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LIPSCHITZ SEMIGROUP FOR AN INTEGRO–DIFFERENTIAL EQUATION FOR SLOW EROSION

By

RINALDO M. COLOMBO (*Dept. of Mathematics, Brescia University, Italy*),

GRAZIANO GUERRA (*Dept. of Mathematics and Applications, Milano-Bicocca University, Italy*),

AND

WEN SHEN (*Dept. of Mathematics, Penn State University, U.S.A.*)

Dedicated to Constantine Dafermos in honor of his 70th Birthday

Abstract. In this paper we study an integro-differential equation describing granular flow dynamics with slow erosion. This nonlinear partial differential equation is a conservation law where the flux contains an integral term. Through a generalized wave front tracking algorithm, approximate solutions are constructed and shown to converge strongly to a Lipschitz semigroup.

1. Introduction. Granular matter is being poured from an uphill location outside the interval of interest, and slides down the hill. As it slides down, it interacts with the standing layer. This interaction is described by the *erosion function* f, which depends only on the slope and denotes the rate of mass being eroded or deposited per unit length and per unit mass passing through. There is a critical slope where no interaction happens and f vanishes. In a normalized model one could choose the critical slope to be 1. If the slope is bigger than 1, then $f > 0$ and erosion happens, so that the moving layer grows. If the slope is smaller than 1, then $f < 0$ and part of the moving layer deposits on the standing bed. Under these assumptions, one obtains a 2×2 system of balance laws, with the heights of the standing and moving layers as the unknowns. This model was originally proposed in [16], and the time-dependent solutions were first studied in [21, 1, 4].

We consider the case where the standing layer is very small, and we refer to it as *slow erosion*. In [22], the following one dimensional model is studied, describing the changes

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The work of this author is partially supported by an NSF grant no DMS-0908047.
E-mail address: rinaldo.colombo@unibs.it
E-mail address: graziano.guerra@unimib.it
E-mail address: shen_w@math.psu.edu

for large times in the standing profile due to the materials sliding on it:

$$U_t(t,x) - \left(\exp \int_x^{+\infty} f\left(U_x(t,y)\right) \mathrm{d}y\right)_x = 0\,. \tag{1.1}$$

Here x is the space variable, U is the height of the profile, and t represents the total mass of moving layer that slid through. The slope of the profile U_x is assumed to remain strictly positive.

We stress that here $U(t,x)$ describes the strictly increasing asymptotic profile at x after t material was slowly poured from $+\infty$. Differentiating (1.1) in x, one obtains a conservation law for the slope U_x

$$(U_x)_t(t,x) + \left(f\left(U_x(t,x)\right) \exp \int_x^{+\infty} f\left(U_x(t,y)\right) \mathrm{d}y\right)_x = 0\,. \tag{1.2}$$

In general, the erosion function f is non-linear, therefore the solutions of (1.1) and (1.2) may well lose regularity. Under suitable assumptions on f, the slope U_x remains uniformly bounded in t. Existence, well-posedness and stability of BV solutions of (1.2) are established, see [2, 3, 4, 5, 6].

Allowing more erosion for large slopes, the solutions of (1.1) can develop various types of singularities, including jumps in the profile U, see [22] for a detailed discussion. Therefore, we expect $U(t)$ to attain values in $\mathbf{BV}$ and its space derivative U_x to be a measure. In this case, it is not suitable to study the equation (1.2). Instead one should study (1.1). The presence of the measure U_x causes additional technical challenges in the convergence analysis for the approximate solutions of (1.1). Under suitable assumptions on the erosion function f and on the initial data, global existence of BV solutions of (1.1) is established in [22]. However, continuous dependence on initial data was not treated in [22] due to technical difficulties caused by the measure U_x.

In this paper we tackle the problem of continuous dependence. We introduce the inverse function $X = X(t,u)$ which is the graph completion of the inverse in space of U,

$$X(t,u) = x \iff u \in \left[U(t,x-), U(t,x+)\right]\,.$$

Note that if U is right continuous, then $u \le U\left(t, X(t,u)\right)$ for all $u \in \mathbb{R}$. Wherever U has a jump, the inverse function X will remain constant over the interval of jump in U. Under the further condition that there exists a positive κ such that

$$U(t,x_2) - U(t,x_1) \ge \kappa\,(x_2 - x_1) \qquad \text{for all } x_1, x_2 \in \mathbb{R}, x_1 \le x_2\,, \tag{1.3}$$

the function X is Lipschitz continuous in u. Define the function $z(t,u)$ to be the u-derivative of $X(t,u)$, i.e.,

$$z(t,u) \doteq X_u(t,u)\,. \tag{1.4}$$

This is a well defined function, and $z \in \mathbf{L}^{\infty}(\mathbb{R};\mathbb{R}^+)$. In the case of a smooth function U, we can rewrite the integral in (1.1) as

$$\int_x^{+\infty} f\left(U_x(t,y)\right) \mathrm{d}y = \int_{U(t,x)}^{+\infty} g\left(z(t,v)\right) \mathrm{d}v \tag{1.5}$$

where the function g is defined by

$$g(s) = s\, f(1/s) \qquad \text{for all } s > 0\,. \tag{1.6}$$

Remark that the right hand side in (1.5) is well defined also if $U(t) \in \mathbf{BV}(\mathbb{R};\mathbb{R})$.

We now formally derive the differential equations that govern $X(t,u)$ and $z(t,u)$. By our variable changes, we have, for smooth solutions,

$$\begin{aligned} X_t(t,u) &= -\frac{U_t\left(t, X(t,u)\right)}{U_x\left(t, X(t,u)\right)} = -X_u(t,u)\, U_t\left(t, X(t,u)\right) \\ &= X_u(t,u) f\left(\frac{1}{X_u(t,u)}\right) \int_u^{+\infty} g(z(t,v))\, \mathrm{d}v \end{aligned}$$

which leads to a conservation law for $X(t,u)$

$$X_t(t,u) + \left(\exp \int_u^{+\infty} g(z(t,v))\, dv\right)_u = 0\,. \tag{1.7}$$

Differentiating (1.7) in u, we obtain a conservation law for $z(t,u)$

$$z_t(t,u) - \left(g(z(t,u)) \exp \int_u^{+\infty} g(z(t,v))\, dv\right)_u = 0\,. \tag{1.8}$$

When a jump in the profile occurs, we have $z = 0$ at the jump. However the solutions of (1.7) or (1.8) could lead to $z < 0$, which does not have physical meaning. Therefore we need to impose the pointwise constraint $z \geq 0$ for (1.7)–(1.8). One can combine the constraint and the equations (1.7)–(1.8) into one single conservation law

$$z_t(t,u) - \left(g(z(t,u)) \exp \int_u^{+\infty} g(z(t,v))\, dv\right)_u = \mu\,, \tag{1.9}$$

where μ is a measure satisfying the following property. For every $t \geq 0$ and $a, b \in \mathbb{R}$ such that $z(t,a) > 0$ and $z(t,b) > 0$, one has

$$\mu([a,b]) = 0, \qquad \int_a^b \mu([a,u])\, du = 0\,. \tag{1.10}$$

Note that the first and second properties in (1.10) are precisely the conservations of z and X over a jump, respectively.

The measure μ yields the projection into the cone of non-negative functions. To understand its effect on the $\mathbf{L}^1$ distance between two solutions, consider z_1 and z_2 as in Figure 1, where $z_1 = 0$ on an interval and $z_2 > 0$. The property (1.10) implies the relation $A_0 = A_1 + A_2$ for the areas, and then

$$\left\|z_1^+ - z_2\right\|_{\mathbf{L^1}} - \left\|z_1 - z_2\right\|_{\mathbf{L^1}} \leq -A_0 + A_1 + A_2 \leq 0\,,$$

formally proving that the measure μ does not increase the $\mathbf{L^1}$ distance between two solutions.

From (1.1), thanks to (1.5) which allows to give a meaning to the nonlinear function f applied to U_x, we are led to consider the conservation law

$$U_t(t,x) - \left(\exp \int_{U(t,x)}^{+\infty} g\left(z(t,v)\right) \mathrm{d}v\right)_x = 0 \tag{1.11}$$

where we treat as unknown the function z. Moreover, to allow the reconstruction of U from z, we have to impose further constraints on z, namely that

$$z(t) \in \mathbf{L^\infty}(\mathbb{R};\mathbb{R}^+) \quad \text{and} \quad (z(t)-1) \in \mathbf{L^1}(\mathbb{R};\mathbb{R})\,. \tag{1.12}$$

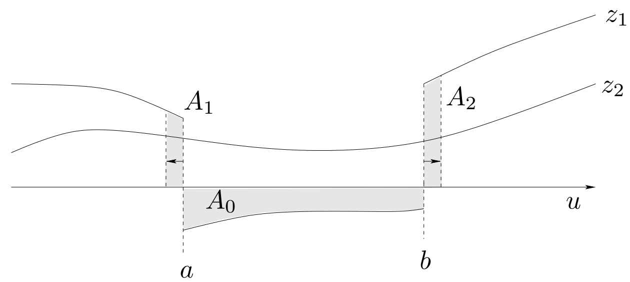


FIG. 1. Effect of the measure μ on the $\mathbf{L^1}$ distance between two solutions.

As a motivation for the conditions above, we first recall the inequality $\|z\|_{\mathbf{L^\infty}} \leq 1/\kappa$, with κ as in (1.3). Secondly, note that $\|z-1\|_{\mathbf{L^1}}$ is related to the height difference between the asymptotic slopes at $+\infty$ and at $-\infty$. Under these conditions, we reconstruct U from z as follows:

$$\begin{aligned} X(t,u) &= u + \int_{-\infty}^{u} (z(t,v)-1)\,\mathrm{d}v \\ U(t,x) &= \max\{v \in \mathbb{R} \colon X(t,v) \leq x\} \,. \end{aligned} \tag{1.13}$$

This paper is thus concerned with the construction of a Lipschitz semigroup of solutions to (1.11)–(1.13).

Other models for granular matter dynamics recently received attention in the mathematical literature. We recall first the well known Savage–Hutter model [19, 20], extended to comprehend deposition and erosion in [9, 14].

From the analytical point of view, the present model (1.11)–(1.13) can be seen as a further step towards the study of conservation laws with nonlocal terms. First, source terms with convolution in space were considered in [13], while [10, 11, 12] deal with memory effects, i.e nonlocalities in time. Then, nonlocal terms in the flow were considered also in [18] in the framework of traffic modeling.

In Section 2 we present the analytical results, collected in Theorem 2.1. In Section 3 we construct the approximate solutions, derive their a priori estimates, and prove the relevant parts in Theorem 2.1. Finally, we establish the Lipschitz dependence on initial data and on the erosion function in Section 4.

2. Main Results and Preliminary Considerations.

We assume that the erosion function g satisfies the following property,

(g): $g \in \mathbf{C^2}((0,+\infty);\mathbb{R}) \cap \mathbf{C^1}([0,+\infty);\mathbb{R})$ satisfies $g(1)=0$, $\sup g'' < 0$, $g(0) \geq 0$.

Note that the above conditions on g are equivalent to the conditions on f in (1.1) used in [22].

Motivated by (1.12), we seek $\mathbf{BV}$ solutions to the Cauchy problem for (1.11) within the class

$$\mathcal{Z} \doteq \left\{ z \in \mathbf{BV}(\mathbb{R};[0,+\infty)) : \begin{array}{l} z \text{ is right-continuous, and} \\ (z-1) \in \mathbf{L^1}(\mathbb{R};[0,+\infty)) \end{array} \right\} \tag{2.1}$$

For notation simplicity, we introduce the map $G\colon \mathcal{Z}\times\mathbb{R}\mapsto\mathbb{R}$ as

$$G\left(z(t,\cdot);u\right)=\exp\int_u^\infty g\left(z(t,v)\right)\,\mathrm{d}v\,. \tag{2.2}$$

Note that G depends on z in a non-local way. We also define the function $\psi\colon [0,+\infty)\to\mathbb{R}$ by

$$\psi(s)=g(s)-s\,g'(s)\,. \tag{2.3}$$

By **(g)**, we have

$$\psi(0)=g(0)\geq 0,\qquad \psi'(s)=-sg''(s)>0\,. \tag{2.4}$$

Therefore, the map ψ is positive, bounded (for bounded s) and strictly increasing.

We now state the main result of our paper.

THEOREM 2.1. Fix $T>0$ and let $\mathcal{Z}$ be as in (2.1). For any g satisfying **(g)**, there exists a map $S^g\colon [0,T]\times\mathcal{Z}\to\mathcal{Z}$ with the following properties:

(1) $S^g_0=\mathbf{Id}$ and for any $t_1,t_2\in[0,T]$ with $t_1+t_2\in[0,T]$, the semigroup property holds: $S^g_{t_1}\circ S^g_{t_2}=S^g_{t_1+t_2}$.
(2) For any $z_o\in\mathcal{Z}$, the orbit $t\to S^g_t z_o$ solves (1.11)–(1.13) in the sense of distributions.
(3) There exists a constant $L>0$ such that for any $g,\bar g$ satisfying **(g)**, for any $z,\bar z\in\mathcal{Z}$ and for any $t,\bar t\in[0,T]$ with $\bar t\leq t$,

$$\left\|S^g_t z-S^{\bar g}_{\bar t}\bar z\right\|_{\mathbf{L^1}}\leq L\,\left(t\|g-\bar g\|_{\mathbf{W^{1,\infty}}}+e^{Lt}\|z-\bar z\|_{\mathbf{L^1}}+|t-\bar t|\right)\,.$$

This result is obtained through piecewise constant approximation generated by a suitable wave front tracking algorithm. Some similar algorithms are used in [22, 5]. For front tracking for conservation laws, see [8, 15].

2.1. *Jump Conditions and Characteristic Speeds.* The propagation speeds of the various waves are basically derived from Rankine-Hugoniot condition. We provide here some heuristic considerations. First we observe that where the unknown $z(t,u)$ is strictly positive and continuous, then $U(t,x)$ is differentiable with

$$\begin{aligned} U_x(t,x) &= \frac{1}{z(t,u)}, \\ U_t(t,x) &= -g\left(z(t,u)\right)G\left(z(t,\cdot);u\right)U_x(t,x) \;=\; -\frac{g\left(z(t,u)\right)G\left(z(t,\cdot);u\right)}{z(t,u)} \end{aligned}$$

where we set $u=U(t,x)$ and used (1.11), (1.13). Consider now the case of a jump discontinuity in the map $x\to U(t,x)$, which we label as a u–*shock*. Then, the Rankine–Hugoniot conditions [8, § 4.2] related to (1.11) impose that the discontinuity's speed Λ in the (t,x) plane satisfies

$$\Lambda\,\left(U(t,x+)-U(t,x-)\right)=G\left(z(t,\cdot);U(t,x+)\right)-G\left(z(t,\cdot);U(t,x-)\right)\,.$$

If we define $u^\pm = U(t,x\pm)$ passing to the speed λ of the same jump in the (t,u) plane (see Figure 2), we have

$$\begin{aligned}\lambda^- &= \frac{G\left(z(t,\cdot);u^+\right)-G\left(z(t,\cdot);u^-\right)}{z\left(t,u^- -\right)\left(u^+-u^-\right)} - \frac{g\left(z(t,u^- -)\right)\, G\left(z(t,\cdot);u^-\right)}{z\left(t,u^- -\right)}\\ \lambda^+ &= \frac{G\left(z(t,\cdot);u^+\right)-G\left(z(t,\cdot);u^-\right)}{z\left(t,u^+ +\right)\left(u^+-u^- -\right)} - \frac{g\left(z(t,u^+ +)\right)\, G\left(z(t,\cdot);u^+\right)}{z\left(t,u^+ +\right)}\,.\end{aligned}$$

Setting $z^\pm = z\left(t,u^\pm \pm\right)$, $G^\pm = G\left(z(t,\cdot);u^\pm\right)$ the expressions above become

$$\begin{aligned}\lambda^- &= -\frac{G^-}{z^-}\left(\frac{e^{-g(0)(u^+-u^-)}-1}{u^+-u^-}+g(z^-)\right)\\ \lambda^+ &= -\frac{G^+}{z^+}\left(\frac{1-e^{g(0)(u^+-u^-)}}{u^+-u^-}+g(z^+)\right).\end{aligned}$$

The classical Lax shock condition [8], when applied in the left extreme of a u-shock, reads $-g'(z^-)\,G^- \geq \lambda^-$, which is equivalent to $\psi(z^-) \geq (1-e^{g(0)(u^+-u^-)})/(u^+-u^-)$ and is always satisfied. In u^+, $-g'(z^+)\,G^+ \leq \lambda^+$ becomes

$$\psi(z^+) \leq \frac{e^{g(0)(u^+-u^-)}-1}{u^+-u^-} \tag{2.5}$$

which selects the admissible upward jumps.

Observe that for $u \in (u^-,u^+)$, the function z satisfies $z(t,u) = z_t(t,u) = 0$.

When z has a discontinuity between two strictly positive values $z^\pm = z(t,u\pm)$, the map U is continuous at $x = X(t,u)$. We obtain, see Figure 2,

$$\lambda = U_t^- - U_x^- \frac{U_t^+ - U_t^-}{U_x^+ - U_x^-} = -G\left(z(t,\cdot);u\right)\frac{g(z^+)-g(z^-)}{z^+-z^-}\,. \tag{2.6}$$

Finally, to complete the definition of approximate solutions to (1.11), we need also to know how z changes along characteristic curves. To this goal, suppose U smooth, by the implicit function theorem and by (1.11) obtain

$$X_t(t,u) = -\frac{U_t\left(t,X(t,u)\right)}{U_x\left(t,X(t,u)\right)} = -G_u\left(z(t,\cdot);u\right).$$

Differentiating by u we get

$$z_t(t,u) = -G_{uu}\left(z(t,\cdot);u\right) = g'\left(z(t,u)\right)G\left(z(t,\cdot);u\right)z_u(t,u) - g\left(z(t,u)\right)^2 G\left(z(t,\cdot);u\right),$$

hence, z satisfies

$$z_t(t,u) - g'\left(z(t,u)\right)G\left(z(t,\cdot);u\right)z_u(t,u) = -g\left(z(t,u)\right)^2 G\left(z(t,\cdot);u\right).$$

This last equation shows that the characteristics speed is $-g'\left(z(t,u)\right)G\left(z(t,\cdot);u\right)$, (see also (2.6) in the limit $u^- \to u^+$), while the change of z along characteristics is

$$\dot z(t,u) = -g\left(z(t,u)\right)^2 G\left(z(t,\cdot);u\right). \tag{2.7}$$

3. Construction and Properties of the Approximate Solutions. In what follows, up to the final limit, we assume that g satisfies **(g)** and moreover $g \in \mathbf{C^2}\left([0,+\infty);\mathbb{R}\right)$. This latter requirement will be removed in the final part of the proof of Theorem 2.1.

