Source Terms versus Entropy for Environmental Waves

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Abstract

For a wide class of 2×2 nonlinear hyperbolic systems involving the transport equation and a source term, the waves are also solutions to a linear homogeneous wave equation with constant coefficients. (Linear Appearance Theorem). The nonlinearities remain in the shock waves, where the entropy reduces the total energy. We consider here the balance between the energy brought by a source term and the part lost as entropy in the shock waves, and the way to take that in account in a numerical scheme

1 The model

We consider 2×2 hyperbolic systems in one space dimension, whose first equation is the transport equation:

$$q_t + m_x = 0 \ ,$$

where q is a quantity transported by the flux m. The flux equation has the general form:

$$m_t + a\ q_x + b\ m_x + S(q,m) = 0$$

where $S(q,m)$ is a source term depending only on q and m.. The parameters a and b are such that the eigenvalues of the matrix of flux are real; they are denoted λ_1 and λ_2 with $\lambda_2 > \lambda_1$. We set $u = (\lambda_1 + \lambda_2)/2\ ,\ c = (\lambda_2 - \lambda_1)/2\ \ (> 0)$, thus $a = c^2 - u^2,\ b = 2u$. Then the flux equation always reads:

$$m_t + 2u\ m_x + \left(c^2 - u^2\right) q_x + S(q,m) = 0\ .$$

The source term $S(q,m)$ is supposed to be continous and such that in any open set of the admissible states (q,m) there is a state (q_0,m_0) for which $S(q_0,m_0) \neq 0$. The parameters u and c continously depend on q and m. The set of the states (q,m) will be called the phase plane.

1.1 The theorem of linear appearance

In the homogeneous case, the rarefaction waves can be characterized by writing $m = m(q)$ in the system which allows to derive the Riemann invariants. Using the same argument "$m = m(q)$" when a source term S ($\neq 0$) is present, we get that the propagation of the corresponding waves have a linear behaviour.

The Theorem of Linear Appearance: For any regular wave for which m depends only on q, that is $m = m(q)$, then q and m are locally solution to the same advection equation

$$q_t + Aq_x = 0 \quad , \quad m_t + Am_x = 0 \ ,$$

and linked together by the relation

$$m = Aq - B,$$

with some constants A and B (to be determined from the context).
Proof: Compute

$$m = m(q) \ \Rightarrow \ m_x = m'(q)q_x \ , \ m_t = -m'(q)^2 q_x \ .$$

Thus $\left(c^2 - (m'(q) - u)^2\right) q_x + S = 0$, or by using the notation

$$\psi'(q) = \frac{(m'(q) - u)^2 - c^2}{S(q, m(q))} \ ,$$

as the derivative of some function $\psi(q)$, we get simply

$$\psi'(q)q_x = 1 \ .$$

An integration with respect to x gives $\psi(q) = x - K(t)$, since the constant of integration may depend on t. Then a derivation with respect to t leads to $\psi'(q)q_t = -K'(t)$, where $q_t = -m'(q)q_x$. Then, using $\psi'(q)q_x = 1$, we get

$$m'(q) = K'(t),$$

and a derivation with respect to x gives $m''(q)q_x = 0$, that is $m''(q) = 0$ since $q_x \neq 0$ from $\psi'(q)q_x = 1$. We deduce that m is an affine function of q that we denote $m = Aq - B$. Next the equation $q_t + m(q)_x = 0$ reads now $q_t + Aq_x = 0$, and $m_t + Am_x = A(q_t + Aq_x) = 0$.

1.2 The source waves

The "**Source waves**" correspond to these waves satisfying $m = Aq - B$. The profiles can be computed by integrating

$$\psi'(q) = \frac{(m'(q) - u)^2 - c^2}{S(q, m(q))} = \frac{(A - u)^2 - c^2}{S(q, Aq - B)},$$

and using

$$x - At = \psi(q)$$

to be inverted to get q. Next m is given by $m = Aq - B$.

In some (very usual) cases this computation is possible explicitly, since $\psi'(q)$ is often a rational fraction.

1.3 Another theorem

In many applications, we have the relation $m = qu$. This is actually a necessary condition under some natural looking conditions of Galilean invariance, conservation of the flux and nullity of the flux when nothing is transported.

Theorem: We suppose these 3 conditions:

- $c = c(q)$, not depending on m, (Galilean invariance)
- if $q = 0$ then $m = 0$ (zero flux vacuum)
- the flux is conservative: $m_t + F(q,m)_x + S = 0, \quad with\ F(q,m) = \frac{m^2}{q} + \int_0^q c(\xi)^2 d\xi$.

Then

$$m = qu \ ,$$

and $\psi'(q) = \dfrac{B^2 - q^2 c(q)^2}{q^2 S(q, Aq - B)}$.

Remark: : Accordingly, u is the velocity, $c(q)$ is the wave celerity (or soundspeed) and $\int_0^q c(\xi)^2 d\xi = P(q)$ is the pressure. We recognize the usual form of the Euler equations.

Proof: We get

$$\frac{\partial^2 F}{\partial q \partial m} = 2c\frac{\partial c}{\partial m} - 2u\frac{\partial u}{\partial m} = 2\frac{\partial u}{\partial q}$$

thus

$$\frac{\partial u}{\partial q} + u\frac{\partial u}{\partial m} = c\frac{\partial c}{\partial m} = 0$$

from the Galilean invariance, and u is a solution of the Burgers equation (in (m,q) instead of the usual (x,t)) along a characteristic line passing throught $(0,0)$ from the zero flux vacuum argument. Thus $u = m/q$ (instead of the usual x/t). The expressions of $\psi'(q)$ and F are immediately derived..

1.4 The shock waves

We use the symbol $\Delta Q = Q_2 - Q_1$ to denote the jump of any variable Q on a shock, from the left value Q_1 to the right value Q_2.

