Admissibility of weak solutions for the compressible Euler equations, $n \geq 2$

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Abstract

This article compares three popular notions of admissibility for weak solutions of the compressible isentropic Euler equations of gas dynamics: (i) the viscosity criterion, (ii) the entropy inequality (the thermodynamically admissible isentropic solutions),(iii) the viscosity-capillarity criterion. An exact summation of the Chapman-Enskog expansion for Grad's moment system suggests that it is the third criterion that is representing the kinetic theory of gases. This in turn may suggest that the cause of non-uniqueness for the weak solutions satisfying the second criterion is that the entropy inequality is not fully capturing information from kinetic theory.

Keywords: Euler equations, gas dynamics, admissible solutions

Introduction

In a recent and noteworthy paper [1] C. de Lellis and L. Szekelyhidi, Jr. have produced an infinite number of weak solutions to the initial value problem for the isentropic Euler equations of gas dynamics in dimension $n \geq 2$. Furthermore these solutions satisfy an "entropy" inequality (termed "the thermodynamically admissible" inequality in the monograph of C. M. Dafermos [2]). Since physical reality would suggest uniqueness we can logically suppose (at least) the following possibilities:

(a) There are no isentropic gases and non-uniqueness is due to an error in this basic (but unrealistic) model in continuum mechanics.

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(b) The admissibility criteria used in [1] is inadequate and does not reflect physical reality.

Since the mathematical community would be hard put to abandon one of its favorite sets of equations (compressible Euler or p-system) it seems to me useful in this article to review three popular admissibility criteria: (i) the viscosity criterion, (ii) the above mentioned "thermodynamically admissible "solutions satisfying an "entropy" (in fact energy) inequality, and (iii) a generalization of (i) obtained by including Korteweg's theory of capillarity [3]. In particular since (i) and (ii) are built upon the compressible Navier Strokes equations which itself is claimed to have derived from the Chapman-Enskog expansion of Boltzmann's kinetic theory, it seems a valuable exercise to review the basis of the Chapman-Enskog expansion. Specifically I note that the Chapman-Enskog expansion for the linearized Grad moment approximation to the Botlzmann equation when exactly summed (following A. Gorban and I. Karlin [4]-[6]) does not yield the compressible Navier Stokes equations but Kosteweg theory. It is my conjecture that the non-uniqueness of C. de Lellis and L. Szekelymdi, Jr. stems from their choice of admissibility criteria. In fact Korteweg theory by definition would require a bounded initial capillarity energy, i.e. gradient estimates on the density for any approximating sequence of initial data. Hence highly oscillatory initial data will be physically excluded on this basis.

The paper is divided into nine sections after this introduction:

1. Balance Laws
2. Admissibility criteria for the compressible isentropic Euler equations, $n \geq 2$
3. A comparison of admissibility criteria
4. The Chapman-Enskog expansion for the Boltzmann equation
5. Grad's 13 moments
6. The Chapman-Enskog expansion for Grad's 10 moment system
7. The dispersion relation, hydrodynamics and the entropy equality
8. Conclusion
9. References

1 Balance Laws

In this section we recall the balance laws of compressible gas dynamics. We denote by

ρ: fluid density, $\rho \in \mathbb{R}$,
u: fluid velocity, $u \in \mathbb{R}^3$,
T: Cauchy stress tensor, $T \in \{\text{symmetric } 3 \times 3 \text{ matrices}\}$.

For the fluid, position in space is given by $x \in \mathbb{R}^3$ and time $t > 0$. We consider for simplicity only mechanical theory since the balance laws of primary interest are the isentropic Euler equations. In this case the relevant balance laws are conservation of mass and linear momentum which in the absence of body forces are

$$\partial_t \rho + \partial_i (\rho u_i) = 0 \tag{1.1}$$

$$\partial_t (\rho u_i) + \partial_j (\rho u_i u_j) = \partial_j T_{ij} \tag{1.2}$$

where the summation convention is used.