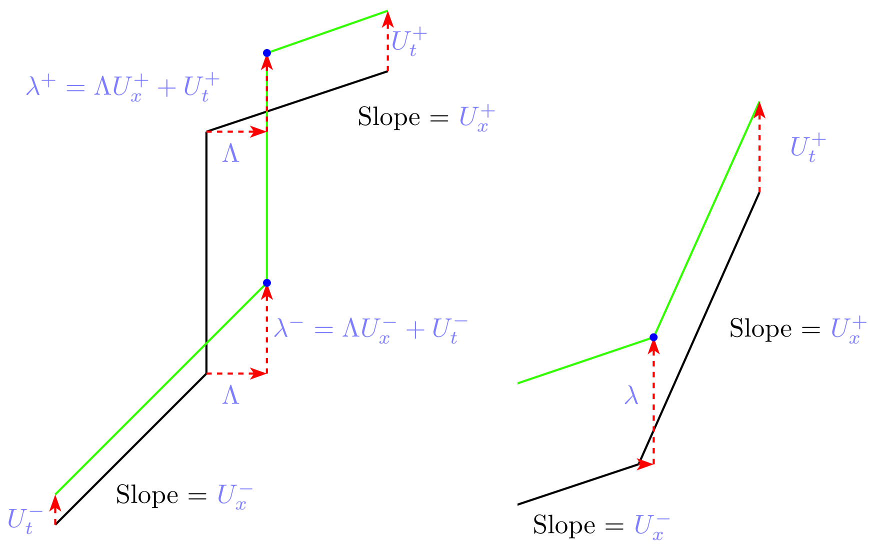


FIG. 2. Relation between the speeds in the (t, x) plane and the ones in the (t, u) plane

3.1. *Construction of the Approximate Solutions.* Piecewise constant approximate solutions are constructed in the style of front tracking, where each discontinuity is treated as a wave front. Let ϵ be the parameter for the approximation, and let $z^\epsilon(t, u)$ denote the piecewise constant approximate solution.

Introduce for later use the map

$$\xi(\Delta) = \frac{e^{g(0)\Delta} - 1}{\Delta}, \tag{3.1}$$

that satisfies $\xi(\Delta) \geq g(0)$, and the function $\zeta(\Delta)$ implicitly by

$$g(\zeta) - \zeta g'(\zeta) = \xi(\Delta). \tag{3.2}$$

The map ζ is a strictly increasing and $\zeta(0) = 0$, see Figure 3.

LEMMA 3.1 (Construction of the discrete initial data). Let $z_o \in \mathcal{Z}$ as defined in (2.1). Then, for every $\epsilon > 0$, there exist $n \in \mathbb{N}$ and real numbers $u_1, \ldots, u_n$; $z_0, \ldots, z_n$ such that setting

$$z_o^\epsilon(u) = z_0\, \mathbf{1}_{(-\infty, u_1)}(u) + \sum_{i=1}^{n-1} z_i\, \mathbf{1}_{[u_i, u_{i+1})}(u) + z_n\, \mathbf{1}_{[u_n, \infty)}(u)$$

the following requirements are met:
1. $\|z_o^\epsilon - z_o\|_{\mathbf{L}^1} < \epsilon$;
2. $\mathrm{TV}(z_o^\epsilon) \leq \mathrm{TV}(z_o)$;
3. $z_i \geq 0$, $z_0 = z_n = 1$;
4. $z_i - z_{i-1} \leq \epsilon$ for all $z_{i-1} > 0$;

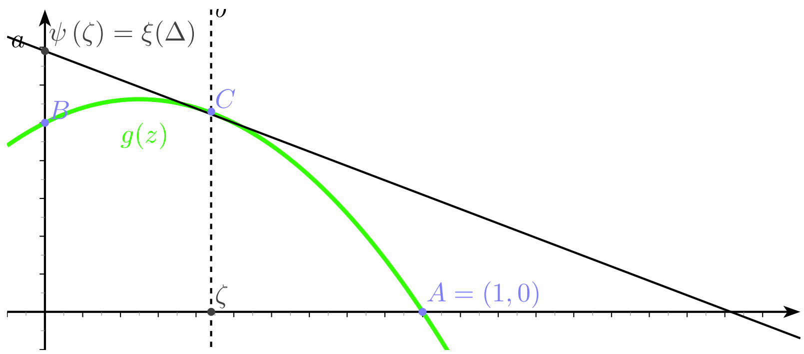


FIG. 3. Geometric interpretation of $\zeta(\Delta)$

5. $z_i \leq \zeta(\Delta_i) + \epsilon$ if $z_{i-1} = 0$, where $\Delta_i = u_i - u_{i-1}$;
6. if $z_i = 0$, then both $z_{i+1} > 0$ and $z_{i-1} > 0$
7. there are no two contiguous states z_{i-1} and z_i such that $0 < z_{i-1} < 1 < z_i$;
8. $u_1 \leq u_2 \leq \ldots \leq u_n$, where $u_i = u_{i+1}$ can happen only if either $0 < z_{i-1} < z_i < z_{i+1}$ or $z_{i-1} = 0$ and $z_i = \zeta(\Delta_i) + \epsilon < z_{i+1}$;
9. $|g(z_i)|\,(u_{i+1} - u_i) < \epsilon$ whenever $z_i > 0$.

Note that 6. ensures that no contiguous u-shock are present. The condition 7. implies that no rarefaction may cross 1. The requirement 8. applies to upward jumps that violate (2.5): it says that the right jump in a non admissible u–shock of width Δ is split starting from $\zeta(\Delta) + \epsilon$.

Proof of Lemma 3.1. Fix $\epsilon > 0$. Let $\tilde{z}_\epsilon$ be any piecewise constant map satisfying 1. and 2., see [8, Lemma 2.2], so that also 3. holds. The conditions 4., 5., 7. and 8. are satisfied adding in a suitable way states z_i. To comply with 6. simply glue adjacent segments where $\tilde{z}^\epsilon$ vanishes. Finally, 9. follows by suitably adding states u_i where $\tilde{z}^\epsilon$ does not vanish. □

3.2. *Evolution of* $u_1, \ldots, u_n$ *and* $z_1, \ldots, z_{n-1}$. To simplify the notation, we assume that ϵ is fixed and omit it.

Now we define a system of ODE which controls the evolution of the two vectors $Z = (z_1, \ldots, z_{n-1})$ and $U = (u_1, \ldots, u_n)$. The discussions in Section 2 lead to the following approximate evolution equations:

$$\begin{cases} \dot{z}_i = 0 & \text{if } z_i = 0 \\ \dot{z}_i = g(z_i)\,\dfrac{G(z, u_{i+1}) - G(z, u_i)}{u_{i+1} - u_i} = -g(z_i)^2\, G(z, \tilde{u}_i) & \text{if } z_i > 0 \end{cases} \tag{3.3}$$

where $\tilde{u}_i$ is a suitable point $u_i < \tilde{u}_i < u_{i+1}$. Moreover, by Rankine–Hugoniot conditions,

$$\begin{cases} \dot{u}_i = -\dfrac{g(z_i) - g(z_{i-1})}{z_i - z_{i-1}} G(z; u_i) & \text{if } z_i, z_{i-1} > 0 \\ \dot{u}_i = -\dfrac{g(z_i) - \xi(\Delta_i)}{z_i} G(z; u_i) & \text{if } z_{i-1} = 0 \\ \dot{u}_i = -\dfrac{g(z_{i-1}) - \xi\left(\Delta_{i+1}\right) e^{-\Delta_{i+1} g(0)}}{z_{i-1}} G(z; u_i) & \text{if } z_i = 0. \end{cases} \tag{3.4}$$

As initial data we take the one defined in Lemma 3.1. Since the right hand sides in (3.3) and (3.4) is locally Lipschitz in a neighborhood of the initial data, there exists a local solution $(Z(t), U(t))$ defined in $[0, \delta)$ for some $\delta > 0$. By the downward convexity of g, for all small times t it holds that $u_1(t) < u_2(t) < u_3(t) < \ldots < u_n(t)$.

Fix now an arbitrary $T > 0$ and define $\varepsilon(t) = \epsilon\, e^{Kt}$ where the constant K will be fixed later and will depend only on T, g, $\mathrm{TV}(z_o)$, $\|z_o - 1\|_{\mathbf{L}^1}$, $\|z_o\|_{\mathbf{L}^\infty}$. Then, proceed up to a time $\tau > 0$, when an interaction of one of the following types takes place:

(I1) one or more wave fronts meet: $u_i = u_{i+1} = \ldots = u_j$. Then, we redefine the indexes so that we have a single wave front (or no wave fronts if $z_{i-1} = z_j = 0$) and use this new (Z, U) as initial data for (3.3)–(3.4), which again admits a solution locally in time;

(I2) $z_i(\tau) = 0$ with $z_i(t) > 0$ for $t < \tau$ and $u_{i+1}(\tau) - u_i(\tau) > 0$ (otherwise we fall in point **(I1)**). Then, continue with $z_i(t) = 0$ for $t > \tau$ according to (3.3). If two or more contiguous states become zero at the same time, then we also erase the intermediate waves.

(I3) for some $z_{i-1}(\tau) = 0$ we have $z_i(\tau) = \zeta(\Delta_i) + 2\varepsilon(\tau)$, with $z_i(t) < \zeta(\Delta_i) + 2\varepsilon(t)$ for $t < \tau$ and $u_{i+1}(\tau) - u_i(\tau) > 0$ (otherwise we are in case **(I1)**), then we split the upward jump $(0, z_i(\tau))$ in two parts: a piece of a u–shock, $(0, \zeta(\Delta_i) + \varepsilon(t))$ and a rarefaction $(\zeta(\Delta_i) + \varepsilon(t), \zeta(\Delta_i) + 2\varepsilon(t))$. If the rarefaction contains the value 1, we split it in two rarefactions in such a way that no new rarefaction crosses 1. Therefore there is the generation of 1 or 2 new rarefactions.

As long as the solution to (3.3)–(3.4) exists or we end up in one of the above cases **(I1)**, **(I2)** or **(I3)**, then an approximate solution z is constructed by the present algorithm. Next, we show that only **(I1)**, **(I2)** or **(I3)** can take place up to time T, which ensures that z can be defined up to that time.

Theorem 3.2. *Fix a positive T. Then, the approximate solution z can be uniquely defined on all $[0, T]$ and for all $t \in [0, T]$ enjoys the following properties:*

a) $z_i(t) \geq 0$ *for* $i = 1, \ldots, n-1$ *and* $z_0(t) = z_n(t) = 1$;
b) $z_i(t) - z_{i-1}(t) \leq \varepsilon(t)$ *as long as* $z_{i-1}(t) > 0$;
c) $z_i(t) \leq \zeta(\Delta_i) + 2\varepsilon(t)$ *whenever* $z_{i-1}(t) = 0$;
d) *if* $z_i(t) = 0$, *then* $z_{i+1}(t) > 0$ *and* $z_{i-1}(t) > 0$;
e) *there are no two contiguous states such that* $0 < z_{i-1}(t) < 1 < z_i(t)$;
f) $|g(z_i(t))|\,(u_{i+1}(t) - u_i(t)) < \varepsilon(t)$ *as long as* $z_i(t) > 0$;

where $\varepsilon(t) = \epsilon\, e^{Kt}$ *with* K *dependent only on an upper bound on* T, $\|g\|_{\mathbf{W}^{2,\infty}}$, $\mathrm{TV}\, z_o$, $\|z_o\|_{\mathbf{L}^\infty}$ *and* $\|z_o - 1\|_{\mathbf{L}^1}$.

Preliminarily, we list the basic properties enjoyed by z as long as it exists.

$\mathbf{L}^\infty$ bound: From (3.3), $\|z(t,\cdot)\|_{\mathbf{L}^\infty} \leq \|z(0,\cdot)\|_{\mathbf{L}^\infty}$.

Rarefactions cannot cross the state 1: It is a straightforward consequence of the evolution (3.3) of the ODE and of the interaction rule **(I3)**

Approximate admissibility of u–shock: If $z_{i-1} = 0$, then by the interaction rule **(I3)** $z_i \leq \zeta(\Delta_i) + 2\varepsilon(t)$.

Exact conservation: Define $\widehat{G}(z;u)$ as the linear interpolation of $G(z;u)$ on the points $u_1, u_2, \ldots, u_n$. If we define

$$F(z;u) = \begin{cases} -g(z_i)\widehat{G}(z;u) & \text{for } u \in (u_i, u_{i+1}) \text{ and } z_i \neq 0 \\ \dfrac{G(z;u_{i+1}) - G(z;u_i)}{u_{i+1} - u_i} & \text{for } u \in (u_i, u_{i+1}) \text{ and } z_i = 0 \end{cases} \tag{3.5}$$

then, direct computations show that $z(t,u)$ turns out to be a weak exact solution to the conservation law

$$z_t + F(z;u)_u = 0, \tag{3.6}$$

this implies that also $z - 1$ is a conserved quantity:

$$(z-1)_t + F(z;u)_u = 0. \tag{3.7}$$

Observe also that, when $z_i = 0$:

$$\begin{array}{rcccl} \dfrac{G(z;u_{i+1}) - G(z;u_i)}{u_{i+1} - u_i} & = & -G(z;u_{i+1})\,\xi\left(\Delta_{i+1}\right) & \leq & -g(0)\,G(z;u_{i+1}) \\ \dfrac{G(z;u_{i+1}) - G(z;u_i)}{u_{i+1} - u_i} & = & -G(z;u_i)\,\frac{1-e^{-g(0)\Delta_{i+1}}}{\Delta_{i+1}} & \geq & -g(0)\,G(z;u_i)\,. \end{array}$$

Changes in the Waves' Nature: Fix positive states z_{i-1} and z_i and, using (3.3), compute

$$\begin{aligned} \frac{d}{dt}(z_i - z_{i-1}) &= -g(z_i)^2\,G(z;\tilde{u}_i) + g(z_{i-1})^2\,G(z;\tilde{u}_{i-1}) \\ &= G(z;\tilde{u}_i)\left[-g(z_i)^2 + g(z_{i-1})^2 \exp\left\{\int_{\tilde{u}_{i-1}}^{\tilde{u}_i} g\left(z(t,v)\right)\,\mathrm{d}v\right\}\right] \end{aligned}$$

where $\tilde{u}_{i-1} \in (u_{i-1}, u_i)$ and $\tilde{u}_i \in (u_i, u_{i+1})$. We consider the following two examples:

- If $1 \leq z_{i-1} \leq z_i$, then $g\left(z(t,v)\right) \leq 0$ for $v \in (u_{i-1}, u_{i+1})$ and

$$\frac{d}{dt}(z_i - z_{i-1}) \leq G(z;\tilde{u}_i)\,\left[-g(z_i)^2 + g(z_{i-1})^2\right] \leq 0 \tag{3.8}$$

since for $z \geq 1$, $z \mapsto g(z)^2$ is strictly increasing. Hence, the strength of rarefactions above 1 can only decrease. They can become shocks, but no shock above 1 can become a rarefaction.

- If $z_{i-1} = z_i < 1$, then $g\left(z(t,v)\right) > 0$ for $v \in (u_{i-1}, u_{i+1})$ and

$$\frac{d}{dt}(z_i - z_{i-1}) > G(z;\tilde{u}_i)\left[-g(z_i)^2 + g(z_{i-1})^2\right] = 0 \tag{3.9}$$

this proves that below 1 no rarefaction becomes a shock. It also suggests that shocks below 1 can become rarefactions and both can increase their strength.

Preliminary Estimate on the Total Variation: Define

$$\hat{z}(t,u) = \max\{1, z(t,u)\}.$$

Both the total variations of z and of $\hat{z}$ do not increase at any interaction. Along the trajectories of the ODE the total variation of z may well increase, due to (3.3). On the contrary, the total variation of $\hat{z}$ may not increase. Indeed, $\hat{z}$ attains the value 1 both at $-\infty$ and at $+\infty$, hence

$$\mathrm{TV}\,\hat{z}(t,\cdot) = 2\sum_{\Delta\hat{z}>0} \Delta\hat{z} \quad \text{and} \quad \frac{d}{dt}\mathrm{TV}\,\hat{z}(t,\cdot) = 2\sum_{\Delta\hat{z}>0}\frac{d}{dt}\Delta\hat{z}. \tag{3.10}$$

Now, if $\Delta\hat{z} > 0$, then $\Delta\hat{z} = z_i - \max\{1, z_{i-1}\}$. If $z_{i-1} \geq 1$, then the wave at u_i is a rarefaction above 1 and (3.8) shows that $\frac{d}{dt}\Delta\hat{z} \leq 0$. If, on the contrary, $z_{i-1} < 1$, then $\frac{d}{dt}\Delta\hat{z} = \frac{d}{dt}z_i \leq 0$. Therefore

$$\mathrm{TV}\,\hat{z}(t,\cdot) \leq \mathrm{TV}\,\hat{z}(0,\cdot) \leq \mathrm{TV}\,z_o.$$

3.3. *Preliminary* $\mathbf{L^1}$ *Estimates.*

LEMMA 3.3. Let $\tau \in (0,\delta)$ and $u \in \mathbb{R}$ be such that $z(\tau, u-) \geq 1 > z(\tau, u+)$. Then

$$\int_u^{+\infty} (1 - z(\tau, v))\,\mathrm{d}v \leq \|z_o - 1\|_{\mathbf{L^1}}.$$

Proof. Observe that the number of times in which $z(t,u) - 1$ changes sign does not increase in time. A shock as that at u may arise neither during the evolution of the ODE, nor during interactions not already containing such a shock. Therefore, we can trace it backward up to time $t = 0$. If two shocks of this type interact, we trace back along, say, the fastest (leftmost) one. For simplicity, to avoid the changes in the indexing at interactions, we call $u(t)$ the support of this shock, $z(t) < 1$ is the state to its right and $z^-(t) \geq 1$ the one to its left.

Because of the conservation (3.7), we can compute (outside interactions):

$$\frac{d}{dt}\int_{u(t)}^{+\infty} [1 - z(t,v)]\,\mathrm{d}v = -(1 - z(t))\,\dot{u}(t) - F\,(z; u(t)+). \tag{3.11}$$

We need now to distinguish two cases: $z(t) > 0$ and $z(t) = 0$.

Case I. $z(t) > 0$:

$$\frac{d}{dt}\int_{u(t)}^{+\infty} [1 - z]\,\mathrm{d}v = (1 - z)\,\frac{g\,(z) - g(z^-)}{z - z^-}\,G(z;u) + g(z)\,G(z;u),$$

(observe that on the discontinuities $\widehat{G} = G$). Now $1 - z > 0$, $z^- \geq 1$, therefore by convexity

$$\text{slope of } b = \frac{g\,(z) - g(z^-)}{z - z^-} \leq \frac{g\,(z) - g\,(1)}{z - 1} = \text{slope of } a,$$

see Figure 4, left. Finally

$$\frac{d}{dt}\int_{u(t)}^{+\infty}[1-z]\,\mathrm{d}v \le -g(z)\,G(z;u)+g(z)\,G(z;u)=0\,,$$

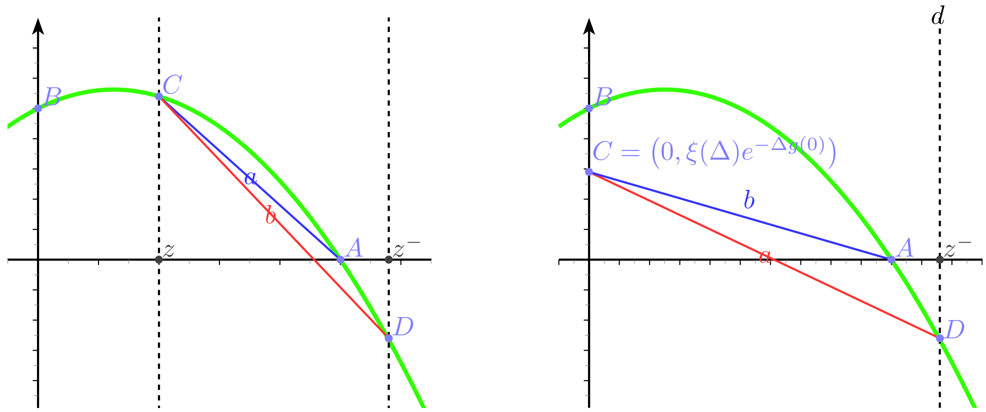


FIG. 4. Comparison between slopes. Left for **I**, i.e. $z(t) > 0$ and, right, for **II**, i.e. $z(t) = 0$.

