# Singular limits for a parabolic-elliptic regularization of scalar conservation laws

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**Abstract**

We consider scalar hyperbolic conservation laws with a nonconvex flux, in one space dimension. Then, weak solutions of the associated initial-value problems can contain undercompressive shock waves. We regularize the hyperbolic equation by a parabolic-elliptic system that produces undercompressive waves in the hyperbolic limit regime. Moreover we show that in another limit regime, called capillarity limit, we recover solutions of a diffusive-dispersive regularization, which is the standard regularization used to approximate undercompressive waves. In fact the new parabolic-elliptic system can be understood as a low-order approximation of the third-order diffusive-dispersive regularization, thus sharing some similarities with the relaxation approximations. A study of the traveling waves for the parabolic-elliptic system completes the paper.

## 1 Introduction

Consider for the unknown $u = u(x,t)$ the homogeneous scalar law

$$u_t + f(u)_x = 0 \tag{1.1}$$

in $\Omega_T := \mathbb{R} \times (0,T)$, $T > 0$. Here, $f : \mathbb{R} \to \mathbb{R}$ is a smooth flux function which we assume to satisfy $f(0) = 0$ without any loss of generality. Provided that $f$ is nonlinear, it is well known that solutions of the Cauchy problem associated to (1.1) can contain discontinuities, even for smooth initial data [6, 25]; such solutions then must be understood in a weak sense. Interesting wave patterns occur if the flux $f$ is not convex or concave, i.e., if $f''$ vanishes at one point at least: in this case it is possible to construct weak solutions that contain *undercompressive waves*.

In order to clarify what we mean by an undercompressive shock wave, consider for $u_\pm \in \mathbb{R}$ and $s = (f(u_+) - f(u_-))/(u_+ - u_-)$ the weak solution

$$U(x,t) = \begin{cases} u_- & \text{if } x - st < 0, \\ u_+ & \text{if } x - st > 0. \end{cases} \tag{1.2}$$

The function $U$ is called a (compressive) Lax shock wave if the inequalities

$$f'(u_-) > s > f'(u_+) \tag{1.3}$$

hold. On the contrary, in this paper we focus on undercompressive shock waves $U$, which must fulfill either

$$f'(u_\pm) < s \quad \text{or} \quad f'(u_\pm) > s. \tag{1.4}$$

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The remaining doubly undercompressive case $f'(u_-) < s < f'(u_+)$ is not taken into account in our study; this case occurs, for instance, in the Chapman-Jouguet theory of deflagration waves [5, 10]. The major interest in undercompressive waves stems from the fact that they appear in several applications as bulk interfaces representing, e.g., phase boundaries [12], saturation fronts [29], precursors in thin film flow [2] and so on. In this framework the scalar case (1.1) considered in this paper must be understood as a simplified model which, however, captures main features of the problem.

In the general theory of conservation laws, a common approach to select meaningful weak solutions consists first in regularizing the system under consideration and then in studying the limit of the solutions obtained for the regularized system when some characteristic parameter vanishes. In the context of this paper the latter step is called the *sharp-interface* (or *hyperbolic*) limit. For a standard viscous regularization of (1.1) only compressive waves can occur in this limit [6, §8.6]. However, undercompressive waves can be driven as well by more refined regularizations. The diffusive-dispersive regularization

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon + \gamma\varepsilon^2 u_{xxx}^\varepsilon \tag{1.5}$$

is by now classical. Here, the singular parameter $\varepsilon$ is assumed to be positive. The parameter $\gamma > 0$ keeps dissipation and dispersion in balance; it plays a role only in the study of traveling waves and then the dependence on $\gamma$ is omitted in the following. For the analysis of the sharp-interface limit $\varepsilon \to 0$ of (1.5) we refer to [24, 11]. A complete study of the traveling waves in the case $f(u) = u^3$, including (1.2) under the condition (1.4), can be found in [12]. Moreover, the analysis in [12] rules out the possibility of the doubly undercompressive case quoted above.

Let $f$ be chosen such that $f'$ is nonnegative or bounded. Denote by $F : \mathbb{R} \to \mathbb{R}$ the primitive of the flux $f$. Then it is readily checked that smooth solutions $u^\varepsilon$ of (1.5) satisfy

$$\frac{d}{dt} E^\varepsilon[u^\varepsilon(.,t)] \leq 0 \tag{1.6}$$

for $t \in (0,T)$, where $E^\varepsilon$ is the van der Waals' type energy

$$E^\varepsilon[u] = \int_{\mathbb{R}} \left( F(u) + \gamma\frac{\varepsilon^2}{2} u_x^2 \right) dx. \tag{1.7}$$

Such energies are used to realize phase separation in multi-phase systems. Indeed, in applications, the energy density $F$ is supposed to have multiple-well structure, in order to determine different phases.

We point out that $E^\varepsilon$ is not the *only* physically relevant choice; then, it is natural to investigate the existence of other energies that are dissipated as well [22] by suitable regularizations of (1.1). The aim of this paper is to analyze the energy

$$E^{\varepsilon,\alpha}[u,\lambda] = \int_{\mathbb{R}} \left( F(u) + \frac{\alpha}{2}(u-\lambda)^2 + \gamma\frac{\varepsilon^2}{2}\lambda_x^2 \right) dx, \tag{1.8}$$

which introduces both the unknown $\lambda$ and the coupling parameter $\alpha > 0$. The functional $E^{\varepsilon,\alpha}$ has been introduced in [26, 27]; we refer to [3, 20, 21] for related functionals. Note that no derivatives of $u$ appear in (1.8), differently from (1.7). We will show in Lemma 3.3 that the energy $E^{\varepsilon,\alpha}$ is dissipated by the following parabolic regularization of (1.1) containing an elliptic constraint, in $\Omega_T$:

$$\left\{ \begin{array}{rcl} u_t^{\varepsilon,\alpha} + f(u^{\varepsilon,\alpha})_x & = & \varepsilon u_{xx}^{\varepsilon,\alpha} - \alpha(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha})_x\,, \\ -\gamma\varepsilon^2\lambda_{xx}^{\varepsilon,\alpha} & = & \alpha(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha})\,. \end{array} \right. \tag{1.9}$$

Indeed, in order to avoid technicalities, most proofs are carried out for the case

$$f(u) = u^3, \tag{1.10}$$

the simplest nonconvex flux leading to undercompressive waves. An extension of our results, either to general smooth fluxes with $f' \geq 0$ and polynomial growth [24] or to globally Lipschitz-continuous fluxes [15], is possible with minor changes.

Here follows a more precise description of the content of the paper. In Section 2 we show that profiles of undercompressive waves can be realized by (1.9), as for the diffusive-dispersive equation (1.5), for suitable values of the parameter $\gamma$. The proof of this result relies on the geometric singular perturbation theory, see [8, 9] or [13] for a detailed introduction, using $\alpha$ as singular parameter. In the following sections we put for simplicity $\gamma = 1$. Section 3 contains some a-priori estimates to be used in the following. We continue in Section 4 with the analysis of the initial-value problem for (1.9); there, we prove the existence of global classical solutions. Finally, in Section 5, we will focus on the behavior of solutions to (1.9) when either $\alpha \to \infty$ or $\varepsilon \to 0$. The results re-display exactly what was proven in [26] by variational calculus for the minimizers of $E^{\varepsilon,\alpha}$. First, for fixed $\varepsilon > 0$, we consider the *diffusive-dispersive* (or *capillarity*) limit $\alpha \to \infty$ and obtain convergence to solutions of the diffusive-dispersive regularization (1.5). Then, for $\alpha > 0$ fixed, in the *sharp-interface limit* $\varepsilon \to 0$ we get convergence to solutions of the hyperbolic equation (1.1). In both cases the crucial point consists in carefully exploiting the dissipation of the energy $E^{\varepsilon,\alpha}$. An appendix shows some connections between our results for (1.9) and the relaxation limits for a class of scaled systems arising in the modeling of flows of radiating gases [16].

# 2 Undercompressive Shock Waves and Dissipative Admissibility

The aim of this section is to prove that, for all $\varepsilon > 0$ and $\alpha$ sufficiently large, the parabolic-elliptic system (1.9)–(1.10) admits smooth traveling-wave solutions, which converge almost everywhere for $\varepsilon \to 0$ to an *undercompressive* shock wave solution of (1.1). This is proved in Theorem 2.2. Therefore, in the regime $\alpha >> 1$ and at least as far as traveling-wave solutions are concerned, the diffusive-dispersive equation (1.5) can be understood as a singular perturbation of (1.9).

Traveling-wave profiles of both Lax and undercompressive shock waves for the equation (1.5) have been proved to exist [12, 11] and behave analogously in the limit $\varepsilon \to 0$. We refrain from making a similar complete traveling-wave analysis for (1.9) because the current paper focuses on undercompressive waves.

The notation in this section slightly differs from that used in the rest of the paper since we use lower indices for simplicity. Both parameters $\varepsilon$ and $\alpha$ are fixed in the following; assumption (1.10).

About the scalar equation (1.1), if $U$ is a shock wave as in (1.2) then $s$ is given by

$$s = u_-^2 + u_- u_+ + u_+^2. \tag{2.1}$$

We only consider the case

$$u_- > 0, \tag{2.2}$$

since the equation (1.5) is invariant under the transformation $u \to -u$. Under (2.2) the first condition in (1.4) is empty while the second one holds if

$$-2u_- < u_+ < -\frac{u_-}{2}. \tag{2.3}$$

Throughout this section we always assume (2.2)–(2.3) and $s$ is given by (2.1).