A classical elastic fluid is given by the constitutive relation

$$T_{ij}^E = -\rho \psi'(\rho) \delta_{ij}$$

when δ_{ij} is the Kronecker delta and $\rho^2 \psi'(\rho) = p(\rho)$ is the pressure. This of course includes the special case of isentropic and isothermal gas dynamics where $p(\rho) = \rho^\gamma, \gamma > 1, \gamma = 1$, respectively. The viscous stress tensor of Cauchy and Poisson is given by

$$T_{ij}^v = \lambda (trD) \delta_{ij} + 2\,\mu D_{ij},$$

where

$$D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

and λ, μ are viscosity coefficients. for simplicity we make the usual choice

$$\lambda = \frac{-2}{3}\mu,\ \mu > 0.$$

The Dutch physicist Korteweg [3] proposed modeling capillarity effects via the capillarity stress tensor for which we take the special form

$$T_{ij}^k = \alpha \rho \partial_i (\rho \partial_j \rho) - \alpha \rho \partial_i \rho \partial_j \rho$$

(see the paper by J.E. Dunn and J. Serrin[7] for a discussion of the general form of Korteweg's theory), where the quantity $\alpha\rho$ is the surface tension coefficient and $\alpha > 0$ is a constant.

We then call the cases
$T = T^E$, an elastic fluid ;
$T = T^E + T^v$, a viscous elastic fluid;
$T = T^E + T^k$, a capillarity elastic fluid;
$T = T^E + T^v + T^k$, a Korteweg fluid.

In each of the above cases the balance laws of mass and momentum imply an additional balance law of mechanical energy (an "entropy" equality). Since the Korteweg fluid's "entropy" equality includes the others as special cases we record only its balance of mechanical energy:

$$\begin{aligned} &\partial_t\left(\frac{1}{2}\rho|u|^2 + \rho\psi(\rho) + \frac{\alpha}{2}\rho\partial_i\rho\partial_j\rho\right) \\ +\partial_j\Big[u_j&\left(\frac{1}{2}\rho|u|^2 + \rho\psi(\rho) - T^k_{ij} - T^v_{ij} - T^E_{ij} - \frac{\alpha}{2}\rho\partial_i\rho\partial_i\rho\right) \\ &+\alpha\rho(\partial_t\rho\partial_j\rho + u_i\partial_i\rho\partial_j\rho) + \mu\partial_i(u_iu_j)\Big] \\ &= -(\lambda+\mu)(\partial_i u_i)^2 - \mu(\partial_j u_i)(\partial_j u_i), \end{aligned} \tag{1.3}$$

where we recall the choice $\lambda = -\frac{2}{3}\mu$ yields $\lambda + \mu = \frac{1}{3}\mu > 0$.

2 Admissibility criteria for the compressible isentropic Euler equations, $n \geq 2$

The balance laws (1.1), (1.2) with $T = T^E$ are the Euler equations of compressible gas dynamics. Local (in time) smooth solutions for the case $p'(\rho) > 0$ are known to exist and be unique (see for example C.M. Dafermos [2]). For the case $n \geq 2$ little is known about weak solutions to the initial value problem. One approach to admissibility of weak solutions is to use our hierarchy of continuum models and make the following definitions.

The **viscosity**(respectively **viscosity-capillarity**) admissibility criterion admits those weak solutions which are limits of smooth solutions of the viscous elastic (respectively Korteweg) fluid which are obtained when $\mu \to 0$ (respectively $\mu, \alpha \to 0$).

An immediate consequence of the viscosity admissibility criterion is that weak solutions satisfying the viscosity admissibility criterion must satisfy the inequality.

$$\partial_t\left(\frac{1}{2}\rho|u|^2+\rho\psi(\rho)\right)+\partial_j\left(\left(\frac{1}{2}\rho|u|^2+\rho\psi(\rho)+p(\rho)\right)u_j\right)\leq 0 \tag{2.1}$$

in the sense of distributions (again we see C.M. Dafermos [2] section 3.3). The"entropy" inequality, more exactly an energy dissipation inequality, can itself be used as admissibility criterion which again following C.M. Dafermos we will call a **thermodynamically admissible solution**.