Case II. $z(t)=0$:

Put $z=0$ in (3.11) and using the corresponding expressions in (3.4) and (3.5), we obtain

$$\frac{d}{dt}\int_{u(t)}^{+\infty}[1-z]\,\mathrm{d}v = G(z;u)\left[\frac{g(z^-)-\xi(\Delta)e^{-\Delta g(0)}}{z^-}+e^{-\Delta g(0)}\xi(\Delta)\right],$$

where Δ is the strength of the u–shock to the right of u. Again by convexity, since $0<\xi(\Delta)e^{-\Delta g(0)}\le g(0)$ (see Figure 4), right, we obtain

$$\frac{g(z^-)-\xi(\Delta)e^{-\Delta g(0)}}{z^- - 0}\le\frac{g\,(1)-\xi(\Delta)e^{-\Delta g(0)}}{1-0}=-\xi(\Delta)e^{-\Delta g(0)}$$

which again implies

$$\frac{d}{dt}\int_{u(t)}^{+\infty}[1-z]\,\mathrm{d}v\le 0$$

The two cases above ensure that, since $\frac{d}{dt}\int_{u(t)}^{+\infty}[1-z]\,\mathrm{d}v\le 0$ always holds, we have

$$\int_{u(t)}^{+\infty}[1-z(t,v)]\,\mathrm{d}v\le\int_{u(0)}^{+\infty}[1-z(0,v)]\,\mathrm{d}v\le\|1-z_o\|_{\mathbf{L}^1}\,,$$

completing the proof. □

LEMMA 3.4. Let $\tau\in(0,\delta)$ and $u\in\mathbb{R}$ be such that

(a): either $z(\tau,u-)\ge 1> z(t,u+)$,

(b): or $z(\tau,u-)<1\le z(t,u+)$.

Then,

$$G(z;u)\le\exp\left(|g'(1)|\;\|1-z_o\|_{\mathbf{L}^1}\right)\,.$$

Proof. Note that in case **(a)**, Lemma 3.3 applies, so that

$$\int_u^{+\infty} (1 - z(\tau, v))\, \mathrm{d}v \le \|z_o - 1\|_{\mathbf{L^1}}\,.$$

The same bound holds also at **(b)**. Indeed, call $A = \{v > u\colon z(\tau, v) < 1\}$. If $A = \emptyset$, then $\int_u^{+\infty} (1 - z(\tau, v))\, \mathrm{d}v \le 0$, while if $A \neq \emptyset$, define $\hat{u} = \inf A$. Apply Lemma 3.3 at $\hat{u}$ and obtain

$$\int_u^{+\infty} (1 - z(\tau, v))\, \mathrm{d}v = \int_u^{\hat{u}} (1 - z(\tau, v))\, \mathrm{d}v + \int_{\hat{u}}^{+\infty} (1 - z(\tau, v))\, \mathrm{d}v \le \|z_o - 1\|_{\mathbf{L^1}}\,.$$

By convexity $g(z) \le g'(1)(z - 1)$, this implies

$$\begin{aligned} G(z; u) &= \exp\left\{\int_u^{+\infty} g\left(z(t, v)\right)\, \mathrm{d}v\right\} \le \exp\left\{g'(1)\int_u^{+\infty} \left[z(t, v) - 1\right]\, \mathrm{d}v\right\} \\ &\le \exp\left\{|g'(1)|\int_u^{+\infty} \left[1 - z(t, v)\right]\, \mathrm{d}v\right\} \\ &\le \exp\left(|g'(1)|\, \|1 - z_o\|_{\mathbf{L^1}}\right) \end{aligned}$$

completing the proof. □

Below, C denotes a constant depending only on an upper bound on T, $\|g\|_{\mathbf{W^{1,\infty}}([0, \|z_o\|_{\mathbf{L^\infty}}])}$, $\mathrm{TV}(z_o)$, $\|z_o - 1\|_{\mathbf{L^1}}$, $\|z_o\|_{\mathbf{L^\infty}}$ while C_* is a constant that depends also on an upper bound on $\|g''\|_{\mathbf{L^\infty}([0, \|z_o\|_{\mathbf{L^\infty}}])}$.

LEMMA 3.5. Let $\tau \in (0, \delta)$ be such that no interaction takes place at time τ. Assume there exist points u_i, u_j with $u_i < u_j$ and, using the notation in Lemma 3.1, such that $z_{i-1} \ge 1 > z_i$ and $z_{j-1} < 1 \le z_j$, with $z(t, v) < 1$ for all $v \in (u_i, u_j)$. Then, using ζ as defined in (3.2),

$$\frac{d}{dt}\int_{u_i}^{u_j} (1 - z(\tau, v))\, \mathrm{d}v \le C_*\, (z_j - 1)\, \left[z_j - \zeta(\Delta_j)\right]^+ \le C_*\, (z_j - 1)\,.$$

In particular, whenever $z_j = 1$, the quantity Δ_j may be not defined but we intend that the right hand side above vanishes.

Above, $[z]^+ = (z + |z|)\,/2$ is the positive part of z.

Proof. By the conservation law (3.7), we can write

$$\frac{d}{dt}\int_{u_i}^{u_j} \left[1 - z(t, v)\right]\, \mathrm{d}v = (1 - z_{j-1})\dot{u}_j + F(z; u_j-) - (1 - z_i)\dot{u}_i - F(z; u_i+)\,.$$

Proceeding as in Lemma 3.3, one proves that $-(1 - z_i)\dot{u}_i - F(z; u_i+) \le 0$. We need to consider the term in the left hand side. If $z_{j-1} > 0$ then it is a rarefaction and we must have $z_j = 1$. By (3.5) and (3.4) we directly have

$$(1 - z_{j-1})\dot{u} + F(z; u_j-) = -(1 - z_{j-1})\, \frac{g\left(1\right) - \left(z_{j-1}\right)}{1 - z_{j-1}}\, G(z; u_j) - g\left(z_{j-1}\right)\, \widehat{G}\left(z; u_j\right) = 0\,.$$

Note that if $z_{j-1} = 0$, then in u_j there is the right part of a u–shock and z_j is not necessarily 1, but greater or equal to 1. By (3.5) and (3.4) we have

$$\begin{aligned}(1-z_{j-1})\,\dot{u}_j + F\left(z;u_j-\right) &= -\frac{g(z_j)-\xi(\Delta_j)}{z_j}G(z;u_j) - \xi(\Delta_j)\,G(z;u_j)\\ &= \frac{1}{z_j}\left[\xi\left(\Delta_j\right) - z_j\xi\left(\Delta_j\right) - g\left(z_j\right)\right]G(z;u_j)\\ &= \frac{z_j-1}{z_j}\left[-\xi(\Delta_j) - \frac{g(z_j)}{z_j-1}\right]G(z;u_j)\,.\end{aligned}$$

By the concavity of g and using (2.3), see Figure 5,

$$-\frac{g(z_j)}{z_j-1} \le g(z_j) - z_j\,g'(z_j) = \psi(z_j)\,, \tag{3.12}$$

By definition (3.2), $\xi\left(\Delta_j\right) = g\left(\zeta(\Delta_j)\right) - \zeta(\Delta_j)\,g'\left(\zeta(\Delta_j)\right) = \psi\left(\zeta\left(\Delta_j\right)\right)$, so that

$$\begin{aligned}(1-z_{j-1})\,\dot{u}_j + F\left(z;u_j-\right) &\le \frac{z_j-1}{z_j}\left[\psi\left(z_j\right) - \psi\left(\zeta\left(\Delta_j\right)\right)\right]\,G(z;u_j)\\ &\le \frac{z_j-1}{z_j}\left\|zg''\right\|_{\mathbf{L}^\infty}\left[z_j - \zeta\left(\Delta_j\right)\right]^+ G(z;u_j)\\ &\le \left\|zg''\right\|_{\mathbf{L}^\infty}(z_j-1)\left[z_j-\zeta\left(\Delta_j\right)\right]^+ e^{\left|g'(1)\right|\left\|1-z_o\right\|_{\mathbf{L}^1}}\,,\end{aligned}$$

where we used Lemma 3.4, since u_j is the location of an upward discontinuity crossing 1.

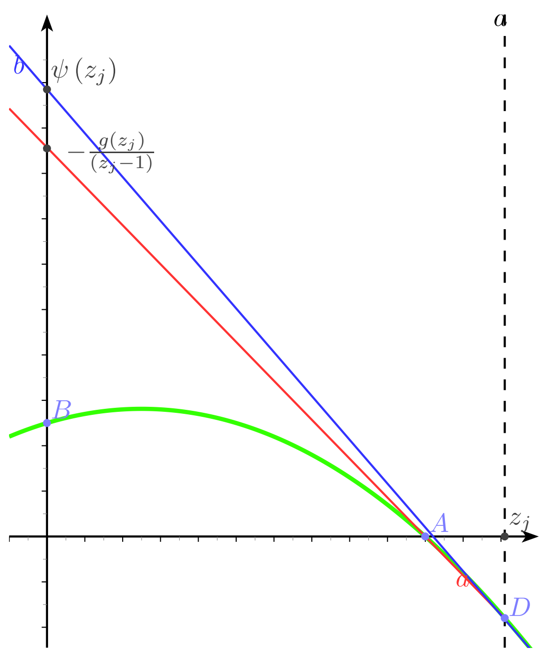


FIG. 5. Geometric justification of the estimate (3.12).

Therefore, the previous equation can be written as

$$(1 - z_{j-1})\ \dot{u}_j + F\,(z; u_j-) \le C_*\,(z_j - 1)\,[z_j - \zeta(\Delta)]^+ \le C_*\,(z_j - 1)$$

completing the proof. □

PROPOSITION 3.6. For all $t \in [0, \delta]$ and all $u \in \mathbb{R}$,

$$\|z(t,\cdot) - 1\|_{\mathbf{L^1}} \le C_* \qquad \text{and} \qquad \frac{1}{C_*} \le G(z; u) \le C_*\,.$$

Proof. Fix a positive time $\tau \in [0, \delta]$ at which no interaction occurs. Observe that

$$\int_{\mathbb{R}} |1 - z(\tau, v)|\ \mathrm{d}v = \int_{\mathbb{R}} [1 - z(\tau, v)]^+\ \mathrm{d}v + \int_{\mathbb{R}} [1 - z(\tau, v)]^-\ \mathrm{d}v$$

while by the conservation law (3.7)

$$\int_{\mathbb{R}} [1 - z(t, v)]\ \mathrm{d}v = \int_{\mathbb{R}} [1 - z(t, v)]^+\ \mathrm{d}v - \int_{\mathbb{R}} [1 - z(t, v)]^-\ \mathrm{d}v = \text{constant.}$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}} [1 - z(t, v)]^+\ \mathrm{d}v = \frac{d}{dt}\int_{\mathbb{R}} [1 - z(t, v)]^-\ \mathrm{d}v$$

and, recalling Lemma 3.5 and (3.10),

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} |1 - z(t, v)|\ \mathrm{d}v &= 2\frac{d}{dt}\int_{\mathbb{R}} [1 - z(t, v)]^+\ \mathrm{d}v \\ &\le C_* \sum_{\text{upward jumps crossing } 1} (z_j - 1) \\ &\le C_*\ \mathrm{TV}\,\hat{z}(t,\cdot) \le C_*\ \mathrm{TV}\,z_o \le C_*\ . \end{aligned}$$

We have finally a uniform bound on the $\mathbf{L^1}$ norm

$$\|z(t,\cdot) - 1\|_{\mathbf{L^1}} \le \|z_o - 1\|_{\mathbf{L^1}} + C_*\,T \le C_*\,,$$

which gives uniform lower and upper bounds on G: $\frac{1}{C_*} \le G(z; u) \le C_*$ for any $u \in \mathbb{R}$. □

3.4. *Existence Between Interactions for the ODE.* Now we prove that the solution to (3.3)–(3.4) can be extended up to the first time $\bar{t}$ at which either **(I1)**, or **(I2)** or **(I3)** occurs. Then, the above algorithm ensures that this solution can be prolonged after $\bar{t}$ up to the next interaction. Standard results on the theory of ordinary differential equations, see [17, Chapter 2, Theorem 3.1], ensure that the solution to (3.3)–(3.4) is defined up to the boundary of the domain where the right hand side is defined. Therefore, the next step consists in proving that none of the functions $u_i(t)$ or $z_i(t)$ tends to $\pm\infty$ and that no value $z_i(t)$ adjacent to a u-shock may vanish at any finite time.

LEMMA 3.7. Let $\bar{t} > 0$ and $s \in (0, \bar{t})$ be fixed. Assume that a solution to (3.3)–(3.4) is defined on $[0, \bar{t})$ and no interaction takes place in the interval $[s, \bar{t})$. Then,

$$\sup_{t\in[s,\bar{t})}\ \max_{i=1,\ldots,n-1} |z_i(t)| < +\infty \qquad \text{and} \qquad \sup_{t\in[s,\bar{t})}\ \max_{i=1,\ldots,n} \{|u_i(t)|\,,\ |\dot{u}_i(t)|\} < +\infty\,.$$

Moreover, if $[u_{i-1}(t), u_i(t)]$ is a u-shock, then

either: $\liminf_{t\to\bar{t}-} z_{i-2}(t) > 0$ and $\liminf_{t\to\bar{t}-} z_i(t) > 0$;

or: $\lim_{t\to\bar{t}} u_{i-1}(t) = \lim_{t\to\bar{t}} u_i(t)$. i.e. at $(\bar{t}, u_i(\bar{t}))$ there is an interaction of type **(I1)**.

Proof. By the rule **(I2)**, $z(t) \geq 0$ and, by (3.3), $\dot{z}_i(t) \leq 0$, so that the upper bound on $z(t)$ is immediately proved.

Consider now the three right hand sides in (3.4). The first one is bounded thanks to Proposition 3.6. To bound the other 2, consider preliminarily a u-shock in the first or last point of jump:

$i = 1$**:** with $z_1 = 0$, then $z_0 = 1$. Then, by (3.4)

$$|\dot{u}_1| = \xi(\Delta_2)\, e^{-\Delta_2 g(0)}\, G(z; u_1) = \frac{1 - e^{-\Delta_2 g(0)}}{\Delta_2}\, G(z; u_1) \leq C_* \,;$$

$i = n$**:** with $z_{n-1} = 0$, then $z_n = 1$, using (3.1),

$$|\dot{u}_n| = \xi(\Delta_n)\, G(z; u_n) \leq C_*\, \xi(\Delta_n) \leq C_*$$

since $\xi(\Delta)$ is an increasing function and Δ_n is bounded by the $\mathbf{L^1}$ norm of $z - 1$. This show that the support of the map $(t, u) \to z(t, u) - 1$ is bounded. Since no interaction takes place in the time interval $[s, \bar{t})$, the maps $t \to u_i(t)$ are bounded, also for $i = 2, \ldots, n-1$.

Consider now u-shocks. If $z_{i-1}(t) = 0$ for $t \in [s, \bar{t})$, by (3.4)

$$\dot{u}_i = \frac{\xi(\Delta_i) - g(z_i)}{z_i} G(z; u_i) \geq \frac{g(0) - g(z_i)}{z_i} G(z; u_i) \geq -\|g'\|_{\mathbf{L^\infty}} G(z; u_i) \geq -C_* \,.$$

Hence this velocity is uniformly bounded from below. Similarly one can show that if $z_i = 0$, then $\dot{u}_i$ is uniformly bounded from above.

Unfortunately, if $i = 2, \ldots, n-1$, no other uniform bound on the velocities of the discontinuities on the sides of any u–shock is available, since the denominator in (3.4) may vanish. Nevertheless, we show that these velocities are bounded between interactions, although this bound does not depend only on T, g, $\mathrm{TV}\, z_o$, $\|z_o - 1\|_{\mathbf{L^1}}$, $\|z_o\|_{\mathbf{L^\infty}}$.

By contradiction, suppose that $\dot{u}_i(t) \to +\infty$ as $t \to \bar{t}-$. The previous considerations imply that $z_{i-1}(t) = 0$ and $z_i(t) > 0$ for $t \in (s, \bar{t})$. Moreover $z_i(t) \to 0+$ as $t \to \bar{t}-$. The differential equation (3.3) implies

$$\dot{z}_i(t) = -g\left(z_i(t)\right)^2\, G(z; \tilde{u}_i) \geq -C_* \,,$$

and hence, integrating with respect to time,

$$z_i(t) \leq C_*(\bar{t} - t) \,.$$

Using the uniform boundedness from above of u_{i-1} and the assumption $\dot{u}_i(t) \to +\infty$,

$$\dot{\Delta}_i = \dot{u}_i - \dot{u}_{i-1} \geq \dot{u}_i - C_* \to +\infty \text{ as } t \to \bar{t}- \ .$$

Therefore, we have $\Delta_i \geq \bar{\Delta} > 0$ in $(s, \bar{t})$. Now, using (3.4), compute

$$\begin{aligned} \dot{u}_i \quad &= \quad \frac{\xi(\Delta_i) - g(z_i)}{z_i}\, G(z; u_i) \ \geq \ \frac{\xi(\bar{\Delta}) - g(0)}{z_i}\, G(z; u_i) + \frac{g(0) - g(z_i)}{z_i}\, G(z; u_i) \\ &\geq \quad \frac{1}{C_*}\, \frac{\xi(\bar{\Delta}) - g(0)}{\bar{t} - t} - C_*\, \|g'\|_{\mathbf{L^\infty}} \ \geq \ \frac{1}{C_*}\, \frac{\xi(\bar{\Delta}) - g(0)}{\bar{t} - t} - C_* \,. \end{aligned}$$

This shows that $\dot{u}_i$ is not integrable in a left neighborhood of $\bar{t}$ and this implies

$$u_i(t) = u_i(s) + \int_s^t \dot{u}_i\,(\sigma)\; d\sigma \to +\infty \text{ as } t \to \bar{t}^- .$$

This implies that $u_i(t)$ has to meet $u_{i+1}(t)$ before the time $\bar{t}$ contradicting the hypothesis that in $(s, \bar{t})$ there are no interaction. A similar argument applies when $u_i(t) \to -\infty$ as $t \to \bar{t}-$.

This completes the proof that the velocities of the wave fronts remain bounded.

Finally, suppose that $z_i(t) > 0$, $z_{i-1}(t) = 0$ for $t \in [s, \bar{t})$, with $z_i(t) \to 0$ as $t \to \bar{t}-$. The speed $\dot{u}_i$ is bounded, hence u_i is Lipschitz continuous and $\lim_{t \to \bar{t}-} u_i(t) = \bar{u}$. By (3.4) and the boundedness of $\dot{u}_i$, we get $\Delta_i \to 0$, therefore the state $z_{i-1} = 0$ disappears, in the sense that $u_{i-1}(\bar{t}) = u_i(\bar{t}) = \bar{u}$. This shows that at time $\bar{t}$ an interaction of type **(I1)** takes place.

