x,p))\right] \\ &- D^2 \cdot \left[\operatorname{sgn}_{\pm}(u-p)(A(x,u)-A(x,p))\right] = \zeta(t,x,p),\end{aligned} \tag{23}$$

where $\zeta \in C(\mathbf{R}_p; w\star\!-\!\mathcal{M}_+(\mathbf{R}^+ \times \mathbf{R}^d))$ we coin as the quasi-entropy defect measure.

Remark 15. Remark that the measure $\zeta(t,x,p)$ can be rewritten in the form $\zeta(t,x,p) = \bar\zeta(t,x,p) + \operatorname{sgn}_{\pm}(u-p)[\omega_q(t,x) + s(t,x,u)] - |\gamma^p_s|$, for a measure $\bar\zeta$. If this measure is positive, then the quasi-solution u is Kruzhkov's entropy solution to (1).

From the latter entropy conditions, the following kinetic formulation can be proved.

Theorem 16. *The function u is a quasi-solution to* (1) *if and only if the functions*

$$h_{\pm}(t,x,\lambda) = \operatorname{sgn}_{\pm}(u(t,x)-\lambda) = \left(|u(t,x)-\lambda|^{\pm}\right)'_{\lambda} \tag{24}$$

are solutions to the following linear equations:

$$\partial_t h_{\pm} + \operatorname{div}\left(F(t,x,\lambda)h_{\pm}\right) - D^2 \cdot [a(x,\lambda)h_{\pm}] = \partial_\lambda \zeta(t,x,\lambda) \tag{25}$$

Proof: It is enough to find derivative of (23) with respect to $p \in \mathbf{R}$ to obtain (25).

Vice versa can be also proven, i.e. that if $h_{\pm}$ satisfies (25) then the function u from (24) represents a quasi-solution to (1). It is enough to integrate (25) over $\lambda \in (-M,p)$, where $-M$ is a lower bound of u. ❑

The main theorem of the paper is the following,

Theorem 17. *Assume that the functions f and A from* (1) *are such that for almost every $(t,x) \in \mathbf{R}^d_+$ and every $\xi \in S^d$ the mapping*

$$\lambda \mapsto \left| i\left(\xi_0 + \sum_{k=1}^d F_k(t,x,\lambda)\xi_k\right) + \langle a(t,x,\lambda)\xi,\xi\rangle \right|, \tag{26}$$

where i is the imaginary unit, is not constant on any set of measure greater than zero.

Assume also that the matrix A is entirely quasi-homogeneous.

Then, a bounded sequence (u_n) of quasi-solutions to (1) *is strongly precompact in $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^d)$.*

Proof: Denote by $h \in L^\infty(\mathbf{R}^d_+)$ an L^∞ weak-$\star$ limit along a subsequence of the sequence $h_+(\lambda - u_n(t,x))$ given by (24). Denote by $\zeta \in C(\mathbf{R}_p; w\star\!-\!\mathcal{M}_+(\mathbf{R}^+ \times \mathbf{R}^d))$ the weak limit of the sequence (ζ_n) of quasi-entropy defect measures corresponding to (v_n). Put $v_n(t,x,\lambda) = h(t,x,\lambda) - h_n(t,x,\lambda)$ and $\sigma_n = \zeta - \zeta_n$. The sequence (v_n) satisfies:

$$\partial_t v_n + \operatorname{div}\left(F(t,x,\lambda)v_n\right) - D^2\cdot[a(t,x,\lambda)h_\pm] = \partial_\lambda \sigma_n(t,x,\lambda) \tag{27}$$

We continue with a special choice of the test function to be applied in (27). We take for a fixed p and $\psi \in C^{\lfloor \frac{d}{2}\rfloor+1}(S^{d-1})$ [1]:

$$\theta(t,x,\lambda,p) = \varphi(t,x)\rho(\lambda,p)\mathcal{A}_{\frac{\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}}\varphi v_n(t,x,p)$$

Now, since $\mathcal{M}_{loc}(\mathbf{R}^d_+)$ is compactly embedded in $W^{1,s}_{loc}(\mathbf{R}^d_+)$, $s\in[1,\frac{d}{d-1}]$, from the Plancherel theorem (left-hand side) and Lemma 7 (right-hand side), we obtain after integrating over $p\in\mathbf{R}$ (i is the imaginary unit below)