The viscosity-capillarity criterion unlike the viscosity criterion will not yield the inequality (2.1) since any weak limiting process will generally provide defect measures from the weak limits of the terms

$$\begin{cases}\alpha\,\rho\;\;\partial_i\rho\;\;\partial_i\rho,\\ \alpha\,\rho\,\partial_t\rho\,\partial_j\rho\\ \alpha\,\rho\,u_i\,\partial_i\rho\,\partial_j\rho\\ \alpha\,\rho\,u_j\,\partial_i\rho\,\partial_i\rho\end{cases} \tag{2.2}$$

as $\alpha\to 0$. Furthermore even to make sense of initial data for an approximating sequence of solutions required by the viscosity-capillarity criterion the initial data would have to satisfy

$$\int\left(\frac{1}{2}\rho|u|^2+\rho\psi(\rho)+\frac{\alpha}{2}\rho\partial_i\rho\partial_i\rho\right)dx<\infty. \tag{2.3}$$

This would clearly penalize high oscillatory initial data. Hence any approximation scheme based on initial data for which (2.3) does not hold would clearly be inadmissible according to the viscosity-capillarity criterion. In fact the only way that solutions of a Korteweg fluid will satisfy the "entropy" inequality (2.1) is if all the terms in (2.2) will approach zero in the sense of distributions.

3 A comparison of admissibility criteria

In section 2 three admissibility criteria for the higher dimensional Euler equations have been presented. Two of them, the viscosity and viscosity-capillarity criteria, require the relevant weak solution to be constructed by very restrictive approximation schemes. On the other hand the thermodynamic admissibility criteria of inequality (2.1) (in the spirit of P.D. Lax [8]) has the distinct advantage of being defined independently of the method of construction of the weak solution. Unfortunately this generality is also its main disadvantage. Specifically the beautiful result of C. de Lellis and L. Szekelyhidi, Jr. [1] tells us:

Theorem. In $n \geq 2$ space dimensions and for any given $p(\rho), p'(\rho) > 0$, there exists bounded initial data (ρ_0, u_0) with $\rho_0 > c > 0$ for which there are infinitely many bounded thermodynamically admissible solutions (ρ, u) of the compressible isentropic Euler equations with $\rho > c > 0$.

The above mentioned theorem seems to reject (at least in its present form) the "thermodynamic admissibility criterion" given by inequality (2.1). What can be said for the other two? We know in one space dimension they have been remarkably successful in ruling out unphysical solutions to gas dynamics and even providing existence of solutions as well (see G.-Q. Chen and M. Perepelitsa [9] for a recent contribution). Furthermore the viscosity-capillarity criterion allowed for a consideration of the case when $p'(\rho)$ may change signs as in the materials exhibiting change of phases (see[10] for a survey and an extensive list of references). One standard argument for preferring the viscosity criterion is based on the Chapman-Enskog expansion for the Boltzmann equation (see for example the book of L. St. Raymond [11], section 2.2.2). Hence it seems reasonable to re-examine the argument based on the Chapman-Enskog expansion and consider the implications.

4 The Chapman-Enskog expansion for the Boltzmann equation

The starting point for the discussion is of course the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = \frac{Q(f,f)}{\varepsilon} \cdot \tag{4.1}$$

Here $f(x,t,\xi)$ denotes the probability of finding a particle of gas at point $x \in \mathbb{R}^3$, at time t , moving with velocity $\xi \in \mathbb{R}^3 \cdot Q(f,f)$ is the collision operator and $\varepsilon > 0$ denotes the Knudson number. As $t \to \infty$ we expect fast decay to a slow invariant manifold which will be governed by the five macroscopic hydrodynamic variables $M : \rho$ (density)$\in \mathbb{R}$, u (velocity)$\in \mathbb{R}^3$, and Θ (temperature)$\in \mathbb{R}$. The Chapman-Enskog expansion is a formal method for computing f on the invariant manifold as a power series in ε, i.e.

$$f_{CE}(M) = f^{(0)}(M) + \varepsilon f^{(1)}(M) + \varepsilon^2 f^{(2)}(M) + \ldots \tag{4.2}$$