An entirely similar argument holds in case $z_{i+1}(t) = 0$ and allows to conclude the proof. □

3.5. *Bound on Rarefactions' Strength.*

LEMMA 3.8. Let $\delta > 0$ be fixed. Assume that the approximate solution z is defined on all the time interval $[0, \delta]$. Then, z enjoys the properties *a), b), . . . , f)* in Theorem 3.2.

Proof. Note that properties a), c), d) and e) are immediate by the construction defined through **(I1)**, **(I2)** and **(I3)**.

Consider now properties b) and f):

$$\begin{array}{lrcll} \text{b)} & z_i(t) - z_{i-1}(t) & \leq & \varepsilon(t) & \text{for all } z_{i-1}(t) > 0\,, \\ \text{f)} & |g(z_i(t))|\,(u_{i+1}(t) - u_i(t)) & \leq & \varepsilon(t) & \text{for all } z_i \neq 0\,. \end{array} \tag{3.13}$$

Note that both properties hold at $t = 0$ by construction.

This proof is divided in two steps.

1. Let $\tau \in [0, \delta]$. If b) and f) hold in $[0, \tau)$, then they hold also on $[0, \tau]$.

If no interaction takes place at time τ, then **1.** trivially holds by continuity. Assume now that an interaction occurs at time τ and consider condition f).

(I1) By continuity, possibly renumbering the various waves, condition f) holds also at τ.

(I2) A new u-shock appears and condition f) trivially holds also at time τ.

(I3) At time τ, a new wave detaches, so that $u_{i-1}(\tau) = u_i(\tau)$ and f) holds.

Passing to condition b):

(I1) If the outgoing wave is a u-shock, then b) is trivial. If the interaction is between the left and right side of a u-shock, then the outgoing wave is a shock and b) holds. Otherwise, if only two waves interact, then they result in a shock and b) holds. If more than two waves interact, then there may not be two adjacent rarefactions. Therefore, grouping the interacting waves in pair, the whole interaction turns out to be equivalent to an interaction between two shocks or between a shock and a rarefaction. In both cases, the resulting wave is either a shock(and b) holds) or a rarefaction weaker than the interacting rarefaction, so that b) still holds.

(I2) Condition b) hols by continuity also at time τ.

(I3) A new rarefaction arises and, by construction it is of size at most ϵ, hence b) holds.

2. Let $\tau \in [0, \delta)$. If b) and f) hold in $[0, \tau]$, then they hold also on $[0, \tau + \bar{\delta}]$ for a positive $\bar{\delta}$.

Consider first b). If $z_{i-1} \geq z_i$, then b) holds by continuity in a right neighborhood of τ. Assume that $0 < z_{i-1} < z_i$ and, using (3.3), compute at time $t = \tau+$:

$$\begin{aligned}
\dot z_i - \dot z_{i-1} - \dot\varepsilon &= -g(z_i)^2 G(z;\tilde u_i) + g(z_{i-1})^2 G\left(z;\tilde u_{i-1}\right) - K\varepsilon \\
&= \left[g(z_{i-1})^2 - g(z_i)^2\right] G(z;u_i) + g(z_i)^2 \left[G(z;u_i) - G\left(z;\tilde u_i\right)\right] \\
&\qquad + g(z_{i-1})^2 \left[G\left(z;\tilde u_{i-1}\right) - G\left(z;u_i\right)\right] - K\varepsilon \\
&\leq C_* |z_i - z_{i-1}| + g(z_i)^3 G(z,u_i^*)(\tilde u_i - u_i) \\
&\qquad + g(z_{i-1})^3\, G(z,u_{i-1}^*)(u_i - \tilde u_{i-1}) - K\,\varepsilon \\
&\leq C_*\,\varepsilon - K\,\varepsilon \ \leq -\varepsilon
\end{aligned}$$

if K is chosen sufficiently large depending only on an upper bound on T, $\|g\|_{\mathbf{W^{2,\infty}}}$, TV z_o, $\|z_o - 1\|_{\mathbf{L^1}}$, $\|z_o\|_{\mathbf{L^\infty}}$. This shows that the strength of any rarefaction cannot exceed $\varepsilon(t)$ for all $t \in [\tau, \tau + \bar\delta]$, for a suitable $\bar\delta > 0$.

Now consider f). Compute again at the time $t = \tau+$ for $z_i > 0$,

$$\frac{d}{dt}\left[|g(z_i)|(u_{i+1} - u_i) - \varepsilon\right] = (u_{i+1} - u_i)\,\frac{d}{dt}|g(z_i)| + |g(z_i)|\,(\dot u_{i+1} - \dot u_i) - K\varepsilon\,.$$

Concerning the first term, we have

$$(u_{i+1} - u_i)\,\frac{d}{dt}|g(z_i)| \leq |g'(z_i)|\, g(z_i)^2\, G(z;\tilde u_i)\,(u_{i+1} - u_i) \leq C_*\,\varepsilon\,,$$

while concerning the second term, we have to distinguish different cases according to the possible presence of a u–shock.

If (u_{i+1}, u_{i+2}) is a u–shock, then, by convexity

$$\begin{aligned}
\dot u_{i+1} &= -\frac{g(z_i) - \xi(\Delta_{i+2})e^{-\Delta_{i+2}g(0)}}{z_i}\, G(z;u_{i+1}) \leq -\frac{g(z_i) - g(0)}{z_i}\, G(z;u_{i+1}) \\
&\leq -g'(z_i)\, G(z;u_{i+1})\,.
\end{aligned}$$

If (u_{i+1}, u_{i+2}) is not a u–shock, then, by convexity

$$\begin{aligned}
\dot u_{i+1} &= -\frac{g(z_{i+1}) - g(z_i)}{z_{i+1} - z_i}\, G(z;u_{i+1}) \\
&\leq \begin{cases} -g'(z_i)\, G(z;u_{i+1}) & \text{if } z_{i+1} \leq z_i \\ -g'(z_{i+1})\, G(z;u_{i+1}) \leq -g'(z_i)\, G(z;u_{i+1}) + C_*\,\varepsilon & \text{if } z_{i+1} > z_i \end{cases} \\
&\leq -g'(z_i)\, G(z;u_{i+1}) + C_*\,\varepsilon\,.
\end{aligned}$$

Now if (u_{i-1}, u_i) is a u–shock, then, by convexity and by (3.2), we have (see Figure 6):

$$\begin{aligned}
-\dot u_i &= \frac{g(z_i) - \xi(\Delta_i)}{z_i}\, G(z;u_i) \\
&\leq \begin{cases} g'(z_i)\, G(z;u_i) & \text{if } z_i \leq \zeta(\Delta) \\ g'\left(\zeta(\Delta)\right)\, G(z;u_i) \leq g'(z_i)\, G(z;u_i) + C_*\,\varepsilon & \text{if } z_i > \zeta(\Delta) \end{cases}
\end{aligned}$$

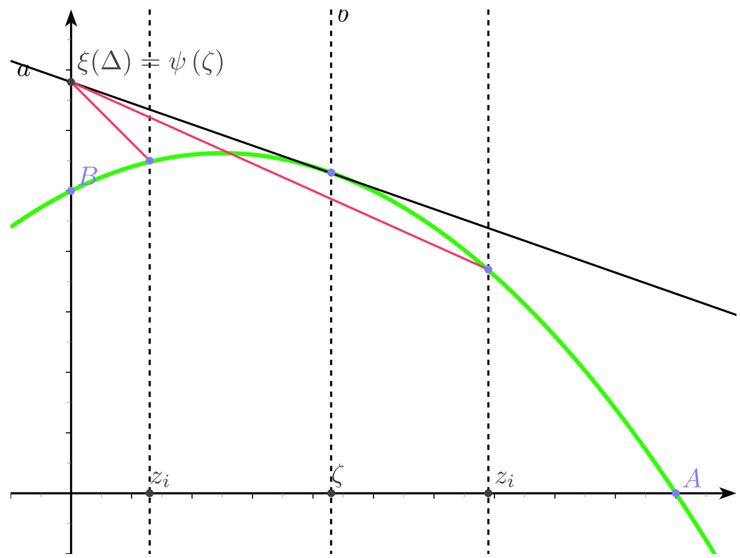


FIG. 6. Comparison between slopes

If (u_{i-1}, u_i) is not a u–shock, then, by convexity

$$\begin{aligned} -\dot{u}_i &= \frac{g(z_i) - g(z_{i-1})}{z_i - z_{i-1}} G(z; u_i) \\ &\leq \begin{cases} g'(z_i)\, G(z; u_i) & \text{if } z_i \leq z_{i-1} \\ g'(z_{i+1})\, G(z; u_i) \leq g'(z_i)\, G(z; u_i) + C_* \, \varepsilon & \text{if } z_i > z_{i-1} \,. \end{cases} \end{aligned}$$

In both cases

$$-\dot{u}_i \leq g'(z_i)\, G(z; u_i) + C_* \, \varepsilon \,.$$

Therefore, we can write

$$\begin{aligned} |g(z_i)|\, (\dot{u}_{i+1} - \dot{u}_i) &\leq |g(z_i)|\, [-g'(z_i)\, G(z; u_{i+1}) + g'(z_i)\, G(z; u_i)] + C_* \, \varepsilon \\ &\leq C_* \left(|g(z_i)|^2\, G(z; \tilde{u}_i)(u_{i+1} - u_i) + \varepsilon \right) \\ &\leq C_* \, \varepsilon \,. \end{aligned}$$

Finally

$$\frac{d}{dt} \left[|g(z_i)|(u_{i+1} - u_i) - \varepsilon \right] \leq C_* \, \varepsilon - K \, \varepsilon \leq -\varepsilon$$

for K sufficiently large depending only on an upper bound on T, $\|g\|_{\mathbf{W^{2,\infty}}}$, $\mathrm{TV}(z_o)$, $\|z_o - 1\|_{\mathbf{L^1}}$, $\|z_o\|_{\mathbf{L^\infty}}$.

This shows that condition f) holds for all $t \in [\tau, \tau + \bar{\delta}]$, completing the proof. □

3.6. *Bound on the Number of Interactions.* If interaction points accumulate at time τ, then the present algorithm can not define an approximate solution after time τ. Therefore, we prove below that there is a finite number of interaction points.

LEMMA 3.9. Fix $\delta > 0$. Assume that the approximate solution z is defined on all the time interval $[0, \delta]$. Suppose that in the interval $[t_1, t_2]$, the interval (u_{i-1}, u_i) is a u–shock with $\Delta_i = u_i - u_{i-1}$ and $z_i(t_1) \le \zeta(\Delta_i(t_1)) + \varepsilon(t_1)$, $z_i(t_2) = \zeta(\Delta_i(t_2)) + 2\varepsilon(t_2)$. Then, $t_2 - t_1 \ge \Delta t_\epsilon$. for a suitable $\Delta t_\epsilon > 0$.

Proof. If at time $t \in (t_1, t_2)$ there is an interaction at u_i with a rarefaction, then necessarily $z_i(t-) \le \zeta(\Delta_i(t))$, so that $z_i(t) \le \zeta(\Delta_i(t)) + \varepsilon(t)$.

Hence, we may assume that u_i interacts only with shocks coming from the right. Therefore, $z_i(t_2) \le z_i(t_1)$, which implies

$$\zeta(\Delta_i(t_1)) - \zeta(\Delta_i(t_2)) \ge 2\,\varepsilon(t_2) - \varepsilon(t_1) = \epsilon\left(2\,e^{Kt_2} - e^{Kt_1}\right) \ge \epsilon\,.$$

Let $\bar t \in (t_1, t_2)$ be the first time such that

$$\zeta(\Delta_i(t_1)) - \zeta(\Delta_i(\bar t)) = \frac{\epsilon}{2}\,. \tag{3.14}$$

From

$$\zeta(\Delta_i(t_1)) - \zeta(\Delta_i(t)) \le \frac{\epsilon}{2} \quad \text{for all } t \in [t_1, \bar t]\,, \text{ and } \zeta(\Delta_i(t_1)) \ge \epsilon$$

we get

$$\zeta(\Delta_i(t)) \ge \frac{\epsilon}{2} \text{ for all } t \in [t_1, \bar t].$$

Considering again (3.14), using (3.2) we compute

$$\frac{\epsilon}{2} = -\int_{t_1}^{\bar t} \frac{d}{dt}\zeta(\Delta_i(t))\,dt = \int_{t_1}^{\bar t} \frac{\xi'(\Delta_i(t))}{\zeta(\Delta_i(t))\,g''(\zeta(\Delta_i(t)))}\,\dot\Delta_i(t)\,dt\,.$$

Observe now that $\Delta_i = u_i - u_{i-1}$ and that we have uniform lower bound on $\dot u_i$ and upper bound on $\dot u_{i-1}$, so that $\dot\Delta_i \ge -C_*$. Since $\sup g'' < 0$, we get

$$\begin{aligned}
\frac{\epsilon}{2} &\le -C_* \int_{t_1}^{\bar t} \frac{\xi'(\Delta_i(t))}{\zeta(\Delta_i(t))\,g''(\zeta(\Delta_i(t)))}\,dt \\
&= C_* \int_{t_1}^{\bar t} \frac{\xi'(\Delta_i(t))}{\zeta(\Delta_i(t))\,[-g''(\zeta(\Delta_i(t)))]}\,dt \\
&\le \frac{C_*}{|\sup g''|} \int_{t_1}^{\bar t} \frac{1}{\zeta(\Delta_i(t))}\,dt \\
&\le \frac{C_*}{|\sup g''|}\frac{2}{\epsilon}(\bar t - t_1) \;\le\; \frac{C_*}{|\sup g''|}\frac{2}{\epsilon}(t_2 - t_1)\,.
\end{aligned}$$

Therefore

$$t_2 - t_1 \ge \frac{|\sup g''|}{4C_*}\,\epsilon^2 = \frac{|\sup g''|}{C_*}\,\epsilon^2 \doteq \Delta t_\epsilon \tag{3.15}$$

completing the proof. □

This last lemma says that between two interactions of type **(I3)** along the same discontinuity, there must be a time interval uniformly bounded from below.

PROPOSITION 3.10. Fix $\delta > 0$. Let the approximate solution z be defined on all the time interval $[0, \delta]$. Then, the number of interaction points is bounded from above by a constant that depends on an upper bound on T, g, on the approximate initial data z_0 and on ϵ, but is independent from δ.

Proof. We first bound the number of **(I3)** interactions. They are the only interactions where new waves arise. They take place only to the right of a u–shock. A u–shock exists or arises only to the right of a shock whose right state is in $(0, 1)$. These shocks are fixed from time 0 since, by (3.9), no new shock can arise below 1 and shocks above 1 cannot cross 1.

Call u_* any one of the, say, m shocks that at time 0 have a right state below 1. No new similar shock may arise at a positive time. Therefore, we trace the evolution of u_* in time, denoting $u_*(t)$ its position at time t. The discontinuity at $u_*(t)$ may interact with a similar shock or may also cease to exist, decreasing the total number of such socks. As long as $u_*(t)$ is defined, call $\bar{u}(t)$ the discontinuity adjacent to the right of $u_*(t)$ and $\bar{z}(t)$ the state to the immediate right of $\bar{u}(t)$. A type **(I3)** interaction may occur exclusively along $\bar{u}(t)$, provided $(u_*(t), \bar{u}(t))$ is a u-shock.

If $(u_*(t), \bar{u}(t))$ becomes a u shock at a positive time $\bar{t}$, then in $\bar{u}(\bar{t}-)$ there is a rarefaction. Hence, $\bar{z}(\bar{t}) \leq \varepsilon(\bar{t}) \leq \zeta\left(\bar{u}(\bar{t}) - u_*(\bar{t})\right) + \varepsilon(\bar{t})$ and one has to wait at least the time Δt_ϵ in (3.15) before a type **(I3)** interaction occurs. On the time interval $[0, \delta]$, the total number of type **(I3)** interactions that may take place along $\bar{u}$ is bounded from above by $\delta/\Delta t_\epsilon \leq T/\Delta t_\epsilon$. Since there are m such shocks, the total number of type **(I3)** interaction is bounded by $mT/\Delta t_\epsilon$.

The total number of waves may increase only at type **(I3)** interactions, decreases at type **(I1))** interactions and remains unchanged at type **(I2)** interactions. Since the number of **(I3)** interaction is bounded, so is the number of **(I1)** interactions.

Similarly, the total number intervals where z is strictly positive may increase only at type **(I3)** interactions while it strictly decreases at types **(I2)** and decreases or remains unchanged at **(I1)** interactions. Since the number of **(I3)** interaction is bounded, so is the number of **(I2)** interactions. □

A consequence of the above result is that the present algorithm is able to construct a solution on all the interval $[0, T]$. Hence, Theorem 3.2 is proved.

3.7. $\mathbf{L^1}$ *and* TV *Estimates.* The next proposition improves the estimate in Lemma 3.6.

PROPOSITION 3.11. The approximate solution satisfies the following estimate, for all $t \in [0, T]$:

$$
\begin{gathered}
\int_{\mathbb{R}} |1 - z(t, v)| \, \mathrm{d}v \leq \int_{\mathbb{R}} |1 - z_o(v)| \, \mathrm{d}v + C_* \, \epsilon \,, \\
\frac{1}{C} \, e^{-C_* \epsilon} \leq G(z; u) \leq C \, e^{C_* \epsilon} \,,
\end{gathered}
$$

where, as usual, C depends only on an upper bound on T, $\|g\|_{\mathbf{W^{1,\infty}}}$, $\mathrm{TV}(z_o)$, $\|z_o - 1\|_{\mathbf{L^1}}$ and $\|z_o\|_{\mathbf{L^\infty}}$, while C_* depends also on $\|g''\|_{\mathbf{L^\infty}}$.

Proof. With the value of K chosen in Lemma 3.8, use Lemma 3.5, the relations (3.10) and the bound $[z_j - \zeta(\Delta)]^+ \leq 2\varepsilon$, see point c) in Theorem 3.2, to obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |1 - z(t,v)| \, \mathrm{d}v &= 2\,\frac{d}{dt}\int_{\mathbb{R}} [1 - z(t,v)]^+ \, \mathrm{d}v \\
&\leq C_*\,\varepsilon(t) \sum_{\text{upward jumps crossing } 1} (z_j - 1) \\
&\leq C_*\,\varepsilon(t)\ \mathrm{TV}\left(\hat{z}(t,\cdot)\right) \\
&\leq C_*\,\varepsilon(t)\ \mathrm{TV}(z_o) \leq C_*\,\epsilon\, e^{KT} \ \leq\ C_*\,\epsilon\,.
\end{aligned}$$

The second chain of inequalities is now straightforward, concluding the proof. □

Our next step consists in obtaining a uniform bound on $\mathrm{TV}(z)$.

