High resolution schemes: total variation stability and its dependence on smoothness parameter

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Abstract

A classification in terms of accuracy of flux limited high resolution schemes in steep gradient region is done based on two different total variation (TV) stability region. The dependence of TV stability region on the smoothness parameter is discussed which relates these two TV stability regions. A common unifying TV stability region is proposed for both class of schemes. Moreover new flux limiters, satisfying the unifying TV stability region are also proposed which are robust and work efficiently for both backward (left) and forward (right) moving solution profile. Main significant feature of this work is that it can improve accuracy of all existing flux limiters based schemes. Numerical results on linear test problems are given to support the theoretical discussion.

Keywords High resolution schemes; Flux limiters, TVD region; Smoothness parameter; Hyperbolic equations.

1 Introduction

It has been around three decade since CFD community is celebrating the class of high resolution schemes (HRS). The term high resolution scheme coined by Harten represents a class of conservative schemes which crisply resolve discontinuities like contact, shocks without exhibiting spurious oscillations and give at least second order of accuracy for smooth solution [4]. Various approaches and methods have been proposed to design such schemes e.g. essentially non-oscillatory (ENO) [18], weighted essential non-oscillatory schemes [5] and high resolution TVD scheme using flux limiters [20, 7]. A good detail on these methods can be found in [12, 22, 11]. The high resolution total variation diminishing (HRTVD) schemes have been used extensively for excellent results and theoretical support. In this work, focus is on the HRTVD schemes using flux limiters.

Among all, the approach proposed by Sweby can be considered as a representative framework for designing Lax-Wendroff type HRTVD schemes using flux limiters [20, 2, 26, 21]. More importantly a TV stability region is given for flux limiters to yield total variation diminishing schemes in [20]. Later a variety of high resolution schemes as well new flux limiters [16, 17, 9, 13, 8] are proposed to improve numerical results in one way or other. Recently a comparative study of various flux limiters is done for solid gas reaction problem in [6]. In fact a detailed study on desirable properties of these flux limiters to yield high resolution TVD schemes are given in [1, 14, 24]. The most unique feature for all these limiters is that they satisfy either completely or for up to a finite positive value of smoothness parameter, the TVD stability region proposed in [20]. Despite of such tremendous development in the area of TVD schemes, it seems that attempts are not made to find alternate TVD region except in [10, 21]. Using diffusive centered difference first order flux, centered high resolution TVD schemes and flux limiters are proposed in [21]. Also a Courant number dependent TV stability region is given for centered limiters which reduces to Courant number independent TV stability region given in [20]. In [10], a general framework for constructing second order upwind high resolution TVD schemes using flux limiters. Also an entirely new TV stability region and a class of new limiters for proposed schemes are designed which satisfies the proposed TVD region. As far the classification of such high resolution schemes as well for flux limiters are concern few attempts are made. Such classifications, (to the best of knowledge of author) are mostly based on central and upwind nature of discretization [26, 21].

The flow of the paper goes as follows: In section 2, we give a compact detail on construction of flux limiter based high resolution schemes. In section 3, we characterize high resolution TVD schemes based on TV stability region for flux limiters in to two class. This classification make sense as schemes of one class approximate the solution in the region of steep gradient in opposite way to the scheme of another class. An investigation on the relation between the TVD region of high resolution schemes using flux limiters and its dependence on the smoothness parameter is done in section 4 which give an unifying relation for the known TVD region for flux limiter proposed in [20] and [10]. This unification enable the chance to use any flux limiters with both class of high resolution schemes by treating smoothness parameter accordingly. An unifying universal TV stability region is proposed along with new universal flux limiters which are shown to works efficiently and are independent of choice of measure of smoothness for both class of schemes.

2 High resolution TVD schemes using flux limiters

In this section, we give a brief idea on the construction of high resolution schemes using flux limiters for the completeness and clarity on the classification of these schemes. For present work we discuss the idea for linear hyperbolic problem though it holds for non-linear case also.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, f(u) = au, \quad 0 \neq a \in \mathbb{R}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{1}$$

where u denotes convection variable and characteristics speed associated with (1) a is constant. Divide the spatial and temporal space into N equal length cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 0, 1, \dots N$ and M intervals $[t^n, t^{n+1}]$, $n = 0, 1, \dots, M$ respectively, where $x_{i\pm\frac{1}{2}}$ is called cell interface and t^n denotes the n^{th} time level. We know that conservative numerical approximation for above equation is obtained by

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \lambda \left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} \right) \tag{2}$$

where $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, $\Delta t = t^{n+1} - t^n$ and $\lambda = \frac{\Delta t}{\Delta x}$. $\mathcal{F}_{i+\frac{1}{2}}$ is time-integral average of flux function at cell interface and $\bar{u}_i^n$ is spatial cell-integral average defined as,

$$\bar{u}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)dx, \quad \mathcal{F}_{i+\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))dt. \tag{3}$$

The choice of numerical flux function $\mathcal{F}_{i\pm\frac{1}{2}}$ govern the spatial performance like accuracy, dissipation, numerical oscillations or shock capturing feature of resulting conservative scheme.