$$\begin{aligned} i\int_{\xi,\lambda}\Big(&\frac{\xi_0\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\rho(\lambda,p)\mathcal{F}(\varphi v_n(\cdot,\lambda)) \\ &+\sum_{k=1}^d \frac{\xi_k\psi(\xi/|\xi|)}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\rho(\lambda,p)\mathcal{F}(F_k(\cdot,\lambda)\varphi v_n(\cdot,\lambda))\Big)\mathcal{F}(\varphi v_n(\cdot,p)) \\ -\int_{\xi,\lambda}&\rho(\lambda,p)\mathcal{F}\left(\frac{\langle a(t,x,\lambda)\xi,\xi\rangle}{|\xi|+\langle q(\lambda)\xi,\xi\rangle}\varphi(t,x)v_n(t,x,\lambda)\right)\mathcal{F}(\varphi(t,x)v_n(t,x,p)) = o_n(1), \end{aligned} \tag{28}$$

as $n\to\infty$. Next, notice that

$$v_n(t,x,\lambda) = \mathcal{A}_{\chi_0}v_n + \mathcal{A}_{1-\chi_0}v_n,$$

and let $n\to\infty$ in (28). We get after taking into account Lemmas 10-13:

$$\begin{aligned} &i\int_{p,\lambda}\int_{\mathbf{R}^{d+1}_+\times S^d}\left(\xi_0+\sum_{k=1}^d F_k(x,\lambda)\xi_i\right)\rho(\lambda)\varphi(t,x)d\mu_{0a}d\lambda dp \\ &-\int_{p,\lambda}\int_{\mathbf{R}^{d+1}_+\times S^d}\frac{\langle a(t,x,\xi)\xi,\xi\rangle}{\langle q(\lambda)\xi,\xi\rangle}\rho(\lambda)\varphi(t,x)d\mu_{1a}d\lambda dp = 0. \end{aligned}$$

From here and Lemma 9, we conclude

$$\left(\xi_0+\sum_{k=1}^d F_k(x,\lambda)\xi_i\right)d\mu_{0a} = \left(\xi_0+\sum_{k=1}^d F_k(x,\lambda)\xi_i\right)d\mu_{00} = 0 \tag{29}$$

$$\frac{\langle a(t,x,\xi)\xi,\xi\rangle}{\langle q(\lambda)\xi,\xi\rangle}d\mu_{1a} = \frac{\langle a(t,x,\xi)\xi,\xi\rangle}{\langle q(\lambda)\xi,\xi\rangle}d\mu_{11} = 0. \tag{30}$$

From (30), it follows

$$0 = \frac{\langle a(t,x,\xi)\xi,\xi\rangle}{\langle q(\lambda)\xi,\xi\rangle}d\mu_{11} \geq \frac{c\langle q(\lambda)\xi,\xi\rangle}{\langle q(\lambda)\xi,\xi\rangle}d\mu_{11} = cd\mu_{11} \tag{31}$$

implying that $d\mu_{11}\equiv 0$.

Next, notice that

$$\langle q(\lambda)\xi,\xi\rangle d\mu_{00}\equiv 0,$$

---

[1] Notice that we need ψ of higher regularity in order to apply the Hörmander-Mikhlin theorem. Eventually, it does not affect the procedure since once we let $n\to\infty$, we are in the realm of measures and we can replace such ψ by a $C(S^{d-1})$-function

since μ_{00} is defined via $\mathcal{A}_{\chi_0}$ and $\chi_0\langle q(\lambda)\xi,\xi\rangle \equiv 0$. Thus, from (29), we conclude

$$0 = i\left(\xi_0 + \sum_{k=1}^d F_k(x,\lambda)\xi_i\right) d\mu_{0a} = \left(i\left(\xi_0 + \sum_{k=1}^d F_k(x,\lambda)\xi_i\right) - \langle q(\lambda)\xi,\xi\rangle\right) d\mu_{00}$$
$$\implies \; d\mu_{00} = 0,$$

due to assumption (26). Indeed, according to the representation theorem Proposition 5, we can rewrite the last formula as

$$\left(i\left(\xi_0 + \sum_{k=1}^d F_k(x,\lambda)\xi_i\right) - \langle q(\lambda)\xi,\xi\rangle\right) f_{00}(p,\lambda,t,x,\xi) d\nu_{00}(t,x,\xi) = 0. \tag{32}$$