PROPOSITION 3.12. The approximate solution satisfies the following estimate, for all $t \in [0,T]$:

$$\mathrm{TV}\left(z(t)\right) \leq -1 + \left(1 + \mathrm{TV}(z_o)\right) \exp\left(C\, e^{C_*\epsilon}\, t\right) \leq C\, \exp(C\, e^{C_*\epsilon})\,.$$

Proof. It is immediate to prove that in each of the interactions **(I1)**, **(I2)** and **(I3)** the total variation does not increase. Out of any interaction time, we can estimate the rate of increase of $\mathrm{TV}\left(z(t)\right)$ as follows:

$$\frac{d}{dt}\,\mathrm{TV}\left(z(t)\right) = \frac{d}{dt}\left(\sum_{i:\ z_{i-1}z_i \neq 0} + \sum_{i:\ z_{i-1}=0, z_i\neq 0} + \sum_{i:\ z_i=0, z_{i-1}\neq 0}\right)\left|z_i(t) - z_{i-1}(t)\right|. \quad (3.16)$$

The latter terms are non positive. Indeed, if $z_{i-1}(t) = 0$, then

$$\frac{d}{dt}|z_i(t) - z_{i-1}(t)| = \frac{d}{dt}|z_i(t)| = \frac{d}{dt}z_i(t) \leq 0$$

by (3.3). The case $z_i(t) = 0$ is identical. To conclude the proof, estimate each term in the first sum in (3.16) using (3.3) as follows:

$$\begin{aligned}
&\frac{d}{dt}|z_i(t) - z_{i-1}(t)| \ \leq\ |\dot{z}_i(t) - \dot{z}_{i-1}(t)| \\
&= \ \left|-g(z_i)^2\, G(z,\tilde{u}_i) + g(z_{i-1})^2\, G(z,\tilde{u}_{i-1})\right| \\
&\leq \ g(z_i)^2\, |G(z;\tilde{u}_i) - G(z,u_i)| + g(z_{i-1})^2\, |G(z;\tilde{u}_{i-1}) - G(z;u_i)| \\
&\qquad + G(z;u_i)\left|g(z_{i-1})^2 - g(z_i)^2\right| \\
&\leq \ C\, e^{C_*\epsilon}\left(|g(z_i)|\,(u_{i+1} - u_i) + |g(z_{i-1})|\,(u_i - u_{i-1})\right) + Ce^{C_*\epsilon}\left|z_i - z_{i-1}\right|.
\end{aligned}$$

Adding over i we obtain

$$\begin{aligned}
\frac{d}{dt}\,\mathrm{TV}\left(z(t)\right) &\leq C\, e^{C_*\epsilon}\left(\|z(t) - 1\|_{\mathbf{L^1}} + \mathrm{TV}\left(z(t)\right)\right) \\
&\leq C\, e^{C_*\epsilon}\left(\|z_o - 1\|_{\mathbf{L^1}} + C_*\,\epsilon + \mathrm{TV}\left(z(t)\right)\right) \\
&\leq C\, e^{C_*\epsilon}\ \left(1 + \mathrm{TV}\left(z(t)\right)\right)
\end{aligned}$$

and solving this differential inequality completes the proof. □

We state without proof the following basic result that will be used below.

LEMMA 3.13. Let X be a normed vector space, $T > 0$ and $v \in \mathbf{C^0}([0,T]; X)$. If there exists a $\mathcal{L} > 0$ such that for all $t \in [0,T)$

$$\limsup_{h\to 0+} \frac{1}{h} \|v(t+h) - v(t)\|_X \le \mathcal{L}\,,$$

then $v \in \mathbf{C^{0,1}}([0,T]; X)$ with Lipschitz constant $\mathcal{L}$.

The next result proves the Lipschitz continuity in time of the approximate solution.

PROPOSITION 3.14. The approximate solution is Lipschitz continuous in time, i.e.

$$\|z(t_1,\cdot) - z(t_2,\cdot)\|_{\mathbf{L^1}} \le C\,\exp\left(C\,e^{C_*\epsilon}\right)\,|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0,T]\,.$$

Proof. Fix any $t \in [0,T)$ and $h \in (0, T-t)$ such that no interaction in z takes place in the interval $(t, t+h]$. We also require that $u_i(t+s) < u_{i+1}(t)$ for all $t \in [0,h)$ and all i. To simplify the notation, we denote $u_0 = -\infty$, $z_0 = 1$, $u_{n+1} = +\infty$ and $z_n = 1$, where $n = n(t)$ is the number of jumps in $z(t,\cdot)$. Then,

$$\begin{aligned}
\|z(t+h,\cdot) - z(t,\cdot)\|_{\mathbf{L^1}} &\le \int_{\mathbb{R}} \left| \sum_{i=0}^{n} z_i(t+h)\mathbf{1}_{[u_i(t+h),u_{i+1}(t+h))}(u) - z_i(t)\mathbf{1}_{[u_i(t),u_{i+1}(t))}(u) \right| \mathrm{d}u \\
&\le \int_{\mathbb{R}} \sum_{i=0}^{n} |z_i(t+h) - z_i(t)|\,\mathbf{1}_{[u_i(t+h),u_{i+1}(t+h))}(u)\,\mathrm{d}u \\
&\quad + \int_{\mathbb{R}} \left| \sum_{i=0}^{n} z_i(t)\left(\mathbf{1}_{[u_i(t+h),u_{i+1}(t+h))}(u) - \mathbf{1}_{[u_i(t),u_{i+1}(t))}(u)\right) \right| \mathrm{d}u\,.
\end{aligned}$$

Consider the two last terms separately. Start with the first one:

$$\begin{aligned}
&\limsup_{h\to 0+} \frac{1}{h} \int_{\mathbb{R}} \sum_{i=0}^{n} |z_i(t+h) - z_i(t)|\,\mathbf{1}_{[u_i(t+h),u_{i+1}(t+h))}(u)\,\mathrm{d}u \\
\le\;& \int_{\mathbb{R}} \sum_{i=1}^{n-1} |\dot z_i(t)|\,\mathbf{1}_{[u_i(t),u_{i+1}(t))}(u)\,\mathrm{d}u \\
\le\;& \sum_{i=1}^{n-1} g^2\left(z_i(t)\right)\,G(z,\tilde u_i)\,\left(u_{i+1}(t) - u_i(t)\right) \\
\le\;& C\,e^{C_*\epsilon}\,\|z-1\|_{\mathbf{L^1}} \\
\le\;& C\,e^{C_*\epsilon}\,.
\end{aligned}$$

Passing now to the latter term:

$$
\begin{aligned}
& \limsup_{h\to 0+} \frac{1}{h} \int_{\mathbb{R}} \left| \sum_{i=0}^{n} z_i(t) \left(\mathbf{1}_{[u_i(t+h),u_{i+1}(t+h))}(u) - \mathbf{1}_{[u_i(t),u_{i+1}(t))}(u) \right) \right| \mathrm{d}u \\
\leq \; & \limsup_{h\to 0+} \frac{1}{h} \sum_{i=1}^{n} |z_i(t) - z_{i-1}(t)| \, |u_i(t+h) - u_i(t)| \\
= \; & \sum_{i=1}^{n} |z_i(t) - z_{i-1}(t)| \, |\dot u_i(t)| \\
= \; & \left(\sum_{i\colon z_{i-1}z_i \neq 0} + \sum_{i\colon z_{i-1}=0, z_i\neq 0} + \sum_{i\colon z_i=0, z_{i-1}\neq 0} \right) |z_i(t) - z_{i-1}(t)| \, |\dot u_i(t)| \,. \qquad (3.17)
\end{aligned}
$$

Note that Proposition 3.11 ensures that the right hand side in the first of (3.4) is uniformly bounded by $Ce^{C_*\epsilon}$, whenever $z_{i-1} \neq 0$ and $z_i \neq 0$. Hence, the former sum above is bounded by $C\, e^{C_*\epsilon}\, \mathrm{TV}(z) \leq C \exp\left(Ce^{C_*\epsilon}\right)$. To bound the second sum in (3.17), use (3.4) to obtain:

$$
\begin{aligned}
& \sum_{i\colon z_{i-1}=0, z_i\neq 0} |z_i(t) - z_{i-1}(t)| \, |\dot u_i(t)| \\
= \; & \sum_{i\colon z_{i-1}=0, z_i\neq 0} |z_i(t)| \left| \frac{g(z_i) - \xi(u_{i+1} - u_i)}{z_i} \, G(z, u_i) \right| \\
\leq \; & C\, e^{C_*\epsilon} \sum_{i\colon z_{i-1}=0, z_i\neq 0} \left(|g(z_i) - g(z_{i-1})| + |g(0) - \xi(u_{i+1} - u_i)| \right) \\
\leq \; & C\, e^{C_*\epsilon} \left(\mathrm{TV}(z) + \sum_{i\colon z_{i-1}=0, z_i\neq 0} (u_{i+1} - u_i) \right) \\
\leq \; & C\, e^{C_*\epsilon} \left(\mathrm{TV}(z) + \|z - 1\|_{\mathbf{L^1}} \right) \\
\leq \; & C \, \exp\left(C\, e^{C_*\epsilon} \right) \,.
\end{aligned}
$$

The latter sum in (3.17) is estimated analogously. The proof is then completed applying Lemma 3.13. □

4. Stability and Uniqueness of the Limit. In this section we will compare any two approximate solutions obtained in Section 1. Let $z = z(t,u)$ be an ϵ-approximate solution to

$$
z_t(t,u) + \left[g\left(z(t,u)\right) \left(\exp \int_u^{+\infty} g\left(z(t,v)\ \mathrm{d}v\right) \right) \right]_u = 0 \,, \qquad z(t,u) \geq 0 \qquad (4.1)
$$

and $\bar z = \bar z(t,u)$ be a $\bar\epsilon$-approximate solution to

$$
\bar z_t(t,u) + \left[\bar g\left(\bar z(t,u)\right) \left(\exp \int_u^{+\infty} \bar g\left(\bar z(t,v)\ \mathrm{d}v\right) \right) \right]_u = 0, \qquad \bar z(t,u) \geq 0
$$

with g and $\bar g$ being two (possibly different) erosion functions, both satisfying **(g)** and in $\mathbf{C^2}\left([0,+\infty);\mathbb{R}\right)$.

As in the previous section we use the notation

$$G(z;u) = \exp \int_u^{+\infty} g\left(z(t,v)\right)\, dv \qquad \text{and} \qquad \bar G(\bar z;u) = \exp \int_u^{+\infty} \bar g\left(\bar z(t,v)\right)\, dv$$

and, from now on, C is a positive constant dependent only on upper bounds on T, $\|g\|_{\mathbf{W^{1,\infty}}}$, $\|\bar g\|_{\mathbf{W^{1,\infty}}}$, $\mathrm{TV}(z_o)$, $\mathrm{TV}(\bar z_o)$, $\|z_o\|_{\mathbf{L^\infty}}$, $\|\bar z_o\|_{\mathbf{L^\infty}}$, $\|z_o - 1\|_{\mathbf{L^1}}$ and $\|\bar z_o - 1\|_{\mathbf{L^1}}$. Similarly, C_* depends also on upper bounds on

$$\|g''\|_{\mathbf{L^\infty}([0,\max\{\|z_o\|_{\mathbf{L^\infty}},\|\bar z_o\|_{\mathbf{L^\infty}}\}])} \quad \text{and} \quad \|\bar g''\|_{\mathbf{L^\infty}([0,\max\{\|z_o\|_{\mathbf{L^\infty}},\|\bar z_o\|_{\mathbf{L^\infty}}\}])} \,.$$

The discontinuity curves of z and $\bar z$ are Lipschitz continuous. By [7, Theorem 2.71], there exists a sequence of positive δ_n converging to 0 such that the discontinuity curves of z^{δ_n} cross the discontinuity curves of $\bar z$ only in a finite numbers of points. Here, $z^{\delta_n}(t,u) = z(t, u+\delta_n)$ is also an approximate solution to (4.1) corresponding to the translated initial datum $z_o^{\delta_n}$, where $z_o^{\delta_n}(u) = z_o(u+\delta_n)$. Therefore, we consider now the case in which the discontinuity curves of z and $\bar z$ have at most a finite number of points in common. The general case will then follow through a standard $\mathbf{L^1}$ continuity argument.

4.1. *Conservation Law Type Estimates.* Fix now a time $\tau > 0$ at which both in z and in $\bar z$ there are no interactions and at which the discontinuity curves of the two approximate solutions do not cross. Assume that to the left of $u(\tau)$ the function z is below $\bar z$, whereas to the right of $u(\tau)$ the map $\bar z$ is above z, in the sense rigorously described by *a)*, *b)* and *c)* below. Then, a first $\mathbf{L^1}$ type estimate is available.

THEOREM 4.1. Let $u = u(t)$ be a discontinuity curve of z or of $\bar z$ defined in a neighborhood of τ and such that
a) $z(t,u) < \bar z(t,u)$ for all $(t,u) \in (\tau, \tau+\delta) \times (u(t)-\delta, u(t))$ for some $\delta > 0$;
b) $z\left(\tau, u(\tau)+\right) \geq \bar z\left(\tau, u(\tau)+\right)$;
c) if $z\left(\tau, u(\tau)+\right) = 0$, then the first upward jump to the right of $u(\tau)$ is in z.
Then,

$$\begin{aligned}\frac{d}{dt}\left(\int_{u(t)}^{+\infty} \left[z(t,u) - \bar z(t,u)\right]\, du\right)_{\Big|_{t=\tau}} \quad \leq \quad & \bar\varphi(z^*)\, \bar G\left(\bar z; u(\tau)\right) - \varphi(z^*)\, G\left(z; u(\tau)\right) \\ & + C_*\, \epsilon \left|z\left(\tau, u(\tau)+\right) - z\left(\tau, u(\tau)-\right)\right|\end{aligned} \tag{4.2}$$

where $z^* = z\left(\tau, u(\tau)\right)$ if the jump in $u(t)$ is a discontinuity of $\bar z$, while $z^* = \bar z\left(\tau, u(\tau)\right)$ if the jump in $u(t)$ is a discontinuity of z. The functions $\bar\varphi$ and φ are defined as

$$\varphi(z^*) = \begin{cases} g(z^*) & \text{if } z^* > 0 \\ \frac{1-e^{-\bar\Delta g(0)}}{\bar\Delta} & \text{if } z^* = 0 \end{cases}, \qquad \text{and} \qquad \bar\varphi(z^*) = \begin{cases} \bar g(z^*) & \text{if } z^* > 0 \\ \frac{1-e^{-\bar\Delta \bar g(0)}}{\bar\Delta} & \text{if } z^* = 0 \end{cases} \tag{4.3}$$

where $\bar\Delta$ is the strength of the u–shock in $\bar z$ starting at $u(\tau)$ when $z^* = 0$.

Observe that, under the hypotheses of the above Theorem, if $z^* = 0$, then the jump in $u(\tau)$ is in $\bar z$ and $\bar z\left(\tau, u(\tau)+\right) = 0$.

Proof. First observe that, by continuity, condition a) implies

$$z(\tau, u) \leq \bar z(\tau, u) \quad \text{for all} \quad u \in \left(u(\tau) - \delta, u(\tau)\right) \,.$$

Moreover, if $\bar z(\tau,u)=0$ then (3.3) implies that $\bar z(t,u)=0$ also for $t>\tau$, hence a) also implies

$$\bar z(\tau,u)>0 \quad \text{for all} \quad u\in(u(\tau)-\delta,u(\tau))\ .$$

Thanks to the conservation law (3.6), we have

$$\begin{aligned}\frac{d}{dt}\left(\int_{u(t)}^{+\infty}\left[z(t,u)-\bar z(t,u)\right]\,du\right)\Big|_{t=\tau} \quad &= \quad -\dot u(\tau)\left[z\left(\tau,u(\tau)+\right)-\bar z\left(\tau,u(\tau)+\right)\right] \\ &\quad +F\left(z;u(\tau)+\right)-\bar F\left(\bar z;u(\tau)+\right)\end{aligned} \tag{4.4}$$

where F is defined in (3.5) and $\bar F$ is the analogous flow associated to $\bar z$.

We have now two possibilities: the discontinuity in $u(\tau)$ is in z or in $\bar z$.

1. $u(t)$ is a discontinuity curve for z. Then, at $u(t)$ there is an upward jump in z. This upward jump can be a rarefaction or the right side of a u–shock, see Figure 7, left.

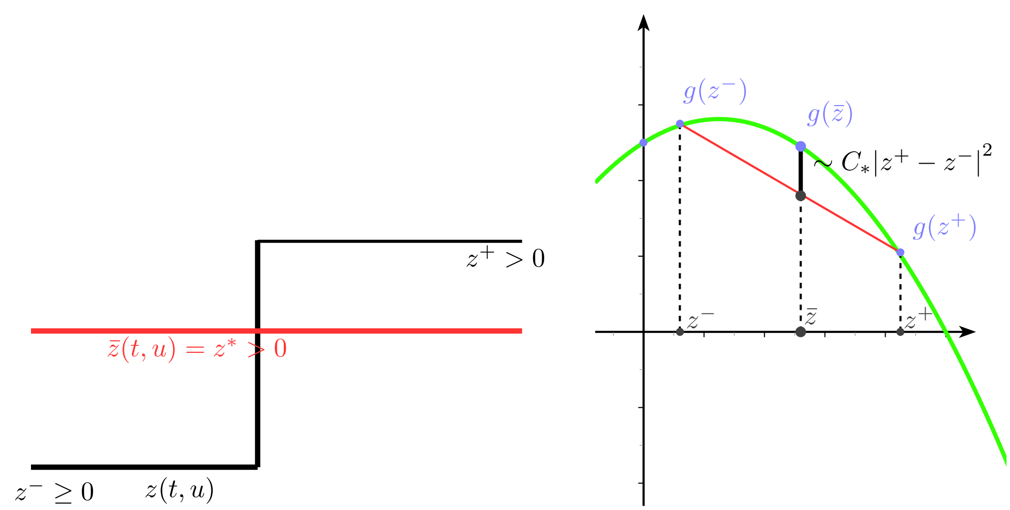


FIG. 7. Left, estimate for rarefactions. right, upward jump in z.

1.1 $u(t)$ is a rarefaction for z. Then, $z(\tau,u(\tau)-)>0$. For simplicity we define $z^\pm = z\left(\tau,u(\tau)\pm\right)$, $\bar z=\bar z\left(\tau,u(\tau)\right)$ and we omit the explicit dependence on time.