Definition 2.1. *A conservative scheme (2) is said to be TVD if* $TV(\bar{u}^{n+1}) \leq TV(\bar{u}^n)$, $\forall n$, *where total variation of grid function* $\bar{u}$ *at time level* n *is defined as* $TV(\bar{u}^n) = \sum_{i=-\infty}^{+\infty} \left|\bar{u}_{i+1}^n - \bar{u}_i^n\right|$.

The general idea of constructing high resolution scheme is to define its numerical flux $\mathcal{F}_{i+\frac{1}{2}}^{hrs}$ as a combination of a dissipative non-oscillatory low order numerical flux $\mathcal{F}_{i+\frac{1}{2}}^l$ and non-dissipative oscillatory high order flux $\mathcal{F}_{i+\frac{1}{2}}^h$ using flux limiter function ϕ as follows,

$$\mathcal{F}_{i+\frac{1}{2}}^{hrs}(r_i) = \mathcal{F}_{i+\frac{1}{2}}^l + \phi(r_i)\left(\mathcal{F}_{i+\frac{1}{2}}^h - \mathcal{F}_{i+\frac{1}{2}}^l\right). \tag{4}$$

The main characteristic of ϕ is that it measures the smoothness of the solution profile. In general limiter ϕ is taken as function of smoothness parameter r which is again a function of ratio of consecutive gradients of the solution of (1).

The flux limiter ϕ is defined in such a way that it diminish to zero in the solution region with extreme points or discontinuities, hence scheme results into first order dissipative approximation for such solution region. Limiter ϕ takes a value close to one to give high accurate approximation scheme for smooth region of solution.

3 Classification

Few attempts are made to classify high resolution TVD scheme and flux limiters but mostly are based on differencing. They are classified into symmetric schemes based on central difference in [26]. Similarly upwind schemes can be classified based on up-winding [23, 7, 10]. In [25], flux limiters are also classified as symmetric and upwind type and few symmetric flux limiters are proposed. A qualitative and quantitative comparison is done on some TVD Lax-Wendroff methods using centered and upwind biased flux limiters in [15]. In [21], a classification of HR scheme is done in to centered and upwind TVD schemes which is based on the choice of diffusive first order accurate flux $\mathcal{F}_{i+\frac{1}{2}}^l$ in the construction (4). In case $\mathcal{F}_{i+\frac{1}{2}}^l$ is chosen to be a first order upwind flux then the resulting HR scheme is classified as upwind TVD scheme using upwind limiter ϕ^u whilst called centered TVD scheme using centered limiter ϕ^c if $\mathcal{F}_{i+\frac{1}{2}}^l$ is taken as numerical flux of any first order centered monotone scheme such as Lax-Friedrichs, FORCE [3] and Godunov's first order centered scheme [19]. Moreover Courant number dependent TVD stability region for these centered HR schemes is also derived. The key of construction of centered flux limiter ϕ^c is the following explicit relation of it with upwind flux limiter ϕ^u

$$\phi^c = \hat{\phi}_g + (1-\hat{\phi}_g)\phi^u, \tag{5}$$

with

$$\hat{\phi}_g = \begin{cases} 0, & r \leq 1, \\ \phi_g, & r > 1. \end{cases}$$

where ϕ_g is defined as Godunov point which depends on Courant number and varies with the choice of first order monotone centered flux $\mathcal{F}^l$ [21]. Note that in [21], it is shown that the TVD stability region for proposed centered TVD schemes can be reduce to Courant number independent TVD region given by Sweby. Hence one can consider the TV stability region in [20] a generic TV stability region for most centered Lax-Wendroff type TVD schemes. Recently a new TV stability region is given for proposed upwind based HR schemes with a class of flux limiters in [7]. This TVD region is entirely different from the one in [20] for same smoothness parameter. In the following we discuss both different stability region and formulation.

3.1 Classification based on two TV stability regions

We can classify the HRTVD schemes based on their TV stability region. This classification quantifies the schemes in terms of order of accuracy as scheme of one class approximates the steep gradients solution region in opposite way compared to the scheme of other class. For discussion we take combination of numerical flux function of first order upwind and three representative second order accurate schemes viz Lax-Wendroff, second order upwind and Beam-Warming schemes respectively to construct flux limited schemes for linear problem (1)

The numerical flux function Lax-Wendroff flux limited high resolution scheme can be constructed as in [20]

$$\mathcal{F}^{LxWflm}_{i+\frac{1}{2}}(r_i) = a\bar{u}_i + \frac{1}{2}a(1-a\lambda)\phi(r_i)\left(\bar{u}_{i+1} - \bar{u}_i\right), \ a > 0, \tag{6}$$