If we denote

$$\Xi(\lambda,t,x,\xi) = \begin{cases} 1, & \left|i\left(\xi_0 + \sum\limits_{k=1}^d F_k(x,\lambda)\xi_i\right) - \langle q(\lambda)\xi,\xi\rangle\right| \neq 0 \\ 0, & \left|i\left(\xi_0 + \sum\limits_{k=1}^d F_k(x,\lambda)\xi_i\right) - \langle q(\lambda)\xi,\xi\rangle\right| = 0 \end{cases},$$

then from (32), after integration over $\mathbf{R}^2 \times \mathbf{R}^d_+ \times S^{d-1}$ we conclude that for any $\rho \in C_c(a,b)$, $\varphi \in C_c(\mathbf{R}^d_+ \times S^{d-1})$:

$$\begin{aligned} &\int_{t,x,\xi}\int_{p,\lambda} \Xi(\lambda,t,x,\xi) f_{00}(p,\lambda,t,x,\xi) dp d\lambda d\nu_{00}(t,x,\xi) \\ &= \int_{t,x,\xi}\int_{p,\lambda} \rho(p)\rho(\lambda)\varphi(t,x,\xi) f_{00}(p,\lambda,t,x,\xi) dp d\lambda d\nu_{00}(t,x,\xi) = 0, \end{aligned} \tag{33}$$

since for almost every $(t,x) \in \mathbf{R}^d_+$ and every $\xi \in S^{d-1}$, the mapping $\lambda \mapsto \Xi(\lambda,t,x,\xi)$ equals one almost everywhere. One should also keep in mind that the H-measure μ_{00} is absolutely continuous with respect to the Lebesgue measure (see Remark 4).

Thus, since $d\mu_{11} = d\mu_{1a} \equiv 0$ and $d\mu_{00} = d\mu_{0a} = 0$ (under the assumption (26)), we conclude (Lemma 9)

$$d\mu_a = d\mu_{1a} + d\mu_{0a} \equiv 0,$$

in a neighborhood of any point where the diffusion matrix is quasi-homogeneous. Thus, the sequence $(\int_\lambda \rho(\lambda) v_n(t,x,\lambda))$ converges strongly in $\Omega \subset \mathbf{R}^d_+$ (see e.g. [13, p. 281]).

Now, we simply choose a countable dense subset $\hat{Q}$ of the set $Q \subset \mathbf{R}^d_+ \times \mathbf{R}$ where the quasi-homogeneity is fulfilled and take the subsequence $(v^m_{n'})$ of the sequence (v_n) such that $(\int_\lambda \rho(\lambda) v^m_{n'}(t,x,\lambda))$ converges in a neighborhood $Proj_{t,x}(\hat{Q}_m)$ of $Proj_{t,x}(\hat{Q})$ for any ρ supported in $Proj_\lambda(\bar{Q})m)$, where $m \in \mathbf{N}$ is such that $meas(\mathbf{R}^d_+ \times \mathbf{R}\backslash\bar{Q}_m) < \frac{1}{m}$. By letting $m \to \infty$ we see that we can choose a subsequence of (v_n) such that $\int_\lambda \rho(\lambda) v_n(t,x,\lambda)$ strongly converges along the subsequence on entire $\mathbf{R}^d_+ \times \mathbf{R}$.

From here, it is standard to conclude that the sequence (u_n) strongly converges in $L^1_{loc}(\mathbf{R}^d_+)$ (see e.g. [19, 1]).

❑

An obvious consequence of the proof of the last theorem is:

Corollary 18. *Assume that a bounded sequence of functions* (u_n) *satisfies for almost every* $p \in \mathbf{R}$*;*

$$\partial_t |u-p|^{\pm} + \operatorname{div}\left[\operatorname{sgn}_{\pm}(u-p)(f(t,x,u)-f(t,x,p))\right]$$
$$- D^2 \cdot \left[\operatorname{sgn}_{\pm}(u-p)(A(x,u)-A(x,p))\right] \quad \text{is strongly precompact in } W^{-1,s}$$

for some $s > 1$*, where the flux* f *and the diffusion matrix* A *satisfy the genuine-nonlinearity conditions* (26)*, and the matrix* A *is entirely quasi-homogeneous. Then, the sequence* (u_n) *is strongly* L^1_{loc}*-precompact.*

A direct consequence of Corollary 18 is the following existence statement.

Theorem 19. *There exists a weak solution to* (1) *augmented with the initial conditions*

$$u|_{t=0} = u_0 \in L^1 \cap L^\infty(\mathbf{R}^d_+) \tag{34}$$

providing that the genuine nonlinearity conditions (26) *are satisfied, and that there exist smooth regularization* (f_n) *and* (A_n) *with respect to* $(t,x) \in \mathbf{R}^d_+$ *of the flux* f *and the diffusion matrix* A *such that the conditions of* [8, Theorem 6.1] *are satisfied.*

Proof: It is enough to consider the regularization of problem (1), (34):

$$\begin{aligned} &\partial_t u_n + \operatorname{div}_x f_n(t,x,u_n) = D^2 \cdot A_n(t,x,u_n) + s(t,x,u_n) \\ &u_n|_{t=0} = u_0(x) \end{aligned} \tag{35}$$

According to [8, Theorem 6.1] the latter problem generates the sequence (u_n) of entropy solutions to (35). It is not difficult to see that the sequence (u_n) satisfies conditions of Corollary 18 which in turn implies strong $L^1_{loc}(\mathbf{R}^d_+)$ precompactness of (u_n) whose limit along a subsequence represents the weak solution to (1), (34).