From (4.1) we see

$$Q(f^{(0)}, f^{(0)}) = 0 \tag{4.3}$$

so that $f^{(0)}(M)$ is the equilibrium Maxwellian distribution. Historically truncating the Chapman- Enskog expansion at order $1, \varepsilon, \varepsilon^2, \varepsilon^3$ are called the Euler, Navier-Stokes-Fourier, Burnett, and super-Burnett approximations, respectively. The apparent ability to derive the Navier-Stokes-Fourier theory from the kinetic theory of gases was of course a strong motivation to continue the expansion to higher orders of ε . However as noted by Bobylev[12],[13] truncation at Burnett order yields instability of fluid equilibrium, a decidedly unphysical result.(See Bobylev's article [13] and book of Struchtrup [15] for recent discussions). As had been noted by P. Rosenau [15]-[17] and more recently by Bobylev in [13], [14] the problem with Chapman-Enskog expansion is not the expansion itself but its truncation. Hence the expansion must be exactly or approximately summed to get an accurate description of the desired invariant hydrodynamic manifold. To validate the Fourier-Navier-Stokes theory based on truncation at the order ε is questionable mathematics at best.(The Latin phrase *vaticinium ex eventu* readily comes to mind, meaning a pseudo-prophecy that was written after the event (see J. Kugel [19], p.145)).

The Chapman-Enskog procedure becomes more and more computationally tedious as we proceed to higher and higher order terms in ε. Hence to get a quantitative picture of the process A. Gorban and I. Karlin [4], [5], [6] applied the technique to the linearized (about the rest state) Grad 13 moment approximation to the Boltzmann equation. It is the remarkable observation of Gorban and Karlin that in this special case the Chapman-Enskog expansion can be exactly summed.

5 Grad's 13 moments

The linearized Grad 13 moment equations are obtained by

(a) computing the first 13 moment equations from the Botzmann equation,

(b) invoking Grad's closure rule for the distribution function f,

(c) linearizing about the rest state of constant density $\rho_o > 0$, constant temperature $\Theta_0 > 0$, and velocity $u = 0$.

In appropriate non-dimensional form the linearized Grad 13 moment equations are

$$\partial_t \rho = -\nabla \cdot u, \tag{5.1}$$

$$\partial_t u = -\nabla \rho - \nabla \Theta - \nabla \cdot \sigma, \tag{5.2}$$

$$\partial_t \Theta \ = -\frac{2}{3}(\nabla \cdot u + \nabla \cdot q), \tag{5.3}$$

$$\partial_t \sigma = -\left((\nabla u) + (\nabla u)^T - \frac{2}{3} I \nabla \cdot u \right) - \frac{2}{3}\left((\nabla q) + (\nabla q)^T - \frac{2}{3} I \nabla \cdot q \right), \tag{5.4}$$

$$\partial_t q = -\frac{5}{3}\nabla \Theta - \nabla \cdot \sigma - \frac{2}{3} q, \tag{5.5}$$

where the pressure $p = \rho + \Theta$ in linear theory, σ is the extra stress, $q \in \mathbb{R}^3$ is the heat flux.

An even simpler set of equations is obtained by using only the first 10 moments in one space dimension. This formally amounts to using (5.1)(5.4)with $q = 0$, i.e.

$$\partial_t \rho = -\partial_x u \tag{5.6}$$

$$\partial_t u = -\partial_x p - \partial_x \sigma \tag{5.7}$$

$$\partial_t \Theta = -\frac{2}{3}\partial_x u, \tag{5.8}$$

$$\partial_t \Theta = -\frac{4}{5}\partial_x u - \sigma. \tag{5.9}$$

By addition of (5.6), (5.8) (recall $p = \rho + \Theta$) we obtain the system of the three balance laws

$$\partial_t \rho = -\frac{5}{3}\partial_x u, \tag{5.10}$$

$$\partial_t u = -\partial_x p - \partial_x \sigma, \tag{5.11}$$

$$\partial_t \sigma = -\frac{4}{5}\, \partial_x u - \sigma. \tag{5.12}$$

If we rescale space and time, $x = \frac{x'}{\varepsilon}, t = \frac{t'}{\varepsilon}$, and drop the primes, we introduce the Knudsen number ε into the the system, i.e.