Numerical flux function of second order upwind flux limited method in [7] can be written as

$$\mathcal{F}^{IIupflm}_{i+\frac{1}{2}}(r_i) = a\bar{u}_i + \frac{1}{2}a\lambda\psi(r_i)\left(\bar{u}_i - \bar{u}_{i-1}\right), \ a > 0. \tag{7}$$

Similar to second order upwind flux limited method one can obtain Beam-Warming flux limited method

$$\mathcal{F}^{BWflm}_{i+\frac{1}{2}}(r_i) = a\bar{u}_i + \frac{1}{2}a(1-a\lambda)\psi(r_i)\left(\bar{u}_i - \bar{u}_{i-1}\right), \ a > 0. \tag{8}$$

where ϕ, ψ are flux limiters and r_i is the smoothness parameter which measure the smoothness of solution and defined as function of consecutive gradients. On Uniform grid it is,

$$r_i = \frac{\bar{u}_i - \bar{u}_{i-1}}{\bar{u}_{i+1} - \bar{u}_i}. \tag{9}$$

In order to ensure the TV stability of resulting HR schemes following conditions are given on flux limiters ϕ and ψ respectively in [20, 7] as follows.

Theorem 3.1. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{LxWflm}_{i+\frac{1}{2}}(r_i)$ *is TV stable under the CFL condition* $0 \leq a\lambda \leq 1$, $a \geq 0$ *if the flux limiter* $\phi(r)$ *satisfy,*

$$0 \ \leq \frac{\phi(r)}{r} \leq \ 2 \ \text{and} \ 0 \ \leq \phi(r) \ \leq 2. \tag{10}$$

Theorem 3.2. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{IIupflm}_{i+\frac{1}{2}}(r_i)$ *is TV stable under the CFL condition* $0 \le a\lambda \le \frac{1}{2},\ a \ge 0$ *if the flux limiter* $\psi(r)$ *satisfy,*

$$0 \ \le r\psi(r) \le \ 2 \ \textit{and} \ 0 \ \le \psi(r) \ \ \le 2. \tag{11}$$

Remark 1. It can be shown that the Beam-Warming flux limited scheme using numerical flux function $\mathcal{F}^{BWflm}_{i+\frac{1}{2}}(r_i)$ is TV stable under the CFL condition $0 \le a\lambda \le 1,\ a \ge 0$ if the flux limiter $\psi(r)$ satisfy,

$$0 \ \le r\psi(r) \le \ 2 \ \text{and} \ 0 \ \le \psi(r) \ \ \le 2. \tag{12}$$

The distinct TV stability region for the flux limiter ϕ and ψ can rewritten as,

$$\mathcal{R}_1 = \{(r,\phi) \in R \times R : 0 \ \ \le \phi(r) \le \ \ 2\max(r,0) \ \text{ and } \ 0 \ \ \le \phi(r) \ \ \le 2\}\,. \tag{13}$$

$$\mathcal{R}_2 : \left\{(r,\psi) \in R \times R : 0 \ \ \le \psi(r) \le \ \ \frac{2}{\max(r,0)} \ \text{ and } \ 0 \ \ \le \psi(r) \ \ \le 2\right\}. \tag{14}$$

Note that for same measure of smoothness parameter r, TV stability regions in (13) and (14) (shown in Figure 1) are different. Let class of high resolution total variation stable schemes with $\mathcal{R}_1$ and $\mathcal{R}_2$ stability region be denoted by $\mathcal{C}^{\mathcal{R}_1}$ and $\mathcal{C}^{\mathcal{R}_2}$ respectively. This classification make sense as in rapidly monotone increasing solution region i.e. $1 >> r \to 0^+$, limiters for $\mathcal{C}^{\mathcal{R}_1}$ schemes must tend to 0 and give first order approximation whereas limiters for $\mathcal{C}^{\mathcal{R}_2}$ schemes can give at least second order accurate approximation. On the other hand rapidly monotone decreasing solution region i.e. $1 << r \to +\infty$, limiters for $\mathcal{C}^{\mathcal{R}_2}$ class tend to 0 and result into first order approximation whilst limiters for $\mathcal{C}^{\mathcal{R}_1}$ can give higher accuracy (See Figure 3 for numerical illustration). A class of flux limiters ($\delta-$limiters, say) which ensure second order of accuracy in smooth region and satisfy TV stable region $\mathcal{R}_2$ is also given in [7] as follows,

$$\psi^{\delta}(r) = \begin{cases} 0, & r \le \ 0, \\ \min\left[2, \frac{2}{r}, \frac{1+\delta}{\delta+r}\right], & r > \ 0, \ \text{ for any fixed } \ \delta \in [0,\infty). \end{cases} \tag{15}$$

Also analogous to classical Minmod limiter i.e,

$$\phi^{mm}_b(r) = max\left(min\left(1,\ br\right),0\right),\ 1 \le b \le 2 \tag{16}$$

one can define a diffusive limiter say, $Rk - minmod$, which satisfy (14),

$$\psi^{mm}_b(r) = max\left(min\left(1,\ \frac{b}{r}\right),0\right),\ 1 \le b \le 2 \tag{17}$$

Geometrically TV stability region (13) along with some of the flux limiters is drawn in Figure 1(a). The region (14) with limiters (15) for $\delta = 1, 9$ and limiter (17) is shown in Figure 1 (b).