Inserting (3.4) and (3.5) in (4.4), we obtain (4.2) as follows:

$$\begin{aligned}&-\dot u\left[z^+-\bar z\right]+F(z;u+)-\bar F(\bar z;u+)\\ =\;& G(z;u)\frac{g(z^-)-g(z^+)}{z^--z^+}(z^+-\bar z)-g(z^+)G(z;u)+\bar g(\bar z)\bar G(\bar z;u)\\ =\;& -G(z;u)\left[g(z^+)+\frac{g(z^-)-g(z^+)}{z^--z^+}\left(\bar z-z^+\right)-g(\bar z)\right]+\bar g(\bar z)\bar G\left(\bar z;u\right)-g(\bar z)G(z;u)\\ \leq\;& \bar\varphi(z^*)\,\bar G(\bar z;u)-\varphi(z^*)\,G(z;u)+C\,e^{C_*\epsilon}\,\epsilon\left|z^+-z^-\right|\end{aligned}$$

where we used (4.3), the relation $\bar z = z^* > 0$ and the following inequality (see Figure 7, right):

$$\begin{aligned}\left|g(z^+) + \frac{g(z^-) - g(z^+)}{z^- - z^+}(\bar z - z^+) - g(\bar z)\right| &\le \left|g'(\tilde z) - g'\left(\tilde{\tilde z}\right)\right| \cdot \left|\bar z - z^+\right| \\ &\le C_* \left|z^+ - z^-\right|^2 \le C_*\, \epsilon \left|z^+ - z^-\right| .\end{aligned}$$

1.2 $u(t)$ is at the right of a u–shock for z. Then, $z^- = 0$ and from (3.4) and (3.5), denoting by Δ the strength of the u–shock, we obtain (4.2):

$$\begin{aligned} & -\dot u \left[z^+ - \bar z\right] + F(z; u+) - \bar F(\bar z; u+) \\ = \;& G(z;u)\, \frac{g(z^+) - \xi(\Delta)}{z^+}(z^+ - \bar z) - g(z^+)\, G(z;u) + \bar g(\bar z)\, \bar G(\bar z; u) \\ = \;& G(z;u) \left\{ g\left(\bar z\right) - \left[g(z^+) + \frac{g(z^+) - \xi(\Delta)}{z^+}\left(\bar z - z^+\right)\right]\right\} + \bar g(\bar z) \bar G(\bar z; u) - g(\bar z) G(z;u). \end{aligned}$$

As before $\bar g(\bar z)\bar G(\bar z;u) - g(\bar z)G(z;u) = \bar\varphi(z^*)\bar G(\bar z;u) - \varphi(z^*)G(z;u)$, and we are left with the quantity between braces. We distinguish two cases.
1.2.1 $0 < \bar z \le z^+ \le \zeta(\Delta)$. Then $g(\bar z) - \left[g(z^+) + \frac{g(z^+)-\xi(\Delta)}{z^+-0}(\bar z - z^+)\right] \le 0$, see Figure 8, left.

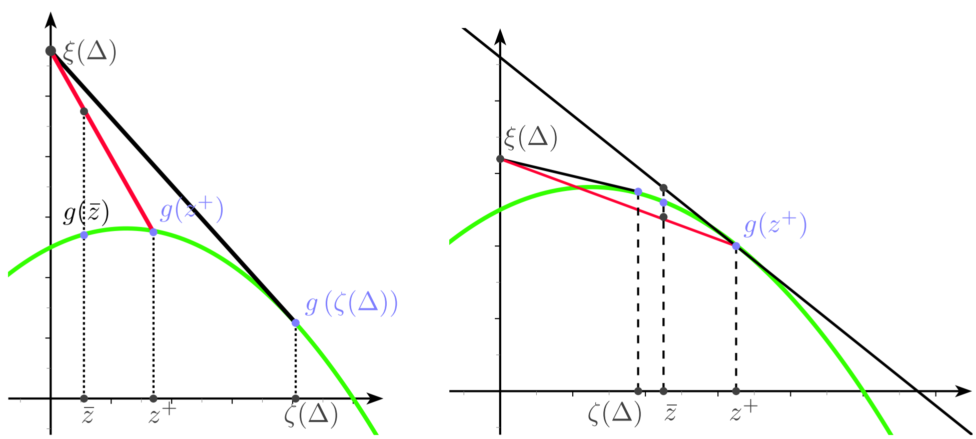


FIG. 8. Left, case 1.2.1 and, right, case 1.2.2.

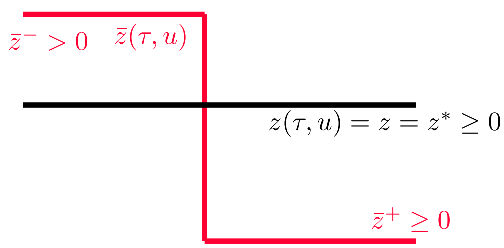
1.2.2 $0 < \bar z \le z^+ \le \zeta(\Delta) + 2\varepsilon$. Then, u–shock is only approximately admissible and, using the equality $\xi(\Delta) = g\left(\zeta(\Delta)\right) - \zeta(\Delta) g'\left(\zeta(\Delta)\right)$, we compute

$$\begin{aligned}\left|\frac{g(z^+) - \xi(\Delta)}{z^+} - g'(z^+)\right| &= \left|\frac{g(z^+) - z^+ g'(z^+) - \left[g\left(\zeta(\Delta)\right) - \zeta(\Delta) g'\left(\zeta(\Delta)\right)\right]}{z^+}\right| \\ &\le \frac{\tilde\zeta}{z^+}\left|g''(\tilde\zeta)\right| \left|\zeta(\Delta) - z^+\right| \\ &\le \left\|g''\right\|_{\mathbf{L}^\infty} \left|\zeta(\Delta) - z^+\right| ,\end{aligned}$$

where $\zeta(\Delta) \leq \tilde\zeta \leq z^+$. So that, by convexity (see Figure 8, right):

$$\begin{aligned}
& g(\bar z) - \left[g(z^+) + \frac{g(z^+) - \xi(\Delta)}{z^+}\left(\bar z - z^+\right)\right] \\
\leq\ & g(\bar z) - \left[g(z^+) + g'(z^+)(\bar z - z^+)\right] + C_* \left|z^+ - \zeta(\Delta)\right| \cdot \left|z^+ - \bar z\right| \\
\leq\ & C_* \,\epsilon \left|z^+ - z^-\right| .
\end{aligned}$$

2. $u(t)$ is a discontinuity curve for $\bar z$. Then, $u(t)$ is a downward jump in $\bar z$ and it can be a shock or the left side of a u–shock, see Figure 9, left. As above, we define



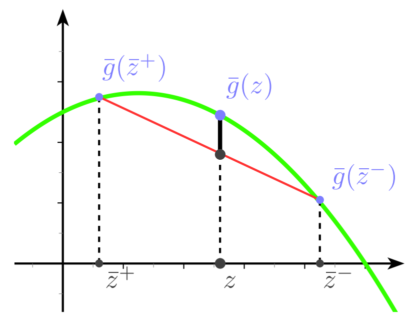


FIG. 9. Left, a discontinuity in $\bar z$. Right, the case of a normal shock.

$\bar z^\pm = \bar z\left(\tau, u(\tau)\pm\right)$, $z = z\left(\tau, u(\tau)\right)$ and we omit the explicit dependence on time. We distinguish three cases.

2.1 $u(\tau)$ is a shock for $\bar z$. Then, $\bar z^+ > 0$. From (3.4), (3.5) and by convexity (see Figure 9, right), we compute

$$\begin{aligned}
& -\dot u\left[z - \bar z^+\right] + F(z; u+) - \bar F(\bar z; u+) \\
=\ & \bar G(\bar z; u) \frac{\bar g(\bar z^-) - \bar g(\bar z^+)}{\bar z^- - \bar z^+}(z - \bar z^+) - g\left(z\right) G(z; u) + \bar g(\bar z^+) \bar G(\bar z; u) \\
=\ & \bar G(\bar z; u) \left\{\bar g(\bar z^+) + \frac{\bar g(\bar z^-) - \bar g(\bar z^+)}{\bar z^- - \bar z^+}(z - \bar z^+) - \bar g\left(z\right)\right\} + \bar g\left(z\right) \bar G(\bar z; u) - g\left(z\right) G(z; u) \\
\leq\ & \bar\varphi\left(z\right) \bar G(\bar z; u) - \varphi\left(z\right) G(z; u) .
\end{aligned}$$

2.2 $u(\tau)$ is the left side of a u–shock for $\bar z$ and $z > 0$. Then, $\bar z^+ = 0$. Call Δ the strength of this u–shock. From (3.4), (3.5) and by convexity (see Figure 10, left), we obtain

$$\begin{aligned}
& -\dot u\left[z - \bar z^+\right] + F(z; u+) - \bar F(\bar z; u+) \\
=\ & \bar G(\bar z; u) \frac{\bar g(\bar z^-) - \xi(\Delta) e^{-\Delta \bar g(0)}}{\bar z^-} z - \frac{e^{-\Delta \bar g(0)} - 1}{\Delta} \bar G(\bar z; u) - g\left(z\right) G(z; u) \\
=\ & \bar G(\bar z; u) \left\{\frac{\bar g\left(\bar z^-\right) - \xi(\Delta) e^{-\Delta \bar g(0)}}{\bar z^-} z + \xi(\Delta) e^{-\Delta \bar g(0)}\right\} - g\left(z\right) G(z; u) \\
\leq\ & \bar G(\bar z; u) \left\{\frac{\bar g\left(z\right) - \xi(\Delta) e^{-\Delta \bar g(0)}}{z} z + \xi(\Delta) e^{-\Delta \bar g(0)}\right\} - g\left(z\right) G(z; u) \\
=\ & \bar g\left(z\right) \bar G(\bar z; u) - g\left(z\right) G(z; u) \\
=\ & \bar\varphi(z^*) \bar G(\bar z; u) - \varphi(z^*) G(z; u) .
\end{aligned}$$

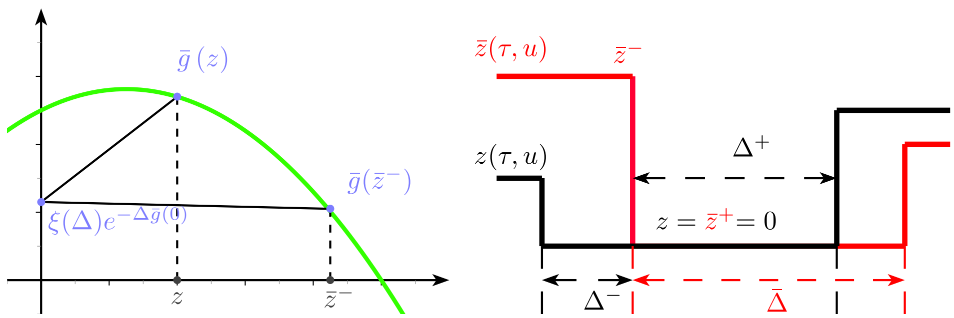


FIG. 10. Left, first u–shock case. Right, two interlaced u–shocks

2.3 $u(\tau)$ is a u–shock for $\bar{z}$ and $z = 0$. By condition c), the u–shock in z ends before the u–shock in $\bar{z}$, see Figure 10, right. We denote by $\bar{\Delta}$ the strength of the u–shock in $\bar{z}$, by Δ^- and Δ^+ respectively the strength of the u–shock in z to the left and the right of $u(\tau)$, see Figure 10, right. Using the expressions (3.5) for the fluxes, denoting by u_l and u_r the left and the right limits of the u–shock in z and letting $u(\tau) = u$, we have

$$\begin{aligned} F\left(z; u(\tau)\right) &= \frac{G(z;u_r) - G(z;u_l)}{u_r - u_l} \\ &= G(z;u_r)\,\frac{1 - e^{\left(\Delta^- + \Delta^+\right)g(0)}}{\Delta^+ + \Delta^-} \\ &= G(z;u)\,\frac{1 - e^{\left(\Delta^- + \Delta^+\right)g(0)}}{\Delta^+ + \Delta^-}\,e^{-\Delta^+ g(0)}\,. \end{aligned}$$

Therefore, using also the expression for $\bar{F}(\bar{z}; u+)$ and the fact that $t \mapsto \frac{1 - e^{tg(0)}}{t}$ is decreasing,

$$\begin{aligned} & -\dot{u}\left[z - \bar{z}^+\right] + F(z;u+) - \bar{F}(\bar{z};u+) \\ = & -\bar{F}(\bar{z};u+) + F(z;u+) \\ = & \frac{1 - e^{-\bar{\Delta}\bar{g}(0)}}{\bar{\Delta}}\,\bar{G}(\bar{z};u) + \frac{1 - e^{\left(\Delta^- + \Delta^+\right)g(0)}}{\Delta^+ + \Delta^-}\,G(z;u)\,e^{-\Delta^+ g(0)} \\ \leq & \frac{1 - e^{-\bar{\Delta}\bar{g}(0)}}{\bar{\Delta}}\,\bar{G}(\bar{z};u) + \frac{1 - e^{\Delta^+ g(0)}}{\Delta^+}\,G(z;u)\,e^{-\Delta^+ g(0)} \\ \leq & \frac{1 - e^{-\bar{\Delta}\bar{g}(0)}}{\bar{\Delta}}\,\bar{G}(\bar{z};u) - \frac{1 - e^{-\bar{\Delta}g(0)}}{\bar{\Delta}}\,G(z;u) \\ = & \bar{\varphi}(z^*)\,\bar{G}(\bar{z};u) - \varphi(z^*)\,G(z;u)\,. \end{aligned}$$

with $z^* = 0$ and where we used the fact that $\Delta^+ \leq \bar{\Delta}$. □

4.2. *Grouping Wave Fronts.* In this subsection we show that through a careful grouping of wave fronts, the estimates of the previous sections are all we need to treat the conservation law part of the problem.

Let τ be a time at which no interaction both in z or $\bar{z}$ occur and at which no wave front of z crosses wave fronts of $\bar{z}$. Let $u_1(t), \ldots, u_n(t)$ be the ordered wave fronts of both z and $\bar{z}$ in a neighborhood of τ and let $z_1(t), \ldots, z_n(t)$, $\bar{z}_1(t), \ldots, \bar{z}_n(t)$ be the corresponding states attained by z and $\bar{z}$. To group appropriately the waves, we define the following coefficients

$$c_i = \begin{cases} 1 & \text{if there exists } \delta > 0 \text{ such that } z_i(t) > \bar{z}_i(t) \text{ for all } t \in (\tau, \tau+\delta) \\ -1 & \text{if there exists } \delta > 0 \text{ such that } z_i(t) < \bar{z}_i(t) \text{ for all } t \in (\tau, \tau+\delta) \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Observe that $z_i(t)$, $\bar{z}_i(t)$, and

$$\ell_i(t) = \int_{u_i(t)}^{u_{i+1}(t)} [z_i(t) - \bar{z}_i(t)]\ du = [z_i(t) - \bar{z}_i(t)] \cdot [u_i(t) - u_{i+1}(t)]$$

are differentiable in a neighborhood of τ, therefore if $c_i = 0$ we have:

- $z_i(\tau) = \bar{z}_i(\tau)$;
- $\frac{d}{dt}\ell_i(t)\Big|_{t=\tau} = \frac{d}{dt}\int_{u_i(t)}^{u_{i+1}(t)} ||z_i(t) - \bar{z}_i(t)|\, du\Big|_{t=\tau} = 0.$

The last equality is due to the fact that if $c_i = 0$, $l_i(t_\nu) = 0$ for a suitable sequence t_ν converging to 0^+.

If $c_i = \pm 1$, then $|z_i(t) - \bar{z}_i(t)| = c_i\,[z_i(t) - \bar{z}_i(t)]$ for all $t \in (\tau, \tau+\delta)$, hence denoting by $\frac{d^+}{dt}$ the left derivative, which obviously coincides with the derivative for differentiable functions, we have that for any value of c_i:

$$\frac{d^+}{dt}\int_{u_i(t)}^{u_{i+1}(t)} |z_i(t) - \bar{z}_i(t)|\ du\Big|_{t=\tau} = c_i \cdot \frac{d^+}{dt}\int_{u_i(t)}^{u_{i+1}(t)} [z_i(t) - \bar{z}_i(t)]\ du\Big|_{t=\tau}, \tag{4.6}$$

which implies

$$\begin{aligned} \frac{d^+}{dt}\left(\int_{\mathbb{R}} |z_i(t) - \bar{z}_i(t)|\, du\right)_{\Big|_{t=\tau}} &= \sum_{i=1}^{n-1} c_i\, \frac{d^+}{dt}\left(\int_{u_i(t)}^{u_{i+1}(t)} (z_i(t) - \bar{z}_i(t))\ du\right)_{\Big|_{t=\tau}} \\ &= \sum_{i=1}^{n-1}([c_i]^+ - [c_i]^-)\, \frac{d^+}{dt}\left(\int_{u_i(t)}^{u_{i+1}(t)} (z_i(t) - \bar{z}_i(t))\ du\right)_{\Big|_{t=\tau}}. \end{aligned}$$

Since $\ell_i'(\tau) = 0$ whenever $c_i = 0$, by conservation

$$0 = \frac{d^+}{dt}\left(\int_{\mathbb{R}} (z_i(t) - \bar{z}_i(t))\ du\right)_{\Big|_{t=\tau}} = \sum_{i=1}^{n-1}([c_i]^+ + [c_i]^-)\, \frac{d^+}{dt}\left(\int_{u_i(t)}^{u_{i+1}(t)} (z_i(t) - \bar{z}_i(t))\ du\right)_{\Big|_{t=\tau}},$$

which leads to

$$\frac{d^+}{dt}\left(\int_{\mathbb{R}} |z_i(t) - \bar{z}_i(t)|\, du\right)_{\Big|_{t=\tau}} = 2\sum_{i=1}^{n-1}[c_i]^+\, \frac{d^+}{dt}\left(\int_{u_i(t)}^{u_{i+1}(t)} (z_i(t) - \bar{z}_i(t))\ du\right)_{\Big|_{t=\tau}}.$$

Now we erase from the last sum some (but not all) of the term with the coefficients $[c_i]^+ = 0$. First, we erase the terms corresponding to coefficients $c_i = 0$ which are in between two other coefficients equal to -1 or to 1 on the left and equal to -1 to the right. Then, we erase all the terms corresponding to coefficients equal to -1. Let

$\{I_k, I_k+1, \ldots, J_k-1\}$ for $k=1,\ldots,m$ be the sets of contiguous indexes left in the summation, so that we can write

$$\begin{aligned} \frac{d^+}{dt}\left(\int_{\mathbb{R}} |z_i(t)-\bar z_i(t)|\,du\right)_{\big|_{t=\tau}} &= 2\sum_{k=1}^{m}\sum_{i=I_k}^{J_k-1}[c_i]^+\,\frac{d^+}{dt}\left(\int_{u_i(t)}^{u_{i+1}(t)}(z_i(t)-\bar z_i(t))\,du\right)_{\big|_{t=\tau}} \\ &= 2\sum_{k=1}^{m}\sum_{i=I_k}^{J_k-1}\frac{d^+}{dt}\left(\int_{u_i(t)}^{u_{i+1}(t)}(z_i(t)-\bar z_i(t))\,du\right)_{\big|_{t=\tau}} \end{aligned}$$

since when $c_i=0$, also $\ell_i'(\tau)=0$. We finally obtain

$$\begin{aligned} &\frac{d^+}{dt}\left(\int_{\mathbb{R}} |z_i(t)-\bar z_i(t)|\,du\right)_{\big|_{t=\tau}} \\ = \;& 2\sum_{k=1}^{m}\frac{d^+}{dt}\left(\int_{u_{I_k}(t)}^{u_{J_k}(t)}(z_i(t)-\bar z_i(t))\,du\right)_{\big|_{t=\tau}} \\ = \;& 2\sum_{k=1}^{m}\left\{\frac{d^+}{dt}\left(\int_{u_{I_k}(t)}^{+\infty}(z_i(t)-\bar z_i(t))\,du\right)_{\big|_{t=\tau}} - \frac{d^+}{dt}\left(\int_{u_{J_k}(t)}^{+\infty}(z_i(t)-\bar z_i(t))\,du\right)_{\big|_{t=\tau}}\right\} \\ = \;& 2\sum_{k=1}^{m}\left\{\frac{d^+}{dt}\left(\int_{u_{I_k}(t)}^{+\infty}(z_i(t)-\bar z_i(t))\,du\right)_{\big|_{t=\tau}} + \frac{d^+}{dt}\left(\int_{u_{J_k}(t)}^{+\infty}(\bar z_i(t)-z_i(t))\,du\right)_{\big|_{t=\tau}}\right\}. \end{aligned}$$