When the model is of conservative flux, the jump of the velocity u is linked to the jump of the transported quantity q by the Rankine Hugoniot condition: :

$$\Delta u = \mp\sqrt{\frac{\Delta P \ \Delta q}{q_1 \ q_2}} \quad .$$

The velocity of the shock is

$$x'(t) = \frac{\Delta m}{\Delta q} = \frac{u_1 + u_2}{2} \mp \frac{q_1 + q_2}{2\sqrt{q_1 q_2}} \sqrt{\frac{\Delta P}{\Delta q}} \quad .$$

The entropy conditions rule the meeting of the characteristic curves:

for a "$u - c$" $-$ *shock* : $u_1 - c(q_1) \geq x'(t) \geq u_2 - c(q_2)$
for a "$u + c$" $-$ *shock* : $u_1 + c(q_1) \geq x'(t) \geq u_2 + c(q_2)$.

1.5 Relaxation by contact discontinuities

Since $P_t = c(q)^2 q_t$, we get a new equation for the pressure, which looks like a Hooke law:

$$P_t + uP_x + qc(q)^2 u_x = 0 \ .$$

The flux equation gives another equation for the velocity

$$u_t + uu_x + \frac{1}{q}P_x + \frac{S(q, qu)}{q} = 0 \ .$$

Together with the transport equation $q_t + (qu)_x = 0$, we get a 3×3 system with 3 wave velocities:

$$u - c(q), \ u, \ u + c(q).$$

The new waves, of velocity u, correspond to the **contact discontinuities**, satisfying

$$\Delta u = 0 \ , \ \Delta P = 0 \ .$$

Taking these waves in account allows sometimes to **relax** the previous 2×2 system, by breaking a regular wave into two parts.

1.6 Elementary waves

For a given state $M_0 = (q_0, m_0)$, $u_0 = m_0/q_0$, we look for the attainable sets from M_0. The attainable states throught a **shock wave** are located on the Hugoniot curves, of equations

$$RH_{\mp}(M_0) = \left\{ M = (q, m) \ \mid \ m = qu_0 \mp \sqrt{\frac{q}{q_0}\Delta P \Delta q} \ \right\},$$

where the sign $\mp$ corresponds to the wave velocity $(u \mp c)$. The attainable states throught a **source wave** corresponds to the straight lines

$$D_+(M_0) = \{ m = (u_0 + c_0)q - q_0 c_0 \ \} \ , \ D_-(M_0) = \{ m = (u_0 - c_0)q + q_0 c_0 \ \} \ ,$$

or to the (inverse) sets $K_{\mp}(M_0) = \{ M \ \mid \ M_0 \in D_{\mp}(M) \ \}$. We also recall the Riemann Invariants passing throught M_0, of equations:

$$RI_{\mp}(M_0) = \left\{ M = (q, u) \ \mid \ m = qu_0 \mp q \int_{q_0}^{q} \frac{c(\xi)}{\xi} d\xi \ \ \right\} \ ,$$

which correspond to the **Rarefaction waves** in the homogeneous case. We shall see that these Riemann invariants will still play a role even when a source term is present.
These curves are presented on Figure 1 for the wave velocity $u - c$ on the left side and for the wave velocity $u + c$ on the right side.

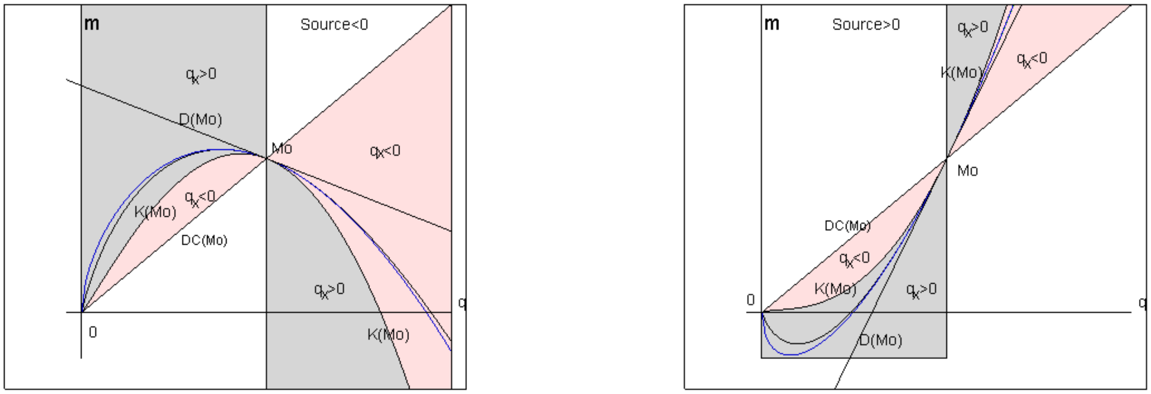

Figure 1: Elementary waves in the phase plane

1.7 Sequence of elementary waves and Saw waves

We can shape some composite waves by concatenating a sequence of sources waves and shock waves, as shown on figure 2.

Such a sequence of elementary waves can be shaped along a Riemann invariant, for example. From a physical point of view, the source waves are often unable to grow up from different reasons, as for a lack of mass or of energy. Then a return to the invariant is done throught a shock wave, near the starting state. Such a wave with small sawteeth will be called a **Saw wave**. Such a saw wave is shown on figure 2.

That way, the Riemann invariant may be considered as the limit of a sequence of saw waves. The same situation occurs when we consider for example the minimal positive solution of the equation $|y'| = 1,\ y(0) = 0$ in the set of continuous functions defined on $[0, 1]$, with a bounded derivative. This minimal solution is obviously zero, which does not satisfy the equation, but satisfy the homogeneous form of the equation. Here, the Riemann invariant corresponds to a rarefaction wave which is solution to the homogeneous form of the system.