4 TV stable region: Dependence on smoothness parameter

The smoothness parameter r taken in the formulation of HR scheme in [20] and [7] is defined exactly same way as in (9) and despite of same r both schemes have different TV

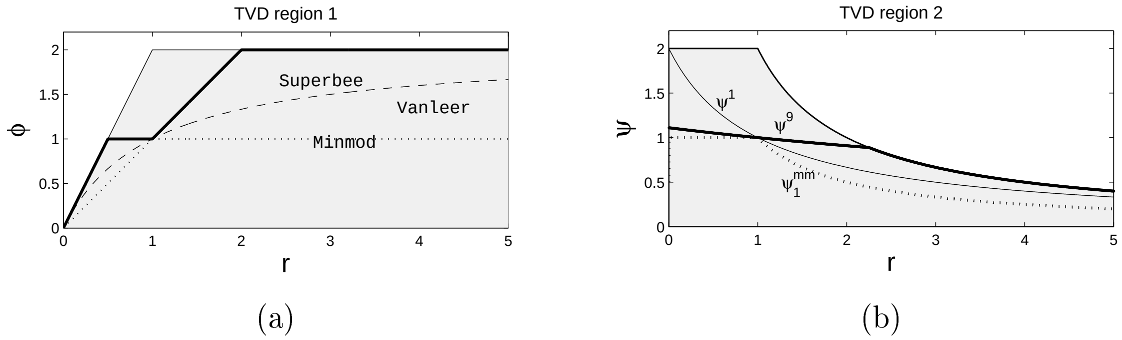


Figure 1: TVD regions and some limiter passing through them: $\mathcal{R}_1$ (left) and $\mathcal{R}_2$ (right)

stability region. In fact one can also take the measure of smoothness as $s_i = \dfrac{1}{r_i}$ i.e.,

$$s_i = \frac{\bar{u}_{i+1} - \bar{u}_i}{\bar{u}_i - \bar{u}_{i-1}}. \tag{18}$$

This smoothness measure s is indeed taken in literature but to preserve TV stability region $\mathcal{R}_1$ or $\mathcal{R}_2$ for $\mathcal{C}^{\mathcal{R}_1}$ or $\mathcal{C}^{\mathcal{R}_2}$ schemes to construct variant of these respective HR schemes for convection equation with negative characteristic speed. In the following, TV stability region for resulting HR schemes is investigated while s is used instead of r as measure of smoothness in (6) and (7) in case of positive characteristic speed.

Theorem 4.1. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{LxWflm}_{i+\frac{1}{2}}(s_i)$ *is TV stable under the CFL condition* $0 \le a\lambda \le 1,\ a > 0$ *if the flux limiter* $\phi(s)$ *satisfy,*

$$0 \ \le s\phi(s) \le \ 2 \ \textit{and} \ 0 \ \le \phi(s) \ \le 2. \tag{19}$$

Proof. The resulting scheme can be written in following conservative I-form,

$$\bar{u}_i^{n+1} = \bar{u}_i^n + \alpha_{i+\frac{1}{2}} \Delta_+ \bar{u}_i^n - \alpha_{i-\frac{1}{2}} \Delta_- \bar{u}_i^n, \tag{20}$$

where $\Delta_+ \bar{u}_i^n = \Delta_- \bar{u}_{i+1}^n = \bar{u}_{i+1}^n - \bar{u}_i^n,\ \alpha_{i+\frac{1}{2}} = 0$ and $\beta_{i-\frac{1}{2}} = \lambda\, a \left(1 - \dfrac{1}{2}(1 - \lambda\, a)(\phi_{i-1} - \phi_i s_i) \right)$.
A sufficient condition for any scheme of the form (20) to be TVD is given in [4] as follows,

$$\alpha_{i+\frac{1}{2}} \ge 0,\ \beta_{i-\frac{1}{2}} \ge 0,\ 0 \le \alpha_{i+\frac{1}{2}} + \beta_{i-\frac{1}{2}} \le 1. \tag{21}$$

Under the linear stability condition $0 \le \lambda a \le 1$, inequalities (21) satisfy if,

$$-2 \le -\phi_{i-1} + \phi_i s_i \le 2,\ \forall i.$$

which satisfies (after dropping out index i) if,

$$0 \ \le s\phi(s) \le \ 2 \ \text{and} \ 0 \ \le \phi(s) \ \le 2.$$

which completes the proof. □

Similarly following theorem can be proved

Theorem 4.2. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{IIupflm}_{i+\frac{1}{2}}(s_i)$ *is TV stable under the CFL condition* $0 \le a\lambda \le \frac{1}{2}$*,* $a > 0$ *if the flux limiter* $\psi(s)$ *satisfy,*

$$0 \le \frac{\psi(s)}{s} \le 2 \text{ and } 0 \le \psi(s) \le 2. \tag{22}$$

In case of negative characteristic speed i.e., $a \le 0$ in (1) we have following analogous results to ensure the TV stability of schemes.