## 2.1 Traveling Waves for the Diffusive-Dispersive Regularization

We search for traveling waves, i.e., solutions

$$U^\varepsilon(x,t) = u\left(\frac{x - st}{\varepsilon}\right)$$

that satisfy (1.5) together with $u(\pm\infty) = u_\pm$, $u'(\pm\infty) = 0$ and $u''(\pm\infty) = 0$. The search for a traveling wave solution $U^\varepsilon$ can be formulated as the ordinary boundary-value problem

$$\left\{\begin{array}{rcll} u' &=& z, & u(\pm\infty) = u_\pm, \\ \gamma z' &=& -z - su + f(u) + c, & z(\pm\infty) = 0, \end{array}\right. \tag{2.4}$$

with unknowns $(u, z) : \mathbb{R} \to \mathbb{R}^2$, where

$$c := su_- - f(u_-) = su_+ - f(u_+). \tag{2.5}$$

There are three rest points $(u, 0)$ for the flow in (2.4), namely, for $u$ assuming the values

$$u_+ < u_0 = -(u_+ + u_-) < u_-,$$

where the inequalities follow from (2.3). It is useful to introduce the cubic polynomial

$$p(u) := su - f(u) - c,$$

which vanishes at the rest points; the dependence on $u_\pm$ is dropped for simplicity. Then, $p$ is bistable and the complete problem (2.4) is analogous to that studied in the classical paper [1].

The eigenvalues at the rest points $(u, 0) = (u_\pm, 0)$ are $\lambda = [-1 \pm \sqrt{1 - 4\gamma p'(u)}\,]/(2\gamma)$ and $u_\pm$ are both saddles if and only if $p'(u_\pm) < 0$. This condition is equivalent to (2.3).

Therefore we cannot expect the existence of a traveling wave for arbitrary $\gamma \in \mathbb{R}$. With this in mind we consider $\gamma$ as an additional real-valued unknown function and augment (2.4) by a trivial equation, obtaining

$$\left\{\begin{array}{rcll} u' &=& z, & u(\pm\infty) = u_\pm, \\ \gamma z' &=& -z - p(u), & z(\pm\infty) = 0, \\ \gamma' &=& 0, & \gamma(0) = l. \end{array}\right. \tag{2.6}$$

The sets

$$\mathcal{M}_0^\pm = \{(u_\pm, 0, l) \,|\, l \in \mathbb{R},\, l \neq 0\} \subset \mathbb{R}_u \times \mathbb{R}_z \times \mathbb{R}_\gamma \tag{2.7}$$

are one-dimensional submanifolds of the critical manifold of rest points to (2.6). In view of the hyperbolicity of $(u_\pm, 0)$ with respect to (2.4), the linearization of the flow of (2.6) at any point of $\mathcal{M}_0^\pm$ has exactly one eigenvalue, namely 0, on the imaginary axis. Therefore the manifolds $\mathcal{M}_0^\pm$ are normally hyperbolic [13].

Motivated by the change of variables (2.14) presented in the next subsection for (1.9), we rewrite (2.6) in an equivalent form. We define

$$w := z + p(u) \tag{2.8}$$

so that $z = w - p(u)$. This change of variables is a diffeomorphism with unit Jacobian determinant. We then obtain for the variables $(u, w)$ the problem

$$\left\{\begin{array}{rcll} u' &=& w - p(u), & u(\pm\infty) = u_\pm, \\ w' &=& -\dfrac{w}{\gamma} + p'(u)\left(w - p(u)\right), & w(\pm\infty) = 0, \\ \gamma' &=& 0. & \end{array}\right. \tag{2.9}$$

We denote

$$M_0^\pm = \{(u_\pm, 0, l) \,|\, l \in \mathbb{R},\, l \neq 0\} \subset \mathbb{R}_u \times \mathbb{R}_w \times \mathbb{R}_\gamma$$

the transformed manifolds of (2.7), which are still normally hyperbolic. From [12] the following results can be collected.

**Theorem 2.1** *Consider the boundary-value problem* (2.4)*, or equivalently* (2.9)*, under assumptions* (1.10) *and* (2.2)*. Then the following holds.*

*(i) If*

$$-u_- < u_+ < -\frac{u_-}{2}\,, \tag{2.10}$$

*then there is a unique number* $\bar\gamma > 0$ *such that, up to shifts, there is a unique solution of* (2.4) *(and* (2.9) *with* $\gamma = \bar\gamma$*).*

*(ii) The intersection of the unstable manifold* $W^u(u_-, \bar\gamma)$ *emanating from* $(u_-, 0, \bar\gamma) \in M_0^-$ *and the stable manifold* $W^s(u_+, \bar\gamma)$ *from* $(u_+, 0, \bar\gamma) \in M_0^+$ *is transverse with respect to the flow of the augmented system* (2.9).

*Proof.* Concerning *(i)*, in [12] it is proved that if $u_- > \frac{2\sqrt{2}}{3\sqrt{\gamma}}$ then there is a saddle to saddle connection to $u_+ = -u_- + \frac{\sqrt{2}}{3\sqrt{\gamma}}$. The statement above is deduced estimating $\gamma$ in terms of $u_-$ and then estimating consequently $u_+$. Remark that the weaker (2.3) simply expresses the undercompressive condition, while the stronger (2.10) is a consequence of the assumptions for the existence of an invariant parabola [12, Theorem 3.4].

About *(ii)*, the heteroclinic trajectory joining $(u_-, 0)$ with $(u_+, 0)$ of the previous item can be viewed as the intersection of the unstable manifold of the line $M_0^-$ of critical points with the stable manifold of the line $M_0^+$, in both cases at least for $\gamma$ in a small neighborhood of $\bar\gamma$. Transversality is then proved as in [12, (3.22)]; see also [13, §4.5] for a different proof. □

Since (2.10) implies (2.3), the solution provided by Theorem 2.1 is undercompressive in the sense that, in the limit $\varepsilon \to 0+$, it provides an undercompressive shock wave to (1.1) as in (1.2).

## 2.2 Traveling Waves for the Parabolic-Elliptic Regularization

Now, we return to the system (1.9). A traveling-wave solution to (1.9) with speed $s$ is a solution to (1.9) of the form

$$\big(U_{\varepsilon,\alpha}(x,t), L_{\varepsilon,\alpha}(x,t)\big) = \left(u_\alpha\left(\frac{x-st}{\varepsilon}\right), \lambda_\alpha\left(\frac{x-st}{\varepsilon}\right)\right) \tag{2.11}$$

satisfying $(u_\alpha(\pm\infty), \lambda_\alpha(\pm\infty)) = (u_\pm, \lambda_\pm)$ and $u_\alpha'(\pm\infty) = \lambda_\alpha'(\pm\infty) = 0$. The states $u_\pm$ and $\lambda_\pm$ can depend on $\alpha$, but we dropped this index for simplicity of notation. By construction, the existence of a traveling wave (2.11) to (1.9) implies the convergence almost everywhere of $\{U_{\varepsilon,\alpha}\}_{\varepsilon>0}$ for $\varepsilon \to 0$ to the undercompressive shock wave $U$ to (1.1) as in (1.2).

By plugging the previous ansatz about $(U_{\varepsilon,\alpha}, L_{\varepsilon,\alpha})$ into (1.9) we see that $(u_\alpha, \lambda_\alpha)$ must solve the system

$$\begin{cases} u_\alpha' &= \alpha(u_\alpha - \lambda_\alpha) - p(u_\alpha), \\ -\gamma\lambda_\alpha'' &= \alpha(u_\alpha - \lambda_\alpha). \end{cases} \tag{2.12}$$

For $(u_\pm, \lambda_\pm)$ to be rest points of the flow in (2.12) we must have $\lambda_\pm = u_\pm$. Therefore the assumption $u_\alpha'(\pm\infty) = 0$ implies that $c$ in (2.12) is still defined by (2.5). In conclusion, system (2.12) is completed by the boundary conditions

$$u_\alpha(\pm\infty) = \lambda_\alpha(\pm\infty) = u_\pm\,, \quad \lambda_\alpha'(\pm\infty) = 0. \tag{2.13}$$

We make the change of variables

$$w_\alpha := \alpha(u_\alpha - \lambda_\alpha). \tag{2.14}$$

Equation $(2.12)_1$ now reads $u_\alpha' = w_\alpha - p_\alpha$; then $u_\alpha'' = w_\alpha' - p_\alpha'(w_\alpha - p_\alpha)$, where we wrote $p_\alpha = p(u_\alpha)$ for short. By $(2.12)_2$ we deduce

$$w_\alpha'' = \alpha\left(\frac{w_\alpha}{\gamma} + u_\alpha''\right) = \frac{\alpha}{\gamma}\left(w_\alpha - \gamma p_\alpha'(w_\alpha - p_\alpha) + \gamma w_\alpha'\right).$$

Denoting $v_\alpha = w'_\alpha$ and

$$G(u,w,\gamma,v) = \frac{1}{\gamma}\Big(w - \gamma p'(u)\left(w-p(u)\right) + \gamma v\Big),$$

we obtain that (2.12)–(2.13) is equivalent to

$$\begin{cases} u'_\alpha &= w_\alpha - p(u_\alpha), & u_\alpha(\pm\infty) = u_\pm, \\ w'_\alpha &= v_\alpha, & w_\alpha(\pm\infty) = 0, \\ \gamma'_\alpha &= 0, & \\ \frac{1}{\alpha}v'_\alpha &= G(u_\alpha, w_\alpha, \gamma_\alpha, v_\alpha), & v_\alpha(\pm\infty) = 0, \end{cases} \tag{2.15}$$

where we again understood $\gamma_\alpha$ as an unknown.