1.8 The bow pattern

We suppose that the source term is zero along a curve of the phase plane, and consider a state $M_0(q_0, m_0)$ on this curve: $S(q_0, m_0) = 0$. We look for the source waves starting from M_0, whose velocity has the form "$u + c$", thus satisfying $\psi'(q)q_x = 1$, with

$$\psi'(q) = \frac{B^2 - q^2 c(q)^2}{q^2 S(q, Aq - B)} \ \ , \ m_0 = Aq_0 - B \ , \ B > 0.$$

For $q = q_0$, the denominator is null and the only way to have $\psi'(q)$ bounded is to impose $B = q_0 c(q_0)$. Then we deduce $A = m_0/q_0 + c(q_0)$. This is asserted as follows:

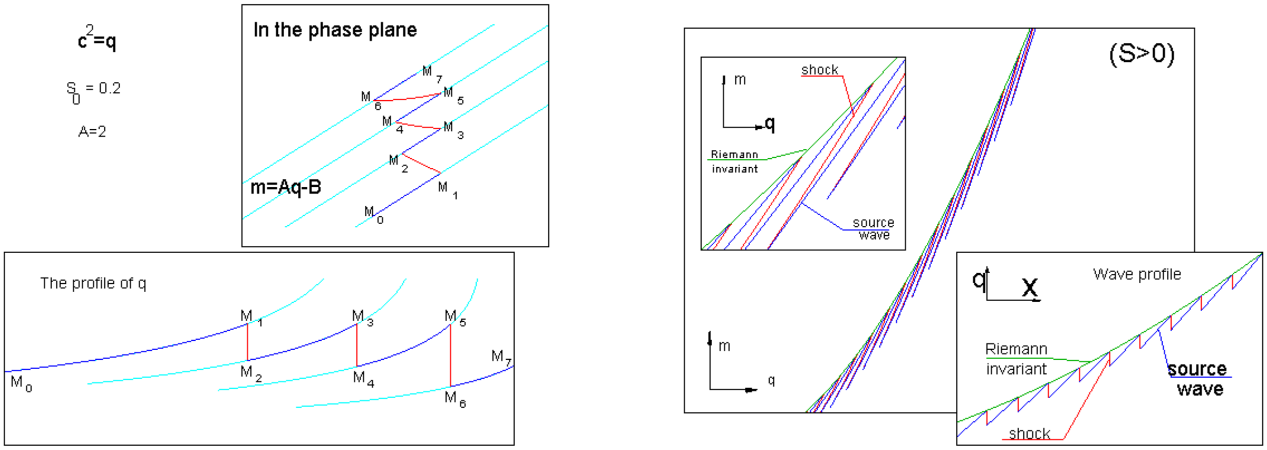

Figure 2: Sequence of elementary waves and Saw waves

Proposition: The only way to cross the curve $S(q,m)=0$ for a source wave of straight line $m = Aq - B$, at a state $M_0 = (q_0, m_0)$ such that $S(q_0, m_0) = 0$ is for $A = m_0/q_0 \mp c(q_0)$, $B = \mp q_0 c(q_0)$, and q increasing if $S(q, Aq - B)$ decreases, or q decreasing if $S(q, Aq - B)$ increases.
Proof: The first part was proved above. From $\psi'(q)q_x = 1$, the signs of q_x and $\psi'(q)$ must be the sames, which completes the proof.

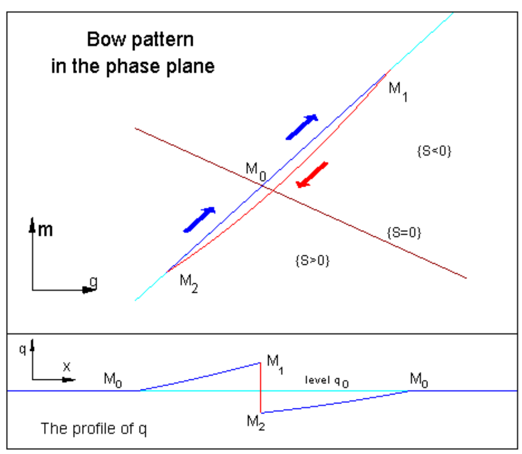

Figure 3: A Bow pattern (Roll wave)

Now, consider the same state M_0 on the curve $S(q,m)=0$, and two other states $M_1 = (q_1, m_1)$ and $M_2 = (q_2, m_2)$ as values of a source wave determined by the line $m = A_0 q - B_0$, with $A_0 = m_0/q_0 + c(q_0)$, $B_0 = q_0 c(q_0)$, and $q_1 < q_0 < q_2$. We suppose that $S(q_1, m_1)$ and $S(q_2, m_2)$ are of opposite signs. One can build a source wave starting from M_0 and joining M_2 and another source wave joining M_1 to M_0. However it is not possible to join M_2 to M_1 using a regular wave, with q decreasing.. This is only possible throught a **shock wave** when the states M_1 and M_2 are on the same Hugoniot curve. Such a situation occurs for the **Roll waves** in hydraulics. In the phase plane, the values of such a wave are shaped as a **bow** whose string is the straight line $m = A_0 q - B_0$ and the arc is the Hugoniot curve. This **bow pattern** is pictured on Figure 3.

2 The Saint-Venant model

In this model the transported quantity is $q = h$, the height of water in the flow. The model is:

$$h_t + m_x = 0 \ , \ m_t + 2um_x + \left(c^2 - u^2\right) h_x + S(h,m) = 0 \ ,$$

where

$$S(h,m) = ghp + k\,|u|\,u \ ,$$

with p the slope supposed constant and negative here, $k > 0$ the Strickler friction coefficient and g the gravity constant. For a motionless state, this model reduces to $c^2 h_x = ghp$, where $h + px = constant$, since the water surface is flat. We get that way the wave velocity

$$c = \sqrt{gh} \ .$$

This model is also called the **shallow water model** since it can be created by using an hypothesis of weak depth, which is not necessary here.

A **permanent regime** appears when all the profiles are constant. The time derivatives are and remain null, and h is constant since the bottom slope p is constant.