Corollary 4.3. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{LxWflm}_{i+\frac{1}{2}}(r_{i+1})$ *is TV stable under the CFL condition* $-1 \le a\lambda \le 0$*,* $a < 0$*, if the flux limiter* $\phi(r)$ *satisfy,*

$$0 \le r\phi(r) \le 2 \text{ and } 0 \le \phi(r) \le 2. \tag{23}$$

Corollary 4.4. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{LxWflm}_{i+\frac{1}{2}}(s_{i+1})$ *is TV stable under the CFL condition* $-1 \le a\lambda \le 0$*,* $a < 0$*, if the flux limiter* $\phi(s)$ *satisfy,*

$$0 \le \frac{\phi(s)}{s} \le 2 \text{ and } 0 \le \phi(s) \le 2. \tag{24}$$

Corollary 4.5. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{IIupflm}_{i+\frac{1}{2}}(r_{i+1})$ *is TV stable under the CFL condition* $-\frac{1}{2} \le a\lambda \le 0$*,* $a < 0$ *if the flux limiter* $\psi(r)$ *satisfy,*

$$0 \le \frac{\psi(r)}{r} \le 2 \text{ and } 0 \le \psi(r) \le 2. \tag{25}$$

Corollary 4.6. *The resulting conservative scheme using numerical flux function* $\mathcal{F}^{IIupflm}_{i+\frac{1}{2}}(s_{i+1})$ *is TV stable under the CFL condition* $-\frac{1}{2} \le a\lambda \le 0$*,* $a < 0$ *if the flux limiter* $\psi(r)$ *satisfy,*

$$0 \le s\psi(s) \le 2 \text{ and } 0 \le \psi(s) \le 2. \tag{26}$$

Remark 2. All the above results for second order upwind flux limited method hold true for Beam-Warming flux limited method under the CFL condition $|a\lambda| \le 1$.

Remark 3. Based on above theorems and analogous corollaries, dependence of TV stability region for flux limiter on the measure of smoothness parameter is evident. It can be observed that if a fix measure of smoothness i.e, r (or s) is taken for convection equation irrespective of sign of characteristics speed then HR schemes which belong to class $\mathcal{C}^{\mathcal{R}_1}$ (or $\mathcal{C}^{\mathcal{R}_2}$) for positive characteristic speed will fall in to class $\mathcal{C}^{\mathcal{R}_2}$(or $\mathcal{C}^{\mathcal{R}_1}$) respectively for negative characteristic speed.

4.1 Unifying TV stability region

It can be easily observed that all the flux limiters developed so far for TV stability region $\mathcal{R}_1$ to yield TV stable $\mathcal{C}^{\mathcal{R}_1}$ schemes fail to preserve TV stability when applied on the schemes of class $\mathcal{C}^{\mathcal{R}_2}$. Similarly limiters (15) and (17) developed for TV region $\mathcal{R}_2$ to yield TV stable $\mathcal{C}^{\mathcal{R}_2}$ schemes fail to give TV stability for schemes of class $\mathcal{C}^{\mathcal{R}_1}$. A unifying TV stable region can be deduce for both class of HR schemes as follows,

$$\mathcal{R}_c = \left\{(\theta, \xi) \in R \times R : 0 \le \theta\,\xi(\theta) \le 2 \text{ and } 0 \le \frac{\xi(\theta)}{\theta} \le 2\right\}, \tag{27}$$

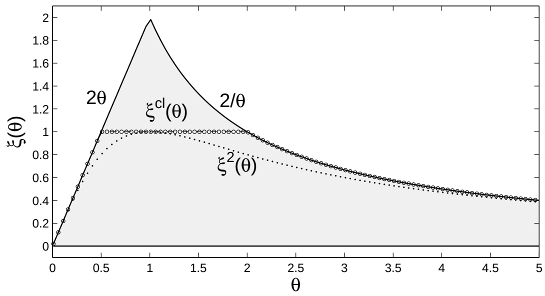


Figure 2: Unifying TVD region along with flux limiters

where $\theta = r, s$ and $\xi = \phi, \psi$. Note that, the above unifying common TV stability region R_c shown in Figure 2 is more restrictive compared to $\mathcal{R}_1$ or $\mathcal{R}_2$ but fortunately give enough freedom to design flux limiters which satisfy this common TV stable region and guarantee second order of accuracy for smooth solution profile i.e near $r \approx 1$. We propose flux limiters as follows,

$$\xi(\theta) = \xi^n(\theta) = \frac{\theta + |\theta|}{\theta^n + 1}, \; n \geq 2, \; \lim_{\theta \to +\infty} \xi \to 0. \tag{28}$$