$$\partial_t p = -\frac{5}{3}\partial_x u, \tag{5.13}$$

$$\partial_t u = -\partial_x p - \partial_x \sigma, \tag{5.14}$$

$$\partial_t \sigma = -\frac{4}{5}\,\partial_x u - \frac{\sigma}{\varepsilon} \tag{5.15}$$

Of course we could solve for σ in (5.15) and obtain "visco-elastic dynamics" of Maxwell type (see for example D.D. Joseph [20]). This again will reflect the rapid decay to the invariant hydrodynamic manifold but not provide a computation of the invariant manifold.

6 The Chapman-Enskog expansion for the Grad 10 moment system

In this section we recall the results of Gorban and Karlin [4],[5],[6] for the exact summation of the Chapman-Enskog expansion for the Grad 10 moment system in one space dimension. In fact Gorban and Karlin presented an exact summation for the full 13 moments in three space dimensions but for simplicity only their more restricted theory is presented here. More details may be found in their original articles and my review [21].

For the Grad 10 moment system write the expansion

$$\sigma_{CE} = \varepsilon\sigma^{(0)} + \varepsilon^2\sigma^{(1)} + \varepsilon^3\sigma^{(2)} + \cdots , \tag{6.1}$$

where $\sigma^{(n)}$ depend on the current values of p, u and their space derivatives. Substitute (6.1) into (5.13), (5.14), balance orders of ε, and use the equations themselves to eliminate time derivatives $\partial_t u, \partial_t p$ in favor of space derivatives. This yields the form of σ_{CE}:

$$\sigma_{CE} = -\frac{4}{3}(\varepsilon\partial_x u + \varepsilon^2\partial_x^2 p + \frac{\varepsilon^3}{3}\partial_x^3 u + \cdots). \tag{6.2}$$

If we truncate at order $\varepsilon, \varepsilon^2, \varepsilon^3$ and compute the spectrum of (5.13)-(5.15) in the Fourier frequency domain we recover the dispersion relations:

Navier-Stokes

$$w_{\pm} = -\frac{2}{3}k^2 \pm i|k|\sqrt{4k^2 - 15};$$

Burnett

$$w_{\pm} = -\frac{2}{3}k^2 \pm i|k|\sqrt{8k^2 - 15};$$

Super-Burnett

$$w_{\pm} = \frac{2}{9}k^2(k^2 - 3) \pm \frac{i}{9}|k|\sqrt{4k^6 - 24k^4 - 72k^2 - 135}.$$

Hence $Re w_{\pm}(k) \leq 0$ for the Navier-Stokes and Burnett truncations but for wave number $|k| > \sqrt{3}$ the Super-Burnett truncation yields a Bobylev instability.

In fact Gorban and Karlin have shown that (6.2) is indeed representative of the entire expansion, i.e.

$$\sigma_{CE} = \sum_{n=0}^{\infty} \varepsilon^{2n+1} a_n \, \partial_x^{2n+1} p + \sum_{n=0}^{\infty} \varepsilon^{2n+2} \, b_n \, \partial_x^{2(n+1)} u. \tag{6.3}$$

These expansions are in primed variables (x, t) and if we rescale back to the unprimed variables we have

$$\sigma_{CE} = \sum_{n=0}^{\infty} a_n \, \partial_x^{2n+1} p + \sum_{n=0}^{\infty} b_n \, \partial_x^{2(n+1)} \ u. \tag{6.4}$$

The coefficients a_n, b_n satisfy the recursion relations

$$a_{n+1} = \frac{5}{3} b_n + \sum_{m=0}^{n} a_m a_{n-m},$$

$$b_{n+1} = a_{n+1} + \sum_{m=0}^{n} a_{n-m} b_m. \tag{6.5}$$

Fortunately the relations (6.5) are in convolution form which just as in continuous Fourier theory makes their (discrete) transform elementary. First let us agree on the definition of the Fourier transform