Theorem: In permanent regime the Froude number $|u|/c$ satisfies

$$\frac{|u|}{c} = \lambda \equiv \sqrt{-\frac{p}{k}} \ .$$

Proof: The transport equation reduces to $m_x = 0$. Then the dynamic equation, since $h_x = 0$, becomes $S(h,m) = 0$, that is $c^2 p = -ku^2$, with $u > 0$, which is the expected result.

Remark: This theorem provides a convenient way to estimate the Strickler coefficient, since u, p and h are easy to measure on the ground.

2.1 The Roll Waves

The set $S(h,m) = 0$ corresponds to the equation $m = \sqrt{|p|\,g/k}\ h^{3/2} = \lambda\sqrt{g}\ h^{3/2}$. Let $M_0 = (h_0, m_0)$ belonging to this curve. We set $u_0 = m_0/h_0\,(= \lambda c(h_0))$. A bow pattern is expected when the straight line passing throught M_0 with the slope $A_0 = u_0 + c(h_0)$ crosses this line from above $(S(h,m) > 0,\ h < h_0)$ to below $(S(h,m) < 0, h > h_0)$. Hence $\frac{dm}{dh}(h_0) > A_0$ that is $\frac{3}{2}\lambda c(h_0) > u_0 + c(h_0) = (\lambda + 1)\,c(h_0)$, which gives

$$\lambda > 2.$$

A bow pattern is only possible when $\lambda > 2$. We get the profile of h by integrating

$$\psi'(h) = \frac{gh_0^2 - gh^3}{h^2\left(ghp + ku^2\right)} \ ,$$

which can be done by hand.
Then, with a constant C,

$$\psi(h) = x - (u_0 + c_0)\, t + C$$

gives the profile of the wave on both side of the shock linking a state $M_2 = (h_2, m_2)$ to a state $M_1 = (h_1, m_1)$.

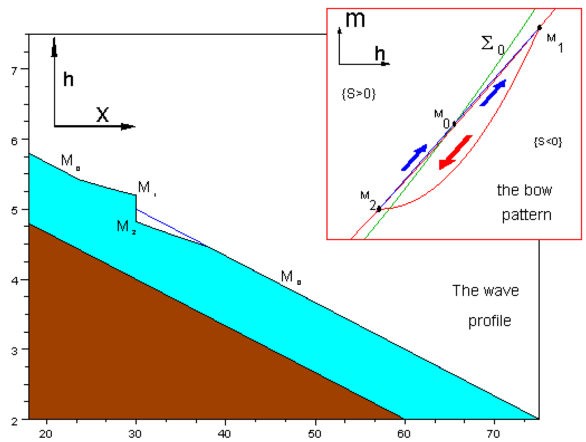

Figure 4: Roll wave profile

Remark: From the Rankine Hugoniot condition we derive a surprising average formula linking $h_0,\ h_1$ and h_2 which reads

$$h_0 = \left(\frac{h_1 + h_2}{2} h_1 h_2\right)^{1/3} .$$

The amount of water on the left of the shock is greater than the lack of water on the right. Therefore a Roll wave differs from a wavelet.

3 From Gas dynamics to Acoustics

We consider an air flow of density ρ and velocity u passing throught a sound tube of cross section $a(x)$ at point $x \in [0, L]$, where L is the length of the tube. The mass conservation implies:

$$a(x)\rho_t + (a(x)\rho u)_x = 0 \ .$$

We take $q = a(x)\rho,\ m = a(x)\rho u$, in order to transform this equation into a transport equation.

The Euler equation, with a friction term of coefficient $k > 0$ which may depend on x, in a way to be detailed later, reads

$$\rho\left(u_t + uu_x\right) + p(\rho)_x + k(x)\left|u\right|u = 0 \ \ , \ p(\rho) = \rho^\gamma,$$

with $\gamma = 1.4$, the air adiabatic constant. We look for deriving an equation for m. By multipling by $a(x)$, we get

$$m_t + 2um_x + \left(c^2 - u^2\right)q_x + S(q, m, x) = 0,$$

where $c = \sqrt{p'(\rho)}$ and

$$S(q, m, x) = k(x)a(x)\left|u\right|u - \gamma K\frac{a'(x)}{a(x)^\gamma}q^\gamma,$$

which depends on x but if $k(x)a(x)$ is a constant (denoted k_0) and

$$\frac{a'(x)}{a(x)^\gamma} = Constant\ (= \frac{k_0}{\gamma K})\ ,\ then\ a(x) = \left(\frac{\gamma K}{(\gamma - 1)\,k_0\,(x_0 - x)}\right)^{1/(\gamma-1)}.$$

Then Roll wave are possible, and also some continuous waves. The assumption on $a(x)$ corresponds to a differential equation obviously integrated. The profile of the solution is strongly connected to the reality, since the shapes of usual wind instruments are refound, depending on two parameters K and $x_0(> L)$. The parameter k_0 is linked to the material of the instrument, brass or wood for instance.