Note that for $n = 1$, we obtain Vanleer flux limiter while for $r > 0$, $n = 2$ we obtain van-Albada2 type limiter ($\xi^2(\theta)$) [8]. A more compressive limiter ($\xi^{cl}(\theta)$) can be defined as

$$\xi(\theta) = \xi^{cl}(\theta) = max\left[min\left\{2\theta, \; \frac{2}{\theta}, \; 1\right\}, \; 0\right]. \tag{29}$$

In Figure 2 flux limiter (28) for $n = 2$ and limiter (29) are also shown.

Remark 4. It can be easily observed that the proposed unifying stability region R_c is invariant under any transformation i.e. with respect to change in definition of measure of smoothness or change in the sign of characteristic speed, hence can be considered as universal TV stability region for high resolution total variation diminishing schemes using flux limiters.

5 Numerical results

Our aim by giving numerical results is to show the behavior of both class of scheme in near discontinuities or steep gradient region as discussed in section 3.1. We consider the linear convection equation

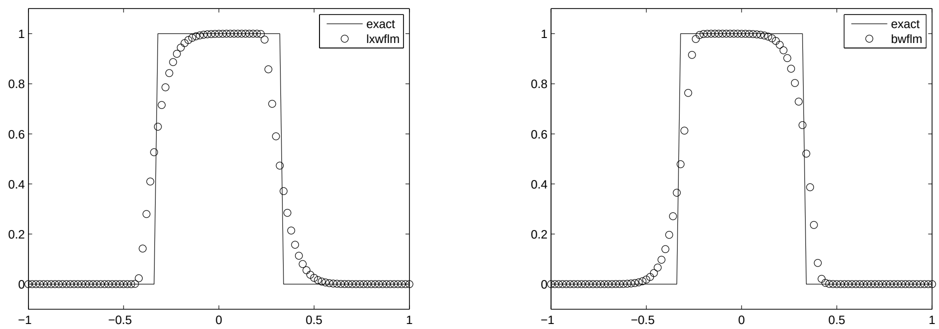
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \tag{30}$$

where a is characteristic speed. In the following test cases we take equation (30) with different initial conditions and characteristic speed to show the numerical result which justify the above discussion. In all the presented Figures following name convention is used: Results by centered TVD high resolution scheme for three choice of $\mathcal{F}^l$ as centered first order monotone flux viz: Lax-Friedrichs, FORCE and Godunov [19] are shown by c-lxf, c-force and c-god respectively. Results obtained by upwind flux limited method (6), (7) and (8) are shown by LxWflm, IIupflm, BWflm.

5.1 Test for accuracy in steep gradient or discontinuous region

Consider (30) with following initial condition $u(x,0) = \begin{cases} 1, & \text{if } |x| \leq \frac{1}{3}, \\ 0, & else. \end{cases}$ and periodic boundary conditions. This test case has two propagating contact discontinuities especially taken to depict the accuracy of both class of schemes on capturing the left and right discontinuities in the solution profile. Numerical results obtained with LxWflm and BWflm using Minmod type limiters (16) and (17) respectively are shown in Figure 3. Parameter $b = 2$ is taken in these limiters as it ensures second order accuracy for maximum range of $r \geq 0$.

It can be easily observe from Figure 3(a) and 3(b) that scheme LxWflm of class $\mathcal{C}^{\mathcal{R}_1}$ give diffusive low order approximation for top of left jump and bottom of right jump whilst bottom of left jump and top of right jump is approximated with higher accuracy. On the other hand scheme BWflm of class $\mathcal{C}^{\mathcal{R}_2}$ show opposite behavior on left and right jumps.



(a) (b) (c)

Figure 3: Numerical result of LxW and BW flux limited method with minmod limiters $\phi_2^{mm}(r)$ and $\psi_2^{mm}(r)$ respectively with $a\,\lambda = 0.8, N = 50, T = 10.0,\ N = 100$.

Remark 5. Numerical results in Figure 3 show that near discontinuities or steep gradient region both class of schemes give opposite approximation in terms of accuracy which supports the discussion in section 3.