If we compare (2.15) with (2.9) we realize that (2.9) is the reduced system, for $\alpha = \infty$, of (2.15), governing the slow flow. Thus the system (2.15), which is written with respect to a slow-time scale, falls into the framework of the geometric singular perturbation theory for $\alpha$ sufficiently large [8, 13]. We now state our final result.

**Theorem 2.2** *Consider the boundary-value problem* (2.12)–(2.13)*, or equivalently* (2.15)*, with* (1.10) *and assume* (2.2)–(2.10)*.*

*Then, for $\alpha >> 1$ there is a unique number $\bar\gamma_\alpha > 0$ such that, up to shifts, there is a unique solution of* (2.15) *with $\gamma_\alpha = \bar\gamma_\alpha$ and thus a solution of* (2.12)–(2.13)*. Moreover, we have $\bar\gamma_\infty = \bar\gamma$.*

*Proof.* We rely on the formulation of the geometric singular perturbation theory provided in [9, Proposition 3.2]. There are two conditions to be checked. First, the equation

$$G(u,w,\gamma,v) = 0 \tag{2.16}$$

must have a manifold $\mathcal{C}_0 \subset \mathbb{R}\times\mathbb{R}\times(0,\infty)\times\mathbb{R}$ of solutions that is the graph of some smooth function $h = h(u,w,\gamma)$, mapping (a subset of) $\mathbb{R}\times\mathbb{R}\times(0,\infty)$ into $\mathbb{R}$. Second, we need that $G_v \neq 0$ in $\mathcal{C}_0$. Under these conditions it follows that, for $\alpha$ sufficiently large:

(a) normally hyperbolic (manifolds of) rest points of the reduced system (2.9) extend to normally hyperbolic (manifolds of) rest points for the singularly perturbed system (2.15);

(b) transverse intersections of the associated stable and unstable manifolds of (2.9) persist for the system (2.15).

In view of the Theorem 2.1(ii) it remains to check the conditions above. This is straightforward: since $\gamma > 0$, the implicit equation (2.16) is uniquely solved as

$$v = -\frac{w}{\gamma} + \left(s - f'(u)\right)\left(w - p(u)\right) =: h(u,w,\gamma),$$

which defines a manifold $\mathcal{C}_0$ on which $G_v \equiv 1$. Thus both conditions above hold and the theorem is proved.

□

# 3 A-Priori Estimates

In the following, in order to simplify notation we let

$$\gamma = 1.$$

First, we briefly discuss the elliptic equation

$$-\nu^2\lambda^\nu_{xx} + \lambda^\nu = w \tag{3.1}$$

for $\nu > 0$. If $w \in L^2(\mathbb{R})$ then the equation (3.1) has a unique solution $\lambda^\nu \in H^2(\mathbb{R})$ such that $\lambda^\nu \to 0$ as $|x| \to \infty$ [4, Exemple 8, Ch. VIII]. An explicit expression for $\lambda^\nu$ is obtained introducing

$$K^\nu(x) \quad = \quad \frac{1}{2\nu}\mathrm{e}^{-\frac{|x|}{\nu}}\,. \tag{3.2}$$

The kernel $K^\nu$ has unit integral, $K^\nu \to \delta$ as $\nu \to 0$ in $\mathcal{D}'$ and in $\mathcal{D}'$ it satisfies

$$-\nu^2K^\nu_{xx} = \delta - K^\nu. \tag{3.3}$$

Then $K^\nu$ is a fundamental solution of the homogeneous part in (3.1) and $\lambda^\nu = K^\nu * w$ solves (3.1); here, and in the following, '$*$' denotes convolution with respect to the space variable $x$. Moreover, $\lambda^\nu$ is continuous and it is precisely the unique solution stated above, because $\lambda^\nu \to 0$ as $|x| \to \infty$ [17, Lemma 2.20]. As a consequence, $\|\lambda^\nu\|_{L^2(\mathbb{R})} \le \|w\|_{L^2(\mathbb{R})}$. In the following we shall always refer to such solution.

The solution of the equation

$$-\varepsilon^2\lambda^{\varepsilon,\alpha}_{xx} = \alpha(u^{\varepsilon,\alpha} - \lambda^{\varepsilon,\alpha}) \tag{3.4}$$

is therefore

$$\lambda^{\varepsilon,\alpha} = K^{\frac{\varepsilon}{\sqrt{\alpha}}} * u^{\varepsilon,\alpha}\,. \tag{3.5}$$

From this formula and (3.3) we deduce that both the equation

$$u^{\varepsilon,\alpha}_t + f(u^{\varepsilon,\alpha})_x = \varepsilon u^{\varepsilon,\alpha}_{xx} + \varepsilon^2\left(K^{\frac{\varepsilon}{\sqrt{\alpha}}} * u^{\varepsilon,\alpha}\right)_{xxx} \tag{3.6}$$

and the equation

$$u^{\varepsilon,\alpha}_t + f(u^{\varepsilon,\alpha})_x = \varepsilon u^{\varepsilon,\alpha}_{xx} + \alpha\left(K^{\frac{\varepsilon}{\sqrt{\alpha}}} * u^{\varepsilon,\alpha} - u^{\varepsilon,\alpha}\right)_x \tag{3.7}$$

are equivalent to (1.9) at least for functions $u^{\varepsilon,\alpha}$ with $u^{\varepsilon,\alpha}(.,t) \in H^2(\mathbb{R})$.

We return to the initial value problem for (1.9). Consider

$$u^{\varepsilon,\alpha}(.,0) = u_0 \tag{3.8}$$

for some function $u_0 \in L^2(\mathbb{R})$ and let $\lambda^{\varepsilon,\alpha}_0$ be the solution of

$$-\varepsilon^2\lambda^{\varepsilon,\alpha}_{0,xx} = \alpha(u_0 - \lambda^{\varepsilon,\alpha}_0). \tag{3.9}$$

We a-priori estimates for solutions of (1.9), (3.8) under the following assumption.

**Assumption 3.1** *Assume* (1.10), $u_0 \in H^3(\mathbb{R}) \cap L^4(\mathbb{R})$ *and* $T > 0$. *For every* $\varepsilon, \alpha > 0$ *there exists a classical solution* $(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha}) : \bar{\Omega}_T \to \mathbb{R}^2$ *of* (1.9), (3.8) *that satisfies*

$$u^{\varepsilon,\alpha} \quad \in \quad C^3_1((0,T]\times\mathbb{R}) \cap L^\infty\left(0,T;H^3(\mathbb{R})\right), \tag{3.10}$$

$$u^{\varepsilon,\alpha} \quad \in \quad L^\infty\left(0,T;L^4(\mathbb{R})\right). \tag{3.11}$$

The requirement of *three* spatial derivatives in Assumption 3.1 seems a quite high regularity condition. In Remarks 3.7 and 4.3 we motivate this choice, which is needed for having estimates independent of $\alpha$. We prove later on in Theorem 4.1 a result on local existence, uniqueness and regularity of solutions to (1.9), (3.8); in Theorem 4.2 such solutions are extended to global solutions and then proved to satisfy Assumption 3.1.

Above, by classical solution we mean a function $u \in C^2_1(\Omega_T)$, the space of functions in $\Omega_T$ having two continuous derivatives in $x$ and one continuous derivative in $t$, which satisfies (1.9) in $\Omega_T$ and (3.8) a.e. in $\mathbb{R}$. We point out that the regularity $H^3$ required for $u_0$ does not depend on the choice of $f$ made in (1.10); on the contrary, the space $L^4$ is motivated precisely by (1.10).

We begin with an $L^\infty(0,T;L^2)$-estimate for solutions of (1.9),(3.8) which is uniform in both parameters $\varepsilon$ and $\alpha$.

**Lemma 3.2** *Let Assumption 3.1 hold. Then, for $t \in [0,T]$ we have*

$$\frac{1}{2}\|u^{\varepsilon,\alpha}(\cdot,t)\|^2_{L^2(\mathbb{R})} + \varepsilon\|u^{\varepsilon,\alpha}_x\|^2_{L^2(\Omega_t)} = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R})}\,. \tag{3.12}$$

*Proof.* We drop the upper indices and write for simplicity $(u,\lambda) = (u^{\varepsilon,\alpha},\lambda^{\varepsilon,\alpha})$. By Assumption 3.1 and Morrey's estimate (see, e.g., [7, §5.6.2] or [4, Cor. VIII.8]), we deduce the decay

$$\lim_{|x|\to\infty}|u(x,t)| = 0$$

for every $t \in (0,T]$. By multiplying $(1.9)_1$ by $u$ and integrating with respect to $x$ we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2\ \mathrm{d}x + \varepsilon\int_{\mathbb{R}}(u_x)^2\ \mathrm{d}x = -\alpha\int_{\mathbb{R}} u\,(u-\lambda)_x\ \mathrm{d}x\ . \tag{3.13}$$

Moreover, we have that $\lambda(.,t) = K^{\frac{\varepsilon}{\sqrt{\alpha}}} * u(.,t) \in H^2(\mathbb{R})$ which implies as above

$$\lim_{|x|\to\infty}|\lambda(x,t)| = \lim_{|x|\to\infty}|\lambda_x(x,t)| = 0.$$

By differentiating $(1.9)_2$ with respect to $x$, multiplying it by $\lambda$ and then integrating with respect to $x$ we find

$$\alpha\int_{\mathbb{R}}\lambda\,(u-\lambda)_x\ \mathrm{d}x = 0\,. \tag{3.14}$$

By summing up (3.13) and (3.14) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^2\ \mathrm{d}x + \varepsilon\int_{\mathbb{R}}(u_x)^2\ \mathrm{d}x = 0\,. \tag{3.15}$$

An integration with respect to time gives (3.12). Remark that in the above proof we needed neither $u_0 \in L^4(\mathbb{R})$ nor (3.11). □

The next result will be crucial in the following. It shows that the system (1.9) dissipates the energy functional $E^{\varepsilon,\alpha}$ in (1.8).