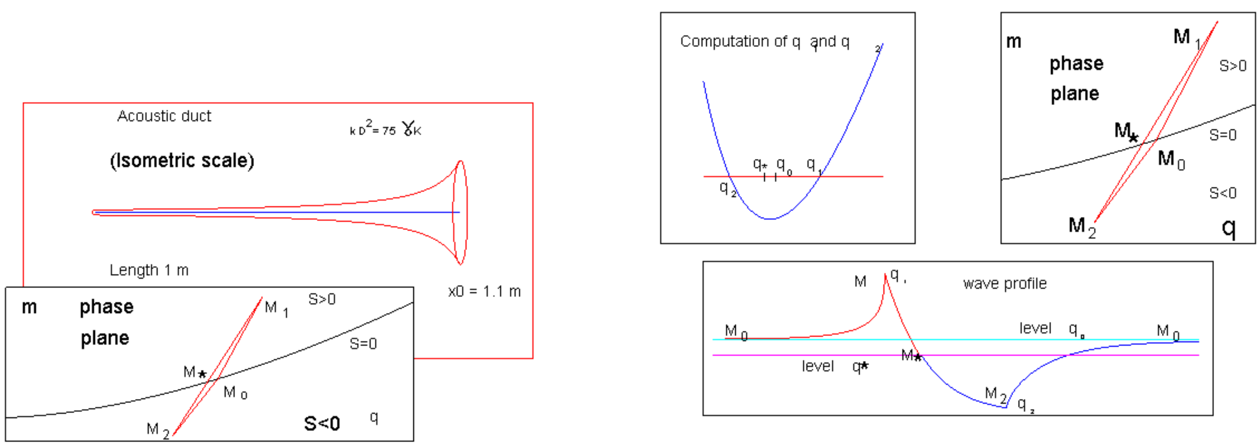

Figure 5: Shape and waves profiles in the tube

The sound velocity as coefficient of q_x can be supposed to be a constant: $c(q) = c_0$. This corresponds to the isothermal hypothesis. A sufficiently long sequence of such waves corresponds to a sound.

4 The Shock tube with friction

We consider the Euler equations with a Strickler friction term applied to the well known Sod shock tube. This is a Riemann problem made of the system

$$\rho_t + m_x = 0, \quad m_t + \left(\frac{m^2}{\rho} + P_0 \rho^\gamma \right)_x + k\,|u|\,u = 0 \ ,$$

and the initial condition $\rho(x,0) = \begin{cases} \rho_L & or\ x < 0 \ , \\ \rho_R & for\ x > 0 \ , \end{cases} , u(x,0) = 0$.
with $\rho_L > \rho_R > 0$ constant. The transported quantity is ρ by the flux m. We have $u = m/\rho$. The constant P_0 is used to fix the pressure units, γ is the usual adiabatic constant of air ($\gamma = 1.4$), and k is the Strickler friction constant.

Just after the initial time, the action of the source term is negligible and the profile of the solution starts like the profile in the homogeneous case. A shock wave propagates towards the right side and a rarefaction wave develops on the left side. However these two waves cannot be connected by a constant state, from the source term, but by source waves. A combination of two source waves is necessary, since a single one cannot join the top of the shock to the lower part of the rarefaction wave. One of them is a stationary wave. The relaxation by a contact discontinuity can solve this case, by allowing a jump from the stationary source wave to the other one. The rarefaction wave becomes a saw wave, but its profile is still the same.

Therefore the solution proposed here combines Saw waves, Stationary waves, Source waves, a contact discontinuity and Shocks.

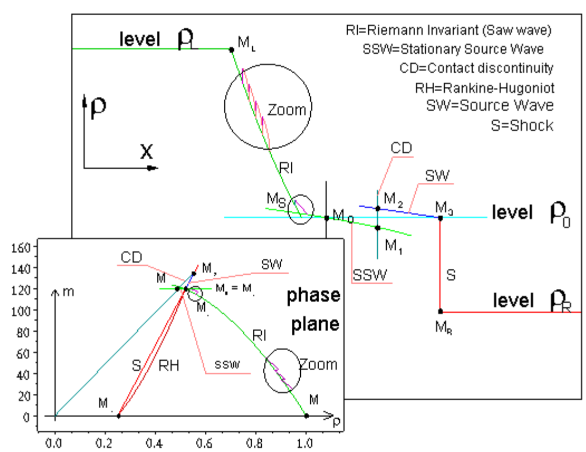

Figure 6: Shock tube with friction

Since the solutions of Riemann problems are used in some numerical schemes, for a short time, we see here that the action of the source term may be neglected. When the rarefaction wave crosses the point $x = 0$, the action of the source term is null.

5 Dam Break

The dam break problem is onother Riemann problem, starting with a null velocity, the water at rest in a reservoir and a dry ground downstream, and the Saint Venant model, with friction and slope.

At $t = 0$, the initial condition for the height of water h and the velocity u is

$$h(x,0) = \begin{cases} H & for\ x < 0, \\ 0 & for\ x > 0\ . \end{cases} \quad , \quad u(x,0) = 0.$$

Since the ground downstream is dry, the solution is made of a single rarefaction wave, which is actually a saw wave.

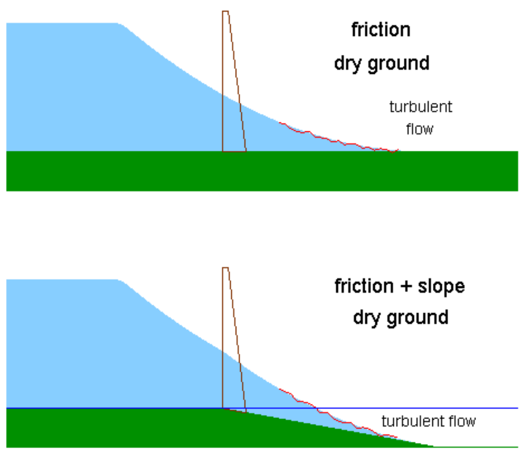

Figure 7: Dam break on a slope

The flow is turbulent in reality, which corresponds to the loss of energy provided by the source term and dissipated by the small shocks along the saw wave profile.

Such a study is useful to prevent a catastrophe and to organize the protection of the population downstream. The result here says that it may be obtained directly by looking a map, just remembering to divide the velocity u, which is the horizontal component of the real velocity, by $cos\theta$, with θ the angle of the slope.

6 Two numerical methods

It is not expected that the usual schemes are always efficient in selecting the real action of a source term. Even the **well balanced schemes** (see [1]) can have some wrong behaviour near zeros of the source term and are unable to select the right velocity of a source wave. Two families of schemes are proposed below.

6.1 The SP schemes

The stationary profiles schemes (=SP-schemes) are adapted to the conservative cases. Let $M_j^n = (q_j^n,\ m_j^n)$ the values of the state in the cell centered at $x_j = j\Delta x$, at time $t_n = n\Delta t$, for $j \in \mathbb{Z},\ n \in \mathbb{N}$??? and a given spacestep Δx and a given time step Δt.