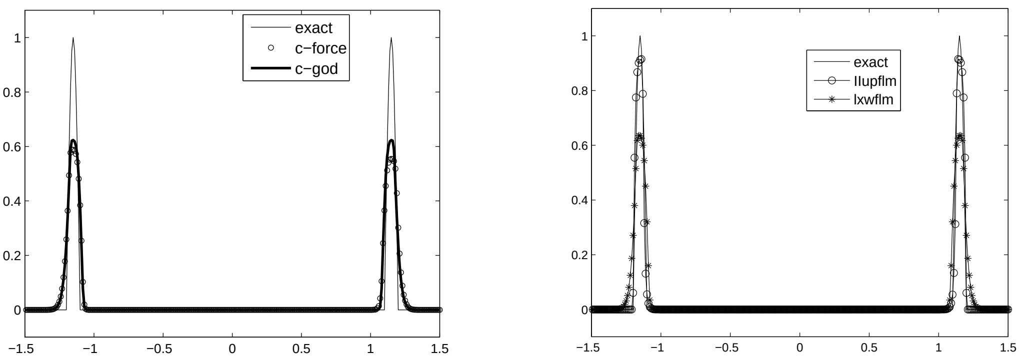
5.2 Results obtained by universal limiters

5.2.1 Smooth solution case: Separating peaks

In this test case we take the characteristic speed given by $a = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$. Consider the following smooth initial condition with steep profile, $u(x,\,0) = \begin{cases} |sin(10\pi\, x)|, & 0.1 \leq |x| \leq 0.2, \\ 0, & else. \end{cases}$. Note that the initial solution has two upward peaks which convect in opposite direction and separate due to positive and negative characteristic speed $a = \pm 1$ on right and left side of the point $x = 0$. For the computation of results given in Figure 4 and Figure 5, same measure of smoothness i.e., $r = \dfrac{\Delta_- \bar{u}_i}{\Delta_+ \bar{u}_i}$ for positive as well negative characteristic speed is used. In Figure 4, numerical results obtained by centered and upwind flux limited methods are given for CFL $a\lambda = 0.45$ at time $T = 1.0$. Note that upto time $T = 1.0$ non-zero initial profile does not touch the computational boundary which make it well-posed. In the computation compressive limiter ξ^{cl} is used. Note that since centered first order flux FORCE is diffusive compared to the GODUNOV and first order upwind flux hence the numerical results c-force is more dissipative compared to the results of upwind TVD schemes in Figure 4(b). Note that the high order Lax-Wendroff numerical flux is taken for all these schemes except in IIupflm which takes second order upwind flux. Figure 5(a) and 5(b) show the results obtained by scheme $LxWflm$ and $IIupflm$ using Rk-minmod ($\psi_1^{mm}(r)$) in (17) and classical minmod ($\phi_1^{mm}(r)$) limiter (17) and (16) for smoothness parameter r defined in (9). Solution is computed for $T = 1.0,\ CFL = 0.25,\ N = 400$. It can be seen that both the schemes captures the right moving peak with a TVD approximation whereas give oscillatory approximation for left moving peak. These results show that the flux limiter designed for TVD region (13) or (14) fails for schemes of class $\mathcal{C}^{\mathcal{R}_1}$ or $\mathcal{C}^{\mathcal{R}_2}$ for a fixed definition of measure of smoothness for both direction of characteristic speed. In Figure 6, we give zoom view of the numerical results obtained by upwind high resolution schemes [7] and [20]. In order to compare the unifying limiters with minmod limiters ($\psi_1^{mm}(\theta)$ and $\phi_1^{mm}(\theta)$) with respective schemes the measure of smoothness is taken depending on direction of flow i.e.,

$$\theta = \begin{cases} r, & \text{if } a \geq 0, \\ s, & \text{if } a < 0, \end{cases} \tag{31}$$

where r and s are defined as (9) and (18) respectively. These results show that the limiter $\xi^2(\theta)$ give comparable results with minmod type limiters whilst $\xi^{cl}(\theta)$ give less dissipation and better approximation for the solution.



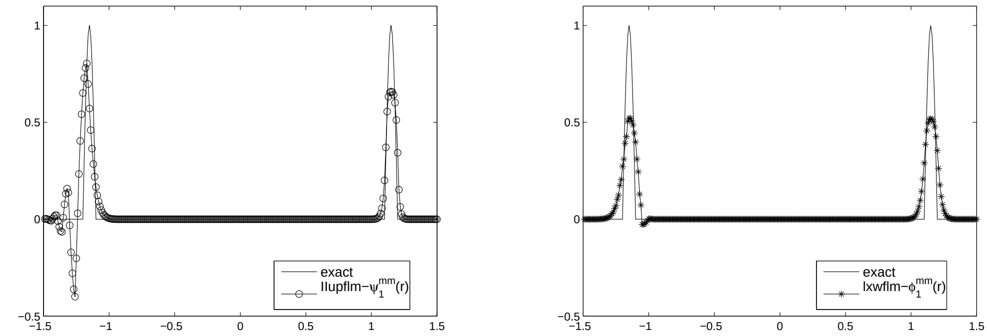
(a) (b)

Figure 4: Numerical result of centered and upwind high resolution scheme using compressive limiter ($\xi^{cl}(r)$) with $a\,\lambda = 0.45, T = 1.0,\ N = 400$.