**Lemma 3.3 (Energy dissipation)** *Let Assumption 3.1 be valid. Then, for $t \in [0,T]$ we have*

$$\begin{aligned} &E^{\varepsilon,\alpha}[u^{\varepsilon,\alpha}(.,t),\lambda^{\varepsilon,\alpha}(.,t)] - E^{\varepsilon,\alpha}[u_0,\lambda_0^{\varepsilon,\alpha}] \\ &\quad= \ -\varepsilon\Big(\|u^{\varepsilon,\alpha}_x f'(u^{\varepsilon,\alpha})\|^2_{L^2(\Omega_t)} + \alpha\|u^{\varepsilon,\alpha}_x - \lambda^{\varepsilon,\alpha}_x\|^2_{L^2(\Omega_t)} + \varepsilon^2\|\lambda_{xx}\|^2_{L^2(\Omega_t)}\Big)\,. \end{aligned} \tag{3.16}$$

*Proof.* We write again $(u,\lambda) = (u^{\varepsilon,\alpha},\lambda^{\varepsilon,\alpha})$ for simplicity. We multiply $(1.9)_1$ by $u^3$; then we multiply again $(1.9)_1$ by $\alpha(u-\lambda)$ and $(1.9)_2$ by $\lambda_t$. Finally, we integrate with respect to $x$ and obtain

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u^4\,\mathrm{d}x + \int_{\mathbb{R}}(u^3)_x u^3\,\mathrm{d}x \ = \ \varepsilon\int_{\mathbb{R}} u_{xx}u^3\,\mathrm{d}x - \alpha\int_{\mathbb{R}}(u-\lambda)_x u^3\,\mathrm{d}x\,, \tag{3.17}$$

$$\alpha\int_{\mathbb{R}} u_t(u-\lambda)\,\mathrm{d}x + \alpha\int_{\mathbb{R}}(u^3)_x(u-\lambda)\,\mathrm{d}x \ = \ \varepsilon\alpha\int_{\mathbb{R}} u_{xx}(u-\lambda)\,\mathrm{d}x$$

$$-\alpha^2 \int_{\mathbb{R}} (u-\lambda)_x (u-\lambda)\,\mathrm{d}x\,, \tag{3.18}$$

$$-\varepsilon^2 \int_{\mathbb{R}} \lambda_{xx}\lambda_t\,\mathrm{d}x \;=\; \alpha\int_{\mathbb{R}} (u-\lambda)\lambda_t\,\mathrm{d}x\,. \tag{3.19}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}} u_{xx}(u-\lambda)\,\mathrm{d}x &= -\int_{\mathbb{R}} (u_x)^2\,\mathrm{d}x - \int_{\mathbb{R}} u\lambda_{xx}\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} (u_x)^2\,\mathrm{d}x + \int_{\mathbb{R}} (\lambda_x)^2\,\mathrm{d}x + \frac{\varepsilon^2}{\alpha}\int_{\mathbb{R}} (\lambda_{xx})^2\,\mathrm{d}x\,.
\end{aligned}$$

The second line above was obtained by plugging the expression of $u$ deduced from $(1.9)_2$. Here and in the following we exploited (3.10) to justify integration by parts.

By summing up (3.17)–(3.19) and taking into account the previous identity, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left\{\frac{u^4}{4}+\frac{\alpha}{2}(u-\lambda)^2+\frac{\varepsilon^2}{2}(\lambda_x)^2\right\}\mathrm{d}x + \int_{\mathbb{R}}\left\{3\varepsilon\,(u_x)^2u^2+\varepsilon\alpha(u_x)^2\right\}\mathrm{d}x \;= \\
&= \varepsilon\int_{\mathbb{R}}\left\{\alpha(\lambda_x)^2+\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x\;.
\end{aligned} \tag{3.20}$$

The last summand on the left-hand side "adsorbs" the right-hand side. Indeed, by using $(1.9)_2$,

$$\begin{aligned}
-\varepsilon^2\int_{\mathbb{R}}(\lambda_{xx})^2\,\mathrm{d}x &= \varepsilon^2\int_{\mathbb{R}}\left\{2\lambda_x\lambda_{xxx}+(\lambda_{xx})^2\right\}\mathrm{d}x \\
&= \int_{\mathbb{R}}\left\{-2\alpha\lambda_x u_x+2\alpha(\lambda_x)^2+\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x\;.
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}}\left\{\alpha(u_x)^2-\alpha(\lambda_x)^2-\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x \;=\; \int_{\mathbb{R}}\left\{\alpha\,(u_x-\lambda_x)^2+\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x\,.$$

Then (3.20) finally writes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left\{\frac{u^4}{4}+\frac{\alpha}{2}(u-\lambda)^2+\frac{\varepsilon^2}{2}(\lambda_x)^2\right\}\mathrm{d}x\,+ \\
&\quad +\varepsilon\int_{\mathbb{R}}\left\{3\,(u_xu)^2+\alpha\,(u_x-\lambda_x)^2+\varepsilon^2(\lambda_{xx})^2\right\}\mathrm{d}x=0\,.
\end{aligned} \tag{3.21}$$

An integration with respect to $t$ gives (3.16). $\square$

**Remark 3.4** *Consider $u_0$ as in Assumption 3.1. The initial energy $E^{\varepsilon,\alpha}[u_0,\lambda_0^{\varepsilon,\alpha}]$ contains terms that depend implicitly on $\varepsilon$ and $\alpha$. However, by (3.9) and (3.5) we have*

$$\begin{aligned}
\|u_0-\lambda_0^{\varepsilon,\alpha}\|_{L^2(\mathbb{R})} &= \frac{\varepsilon^2}{\alpha}\|\lambda_{0,xx}^{\varepsilon,\alpha}\|_{L^2(\mathbb{R})} \;\le\; \frac{\varepsilon^2}{\alpha}\|u_0\|_{H^2(\mathbb{R})}, \\
\|\lambda_{0,x}^{\varepsilon,\alpha}\|_{L^2(\mathbb{R})} &\le \|u_0\|_{H^1(\mathbb{R})}.
\end{aligned}$$

*As a consequence,*

$$E^{\varepsilon,\alpha}[u_0,\lambda_0^{\varepsilon,\alpha}] \le \frac{1}{4}\|u_0\|^4_{L^4(\mathbb{R})}+\frac{\varepsilon^2}{2}\left(\|u_0\|^2_{H^1(\mathbb{R})}+\frac{\varepsilon^2}{\alpha}\|u_0\|^2_{H^2(\mathbb{R})}\right),$$

*showing that $E^{\varepsilon,\alpha}[u_0, \lambda_0^{\varepsilon,\alpha}]$ depends on none of the parameters $\alpha$ and $\varepsilon$ in a critical way. Therefore the quantities*

$$\|u_x^{\varepsilon,\alpha} f'(u^{\varepsilon,\alpha})\|^2_{L^2(\Omega_t)}, \quad \alpha\|u_x^{\varepsilon,\alpha} - \lambda_x^{\varepsilon,\alpha}\|^2_{L^2(\Omega_t)}, \quad \|\lambda_{xx}^{\varepsilon,\alpha}\|^2_{L^2(\Omega_t)}$$

*are bounded uniformly with respect to $\alpha$ by a constant depending on $\varepsilon$ (and $T$).*

The following lemma is a direct consequence of the integral representation (3.5).

**Lemma 3.5** *Let Assumption 3.1 hold. Then we have for $t \in [0, T]$*

$$\begin{aligned} \|\partial_x^l \lambda^{\varepsilon,\alpha}\|_{L^2(\Omega_t)} &\leq \|\partial_x^l u^{\varepsilon,\alpha}\|_{L^2(\Omega_t)} \quad (l = 0, 1, 2, 3), \\ \|\lambda_t^{\varepsilon,\alpha}\|_{L^2(\Omega_t)} &\leq \|u_t^{\varepsilon,\alpha}\|_{L^2(\Omega_t)}. \end{aligned}$$

Now, we show that the $L^2$-norms of both $u^{\varepsilon,\alpha}$ and $u_t^{\varepsilon,\alpha}$ are uniformly bounded with respect to $\alpha$. By Lemma 3.5 the same bounds shall apply to $\lambda^{\varepsilon,\alpha}$, too.