We consider now the two terms in the summation. If $I_k=1$, then by conservation

$$\frac{d^+}{dt}\left(\int_{u_{I_k}(t)}^{+\infty}(z_i(t)-\bar z_i(t))\,du\right)_{\big|_{t=\tau}} = 0 = \bar\varphi_{I_k}(z^*_{I_k})\,\bar G(\bar z;u_{I_k}) - \varphi_{I_k}(z^*_{I_k})\,G(z;u_{I_k})$$

with

$$\bar\varphi_{I_k}(z^*_{I_k}) = \begin{cases} \bar g(z^*_{I_k}) & \text{if } z^*_{I_k}\neq 0 \\ \frac{1-e^{-\bar\Delta_{I_k}\bar g(0)}}{\bar\Delta_{I_k}} & \text{if } z^*_{I_k}=0\,, \end{cases} \qquad \text{and} \qquad \varphi_{I_k}(z^*_{I_k}) = \begin{cases} g(z^*_{I_k}) & \text{if } z^*_{I_k}\neq 0 \\ \frac{1-e^{-\bar\Delta_{I_k} g(0)}}{\bar\Delta_{I_k}} & \text{if } z^*_{I_k}=0\,. \end{cases}$$

where $\bar\Delta_{I_k}$ is the strength of the u–shock in $\bar z$ (which is present when $z^*_{I_k}=0$) beginning at $u_{I_k}(\tau)$. This last equality holds since the function which does not have a jump in the first discontinuity point u_{I_k} attains the value $z^*_{I_k}=1$, which implies that $\bar\varphi(z^*_{I_k})=0$, $\varphi(z^*_{I_k})=0$. If on the other hand $I_k\neq 1$, then by the way we selected the terms in the summation, $c_{I_k-1}=-1$ which means that for the curve $u(t)=u_{I_k}(t)$, the hypothesis *a*) of Theorem 4.1 is satisfied. Moreover, c_{I_k} is equal to 0 or to 1, which implies that also *b*) of Theorem 4.1 is satisfied. Finally, if $z\,(\tau,u_{I_k}+)=0$, then also $\bar z\,(\tau,u_{I_k}+)=0$, hence $c_{I_k}=0$. Then, the first upward jump to the right of u_{I_k} must be an upward jump in z, otherwise c_{I_k} would be -1 and the term with index I_k would have been erased from the summation. This means that also condition *c*) is satisfied and that Theorem 4.1 can be applied to obtain, in any case:

$$\frac{d^+}{dt}\left(\int_{u_{I_k}(t)}^{+\infty}(z_i(t)-\bar z_i(t))\,du\right)_{\Big|_{t=\tau}} \le \bar\varphi_{I_k}(z^*_{I_k})\,\bar G(\bar z;u_{I_k})-\varphi_{I_k}(z^*_{I_k})\,G(z;u_{I_k}) +C_*\,\epsilon\,|z(\tau,u_{I_k}+)-z(\tau,u_{I_k}-)|\,.$$

Similar arguments hold for the other term in the summation, Theorem 4.1 can be applied with $\bar z$ and z exchanged leading to

$$\frac{d^+}{dt}\left(\int_{u_{J_k}(t)}^{+\infty}(\bar z_i(t)-z_i(t))\,du\right)_{\Big|_{t=\tau}} \le \varphi_{J_k}(z^*_{J_k})\,G(z;u_{J_k})-\bar\varphi_{J_k}(z^*_{J_k})\,\bar G(\bar z;u_{J_k}) +C_*\,\bar\epsilon|\bar z(\tau,u_{J_k}+)-\bar z(\tau,u_{J_k}-)|\,.$$

with

$$\bar\varphi_{J_k}(z^*_{J_k})=\begin{cases}\bar g(z^*_{J_k}) & \text{if } z^*_{J_k}\neq 0\\ \frac{1-e^{-\Delta_{J_k}\bar g(0)}}{\Delta_{J_k}} & \text{if } z^*_{J_k}=0\,,\end{cases}\qquad\text{and}\qquad \varphi_{J_k}(z^*_{J_k})=\begin{cases}g(z^*_{J_k}) & \text{if } z^*_{J_k}\neq 0\\ \frac{1-e^{-\Delta_{J_k}g(0)}}{\Delta_{J_k}} & \text{if } z^*_{J_k}=0\,,\end{cases}$$

where Δ_{J_k} is the strength of the u–shock in z (which is present when $z^*_{J_k}=0$) beginning at $u_{J_k}(\tau)$. Putting everything together we have the final estimate

$$\begin{aligned}\frac{d^+}{dt}\left(\int_{\mathbb R}|z_i(t)-\bar z_i(t)|\,du\right)_{\Big|_{t=\tau}} \le\; & 2\sum_{k=1}^{m}\Big\{\bar\varphi_{I_k}(z^*_{I_k})\,\bar G(\bar z;u_{I_k})-\varphi_{I_k}(z^*_{I_k})\,G(z;u_{I_k})\\ &+\varphi_{J_k}(z^*_{J_k})\,G(z;u_{J_k})-\bar\varphi_{J_k}(z^*_{J_k})\,\bar G(\bar z;u_{J_k})\Big\}\\ &+C_*(\epsilon+\bar\epsilon)\,.\end{aligned}\tag{4.7}$$

4.3. *Estimates Related to the Integral Terms.* The following Lemma collects the estimates used to control the right hand side in (4.7).

LEMMA 4.2. Using the notations introduced in Subsection 4.2, if τ is a time at which no interaction occurs and at which no wave front in z crosses wave fronts in $\bar z$, the following estimates hold:

$$\|\bar g(\bar z(\tau,\cdot))-g(z(\tau\cdot))\|_{\mathbf{L^1}} \le Ce^{C_*(\epsilon+\bar\epsilon)}\left(\|\bar g'-g'\|_{\mathbf{L^\infty}}+\|\bar z(\tau,\cdot)-z(\tau,\cdot)\|_{\mathbf{L^1}}\right) \tag{4.8}$$

$$\left\|\bar G(\bar z;\cdot)-G(z;\cdot)\right\|_{\mathbf{L^\infty}} \le Ce^{C_*(\epsilon+\bar\epsilon)}\left(\|\bar g'-g'\|_{\mathbf{L^\infty}}+\|\bar z(\tau,\cdot)-z(\tau,\cdot)\|_{\mathbf{L^1}}\right) \tag{4.9}$$

$$\left|\bar\varphi_{I_k}(z^*_{I_k})-\varphi_{I_k}(z^*_{I_k})\right| \le \|\bar g-g\|_{\mathbf{L^\infty}} \tag{4.10}$$

$$\left|\bar\varphi_{J_k}(z^*_{J_k})-\varphi_{J_k}(z^*_{J_k})\right| \le \|\bar g-g\|_{\mathbf{L^\infty}} \tag{4.11}$$

$$\left|\bar\varphi_{I_k}(z^*_{I_k})-\bar g(z^*_{I_k})\right| \le C\,\bar\Delta_{I_k} \tag{4.12}$$

$$\left|\bar\varphi_{J_k}(z^*_{J_k})-\bar g(z^*_{J_k})\right| \le C\,\Delta_{J_k} \tag{4.13}$$

$$\left|\bar\varphi_{I_k}(z^*_{I_k})-\bar g(z^*_{I_k})-\varphi_{I_k}(z^*_{I_k})+g(z^*_{I_k})\right| \le C\,\bar\Delta_{I_k}\,\|\bar g-g\|_{\mathbf{L^\infty}} \tag{4.14}$$

$$\left|\bar\varphi_{J_k}(z^*_{J_k})-\bar g(z^*_{J_k})-\varphi_{J_k}(z^*_{J_k})+g(z^*_{J_k})\right| \le C\,\Delta_{J_k}\,\|\bar g-g\|_{\mathbf{L^\infty}} \tag{4.15}$$

where $\bar\Delta_{I_k}=0$, respectively $\Delta_{J_k}=0$, when there are no u–shock in $\bar z$ beginning at u_{I_k}, respectively in z beginning at u_{J_k}.

Proof. Omitting the dependence on τ, using Proposition 3.11, we compute

$$\begin{aligned}
\int_{\mathbb{R}} \left|\bar g\left(\bar z(u)\right) - g\left(z(u)\right)\right| du &\leq \int_{\mathbb{R}} \left|\bar g\left(\bar z(u)\right) - \bar g\left(z(u)\right)\right| du \\
&\qquad + \int_{\mathbb{R}} \left|\left[\bar g\left(z(u)\right) - g\left(z(u)\right)\right] - \left[\bar g\left(1\right) - g\left(1\right)\right]\right| du \\
&\leq C \int_{\mathbb{R}} \left|\bar z(u) - z(u)\right| du + \left\|\bar g' - g'\right\|_{\mathbf{L}^\infty} \int_{\mathbb{R}} \left|z(u) - 1\right| du \\
&\leq (C + C_* \epsilon) \left(\|\bar z - z\| + \|\bar g' - g'\|\right)
\end{aligned}$$

which proves (4.8).

The inequality (4.9) follows directly from (4.8) and

$$\left|\bar G(\bar z; u) - G(z; u)\right| = \left|e^{\int_u^{+\infty} \bar g(\bar z(v))\,\mathrm{d}v} - e^{\int_u^{+\infty} g(z(v))\,\mathrm{d}v}\right| \leq C\, e^{C_*(\epsilon+\bar\epsilon)} \left\|\bar g \circ \bar z - g \circ z\right\|_{\mathbf{L}^1}.$$

Concerning (4.10), if $z^*_{I_k} > 0$, $\bar\varphi_{I_k}(z^*_{I_k}) = \bar g(z^*_{I_k})$, $\varphi_{I_k}(z^*_{I_k}) = g(z^*_{I_k})$ and the inequality is trivial. If $z^*_{I_k} = 0$,

$$\left|\bar\varphi_{I_k}(z^*_{I_k}) - \varphi_{I_k}(z^*_{I_k})\right| = \left|\frac{1 - e^{-\bar\Delta_{I_k} \bar g(0)}}{\bar\Delta_{I_k}} - \frac{1 - e^{-\bar\Delta_{I_k} g(0)}}{\bar\Delta_{I_k}}\right| \leq \left|\bar g(0) - g(0)\right| \leq \left\|\bar g - g\right\|_{\mathbf{L}^\infty}.$$

The same argument applies to (4.11).

Consider (4.12). If $z^*_{I_k} > 0$ the inequality is trivial. If $z^*_{I_k} = 0$, compute

$$\left|\bar\varphi_{I_k}(z^*_{I_k}) - \bar g(z^*_{I_k})\right| = \left|\frac{1 - e^{-\bar\Delta_{I_k} \bar g(0)}}{\bar\Delta_{I_k}} - \bar g(0)\right| \leq C\, \bar\Delta_{I_k},$$

since $t \mapsto (1 - e^{-t\bar g(0)})/t$ is Lipschitz continuous for $t \geq 0$. This argument applies to (4.13), too.

Consider finally (4.14). If $z^*_{I_k} > 0$ the inequality is trivial. Let be $z^*_{I_k} = 0$, then compute

$$\begin{aligned}
&\left|\bar\varphi_{I_k}(z^*_{I_k}) - \bar g(z^*_{I_k}) - \varphi_{I_k}(z^*_{I_k}) + g(z^*_{I_k})\right| \\
= &\left|\frac{1 - e^{-\bar\Delta_{I_k} \bar g(0)}}{\bar\Delta_{I_k}} - \bar g(0) - \left[\frac{1 - e^{-\bar\Delta_{I_k} g(0)}}{\bar\Delta_{I_k}} - g(0)\right]\right| \\
= &\left|e^{-\bar\Delta_{I_k} \tilde g} - 1\right| \left|\bar g(0) - g(0)\right| \;\leq\; C\, \bar\Delta_{I_k} \left\|\bar g - g\right\|_{\mathbf{L}^\infty}
\end{aligned}$$

where $\tilde g$ is a suitable point in between $\bar g(0)$ and $g(0)$. The estimate (4.15) is proved analogously. □

Proposition 4.3. Using the notations of Subsection 4.2, if τ is a time at which no interaction occurs and at which no wave front in z crosses wave fronts in $\bar z$, the following estimate holds:

$$\frac{d}{dt}\left(\|\bar z(t) - z(t)\|_{\mathbf{L}^1}\right)_{|t=\tau} \leq C \exp\left(C e^{C_*(\epsilon+\bar\epsilon)}\right)\left(\|\bar g - g\|_{\mathbf{W}^{1,\infty}} + \|\bar z(\tau) - z(\tau)\|_{\mathbf{L}^1}\right) + C_*(\epsilon+\bar\epsilon).$$

Proof. By (4.7),

$$\frac{d}{dt}\left(\|\bar z(t) - z(t)\|_{\mathbf{L^1}}\right)\Big|_{t=\tau} \leq 2\sum_{k=1}^{m} E_k + C_*(\epsilon + \bar\epsilon)\,,$$

where

$$E_k = \bar\varphi_{I_k}(z^*_{I_k})\,\bar G(\bar z; u_{I_k}) - \varphi_{I_k}(z^*_{I_k})\,G(z; u_{I_k}) + \varphi_{J_k}(z^*_{J_k})\,G(z; u_{J_k}) - \bar\varphi_{J_k}(z^*_{J_k})\,\bar G(\bar z; u_{J_k}).$$

For notational convenience we now omit the dependence on τ. Observe that

$$\bar G(\bar z; u_{I_k}) = \bar G(\bar z; u_{J_k})\, e^{\int_{u_{I_k}}^{u_{J_k}} \bar g(\bar z(v))\,\mathrm{d}v}\,,\ G(z; u_{I_k}) = G(z; u_{J_k})\, e^{\int_{u_{I_k}}^{u_{J_k}} g(z(v))\,\mathrm{d}v}\,,$$

hence E_k can be split in the following way

$$\begin{aligned}
E_k &= E^1_k + E^2_k + E^3_k + E^4_k + E^5_k, \qquad \text{with}\\
E^1_k &= \bar G(\bar z; u_{J_k})\left[\bar\varphi_{I_k}(z^*_{I_k}) - \varphi_{I_k}(z^*_{I_k})\right]\left[e^{\int_{u_{I_k}}^{u_{J_k}} \bar g(\bar z(v))\,\mathrm{d}v} - 1\right]\\
E^2_k &= \varphi_{I_k}(z^*_{I_k})\left[\bar G(\bar z; u_{J_k}) - G(z; u_{J_k})\right]\left[e^{\int_{u_{I_k}}^{u_{J_k}} \bar g(\bar z(v))\,\mathrm{d}v} - 1\right]\\
E^3_k &= \varphi_{I_k}(z^*_{I_k})\,G(z; u_{J_k})\left[e^{\int_{u_{I_k}}^{u_{J_k}} \bar g(\bar z(v))\,\mathrm{d}v} - e^{\int_{u_{I_k}}^{u_{J_k}} g(z(v))\,\mathrm{d}v}\right]\\
E^4_k &= \left[\bar\varphi_{I_k}(z^*_{I_k}) - \bar\varphi_{J_k}(z^*_{J_k})\right]\ \left[\bar G(\bar z; u_{J_k}) - G(z; u_{J_k})\right]\\
E^5_k &= G(z; u_{J_k})\left[\bar\varphi_{I_k}(z^*_{I_k}) - \bar\varphi_{J_k}(z^*_{J_k}) - \varphi_{I_k}(z^*_{I_k}) + \varphi_{J_k}(z^*_{J_k})\right]\,.
\end{aligned}$$

We consider the five terms separately. Using (4.10) and the uniform bounds we get

$$\begin{aligned}
\sum_{k=1}^{m} E^1_k &\leq C\,e^{C_*(\epsilon+\bar\epsilon)}\,\|\bar g - g\|_{\mathbf{L^\infty}} \sum_{k=1}^{m}\int_{u_{I_k}}^{u_{J_k}} \left|\bar g\left(\bar z(v)\right)\right|\,\mathrm{d}v\\
&\leq C\,e^{C_*(\epsilon+\bar\epsilon)}\,\|\bar g - g\|_{\mathbf{L^\infty}}\|\bar g\circ\bar z\|_{\mathbf{L^1}}\\
&\leq C\,e^{C_*(\epsilon+\bar\epsilon)}\,\|\bar g - g\|_{\mathbf{L^\infty}}\,\|\bar z - 1\|_{\mathbf{L^1}} \ \leq\ C\,e^{C_*(\epsilon+\bar\epsilon)}\,\|\bar g - g\|_{\mathbf{L^\infty}}\,.
\end{aligned}$$

Using (4.9) we compute

$$\begin{aligned}
\sum_{k=1}^{m} E^2_k &\leq C\,e^{C_*(\epsilon+\bar\epsilon)}\left[\|\bar g' - g'\|_{\mathbf{L^\infty}} + \|\bar z - z\|_{\mathbf{L^1}}\right]\sum_{k=1}^{m}\int_{u_{I_k}}^{u_{J_k}} \left|\bar g\left(\bar z(v)\right)\right|\mathrm{d}v\\
&\leq C\,e^{C_*(\epsilon+\bar\epsilon)}\left[\|\bar g' - g'\|_{\mathbf{L^\infty}} + \|\bar z - z\|_{\mathbf{L^1}}\right].
\end{aligned}$$

Concerning the third term, we apply (4.8) to obtain

$$\begin{aligned}
\sum_{k=1}^{m} E^3_k &\leq \sum_{k=1}^{m}\left|\int_{u_{I_k}}^{u_{J_k}} \left[\bar g\left(\bar z(v)\right) - g\left(z(v)\right)\right]\mathrm{d}v\right|\\
&\leq C\,e^{C_*(\epsilon+\bar\epsilon)}\sum_{k=1}^{m}\int_{u_{I_k}}^{u_{J_k}} \left|\bar g\left(\bar z(v)\right) - g\left(z(v)\right)\right|\,\mathrm{d}v\\
&\leq C\,e^{C_*(\epsilon+\bar\epsilon)}\,\|\bar g\circ\bar z - g\circ z\|_{\mathbf{L^1}} \ \leq\ C\,e^{C_*(\epsilon+\bar\epsilon)}\left[\|\bar g' - g'\|_{\mathbf{L^\infty}} + \|\bar z - z\|_{\mathbf{L^1}}\right]\,.
\end{aligned}$$