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

with inverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx} dk.$$

Then from (6.4) we see

$$\widehat{\sigma}_{CE} = \sum_{n=0}^{\infty} -ik \ \ a_n (-k^2)^n \, \widehat{u} + \sum_{n=0}^{\infty} -k^2 \, b_n (-k^2)^n \widehat{p} \tag{6.6}$$

and if we define

$$A(k^2) = \sum_{n=0}^{\infty} a_n (-k^2)^n, \tag{6.7}$$

$$B(k^2) = \sum_{n=0}^{\infty} b_n(-k^2)^n, \tag{6.8}$$

we can write

$$\widehat{\sigma}_{CE} = -ik \ \ A(-k^2)\widehat{u} - k^2 \, B(k^2)^n \widehat{p} \tag{6.9}$$

Hence if we know $A(k^2), B(k^2)$ we know $\widehat{\sigma}_{CE}$ and the Chapman-Enskog expansion has been summed.

As noted above the key to the computation of A and B is the convolution form of (6.5). Multiply both equations in (6.5) by $(-k^2)^{n+1}$:

$$a_{n+1}(-k^2)^{n+1} = \frac{5}{3} b_n(-k^2)^n(-k^2) - k^2 \sum_{m=0}^{n} a_n(-k^2)^m a_{n-m}(-k^2)^{n-m},$$

$$b_{n+1}(-k^2)^{n+1} = a_{n+2}(-k^2)^{n+1} - k^2 \sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m} b_n(-k^2)^m,$$

then sum from $n = 0$ to ∞ changing the order of summation in the terms on the right hand sides. This yields:

$$A - a_0 = -k^2 \left\{ \frac{5}{3} B + A^2 \right\}, \tag{6.10}$$

$$B - b_0 = A - a_0 - k^2 AB \tag{6.11}$$

where $a_0 = b_0 = -\frac{4}{3}$ so as to agree with the known first two terms in the σ_{CE}. Solve for A in (6.11) to get

$$A = \frac{B}{1 - k^2 B} \tag{6.12}$$

and substitute in (6.10) to obtain a cubic equation for B. If we set $C = k^2 B$ this cubic equation is

$$-\frac{5}{3}(1 - C)^2 \left(C + \frac{4}{5} \right) - \frac{C}{k^2} = 0 \tag{6.13}$$

and an elementary analysis will yield that (6.13) possesses one real non-positive real root $C(k^2)$, monotone decreasing in k^2, $C(0) = 0, C(k^2) \to -\frac{4}{5}$ as $|k| \to \infty$. We can now recover the exact sum of the Chapman-Enskog expansion σ_{CE} via inverse Fourier transform:

$$\sigma_{CE} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-ikA(k^2)\widehat{u} - k^2 B(k^2)\widehat{p} \right) e^{ikx} dk, \tag{6.14}$$

where

$$A = \left(\frac{C}{1 - C} \right) \frac{1}{k^2}, \quad B = \frac{C}{k^2}$$

Thus the extra stress has been represented as a Fourier integral operator.

7 The dispersion relation, hydrodynamics, and the entropy equality

In Fourier space the hydrodynamic equations (5.13), (5.14) with $\sigma = \sigma_{CE}$ become

$$\partial_t \hat{p} = \frac{5}{3} ik\hat{u},$$

$$\partial_t \hat{u} = ik\hat{p} + ik(-ikA(k^2)\hat{u}(t,k) - k^2 B(k^2)\hat{p}(t,k)) \tag{7.1}$$

Set $\hat{p}(t,k) = e^{\varpi} P(k), \hat{u}(t,k) = e^{\varpi} U(k)$ so that

$$\begin{bmatrix} -\varpi & \frac{5}{3}ik \\ ik - ik^3 B & k^2 A - \varpi \end{bmatrix} \begin{bmatrix} P \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7.2}$$

and then set the determinant of the coefficient matrix to zero. This yields the dispersion relation

$$\varpi = \frac{1}{2}\left(\frac{C}{C-1}\right) \pm i|k| \left(\frac{5C^2 - 16C + 20}{3}\right)^{\frac{1}{2}} \tag{7.3}$$

where we have used the fact the C satisfies the cubic (6.13). Again the fact that $C < 0$ and C satisfies (6.13) implies $5C^2 - 16C + 20 > 0$ and hence $Re\varpi < 0$ for $k \neq 0$, $Re\varpi \to -\frac{2}{9}$ as $|k| \to \infty$. Hence the rest state is stable.