**Lemma 3.6 (Uniform boundedness)** *Consider a family of solutions $\{(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})\}_{\varepsilon,\alpha>0}$ to (1.9), (3.8) satisfying Assumption 3.1. Then, there exists a constant $C(\varepsilon) > 0$, which is independent of $\alpha$ and $T$, such that*

$$\|u^{\varepsilon,\alpha}\|_{L^2(0,t;H^3(\mathbb{R}))} + \|u_t^{\varepsilon,\alpha}\|_{L^2(\Omega_t)} \leq C(\varepsilon) t \qquad (t \in [0, T]). \tag{3.22}$$

*Moreover $u^{\varepsilon,\alpha} \in C([0, T]; H^1(\mathbb{R}))$. At last, there is a continuous monotone-increasing function $\mathcal{C}(\|u_0\|_{H^3(\mathbb{R})}, \varepsilon; \cdot) : [0, T] \to [0, \infty)$, which depends on $\|u_0\|_{H^3(\mathbb{R})}$ and $\varepsilon$ but not on $\alpha$, such that*

$$\|u^{\varepsilon,\alpha}(., t)\|_{L^\infty(\mathbb{R})} \leq \mathcal{C}(\|u_0\|_{H^3(\mathbb{R})}, \varepsilon; t) \qquad (t \in [0, T]). \tag{3.23}$$

**Remark 3.7** *We point out that an $L^\infty$-bound for $u^{\varepsilon,\alpha}$ as in (3.23) can be proven more directly using an embedding argument and only an $L^\infty(0, T; H^1(\mathbb{R}))$-regularity for $u^{\varepsilon,\alpha}$. However, using this approach it is hard to check that the function $\mathcal{C}$ is independent of $\alpha$.*

*On one hand, this independence is essential for the singular limit $\alpha \to \infty$ in Section 5. On the other hand, concerning the global existence of solutions, in Theorem 4.2 the parameter $\alpha$ is fixed and plays no role. In the latter case the assumptions can be relaxed, cf. Remark 4.3.*

*Proof of Lemma 3.6.* First, remark that the estimate (3.22) and standard embedding theorems imply $u^{\varepsilon,\alpha} \in C([0, t]; H^2(\mathbb{R}))$ (cf. [28, Lemma 1.2, Ch. III], with $V = H^2(\mathbb{R})$ and $H = H^2(\mathbb{R})$) Note that we only use the $H^1$-bound on $u^{\varepsilon,\alpha}$ and not the $H^3$-bound also contained in (3.22).

Second, the existence of the function $\mathcal{C}$ and the estimate (3.23) are proved again by embedding (apply e.g. [7, Theorem 2, page 286] with $X = H^1(\mathbb{R})$).

So we are left to prove (3.22). We use again the notation $(u, \lambda) = (u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})$. Remark that the uniform estimate on $\|u\|_{L^2(0,T;H^1(\mathbb{R}))}$ is already contained in (3.12).

For the estimate on $\|u_{xx}\|_{L^2(\Omega_T)}$, define $v = u_x$ and $\mu = \lambda_x$. We derive $(1.9)_1$ once with respect to $x$; after multiplication by $v$ and integration with respect to $x$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} v^2 \,\mathrm{d}x + \varepsilon \int_{\mathbb{R}} (v_x)^2 \,\mathrm{d}x = \int_{\mathbb{R}} f(u)_x v_x \,\mathrm{d}x - \alpha \int_{\mathbb{R}} (v_x - \mu_x) v \,\mathrm{d}x\,,$$

where for short we dropped both $x$ and $t$ in the arguments of the functions. Analogously, we differentiate $(1.9)_2$ twice with respect to $x$, multiply by $\mu$ and integrate with respect to $x$; we get

$$0 = \int_{\mathbb{R}} \partial_x(\mu_x)^2 \,\mathrm{d}x = \alpha \int_{\mathbb{R}} (v_x - \mu_x)\, \mu \,\mathrm{d}x\,.$$

Note that in both formulas above we used (3.10) and standard regularity properties of $\lambda$, as solution of the elliptic equation $(1.9)_2$, to perform integration by parts. Altogether we arrive for $s \in [0,t]$ at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (v(\cdot,s))^2 \,\mathrm{d}x + \varepsilon \int_{\mathbb{R}} (v_x(\cdot,s))^2 \,\mathrm{d}x = \int_{\mathbb{R}} f\left(u(\cdot,t)\right)_x v_x(\cdot,s) \,\mathrm{d}x\,.$$

In turn, Cauchy-Schwarz's inequality implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} v^2 \,\mathrm{d}x + \frac{\varepsilon}{2}\int_{\mathbb{R}} (v_x)^2 \,\mathrm{d}x \le \frac{1}{2\varepsilon}\int_{\mathbb{R}} |f'(u)u_x|^2 \,\mathrm{d}x\,. \tag{3.24}$$

The right-hand side of (3.24) is bounded uniformly with respect to $\alpha$ and time because of Lemma 3.3 and Remark 3.4. Thus we have bounded $\|u\|_{L^2(0,t;H^2(\mathbb{R}))}$. The $H^3$-boundedness follows exactly along the same lines by differentiating (1.9) once more.

We turn to estimating the time derivative of $u$. Since $u$ is a classical solution we compute, by squaring $(1.9)_1$ and integrating with respect to $x$,

$$\begin{aligned}\int_{\mathbb{R}} (u_t)^2 \,\mathrm{d}x \;=\;& \varepsilon^2 \int_{\mathbb{R}} (u_{xx})^2 \,\mathrm{d}x + \int_{\mathbb{R}} (f'(u))^2 (u_x)^2 \,\mathrm{d}x + \alpha^2 \int_{\mathbb{R}} (u_x - \lambda_x)^2 \,\mathrm{d}x \\ & - 2\varepsilon \int_{\mathbb{R}} f'(u) u_x u_{xx} \,\mathrm{d}x + 2\alpha \int_{\mathbb{R}} f'(u) u_x (u_x - \lambda_x) \,\mathrm{d}x \\ & - 2\alpha \int_{\mathbb{R}} \varepsilon u_{xx} (u_x - \lambda_x) \,\mathrm{d}x\,.\end{aligned}$$

This leads to

$$\begin{aligned}\|u_t\|^2_{L^2(\Omega_t)} \;\le\;& \varepsilon^2 \|u_{xx}\|^2_{L^2(\Omega_t)} + \|f'(u)u_x\|^2_{L^2(\Omega_t)} + \|\alpha(u-\lambda)_x\|^2_{L^2(\Omega_t)} \\ & + 2\varepsilon \|f'(u)u_x\|_{L^2(\Omega_t)} \|u_{xx}\|_{L^2(\Omega_t)} \\ & + 2\|f'(u)u_x\|_{L^2(\Omega_T)} \|\alpha(u-\lambda)_x\|_{L^2(\Omega_t)} \\ & + 2\varepsilon \|u_{xx}\|_{L^2(\Omega_t)} \|\alpha(u-\lambda)_x\|_{L^2(\Omega_t)}\,,\end{aligned} \tag{3.25}$$

and thus using $(1.9)_2$ and the regularity of $\lambda$ to

$$\begin{aligned}\|u_t\|^2_{L^2(\Omega_t)} \;\le\;& \varepsilon^2 \|u_{xx}\|^2_{L^2(\Omega_t)} + \|f'(u)u_x\|^2_{L^2(\Omega_t)} + \varepsilon^4 \|\lambda_{xxx}\|^2_{L^2(\Omega_t)} \\ & + 2\varepsilon \|f'(u)u_x\|_{L^2(\Omega_t)} \|u_{xx}\|_{L^2(\Omega_t)} \\ & + 2\varepsilon^2 \|f'(u)u_x\|_{L^2(\Omega_t)} \|\lambda_{xxx}\|_{L^2(\Omega_t)} \\ & + 2\varepsilon^3 \|u_{xx}\|_{L^2(\Omega_t)} \|\lambda_{xxx}\|_{L^2(\Omega_t)}\,.\end{aligned}$$

The uniform boundedness of $\|u_t\|_{L^2(\Omega_t)}$ follows now from Lemma 3.3, Remark 3.4, Lemma 3.5, and the uniform bound on $\|u\|_{L^2(0,t;H^3(\mathbb{R}))}$ proved above. □

Let us note that the terms $\|\alpha(u-\lambda)_x\|_{L^2(\Omega_t)}$ in (3.25) can be directly bounded with respect to $\alpha$ by using Remark 3.4. Introducing the third-order derivative $\lambda_{xxx}$ is only needed for the term $\|\alpha(u-\lambda)_x\|^2_{L^2(\Omega_t)}$.

# 4 Wellposedness of Classical Solutions

In this section we consider the initial-value problem (1.9), (3.8) and prove that it has a unique global solution for every positive $\varepsilon$ and $\alpha$, for suitable initial data. Because of (3.5) and (3.9) we focus on (3.6) and only state our results for $u^{\varepsilon,\alpha}$; results for $\lambda^{\varepsilon,\alpha}$ immediately follow. Moreover, since the parameters $\varepsilon$ and $\alpha$ are fixed, we drop the dependence on both of them in the functions below.

Consider the Banach space $\mathcal{B} = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, endowed with the norm

$$\|v\|_{\mathcal{B}} = \max\left\{\|v\|_{L^2(\mathbb{R})},\, \|v\|_{L^\infty(\mathbb{R})}\right\},$$

and $C([0,T];\mathcal{B})$, with the related norm

$$\|u\|_{T,\mathcal{B}} = \sup_{t\in[0,T]} \|u(t)\|_{\mathcal{B}},$$

where $\|u(t)\|_{\mathcal{B}}$ is a shortcut for $\|u(\cdot,t)\|_{\mathcal{B}}$. An analogous notation is used in the following for other functions spaces.

The next result concerns the local existence of classical solutions to (1.9), (3.8). A general flux function $f$ is considered instead of the special case (1.10).