Now, we apply (4.8), (4.11) and (4.12) to compute

$$\begin{aligned}\sum_{k=1}^{m} E_k^4 &\leq C\, e^{C_*(\epsilon+\bar\epsilon)} \left[\|\bar g' - g'\|_{\mathbf{L}^\infty} + \|\bar z - z\|_{\mathbf{L}^1}\right] \sum_{k=1}^{m} \left[\left|\bar g(z^*_{I_k}) - \bar g(z^*_{J_k})\right| + \bar\Delta_{I_k} + \Delta_{J_k}\right] \\ &\leq C\, e^{C_*(\epsilon+\bar\epsilon)} \left[\|\bar g' - g'\|_{\mathbf{L}^\infty} + \|\bar z - z\|_{\mathbf{L}^1}\right] \sum_{k=1}^{m} \left[\left|z^*_{I_k} - z^*_{J_k}\right| + \bar\Delta_{I_k} + \Delta_{J_k}\right] \\ &\leq C\, \exp\left(C e^{C_*(\epsilon+\bar\epsilon)}\right) \left[\|\bar g' - g'\|_{\mathbf{L}^\infty} + \|\bar z - z\|_{\mathbf{L}^1}\right]\end{aligned}$$

since $\sum_{k=1}^m \bar\Delta_{I_k} \leq \|\bar z - 1\|_{\mathbf{L}^1}$, $\sum_{k=1}^m \Delta_{J_k} \leq \|z - 1\|_{\mathbf{L}^1}$ and

$$\left|z^*_{I_k} - z^*_{J_k}\right| \leq \mathrm{TV}\left\{\bar z, [u_{I_k}, u_{J_k}]\right\} + \mathrm{TV}\left\{z, [u_{I_k}, u_{J_k}]\right\} .$$

Finally, we use (4.14) and (4.15) to get

$$\begin{aligned}\sum_{k=1}^{m} E_k^5 &\leq C\, e^{C_*(\epsilon+\bar\epsilon)} \left(\sum_{k=1}^{m} \left|\left[\bar g(z^*_{I_k}) - g(z^*_{I_k})\right] - \left[\bar g(z^*_{J_k}) - g(z^*_{J_k})\right]\right| \right. \\ &\qquad \left. + \|\bar g - g\|_{\mathbf{L}^\infty} \sum_{k=1}^{m} (\bar\Delta_{I_k} + \Delta_{J_k})\right) \\ &\leq C\, e^{C_*(\epsilon+\bar\epsilon)} \left[\|\bar g' - g'\|_{\mathbf{L}^\infty} \sum_{k=1}^{m} \left|z^*_{I_k} - z^*_{J_k}\right| + \|\bar g - g\|_{\mathbf{L}^\infty}\right] \\ &\leq C\, \exp\left(C e^{C_*(\epsilon+\bar\epsilon)}\right) \left[\|\bar g' - g'\|_{\mathbf{L}^\infty} + \|\bar g - g\|_{\mathbf{L}^\infty}\right]\end{aligned}$$

concluding the proof. □

4.4. *The Final Limit.*

THEOREM 4.4. Using the notations introduced in Subsection 4.2, the following estimate holds:

$$\begin{aligned}\int_{\mathbb{R}} \left|\bar z(t,u) - z(t,u)\right| du &\leq \exp\left(C\, \exp\left(C e^{C_*(\epsilon+\bar\epsilon)}\right) t\right) \int_{\mathbb{R}} \left|\bar z(0,u) - z(0,u)\right| du \\ &\quad + \left(\exp\left(C\, \exp\left(C e^{C_*(\epsilon+\bar\epsilon)}\right) t\right) - 1\right) \left(\|\bar g - g\|_{\mathbf{W}^{1,\infty}} + C_*\,(\epsilon + \bar\epsilon)\right) .\end{aligned}$$

Proof. Define $a(t) = \int_{\mathbb{R}} \left|\bar z(t,u) - z(t,u)\right| du$. Then, using Proposition 4.3, for all but a finite number of times,

$$\frac{d}{dt} a(t) = \frac{d^+}{dt} a(t) \leq C\, \exp\left(C\, e^{C_*(\epsilon+\bar\epsilon)}\right) \left(\|\bar g - g\|_{W^{1,\infty}} + a(t)\right) + C_*\,(\epsilon + \bar\epsilon) .$$

Since a is Lipschitz continuous, the above differential inequality leads to complete the proof. □

Proof of Theorem 2.1. Suppose first that $g \in \mathbf{C^2}([0,+\infty))$. Take a sequence $z^\nu(t,u)$ of approximate solutions to (1.11) with approximation parameters $\epsilon_\nu \to 0$, such that $z_o^\nu - 1 \to z_o - 1$ in $\mathbf{L^1}$. Then, using Theorem 4.4, we can compare any two members of this sequence. Since in this case $\bar g = g$ we have for any $t \in [0,T]$

$$\|z^\nu(t,\cdot) - z^\mu(t,\cdot)\|_{\mathbf{L}^1} \leq \exp\left(C\, \exp\left(C e^{C_*(\epsilon+\bar\epsilon)}\right) t\right) \|z_o^\nu - z_o^\mu\| + C_*\,(\epsilon_\nu + \epsilon_\mu) \to 0 ,$$

as $\nu, \mu \to +\infty$. Therefore $z^\nu(t,\cdot)$ is a Cauchy sequence and has a unique limit $z(t,\cdot)$. Theorem 4.4 also ensures that this limit is independent from the particular sequence, as long as $z_o^\nu - 1$ converges to $z_o - 1$ in $\mathbf{L^1}$. It is then possible to define $S_t^g z = z(t,\cdot)$.

By Proposition 3.14, S^g is Lipschitz continuous in time and its Lipschitz constant is independent from $\|g''\|_{\mathbf{L^\infty}}$. The Lipschitz continuous dependence of S from the erosion function g and from the initial datum z_o immediately follows from Theorem 4.4. Observe that, in the limit, the constant C_* that depends on $\|g''\|_{\mathbf{L^\infty}}$ disappear. The proof of point 3. in Theorem 2.1 is thus completed in the case $g \in \mathbf{C^2}([0,+\infty))$.

Note that the approximate solutions do not satisfy the semigroup property 1. in Theorem 2.1. However, this feature is gained in the limit. Indeed, call $(t,u) \to z^\epsilon(t,u;z_o)$ an approximate solution constructed with approximation parameter ϵ at time $t=0$ and initial data z_o. Then, by construction, $z^\epsilon(t+s,u:z_o) = z^{\epsilon(s)}(t,u;z^\epsilon(s))$. Therefore,

$$\begin{aligned}\left\|S_{t+s}^g z_o - S_t^g S_s^g z_o\right\|_{\mathbf{L^1}} &\le \left\|S_{t+s}^g z_o - z^\epsilon(t+s,\cdot:z_o)\right\|_{\mathbf{L^1}} \\ &\quad + \left\|z^{\epsilon(s)}(t,\cdot;z^\epsilon(s,\cdot;z_o)) - S_t^g z^\epsilon(s,\cdot;z_o)\right\|_{\mathbf{L^1}} + \left\|S_t^g z^\epsilon(s,\cdot;z_o) - S_t^g S_s^g z_o\right\|_{\mathbf{L^1}}.\end{aligned}$$

The former summand vanishes as $\epsilon \to 0$ by the above construction. The latter term also, by the Lipschitz continuity of S_t^g. To show that also the second term vanishes in the limit, we use Theorem 4.4 with $\bar\epsilon = 0$ and $\epsilon = \epsilon(s)$, obtaining

$$\left\|z^{\epsilon(s)}(t,\cdot;z^\epsilon(s,\cdot;z_o)) - S_t^g z^\epsilon(s,\cdot;z_o)\right\|_{\mathbf{L^1}} \le \left(\exp\left(C\exp\left(Ce^{C_*\epsilon(s)}\right)t\right) - 1\right)\epsilon(s)$$

which also vanishes as $\epsilon \to 0$, proving 1. in Theorem 2.1.

Consider now point 2. Fix a sequence z^ν of converging approximate solutions with approximation parameter ϵ^ν. Define

$$\begin{aligned}X^\nu(t,u) &= u + \int_{-\infty}^u \left(z^\nu(t,v) - 1\right) \mathrm{d}v \\ U^\nu(t,x) &= \max\left\{v \in \mathbb{R} \colon X^\nu(t,v) \le x\right\}.\end{aligned} \tag{4.16}$$

Let $\phi \in \mathbf{C_c^\infty}((0,T)\times\mathbb{R}^2;\mathbb{R})$ and $M > 0$ such that the support of ϕ is contained in $(0,T)\times(0,M)$, and define

$$I^\nu = \int_0^T \int_{-R}^R \left(U^\nu(t,x)\,\phi_t(t,x) - \exp\left(\int_{U^\nu(t,x)}^{+\infty} g\left(z^\nu(t,v)\right)\mathrm{d}v\right)\phi_x(t,x)\right)\mathrm{d}x\,\mathrm{d}t. \tag{4.17}$$

By the divergence theorem, we get $I^\nu = I_1^\nu + I_2^\nu$ with

$$I_1^\nu = \int_0^T \int_{-R}^R \left(-U_t^\nu(t,x) + \left[\exp\left(\int_{U^\nu(t,x)}^{+\infty} g\left(z^\nu(t,v)\right)\mathrm{d}v\right)\right]_x\right)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{4.18}$$

$$I_2^\nu = \int_0^T \sum_{i=1}^{n(t)} \left[\Lambda_i(t)\Delta U^\nu\left(t,\gamma_i(t)\right) + \Delta\exp\left[\int_{U^\nu(t,\gamma_i(t))}^{+\infty} g\left(z^\nu(t,v)\right)\mathrm{d}v\right]\right]\phi\left(t,\gamma_i(t)\right)\mathrm{d}t \tag{4.19}$$

Consider a single term in the sum above and, for a fixed time t, set $u^\pm = U^\nu\left(t,\gamma(t)\pm\right)$. Then,

$$\Delta\exp\left(\int_{U^\nu(t,\gamma_i(t))}^{+\infty} g\left(z^\nu(t,v)\right)\mathrm{d}v\right) = G(z^\nu;u^+) - G(z^\nu;u^-)$$

Moreover, for any $u \in (u^-, u^+)$ and using (3.5), (3.7),

$$\Lambda_i(t) = X_t^\nu(t,u) = \partial_t \int_{-\infty}^{u} (z^\nu(t,v) - 1)\,\mathrm{d}v = -\frac{G(z^\nu; u^+) - G(z^\nu; u^-)}{\Delta U^\nu}$$

proving that the integral in (4.19) vanishes: $I_2^\nu = 0$.

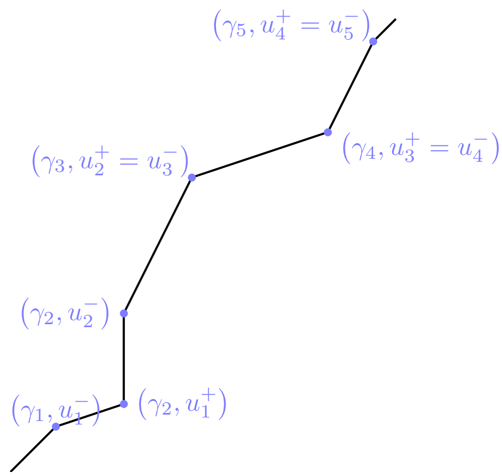


FIG. 11. Graph of $U^\nu(t,\cdot)$.

To estimate the integral (4.18), let $\gamma_i(t)$ $i = 1, \ldots, N(t)$ be the curves along which $U^\nu(t,x)$ has a kink or a jump discontinuity (see Figure 11). Define $u_i^- = U^\nu(t, \gamma_i(t)+)$, $u_i^+ = U^\nu(t, \gamma_{i+1}(t)-)$. If $\gamma_i(t)$ is a discontinuity point for $U^\nu(t,x)$ then $u_{i-1}^+ < u_i^-$. Between $\gamma_i(t)$ and $\gamma_{i+1}(t)$, U^ν is linear, therefore if $x \in (\gamma_i(t), \gamma_{i+1}(t))$, then

$$\begin{aligned} U^\nu(t,x) &\in \left(u_i^-, u_i^+\right); \\ U_x^\nu(t,x) &= \frac{1}{z^\nu(t, U^\nu(t,x))} = \frac{1}{z_i} \text{ with } z_i > 0; \\ U_t^\nu(t,x) &= \frac{-X_t^\nu(t, U^\nu(t,x))}{X_u^\nu(t, U^\nu(t,x))}. = -\frac{g(z_i)\widehat{G}(z^\nu(t,\cdot); U^\nu(t,x))}{z_i} \end{aligned}$$

where we used (3.5), (3.7) and (4.16) to compute

$$\begin{aligned} X_u^\nu(t,u) &= z^\nu(t,u) \\ X_t^\nu(t,u) &= \partial_t \int_{-\infty}^{u} (z^\nu(t,v) - 1)\, dv = g(z_i)\widehat{G}(z^\nu(t,\cdot), u). \end{aligned}$$

Concerning the term with the exponential,

$$\begin{aligned} \left[\exp\left(\int_{U^\nu(t,x)}^{+\infty} g(z^\nu(t,v))\,\mathrm{d}v\right)\right]_x &= \left[G(z^\nu(t,\cdot); U^\nu(t,x))\right]_x \\ &= -g(z^\nu(t, U^\nu(t,x)))\, G(z^\nu(t,\cdot); U^\nu(t,x))\, U_x^\nu(t,x) \\ &= -\frac{g(z_i)\, G(z^\nu(t,\cdot); U^\nu(t,x))}{z_i}. \end{aligned}$$

We write now the integral (4.18):

$$I_1^\nu = \int_0^T \sum_{i=1}^{N(t)-1} \int_{\gamma_i}^{\gamma_{i+1}} \left(-U^\nu(t,x) + \left[G\left(z^\nu(t,\cdot);U^\nu(t,x)\right)\right]_x\right)\phi(t,x)\;\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T \sum_{i=1}^{N(t)-1} \int_{\gamma_i}^{\gamma_{i+1}} \left(\frac{g(z_i)}{z_i}\left[\widehat{G}\left(z^\nu(t,\cdot);U^\nu(t,x)\right) - G\left(z^\nu(t,\cdot);U^\nu(t,x)\right)\right]\right)\phi(t,x)\;\mathrm{d}x\,\mathrm{d}t\,,$$

hence

$$\begin{aligned} |I_1^\nu| &\leq \|\phi\|_{\mathbf{L}^\infty} \int_0^T \sum_{i=1}^{N(t)-1} \int_{\gamma_i}^{\gamma_{i+1}} \frac{|g(z_i)|}{z_i}(u_i^+ - u_i^-)^2 \sup_{u\in[u_i^-,u_i^+]} \left|G_{uu}\left(z^\nu(t,\cdot);u\right)\right|\;\mathrm{d}x\,\mathrm{d}t \\ &\leq \|\phi\|_{\mathbf{L}^\infty}\left\|G\left(z^\nu(t,\cdot)\right)\right\|_{\mathbf{L}^\infty} \int_0^T \sum_{i=1}^{N(t)-1} \frac{\gamma_{i+1}-\gamma_i}{z_i}|g(z_i)|^3(u_i^+ - u_i^-)^2\;\mathrm{d}t \end{aligned}$$

since $G_{uu}\left(z^\nu(t,\cdot);u\right) = g(z_i)^2 G\left(z^\nu(t,\cdot);u\right)$.

If we finally observe that $\frac{\gamma_{i+1}-\gamma_i}{z_i} = u_i^+ - u_i^-$ and that $|g(z_i)|(u_i^+ - u_i^-) \leq C\epsilon^\nu$, we get

$$\begin{aligned} |I_1^\nu| &\leq C(\epsilon^\nu)^2|\phi|_{\mathbf{L}^\infty}\left\|G\left(z^\nu(t,\cdot)\right)\right\|_{\mathbf{L}^\infty} \int_0^T \sum_{i=1}^{N(t)-1} ||gz_i)|(u_i^+ - u_i^-)\;\mathrm{d}t \\ &\leq C(\epsilon^\nu)^2|\phi|_{\mathbf{L}^\infty}\left\|G\left(z^\nu(t,\cdot)\right)\right\|_{\mathbf{L}^\infty} \int_0^T \int_{\mathbb{R}} \left|g\left(z^\nu(t,u)\right)\right|\;\mathrm{d}u\,\mathrm{d}t \\ &\leq C(\epsilon^\nu)^2|\phi|_{\mathbf{L}^\infty}\left\|G\left(z^\nu(t,\cdot)\right)\right\|_{\mathbf{L}^\infty}\left|z^\nu(t,\cdot)-1\right|_{L^1} \to 0 \text{ as } \nu\to+\infty\,. \end{aligned}$$

Therefore, $I^\nu \to 0$ as $\nu \to +\infty$. Moreover, $(z^\nu(t,\cdot)-1) \to (z(t,\cdot)-1)$ in $\mathbf{L^1}$ implies that $G\left(z^\nu(t,\cdot);\cdot\right) \to G\left(z(t,\cdot);\cdot\right)$ and $X^\nu(t,\cdot) \to X(t,\cdot)$ uniformly. Therefore, the corresponding inverse functions U^ν are such $U^\nu(t,x)$ converges to $U(t,x)$ at any x which is a continuity point for $U(t,\cdot)$. Hence, $U^\nu(t,\cdot) \to U(t,\cdot)$ pointwise everywhere outside a countable set. Therefore U^ν, respectively $G\left(z^\nu(t,\cdot);U^\nu\right)$, converges pointwise almost everywhere on the plane (t,x) to U, respectively and $G\left(z(t,\cdot);U\right)$. By the dominated convergence theorem we can pass to the limit in the integral (4.17) and obtain

$$\int_0^T \int_{\mathbb{R}} \left(U(t,x)\,\phi_t(t,x) - \exp\left(\int_{U(t,x)}^{+\infty} g\left(z(t,v)\right)\mathrm{d}v\right)\phi_x(t,x)\right)\mathrm{d}x\,\mathrm{d}t = 0.$$

Suppose now that g satisfies **(g)**. Let p_ν be a sequence of functions in $\mathbf{C^0}\left([0,+\infty);\mathbb{R}\right)$ such that $p_\nu \leq \sup g''$ and $p_\nu \to g''$ in $\mathbf{L^1}\left([0,+\infty);\mathbb{R}\right)$. Define

$$g_\nu(s) = \int_1^s \left(g'(1) + \int_1^\sigma p_\nu(\sigma')\,\mathrm{d}\sigma'\right)\mathrm{d}\sigma\;,$$

so that

$$\begin{aligned} & g_\nu \text{ satisfies } \mathbf{(g)} \\ & g_\nu \in \mathbf{C^2}\left([0,+\infty);\mathbb{R}\right) \\ & g_\nu \to g \text{ in } \mathbf{W^{1,\infty}}\left([0,+\infty);\mathbb{R}\right)\,. \end{aligned}$$

and Theorem 2.1 applies to each g_ν, yielding a semigroup S^{g_ν}. Moreover, the Lipschitz constant L of S^{g_ν} is independent from ν, so that

$$\left\|S_t^{g_\nu} z_o - S_t^{g_\mu} z_o\right\|_{\mathbf{L^1}} \leq L\,t\,\|g_\nu - g_\mu\|_{\mathbf{W^{1,\infty}}}$$

and a straightforward limiting procedure allows to complete the proof of 3.

Concerning 2. observe that the approximating sequence $U^\nu(t,x)$ are exact solutions to the equation

$$\int_0^T \int_{\mathbb{R}} \left(U^\nu(t,x)\,\phi_t(t,x) - \exp\left(\int_{U^\nu(t,x)}^{+\infty} g_\nu\left(z^\nu(t,v)\right) \mathrm{d}v \right) \phi_x(t,x) \right) \mathrm{d}x\,\mathrm{d}t = 0.$$

Since $z^\nu(t) = S_t^{g_\nu} z_o \to z(t) = S_t^g z_o$ in $\mathbf{L^1}$, as before we can pass to the limit inside the integral and conclude the proof of the theorem. □

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