We compute the profile of the source wave with $m = m_j^n,\ \psi(q) = \psi(q_j^n) + x - x_j$.

Next compute $m_{j+1/2,L}^n,\ q_{j+1/2,L}^n$ and $m_{j-1/2,R}^n,\ q_{j-1/2,R}^n$ on both sides of each cell ($L = left,\ R = right$), then compute $m_{j\mp1/2}^n$ and $q_{j\mp1/2}^n$ as solution of the Riemann problem from $L-$ and $R-$ data on each interface. After that, project on the cell at the next time (Godunov scheme) which gives

$$q_j^{n+1} = q_j^n - \frac{\Delta t}{\Delta x}\left(m_{j+1/2}^n - m_{j-1/2}^n\right)$$

and

$$m_j^{n+1} = m_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right) - \Delta t S j^n.$$

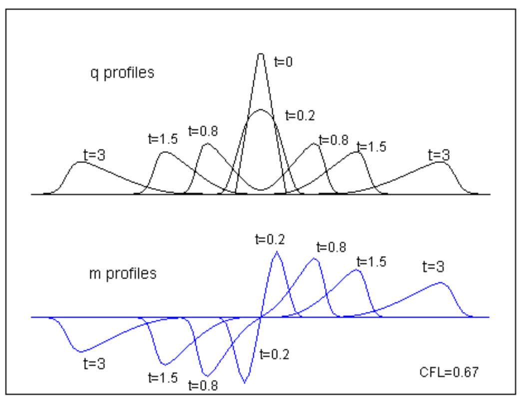

Figure 8: Profiles for the SP scheme

The results on Figure 8 corresponds to $S = k\,|u|\,u$ and $c(q) = \sqrt{q}$. The initial condition is a null velocity, and a concentration of q near $x = 0$. The usual spread of values for the Godunov scheme is still present.

6.2 The DP scheme

The method of **Dynamical Profiles** (=DP) consists into using more parameters in the cell. A state of a wave is characterized by 4 parameters: $q,\ m,\ A$ and $B = m - Aq$, so that 3 of them are necessary. We suppose that the initial condition is made of combination of source waves. Then we only have to discretize the equations:

$$q_t + Aq_x = 0,\ m_t + Am_x = 0,\ A_t + AA_x = 0\ ,$$

which can be done exactly, by using the characteristics.

However, the Burger equation for A must be interpreted as a nonconservative equation. The Shock velocity must satisfy

$$x'(t) = \frac{\Delta m}{\Delta q} \ ,$$

often different from the expected value $(A_1 + A_2)/2$ in the conservative form. For example,the computing of a Roll wave is easy, since A is a constant.

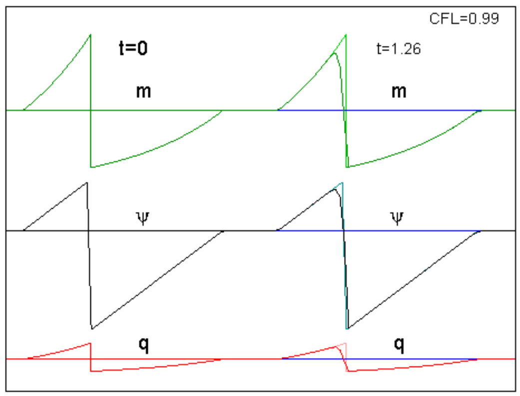

Figure 9: Computing a Roll Wave

We get an exact solution when the CFL number is equal to one. More interesting applications are detailed in the next section.

6.3 The Rogue waves and Tsunamis

Both waves are modelizied by using the Saint-Venant system which is still adapted thought the sea depth is not shallow. This is possible for these waves have a very large wavelength compared to the sea depth. In each case the phenomenons are made of two source waves (named West and East) with a small discontinuity of the velocity A.

The only difficulty is to solve the non conservative Riemann problem

$$A_t + AA_x = 0 \ , \ A(x,0) = \begin{cases} A_{west} & for\ x < 0, \\ A_{east} & for\ x > 0 \ , \end{cases} \ ,$$

with

$$Shock\ velocity = \frac{\Delta m}{\Delta q} \neq \frac{A_{west} + A_{east}}{2}.$$

The slope A_{west} is a little larger than A_{left} and the West wave catches up the East wave, which causes a large local accumulation of water, making a hight wave. This wave is of hight frequency,