**Theorem 4.1 (Local existence and regularity)** *Let $f \in C^1(\mathbb{R})$ and $u_0 \in \mathcal{B}$. Assume that $\|u_0\|_{L^\infty(\mathbb{R})} \le r$ for some $r \ge 0$ and let $L = L(r)$ be the Lipschitz constant of $f$ in the interval $[-2r, 2r]$.*

*(i) The initial-value problem (1.9), (3.8) has a unique classical solution $u \in C([0,T_0];\mathcal{B})$ for*

$$T_0 = T_0(2r) = \frac{\pi}{4}\frac{\varepsilon}{\left(L(r)+\alpha+\sqrt{(L(r)+\alpha)^2+\pi\alpha^{3/2}/2}\right)^2}. \tag{4.1}$$

*(ii) Moreover, assume that $u_0 \in W^{k,2}(\mathbb{R}) \cap W^{k,\infty}(\mathbb{R})$ and $f \in C^k(\mathbb{R})$, for some $k \in \mathbb{N}$. Then the unique solution from (i) satisfies*

$$u \in L^2(0,T_0;W^{k,2}(\mathbb{R}) \cap W^{k,\infty}(\mathbb{R})) \cap C([0,T_0];W^{k-1,2}(\mathbb{R})). \tag{4.2}$$

*Proof.* The proof is classical and goes on, for instance, as in [25, Theorem 14.2], following a slight modification due to [16]. Therefore we only provide a sketch. Note that, analogously to (3.6) and (3.7), the system (1.9) can be written as the scalar equation

$$u_t + h(u)_x = \varepsilon u_{xx} + \alpha H^\nu * u, \tag{4.3}$$

with $h(u) = f(u) + \alpha u$ and $H^\nu(x) = (K^\nu)'(x) = -\frac{\operatorname{sgn} x}{\nu}K^\nu(x)$, for $\nu = \frac{\varepsilon}{\sqrt{\alpha}}$. Remark that $\|H^\nu\|_{L^1(\mathbb{R})} = \frac{1}{\nu}$.

*(i)* Define

$$X = \left\{u \in C([0,T_0];\mathcal{B}) \,:\, \|u - G_\varepsilon * u_0\|_{T_0,\mathcal{B}} \le \|u_0\|_{\mathcal{B}}\right\},$$

where

$$G_\varepsilon(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\,\mathrm{e}^{-\frac{x^2}{4\varepsilon t}}$$

denotes the heat kernel. Clearly, $0 \in X$ and $\|u\|_{T_0,\mathcal{B}} \le 2\|u_0\|_{\mathcal{B}}$ for every $u \in X$. Then we have $\|h(u) - h(v)\|_{T_0,\mathcal{B}} \le (L(r)+\alpha)\|u - v\|_{T_0,\mathcal{B}}$ for any $u, v \in X$ and $t \in [0,T_0]$.

Consider the functional

$$\Phi u = G_\varepsilon * u_0 + \Phi_1 u + \Phi_2 u \tag{4.4}$$

for

$$\begin{aligned}
\Phi_1 u &= \int_0^t \int_{\mathbb{R}} G_\varepsilon(x-y,t-s)\cdot h\left(u(y,s)\right)_y \,\mathrm{d}y\,\mathrm{d}s,\\
\Phi_2 u &= \alpha \int_0^t \int_{\mathbb{R}} G_\varepsilon(x-y,t-s)\cdot (H^\nu * u)(y,s)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

We claim that $\Phi$ has a unique fixed point $u \in X$.

First, we prove that $\Phi X \subset X$, i.e., that for every $u \in X$ we have

$$\|\Phi u - G_\varepsilon * u_0\|_{T_0,\mathcal{B}} = \|\Phi_1 u + \Phi_2 u\|_{T_0,\mathcal{B}} \le \|u_0\|_{\mathcal{B}}.$$

In fact, for $u \in X$ and $t \in [0, T_0]$ we have

$$\begin{aligned} \|\Phi_1 u(t)\|_{L^q(\mathbb{R})} &\le (L(r)+\alpha)C_\varepsilon\sqrt{t} \sup_{s\in[0,T_0]} \|u(s)\|_{L^q(\mathbb{R})}\,, && (4.5)\\ \|\Phi_2 u(t)\|_{L^q(\mathbb{R})} &\le \frac{\alpha}{\nu}\, t \sup_{s\in[0,T_0]} \|u(s)\|_{L^q(\mathbb{R})}\,, && (4.6)\end{aligned}$$

for $q = 2$ or $q = \infty$; here $C_\varepsilon = \frac{2}{\sqrt{\pi\varepsilon}}$, so that $\int_0^t \|G_\varepsilon(s)\|_{L^1(\mathbb{R})} = C_\varepsilon\sqrt{t}$. Then,

$$\begin{aligned} \|\Phi_1 u\|_{T_0,\mathcal{B}} &\le (L(r)+\alpha)C_\varepsilon\sqrt{T_0}\ \|u\|_{T_0,\mathcal{B}} \ \le\ 2(L(r)+\alpha)C_\varepsilon\sqrt{T_0}\ \|u_0\|_{\mathcal{B}}\,,\\ \|\Phi_2 u\|_{T_0,\mathcal{B}} &\le \frac{\alpha}{\nu}\ T_0\|u\|_{T_0,\mathcal{B}} \ \le\ 2\frac{\alpha}{\nu}\ T_0\|u_0\|_{\mathcal{B}}\,.\end{aligned}$$

Then it follows that $\Phi X \subset X$ for $T_0$ by (4.1).

Second, we prove that for $u, v \in X$ and $T_0$ given by (4.1) we have

$$\|\Phi u - \Phi v\|_{T_0,\mathcal{B}} = \|(\Phi_1 u - \Phi_1 v) + (\Phi_2 u - \Phi_2 v)\|_{T_0,\mathcal{B}} \le \frac{1}{2}\|u - v\|_{T_0,\mathcal{B}}\,.$$

Indeed, for $u, v \in X$ and $t \in [0, T_0]$ we have

$$\begin{aligned} \|\Phi_1 u(t) - \Phi_1 v(t)\|_{L^q(\mathbb{R})} &\le (L(r)+\alpha)C_\varepsilon\sqrt{t} \sup_{s\in[0,T_0]} \|u(s)-v(s)\|_{L^q(\mathbb{R})}\,,\\ \|\Phi_2 u(t) - \Phi_2 v(t)\|_{L^q(\mathbb{R})} &\le \frac{\alpha}{\nu}\, t \sup_{s\in[0,T_0]} \|u(s)-v(s)\|_{L^q(\mathbb{R})}\,,\end{aligned}$$

for $q = 2$ or $q = \infty$. This proves our claim.

By construction the fixed point $u$ surely is twice differentiable with respect to space and once with respect to time in $\Omega_{T_0}$; moreover, the initial datum is assumed a.e. Thus $u$ is a classical solution.

*(ii)* We directly obtain that $u \in L^2(0, T_0; W^{k,2}(\mathbb{R}) \cap W^{k,\infty}(\mathbb{R}))$ from the representation formula $\Phi u = u$. Together with the equation (4.3) and embedding for $W^{k,2}(\mathbb{R})$, see [28, Lemma 1.2, Ch. III], gives $u \in C([0, T_0]; W^{k-1,2}(\mathbb{R}))$. □

We finally set $f(u) = u^3$ and prove our global existence result.

**Theorem 4.2 (Global existence)** *Let $T > 0$, assume* (1.10), $u_0 \in H^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R}) \cap L^4(\mathbb{R})$*. Then, for any $\varepsilon, \alpha > 0$ there is a unique classical solution* $(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})$ *of* (1.9), (3.8) *which in addition satisfies* (3.10)–(3.11).

*Proof.* We apply Theorem 4.1 with $r := \mathcal{C}(T) = \mathcal{C}(\|u_0\|_{H^3(\mathbb{R})}, \varepsilon; T)$ and $k = 3$, where $\mathcal{C}$ is defined in Lemma 3.6. Then, we deduce the existence of a unique classical solution in the interval $[0, T_0(2\mathcal{C}(T))]$.

If $T_0\left(2\mathcal{C}(T)\right) \ge T$ holds, we are finished; in particular (3.11) is deduced from the a-priori bound proved above under the assumption (1.10).

Otherwise, assume $T_0\left(2\mathcal{C}(T)\right) < T$. Assumption 3.1 holds and Lemma 3.6 can be applied: with $t$ replaced by $T_0\left(2\mathcal{C}(T)\right)$ formula (3.23) gives

$$\left\|u\left(\cdot, T_0(2\mathcal{C}(T))\right)\right\|_{L^\infty(\mathbb{R})} \le \mathcal{C}\left(T_0(2\mathcal{C}(T))\right) \le \mathcal{C}(T). \tag{4.7}$$

Analogously, Lemma 3.2 shows that

$$\left\|u\left(\cdot, T_0(2\mathcal{C}(T))\right)\right\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}. \tag{4.8}$$

In order to extend the solution $u$ forward in time we apply once more Theorem 4.1: we choose again $r = \mathcal{C}(T)$ and $k = 3$ but take $u\left(\cdot, T_0(2\mathcal{C}(T))\right)$ as initial datum at time $T_0\left(2\mathcal{C}(T)\right)$. This is possible because of (4.7) and (4.8), since $\mathcal{C}$ is an increasing function of time and $T_0\left(2\mathcal{C}(T)\right) < T$. As a consequence, the life span has not changed and we now get the existence of a unique classical solution in $\left[T_0(2\mathcal{C}(T)), 2T_0(2\mathcal{C}(T))\right]$. We proceed until we have reached the end time $T$. $\square$

**Remark 4.3** *In order to obtain the above global existence result we exploited the $L^\infty$-bound* (4.7)*, which is deduced from* (3.23) *in Lemma 3.6. According to Remark 3.7, this last $L^\infty$-bound can be proven by only requiring $u^{\varepsilon,\alpha}(\cdot,t) \in H^2(\mathbb{R})$. In this sense we can relax the assumptions on the initial datum in Theorem 4.2.*

# 5 Singular Limits for the Initial Value Problem

In this section we consider families of classical solutions for (1.9), (3.8). First, we study the diffusive-dispersive limit $\alpha \to \infty$, for fixed $\varepsilon$, and thereafter the sharp-interface limit $\varepsilon \to 0$, now for fixed $\alpha$. The limit function $u$ satisfies the local diffusive-dispersive equation (1.5) in the former case and the hyperbolic equation (1.1) in the distribution sense in the latter.