❑

Acknowledgement Darko Mitrovic is engaged as a part time postdoc at the University of Bergen in the frame of the project "Modeling transport in porous media over multiple scales" of the Research Council of Norway. We gratefully acknowledge the support of the Research Council of Norway.

References

[1] J. Aleksic, D. Mitrovic, *Strong Traces for Entropy Solutions of Heterogeneous Ultra-parabolic Equation*, preprint: www.math.ntnu.no/conservation/2010

[2] N. Antonić, M. Lazar, *H-measures and variants applied to parbolic equations*, J. Math. Anal. Appl., 343 (2008), 207–225.

[3] S. Bianchini and D. Tonon, *A decomposition theorem for BV functions*, Comm. Pure. Appl. Anal. **10** (2011), 1549 – 1566.

[4] M. C. Bustos, F. Concha, R. Burger and E. M. Tory, *Sedimentation and Thickening: Phenomenological Foundation and Mathematical Theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[5] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Rational Mech. Anal., **147** (1999), 269–361.

[6] G.Q. Chen and E. DiBenedetto, *Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations*, SIAM J. Math. Anal., **33** (2001), 751–762.

[7] G.Q. Chen and B. Perthame, *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations*, Ann. Inst. H. Poincare Anal. Non Lineaire, **4** (2002), 645–668.

[8] G. Q. Chen, K. H. Karlsen, *Quasilinear Anisotropic Degenerate Parabolic Equations with Time-Space Dependent Diffusion Coefficients*, Comm. Pure and Appl. Anal., **4** (2005), 241–266.

[9] M. S. Espedal and K. H. Karlsen, *Numerical solution of reservoir flow models based on large time step operator splitting algorithms*, Filtration in Porous Media and Industrial Applications (Cetraro, Italy, 1998), volume 1734 of Lecture Notes in Mathematics, pages 9–77, Springer, Berlin, 2000.

[10] G. B. Folland, *Real Analysis*, Willey, New York, 1984.

[11] P. Gérard, *Microlocal Defect Measures*, Comm. Partial Differential Equations 16(1991), 1761–1794.

[12] H. Holden, K. H. Karlsen, D. Mitrovic, E. Yu. Panov, *Strong Compactness of Approximate Solutions to Degenerate Elliptic-Hyperbolic Equations with Discontinuous Flux Functions*, Acta Mathematica Scientia **29B** (2009), 1573-1612.

[13] M. Lazar, D. Mitrovic, *The velocity averaging for a heterogeneous heat type equation*, Math. Commun. **16** (2011), 271–282.

[14] M. Lazar, D. Mitrovic, *Velocity averaging – a general framework*, preprint.

[15] P. L. Lions, B. Perthame, E. Tadmor, *A kinetic formulation of multidimensional scalar conservation law and related equations*, J. Amer. Math. Soc. **7**, (1994), 169–191.

[16] G. Okikiolu, *Aspects of the Theory of Bounded Integral operators in L^p-Spaces*, Academic Press, 1971.

[17] E. Yu. Panov, *On sequences of measure-valued solutions of a first order quasilinear equations*, Rusiian Acad. Sci. Sb. Math. **81** (1995), 211–227.

[18] E. Yu. Panov, *Ultra-parabolic equations with rough coefficients. Entropy solutions and strong precompactness property*, J. Math. Sci., **159** (2009), 180–228.

[19] S. A. Sazhenkov, *The genuinely nonlinear Graetz-Nusselt ultra-parabolic equation*, Siberian Math. J. **47** (2006), 355–375.

[20] E. M. Stein, *Singular Integrals and Differential Properties of Functions*, Princeton University Press, 1970.

[21] L. Tartar, *H-measures, a new approach for studying homogenisation, oscillation and concentration effects in PDEs*, Proc. Roy. Soc. Edinburgh Sect. A 115:3–4 (1990) 193–230.

[22] E. Yu. Panov, *Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux*, Arch. Rational Mech. Anal. **195** (2010) 643-673.

[23] L. Tartar, *The General Theory of Homogenization: A Personalized Quest*, Springer-Verlag Berlin Heidelberg, 2009.

[24] A.I. Vol'pert and S.I. Hudjaev, *Cauchy's problem for degenerate second order quasilinear parabolic equations*, Math. USSR Sbornik, **7** (1969), 365–387.

Darko Mitrovic, Faculty of Mathematics, University of Montenegro
E-mail address: matematika@t-com.me