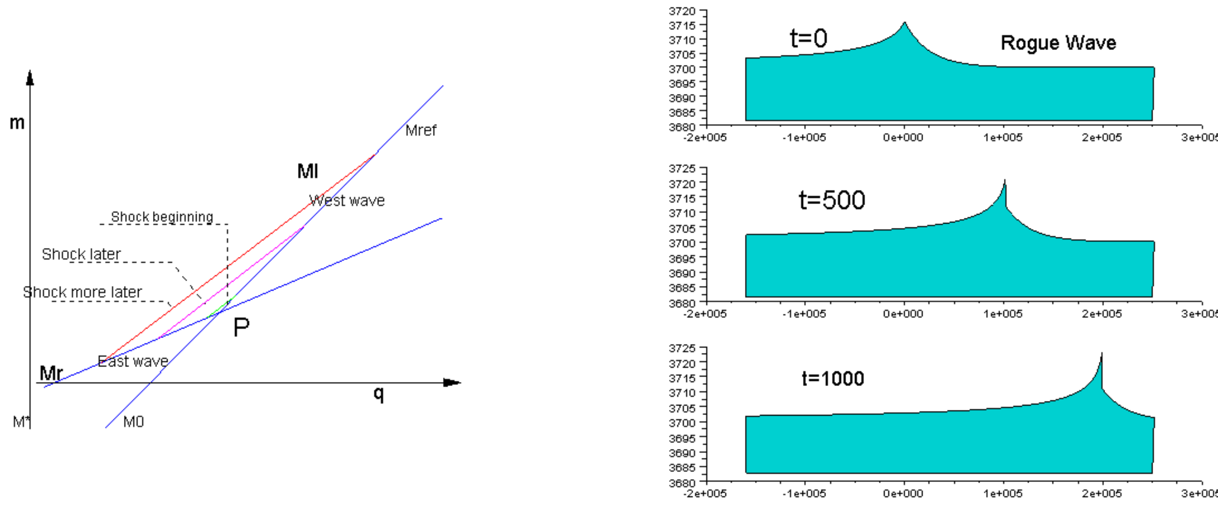

Figure 10: Rogue waves

with short wave lengthes, and disapear after a while, of about a few minuts. The first picture corresponds to the phase plane. The angle between the two lines has been strongly enhanced, in order to separate these lines on the picture. A shock wave starts from the intersection point, and propagates, with a West level along the West line, and an East level along the East line, both linked by a Rankine Hugoniot curve. The value M_{ref} cannot be reached, which explains the collapse of the wave.

The profiles can be computed exactly, as here, with a CFL rate equal to one. A wave of height more than 20 meters has been obtained.

A tsunami wave also corresponds to a Roll wave for the Saint Venant model. The wave front is modelized by a shock. The back and the forerunner trough of the wave correspond to source waves.

This trough of the wave will empty the beach by reaching the shore. On the dry ground, the profile of the front is modified, and given by $\psi'(h)\ h_x = 1$ with

$$\psi'(h) = \frac{B^2 - gh^3}{h^2 S(h,m)}\ ,\ S(h,m) = ku^2 - gph,\ m = Ah - B\ .$$

A negative value of h_x is expected for h small, at the front of the wave. For a flat ground ($p = 0$) we get $\psi'(h) > 0$ when $B \neq 0$. Therefore, near the front, $h \simeq 0$ with $h_x < 0$so that B must be zero, and

$$\psi'(h) = -\frac{gh^3}{km^2} = -\frac{gh^3}{kA^2h^2} = -\frac{g}{kA^2}h.$$

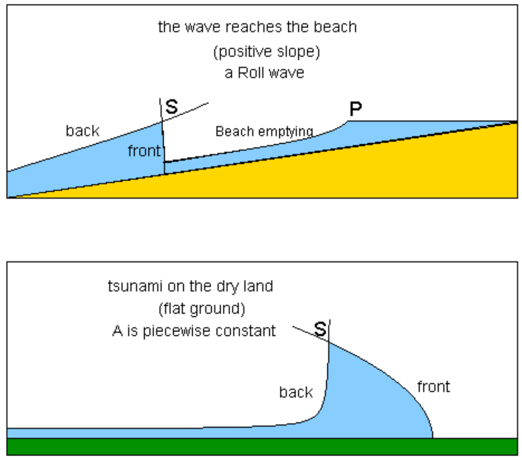

Figure 11: The tsunami model with SV

The integration is obvious and we get $-\dfrac{g}{kA^2}\dfrac{h^2}{2} = x - At - x_0$, or

$$h = \begin{cases} 0 \ , & for \ x > x_0 + At \ , \\ \sqrt{\frac{2kA^2}{g}\left(x_0 + At - x\right)} \ , & for \ x < x_0 + At \\ \end{cases}$$

where x_0 corresponds to the constant of integration. The velocity of the water is equal to A, which means that all objects on the path of the wave will be taken away with this velocity A, and not moved up and down as for a usual wave.

6.4 A wave or not?

The occurence of a tsunami wave after an earthquake depends on the strength of the shock wave starting from the bottom of the sea and also from the depth of the sea. This shock wave is followed by a source wave due to a friction term and the natural water pressure, whose velocity is faster than the velocity of the shock itself. Thus the amplitude of the shock is reduced all along the way to the sea surface, and a tsunami wave will appear only when the remaining amplitude is strong enough. Since the wave travels upwards, the space variable is denoted z and the velacity w. The sea surface corresponds to $z = 0$, and $z < 0$ under the sea surface.

6.4.1 The new transported quantity

Even under strong pressure, the variation of the water density remains insignificant, and the introduction of a new conservative variable is proposed. By denoting $q(p)$ this new quantity, which is supposed to depend on p and to be regular enough, we multiply the Hooke law (as in Section 1.5) by $q'(p)$ and get

$$q(p)_t + wq(p)_z + \rho_0 c(p)^2 q'(p) w_z = 0 \ ,$$

where ρ_0 is the fixed water density ($\rho_0 \simeq 1000$) and $c(p)$ the wave velocity, depending on p too. Next, if $q(p)$ satisfies the differential equation $q(p) = \rho_0 c(p)^2 q'(p)$, we get a new transport equation with $q = q(p)$ as the transported quantity:

$$q_t + (qw)_z = 0 \ .$$

The formulation of $c(p)$ can be obtained from experimental Hugoniot curves

$$c(p) = c_0 \sqrt{1 + \beta_0 p} \ ,$$

as proposed in [1], with $c_0 = 1647 \ ms^{-1}$ and $\beta_0 = 2.84 \ 10^{-9} \ pascal^{-1}$. The previous differential equation reads now

$$\frac{q'(p)}{q(p)} = \frac{1}{\rho_0 c_0^2 \left(1 + \beta_0 p\right)} \ ,$$

to be integrated immediately in

$$q(p) = q_0 \left(1 + \beta_0 p\right)^{1/\rho_0 \beta_0 c_0^2}$$

were q_0 a constant which can be set equal to one . For instance, when p grows from 0 to 10 $kbar$, q varies from 1 to 1.2 which is relatively far more than the variation of the water density. The wave velocity c can be expressed as a function of q which reads

$$c(q) = c_0 q^{\rho_0 \beta_0 c_0^2/2} = c_0 q^{(\gamma_0 - 1)/2},$$

with $\gamma_0 = 1 + \rho_0 \beta_0 c_0^2 \simeq 8.704$. We get the expression of the usual system of Euler equations, but with a larger adiabatic constant γ_0.