## 5.1 The Diffusive-Dispersive Limit $\alpha \to \infty$

In this section the parameter $\varepsilon > 0$ is fixed and we consider the diffusive-dispersive limit $\alpha \to \infty$ for a family of classical solutions $\{u^{\alpha,\varepsilon}, \lambda^{\alpha,\varepsilon}\}_{\alpha>0}$ of the initial value problem for (1.9). For simplicity we use the notation

$$\{u^\alpha, \lambda^\alpha\}_{\alpha>0} := \{u^{\alpha,\varepsilon}, \lambda^{\alpha,\varepsilon}\}_{\alpha>0}, \quad \varepsilon > 0.$$

Our compactness argument relies on the Lions-Aubin lemma which we recall here from [28, Theorem 2.1, Ch. III, §2].

**Lemma 5.1** *Let $B_0 \subset B \subset B_1$ be Banach spaces, $B_0$ and $B_1$ be reflexive. Assume that $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$, $1 < q < \infty$, and define the Banach space*

$$W = \left\{u \in L^p(0,T;B_0) \colon u' \in L^q(0,T;B_1)\right\},$$

*endowed with the norm $\|u\|_{L^p(0,T,B_0)} + \|u'\|_{L^q(0,T,B_1)}$. Then the inclusion $W \hookrightarrow L^p(0,T;B)$ is compact.*

The main result we deduce is the following.

**Theorem 5.2** *Let $T > 0$, $\varepsilon > 0$ be given; assume* (1.10) *and $u_0 \in H^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R}) \cap L^4(\mathbb{R})$. Consider any family $\{(u^\alpha, \lambda^\alpha)\}_{\alpha>0}$ of classical solutions to* (1.9), (3.8) *provided by Theorem 4.2.*
*Then there exists a subsequence of $\{(u^\alpha, \lambda^\alpha)\}_{\alpha>0}$, still denoted by $\{(u^\alpha, \lambda^\alpha)\}_{\alpha>0}$, and a function $u \in L^2(\Omega_T) \cap L^\infty(0,T;L^4(\mathbb{R}))$ such that*

$$u^\alpha \to u, \quad \lambda^\alpha \to u \quad \text{ in } L^2_{loc}(\Omega_T) \text{ for } \alpha \to \infty. \tag{5.1}$$

*Moreover, $u$ is a distributional solution of the initial value problem* (1.5), (3.8)*, i.e.,*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t + f(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0\varphi(.,0)\,\mathrm{d}x = \int_0^T \int_{\mathbb{R}} -\varepsilon u\varphi_{xx} + \varepsilon^2 u\varphi_{xxx}\,\mathrm{d}x\,\mathrm{d}t \tag{5.2}$$

*for all $\varphi \in C_0^\infty\left(\mathbb{R} \times [0,T)\right)$.*

*Proof.* By Lemmas 3.5 and 3.6 we deduce the uniform bound

$$\|\lambda^\alpha\|_{L^2(0,T;H^3(\mathbb{R}))} + \|\lambda_t^\alpha\|_{L^2(\Omega_T)} \le C.$$

It is well known that the inclusion $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is not compact. To overcome this difficulty we introduce an open bounded interval $I \subset \mathbb{R}$. The inclusion $H^1(I) \hookrightarrow L^2(I)$ is compact; then, Lemma 5.1 applies with $p = q = 2$ and $B_0 = H^1(I)$, $B = B_1 = L^2(I)$. We deduce that there is a subsequence of $\{\lambda^\alpha\}_{\alpha>0}$, denoted in the same way, and a function $u \in L^2([0,T) \times I)$ such that

$$\lim_{\alpha\to\infty} \|\lambda^\alpha - u\|_{L^2([0,T)\times I)} = 0. \tag{5.3}$$

By a diagonal process we can extract another subsequence, still denoted by $\{\lambda^\alpha\}$, such that (5.3) holds for every bounded interval $I$; moreover, $u \in L^2(\Omega_T)$ by weak convergence and again passing to a subsequence. From (5.3), the energy estimate in Lemma 3.3 and Remark 3.4 on the initial datum we get immediately

$$\lim_{\alpha\to\infty} \|u^\alpha - u\|_{L^2([0,T)\times I)} = 0. \tag{5.4}$$

The first assertion (5.1) of the theorem is proven.
Using the second equation in (1.9), any classical solution of (1.9), (3.8) satisfies

$$\int_0^T \int_{\mathbb{R}} u^\alpha \varphi_t + f(u^\alpha)\varphi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0\varphi(.,0)\,\mathrm{d}x = -\int_0^T \int_{\mathbb{R}} \varepsilon u^\alpha \varphi_{xx} - \varepsilon^2\lambda^\alpha\varphi_{xxx}\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi \in C_0^\infty(\mathbb{R} \times [0,T))$. The relations (5.3), (5.4) and Lebesgue's dominated convergence theorem prove (5.2).

At last, note that the solutions constructed in Theorem 4.2 have the further property that $\sup_{t\in[0,T]} \|u^\alpha(t)\|_{L^4(\mathbb{R})}$ is uniformly bounded, because of Lemma 3.3. As a consequence, there is a subsequence of $\{u^\alpha\}$ with $u^\alpha \overset{*}{\rightharpoonup} u$, by the uniqueness of the weak limit. Therefore, $u \in L^\infty(0,T;L^4(\mathbb{R}))$. This concludes the proof of the theorem. ☐

## 5.2 The Sharp-Interface Limit $\varepsilon \to 0$

In the previous section we considered the diffusive-dispersive limit $\alpha \to \infty$ in the system (1.9) for $\varepsilon$ fixed. In this section, on the contrary, we focus on the sharp-interface limit $\varepsilon \to 0$ for fixed values of $\alpha > 0$. Let $\{(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})\}_{\varepsilon,\alpha>0}$ be a family of classical solutions for (1.9), (3.8) satisfying Assumption 3.1. This time we use the notation

$$\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0} := \{(u^{\varepsilon,\alpha}, \lambda^{\varepsilon,\alpha})\}_{\varepsilon,\alpha>0}.$$

We shall prove that in the limit $\varepsilon \to 0$ the solutions $\{u^\varepsilon\}$ of (1.9) converge to a weak solution $u$ of the homogeneous equation (1.1) with $f(u) = u^3$. More precisely we have the following result.

**Theorem 5.3** *Let $\alpha > 0$ be given and consider a family $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$ of classical solutions of* (1.9), (3.8) *provided by Theorem 4.2.*
*Then there exists a subsequence of $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$, still denoted as $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$, and a function $u \in L^p(\Omega_T)$, $2 \le p \le 4$, such that*

$$u^\varepsilon \to u \text{ in } L^r_{loc}(\Omega_T) \ (1 \le r < 4). \tag{5.5}$$

*Moreover, $u$ is a weak solution to the initial value problem for* (1.1), *i.e.*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t + f(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0\varphi(.,0)\,\mathrm{d}x = 0$$

*for all $\varphi \in C_0^\infty(\mathbb{R} \times [0,T))$.*

We point out that the limit solution $u$ is not entropic, in general, because of the possible existence of undercompressive waves. We shall use the compensated compactness theory [19] in the $L^p$-framework [24] and in particular we shall refer to the arguments used in [22, §4]. The a-priori estimate provided in (3.12) is crucial also in this part.

Let us note that the right-hand side of (3.16) is estimated uniformly with respect to $\varepsilon$, see Remark 3.4. In particular we deduce

$$\|\lambda^\varepsilon_{xx}\|_{L^2(\Omega_t)} \leq \frac{C_\alpha}{\varepsilon\sqrt{\varepsilon}}, \quad t \in [0,T] \tag{5.6}$$

for a constant $C_\alpha$ depending only on the initial data and $\alpha$. This estimate will be crucial in the following Lemma 5.4. Notice that, for $\varepsilon$ small, the estimate (5.6) refines the estimate $\|\lambda_{xx}\|_{L_2(\Omega_t)} \leq \frac{C\alpha}{\varepsilon^2}$, which can be directly obtained from $(1.9)_2$ and (3.12).

An *entropy pair* $(\eta, q)$ for (1.1) is a couple of functions class $C^2(\mathbb{R})$ satisfying

$$\eta'(w)f'(w) = q'(w)$$

for every $w \in \mathbb{R}$. In the following, we consider entropies satisfying the condition

$$|\eta'(w)| + |\eta''(w)| \leq C_\eta \tag{5.7}$$

for every $w \in \mathbb{R}$.

The following crucial compactness lemma will lead to the proof of Theorem 5.3. We denote by $\mathcal{M}(Q)$ the set of Radon measures on $\Omega$.