5.2.2 Linear convection: Contact Discontinuity case

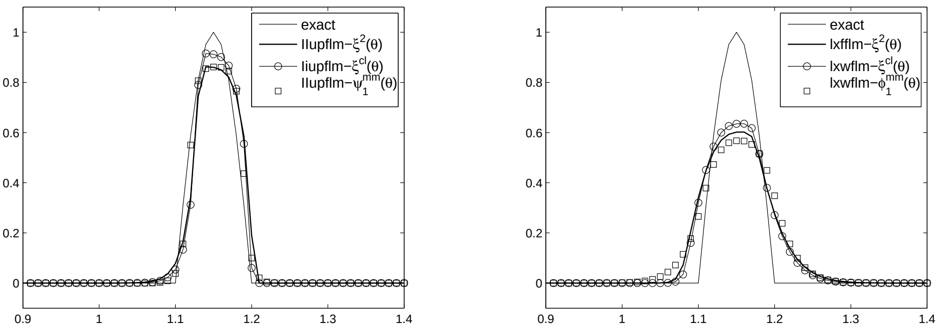
In order to show the performance of proposed limiter (ξ^{cl}) for discontinuous solution profile, we consider the equation (30) with $a = 1.0$ and the following discontinuous initial condition,

$$u(x, 0) = \begin{cases} 0, & \text{if } 0 \leq x \leq 0.4, \\ 1, & \text{else.} \end{cases} \tag{32}$$



(a) (b)

Figure 5: Numerical result of upwind high resolution scheme using minmod type limiter for data $a\,\lambda = 0.45, T = 1.0,\ N = 400$

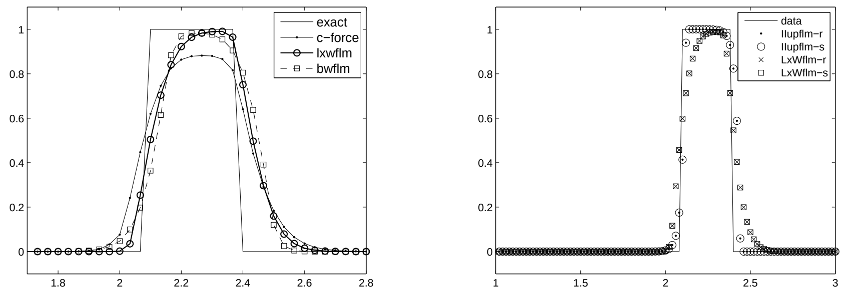


(a) (b)

Figure 6: Solution by upwind high resolution schemes IIupflm and LxWflm using limiter $\xi^2(\theta)$ (28) for $n = 2$, limiter $\xi^{cl}(\theta)$ in (29) and minmod type limiters.

In Figure 7(a), zoom in solution obtained by some high resolution schemes using compressive limiter ϕ^{cl} in (29) at time $T = 2.0$ is given. The computational domain in space $[0, 3]$ is divided into $N = 90$ intervals for $CFL = 0.8$. Result shows that even for limiter ϕ^{cl} scheme LxWflm of class $\mathcal{C}^{\mathcal{R}_1}$ capture the foot of the left discontinuity much crisply compared BWflm of class $\mathcal{C}^{\mathcal{R}_2}$ but for foot of right discontinuity is captured in opposite way. This observation is more prominent in Figure 7(b) as scheme IIupflm of class $\mathcal{C}^{\mathcal{R}_2}$ capture the top most corner of left discontinuity crisply but little dissipation is observed on top of right discontinuity. For the bottom of discontinuities IIupflm shows opposite nature. Similar observation can be deduced for result obtained by LxWflm. In Figure 7(b), for data $N = 150,\ a\,\lambda = 0.45$, we show the result obtained by upwind TVD schemes IIupflm-r (LxWflm-r) and IIupflm-s (LxWflm-s) using universal limiters $\xi^2(\theta)$ for both choice of smoothness parameter $\theta = r$ and $\theta = s$ respectively. Result shows that the universal limiter ξ^2 is independent of choice of smoothness parameter. Note that the change in choice of measure r or s interchange the TV stability region for one class (e.g., $\mathcal{C}^{\mathcal{R}_1}$) to another class (e.g., $\mathcal{C}^{\mathcal{R}_2}$).

Remark 6. Using these universal limiters $\xi^2(\theta)$ and $\xi^{cl}(\theta)$, we obtained exactly same results by centered TVD schemes for both choice of smoothness measure (r or s) .



(a) (b)

Figure 7: Numerical result at $T = 2.0$ obtained by flux limited schemes using limiter (a) $\xi^{cl}(r)$ $a\lambda = 0.8$ and $N = 90$, (b) $\xi^{2}(\theta)$, $a\lambda = 0.45$ and $N = 150$.

6 Conclusion

In this work, flux limiter based high resolution schemes are investigated and classified based on two distinct TV stability regions. It is shown that both class of schemes give crisp resolution for discontinuities in opposite way. Universal TV stability region is proposed along with limiters which are some what diffusive but give better accuracy compared to classical minmod limiter.

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