6.4.2 The profile of the wave

The dynamical equation with constant water density ρ_0 reads

$$\rho_0 \left(w_t + w w_z\right) + p_z + \rho_0 g + k \left|w\right| w = 0 \ ,$$

where g is a gravity constant and k a friction coefficient, corresponding to the Strickler form. We look for an equation for $m = qw$. By using the transport equation, we get

$$m_t + 2w m_z = q \left(w_t + w w_z\right) + w^2 q_z \ .$$

From $q = \left(1 + \beta_0 p\right)^{1/\rho_0 \beta_0 c_0^2}$, we get

$$\frac{q}{\rho_0} p_z = c(q)^2 q_z,$$

which leads to the expected equation

$$m_t + 2w m_z + \left(c^2 - w^2\right) q_z + g q + \frac{k}{\rho_0} q \left|w\right| w = 0 \ .$$

The source term depends only on q and m, then the theorem of linear appearance applies.

The profile at rest, that is for $m = 0$, is given by the reduced equation $c(q)^2 q_z = -g$. Thus $c_0^2 q^{\gamma_0} = -g \gamma_0 z + C$, with a constant C determined from the atmospheric pressure p_a at the level

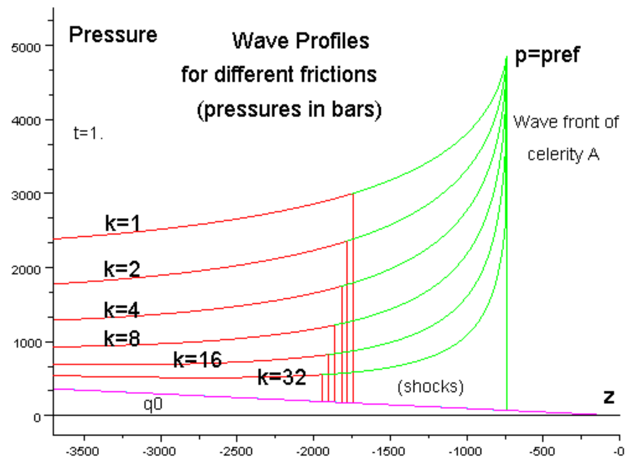

Figure 12: Pressure wave from the bottom

$z = 0$. We get that the profile of q at rest is given by $q_0(z) = (1 + \beta_0 (p_a - gz))^{1/\rho_0\beta_0 c_0^2}$, which increases as z (< 0) decreases.

The front of the wave corresponds to a shock whose position is denoted $z(t)$ at time t. The value of q at the top of the shock is denoted $q_s(t)$, and its value before the shock is $q_0(z(t))$. The velocity of the shock $z'(t)$ is given by the Rankine Hugoniot condition

$$z'(t) = \frac{w(z(t))}{2} + \frac{q_s(t) + q_0(z(t))}{2} \sqrt{\frac{1}{q_s(t)\ q_0(z(t))}\ \frac{P\left(q_s(t)\right) - P\left(q_0(z(t)\right)}{q_s(t) - q_0(z(t))}}\ ,$$

where $P(q) = \frac{c_0^2}{\gamma_0}\ q^{\gamma_0}$. This condition corresponds to a differential equation to be integrated from the bottom, that is $z(0) = z_b$, the depth of the sea $(z_b < 0)$.

The profile of the wave behind the front shock is obtained from a reference state (q_*, m_*) determined by the initial energy at the bottom. We set

$$B = c_0 q_*^{(\gamma_0+1)/2}\ ,\ A = \frac{m_*}{q_*} + c_0 q_*^{(\gamma_0-1)/2}\ .$$

Then the profile q is obtained from the relation

$$\psi(q) = z - At\ +\ \psi(q_*) - z_b\ ,\ with\ \psi'(q) = \frac{B^2 - c_0^2 q^{\gamma_0+1}}{gq^3 + \frac{kq}{\rho_0}\left(Aq - B\right)^2}\ .$$

This wave moves with the velocity A, and since the system is a genuinely hyperbolic one we have $z'(t) < A$ (see Section 1.4). Thus the shock amplitude $q_s(t) - q_0(z(t))$ will decrease during the ascension of the wave. A tsunami is expected only when this amplitude is still significant when reaching the sea surface. In this case a wide part of the sea level will lifted up, with a sufficient wavelength, and a hydraulic wave will propagate, ruled by the Saint Venant model. The value of the friction coefficient plays an important role as shown on Figure 12.

7 Conclusion

Many other applications are possible, even in life modelling or in social/human sciences (stampede), even for non conservative systems. For a given initial condition, how to catch these source waves remains the major difficulty.

Some papers on these topics are published on the Conservation Laws Preprint Server; see [5]. The Theorem of Linear Appearence is also given in [4]. A first paper on Roll Waves was written by R.F.Dressler [1], who had noticed the necessity to have the same roots in the numerator and the denominator in the expression of $\psi'(q)$. However the constant velocity seems to be untapped.

8 Bibliography

[1] R.F.Dressler, Mathematical solution of the problem of RollWaves in inclived open Channel, Comm.Pure Appl.Math. Vol II N2-3, pp149-194 (1949).
[2] J.M.Greenberg, A-Y.LeRoux, A well balanced scheme for the numerical processing of source terms in hyperbolic equations. SIAM Journal on Numerical Analysis Vol.33 N1, pp1-16 (1996).
[3] Group GMX-6, Selected Hugoniot, Report LA-4167, Los Alamos Scientific Laboratory (1969).
[4] L3-Mathématiquess Appliquées A.Yger and J.A.Weil, directors, Pearson Ed. (2009).
[5] Papers on the Conservation Laws Preprint Server: www.ntnu.no/conservation, 2000/029, 2004/045, 2005/048, 2008/010, 2008/019, 2009/034.