**Lemma 5.4** *Let $\alpha > 0$ be given and let a family of classical solutions $\{(u^\varepsilon, \lambda^\varepsilon)\}_{\varepsilon>0}$ of* (1.9)*,* (3.8) *be given such that Assumption 3.1 holds.*
*Then, for every open bounded set $Q \subset \Omega_T$ there exist a compact set $\mathcal{K} \subset W^{-1,2}(Q)$ and a bounded set $\mathcal{B} \subset \mathcal{M}(Q)$ such that*

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \subset \mathcal{K} + \mathcal{B}\,, \tag{5.8}$$

*for every entropy pair $(\eta, q)$ satisfying* (5.7)*.*

*Proof.* By multiplying (1.9) by $\eta'(u^\varepsilon)$ we obtain

$$\begin{aligned}
&\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \\
&\quad = \quad \varepsilon\eta(u^\varepsilon)_{xx} - \varepsilon\eta''(u^\varepsilon)(u^\varepsilon_x)^2 - \alpha\left(\eta'(u^\varepsilon)(u^\varepsilon - \lambda^\varepsilon)\right)_x + \alpha\eta''(u^\varepsilon)u^\varepsilon_x(u^\varepsilon - \lambda^\varepsilon) \\
&\quad = \quad A^\varepsilon_1 + A^\varepsilon_2 + A^\varepsilon_3 + A^\varepsilon_4\,.
\end{aligned}$$

The condition (5.7) is used several times in the following and we omit to mention it explicitly. We denote by $\langle\cdot,\cdot\rangle$ both the duality between $W^{-1,2}(Q)$ and $W^{1,2}_0(Q)$ and between $\mathcal{M}(\mathcal{Q})$ and $C_0(Q)$.

We prove first that $A^\varepsilon_1, A^\varepsilon_3 \subset \mathcal{K}$. For every $\varphi \in W^{1,2}_0(Q)$ we have

$$\begin{aligned}
\left|\langle A^\varepsilon_1, \varphi\rangle\right| &\leq \varepsilon\int_Q \left|\eta'(u^\varepsilon)u^\varepsilon_x\varphi_x\right| \mathrm{d}t\,\mathrm{d}x \\
&\leq C_\eta\varepsilon\,\|u^\varepsilon_x\|_{L^2(Q)}\|\varphi_x\|_{L^2(Q)} \\
&\leq C_\eta\|u_0\|_{L^2}\sqrt{\varepsilon}\,\|\varphi\|_{W^{1,2}(Q)} \to 0\,,
\end{aligned}$$

because of (3.12), for $\varepsilon \to 0$. Analogously, because of (5.6),

$$\begin{aligned}
\left|\langle A^\varepsilon_3, \varphi\rangle\right| &\leq \alpha\int_Q \left|\eta'(u^\varepsilon)(u^\varepsilon - \lambda^\varepsilon)\varphi_x\right| \mathrm{d}t\,\mathrm{d}x \\
&\leq C_\eta\varepsilon^2\int_Q \left|\lambda^\varepsilon_{xx}\varphi_x\right| \mathrm{d}t\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq C_\eta \varepsilon^2 \|\lambda^\varepsilon_{xx}\|_{L^2(Q)} \|\varphi_x\|_{L^2(Q)} \\
&\leq C_\eta C_\alpha \sqrt{\varepsilon}\, \|\varphi\|_{W^{1,2}(Q)} \to 0 \,,
\end{aligned}
$$

for $\varepsilon \to 0$. Then both $A_1^\varepsilon$ and $A_3^\varepsilon$ are contained in $\mathcal{K}$.

Next, we prove that $A_2^\varepsilon, A_4^\varepsilon \subset \mathcal{B}$. For every $\psi \in C_0(Q)$ we have

$$
\left|\langle A_2^\varepsilon, \psi\rangle\right| \leq \varepsilon \int_Q \left|\eta''(u^\varepsilon)(u_x^\varepsilon)^2\psi\right| \mathrm{d}t\,\mathrm{d}x \leq C_\eta \|u_0\|_{L^2}^2 \|\psi\|_{L^\infty(Q)} \,,
$$

because of (3.12). Moreover,

$$
\begin{aligned}
\left|\langle A_4^\varepsilon, \psi\rangle\right| &\leq \alpha \int_Q \left|\eta''(u^\varepsilon)u_x^\varepsilon(u^\varepsilon - \lambda^\varepsilon)\psi\right| \mathrm{d}t\,\mathrm{d}x \\
&\leq C_\eta \varepsilon^2 \int_Q |\lambda^\varepsilon_{xx} u^\varepsilon_x \psi|\, \mathrm{d}t\,\mathrm{d}x \\
&\leq C_\eta \varepsilon^2 \|\lambda^\varepsilon_{xx}\|_{L^2(Q)} \|u^\varepsilon_x\|_{L^2(Q)} \|\psi\|_{L^\infty(Q)} \\
&\leq C_\eta \|u_0\|_{L^2} \|\psi\|_{L^\infty(Q)} \,,
\end{aligned}
$$

because of (5.6) and (3.12). Then both $A_2^\varepsilon$ and $A_4^\varepsilon$ are contained in $\mathcal{B}$. This proves the lemma. □

With this compactness result we can finally prove Theorem 5.3.

*Proof of Theorem 5.3.* The family of norms $\|u^\varepsilon\|_{L^p(\Omega_T)}$ is uniformly bounded for $2 \leq p \leq 4$, because of Lemmata 3.2 and 3.3 and Riesz-Thorin theorem. Then the first statement of the theorem on the family $\{u^\varepsilon\}_{\varepsilon>0}$ is a consequence of Lemma 5.4 and the results in [24, 19].

In order to prove that $u$ solves (1.1), consider any $\varphi \in C_0^\infty(\mathbb{R} \times [0, T))$; then

$$
\begin{aligned}
&\int_{\Omega_T} \big(u^\varepsilon \varphi_t + f(u^\varepsilon)\varphi_x\big)\, \mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} u_0 \varphi(.,0)\, \mathrm{d}x \\
&\quad = \; -\varepsilon \int_{\Omega_T} u^\varepsilon \varphi_{xx}\, \mathrm{d}t\,\mathrm{d}x - \alpha \int_{\Omega_T} (u^\varepsilon - \lambda^\varepsilon)\varphi_x\, \mathrm{d}t\,\mathrm{d}x \,.
\end{aligned}
$$

The sequence $\{u^\varepsilon\}$ converges both in $L^1_{loc}(\Omega_T)$ and in $L^3_{loc}(\Omega_T)$; therefore the left side of the identity above converges to

$$
\int_{\Omega_T} \Big(u\varphi_t + u^3\varphi_x\Big)\, \mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} u_0\varphi(.,0)\, \mathrm{d}x \;.
$$

By the same reason, the first term on the right side vanishes in the limit. At last, by the second equation in (1.9), the second term equals

$$
\varepsilon^2 \int_{\Omega_T} \lambda^\varepsilon \varphi_{xxx}\, \mathrm{d}t\,\mathrm{d}x \;.
$$

Also this term vanishes in the limit because the sequence $\{\lambda^\varepsilon\}$ is uniformly bounded in $L^2(\Omega_T)$. □

# A Scaling and the Diffusive Case

The system (1.9) is somewhat related to the system

$$
\left\{
\begin{array}{l}
u_t + \big(u^2/2\big)_x = -q_x \,, \\
-q_{xx} + q = -u_x \,,
\end{array}
\right. \tag{A.1}
$$

which arises as a model for the flow of a radiating gas and reduces to the equation

$$u_t + (u^2/2)_x = K_1 * u - u\,.$$

This system has been widely studied in the last years; we just quote [14, 16, 18, 23] and refer the reader to the references provided there. The paper [16] contains some results on the viscous approximation of the former equation, namely,

$$u_t + (u^2/2)_x = \varepsilon u_{xx} + K_1 * u - u\,,$$

which is similar to (3.7). Notice that, if the dispersive term in (1.5) is missing, then the system (1.9) can be obtained by (A.1) through a hyperbolic-parabolic scaling [16] and a change of variables. In this appendix we show how scaling may be used to deduce asymptotic systems of (A.1), and then compare the systems obtained in [16] with (1.9). Below, we write $f(u)$ for $u^2/2$.

With the scaling $\tilde{u}(x,t) = \frac{1}{\delta}\, u\left(\frac{x}{\delta}, \frac{t}{\delta^2}\right)$, $\tilde{q}(x,t) = \frac{1}{\delta^2}\, q\left(\frac{x}{\delta}, \frac{t}{\delta^2}\right)$, the system (A.1) writes [16], omitting the ˜'s,

$$\begin{cases} u_t + f(u)_x = -q_x\,, \\ -\delta^2 q_{xx} + q = -u_x\,, \end{cases} \qquad \text{or} \qquad u_t + f(u)_x = \frac{1}{\delta^2}\left(K^\delta * u - u\right)\,. \tag{A.2}$$

The scaled solution converges, for $\delta \to 0$ (the hyperbolic-parabolic relaxation limit), toward the solution of the viscous Burgers equation with unit viscosity coefficient, [16]. In the special case the dispersive term is missing, the equation (1.5) reduces to

$$u_t + f(u)_x = \varepsilon u_{xx} \tag{A.3}$$

and the approximating system, analogous to (1.9), is

$$\begin{cases} u_t + f(u)_x = -\alpha(u - \lambda)\,, \\ -\varepsilon \lambda_{xx} = \alpha(u - \lambda)\,. \end{cases} \tag{A.4}$$

By the change of dependent variables $u(x,t) = \tilde{u}(x,t)$, $\lambda(x,t) = \tilde{\lambda}(x/\sqrt{\varepsilon}, t/\sqrt{\varepsilon})$ the system (A.4) reads, dropping again for simplicity the ˜'s,

$$\begin{cases} u_t + f(u)_x = -\alpha\,(u - \lambda)\,, \\ -\lambda_{xx} = \alpha\,(u - \lambda)\,. \end{cases} \tag{A.5}$$

By (3.5), the system (A.5) can be written exactly as the scalar equation in (A.2) for $\alpha = \frac{1}{\delta^2}$. This shows that the approximation (A.4) to the merely diffusive equation (A.3) is covered by [16]. Remark however that (A.5) is simpler than the system in (A.2) since no derivatives appear in the right side.

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