Furthermore multiplication of (7.1a) by $\frac{3}{5}\overline{\hat{p}}$, (7.1b) by $\overline{\hat{u}}$ (where the over-bar denotes complex conjugation) yields

$$\frac{1}{2}\partial_t \left(\frac{3}{5}|\hat{p}|^2 + |\hat{u}|^2\right) - ik(\overline{\hat{p}}\hat{u} + \hat{p}\hat{u})$$

$$= k^2 A(k^2)|\hat{u}| + ik\overline{\hat{u}}(-k^2 B(k^2)\hat{p}).$$

Now use the relation $\frac{3}{5}\partial_t \overline{\hat{p}} = -ik\overline{\hat{u}}$ to write the above equality as

$$\frac{1}{2}\partial_t \left(\frac{3}{5}|\hat{p}|^2 + |\hat{u}|^2\right) - \frac{3}{5}k^2 B(k^2)|\hat{p}|^2 - ik(\overline{\hat{p}}\hat{u} + \hat{p}\overline{\hat{u}}) = k^2 A(k^2)|\hat{u}|^2. \tag{7.4}$$

This is the entropy equality in Fourier space. Note the "entropy"

$$\frac{1}{2}\left(\frac{3}{5}|\hat{p}|^2 + |\hat{u}|^2 - \frac{3}{5}k^2 B(k^2)|\hat{p}|^2\right)$$

and the "dissipation" $k^2 A(k^2)|\hat{u}|^2$ are respectively positive and negative,$k \neq 0$, since A, B are both negative for $k \neq 0$. Integration of (7.4) in k, $-\infty <$

$k < \infty$, and application of Parseval's identity yields

$$\frac{1}{2}\partial_t \int_{-\infty}^{\infty} \frac{3}{5}|\hat{p}|^2 + |u|^2 dx + \frac{1}{2}\partial_t \int_{-\infty}^{\infty} \frac{-3}{5} k^2 B(k^2)|\hat{p}|^2 dk = \int_{-\infty}^{\infty} k^2 A(k^2)|\hat{u}|^2 dk \tag{7.5}$$

The term $\int_{-\infty}^{\infty} \frac{-3}{5} k^2 B(k^2)|\hat{p}|^2 dk$ represents a non-local version of the capillarity energy, where as

$$\int_{-\infty}^{\infty} k^2 A(k^2)|\hat{u}|^2 dk$$

is a non-local version of the viscous dissipation. Thus the exact sum of the Chapman-Enskog expansion for the linearized Grad 10 moment equations yields a non-local version of Korteweg's theory and not Navier-Stokes theory. Furthermore the viscosity and capillarity coefficients A and B are inseparable since they are linked by equation (6.12).

8 Conclusion

The moral of the story, I believe, is as follows. If the Grad truncation is reflecting the qualitative features of the Boltzmann equation then the Chapman-Enskog expansion for the Grad system should be reflecting qualitative feature of the Chapman- Enskog for the Boltzmann equation. In this scenario we see that it is Korteweg theory and not Navier-Stokes theory that should be basis for admissibility criteria. Moreover since Korteweg theory predicts a capillarity energy term of the form $\alpha\rho|\nabla\rho|^2$ in the energy balance equation it seems that any hydrodynamic limit theory for the Boltzmann equation that attempts to provide the classical Euler equations will have to force this capillarity term to vanish. Indeed this is precisely the state of the art for the incompressible Euler limit as given in the paper and book of L.St. Raymond [11], [22], where assumptions must be made on the both data and the desired limit. Otherwise the weak limit will yield an additional and unavoidable defect measure.

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