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An Integro-Differential Conservation Law Arising in a Model of Granular Flow

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We study a scalar integro-differential conservation law. The equation was first derived in [2] as the slow erosion limit of granular flow. Considering a set of more general erosion functions, we study the initial boundary value problem for which one can not adapt the standard theory of conservation laws. We construct approximate solutions with a fractional step method, by recomputing the integral term at each time step. A-priori $\mathbf{L}^\infty$ bounds and BV estimates yield convergence and global existence of BV solutions. Furthermore, we present a well-posedness analysis, showing that the solutions are stable in $\mathbf{L}^1$ with respect to the initial data.

1. Introduction

We consider the initial boundary value problem for the scalar integro-differential equation

$$q_t + \left(\exp\left\{ \int_x^0 f(q(t,\xi))\, d\xi \right\} f(q) \right)_x = 0\,, \qquad t \geq 0\,, \quad x \leq 0\,, \tag{1.1}$$

with initial condition

$$q(0,x) = \bar{q}(x)\,, \qquad\qquad x \leq 0\,. \tag{1.2}$$

Note that the flux includes a non-local integral term. For notational convenience, we introduce

$$K(q(t,\cdot))(x) \doteq \exp\left\{ \int_x^0 f(q(t,\xi))\, d\xi \right\}\,. \tag{1.3}$$

The function $f : (-1,+\infty) \to \mathbb{R} \in C^2(\mathbb{R})$ is called the *erosion function*. The following assumptions apply to f:

$$f(0) = 0\,, \quad f' > 0\,, \quad f'' < 0\,, \quad \lim_{q\to -1} f(q) = -\infty\,, \quad \lim_{q\to+\infty} \frac{f(q)}{q} = 0\,. \tag{1.4}$$

We remark that the characteristic speed of (1.1) is

$$\dot{x} = f'(q)K\,.$$

By (1.3) and (1.4), the characteristic speed is always positive, therefore no boundary condition is assigned at $x = 0$ for (1.1).

The equation (1.1) arises as the *slow erosion limit* in a model of granular flow, studied in [2], with a specific erosion function

$$f(q) = \frac{q}{q+1}\,. \tag{1.5}$$

Note that this function satisfies all the assumptions in (1.4). In more details, let h be the height of the moving layer, and p be the slope of the standing profile. Assuming $p > 0$, the following 2×2 system of balance laws was proposed in [15]

$$\begin{cases} h_t - (hp)_x &= (p-1)h\,, \\ p_t + \big((p-1)h\big)_x &= 0\,. \end{cases} \tag{1.6}$$

This model describes the following phenomenon. The material is divided in two parts: a moving layer with height h on top and a standing layer with slope $p > 0$ at the bottom. The moving layer slides downhill with speed p. If the slope $p = 1$ (the critical slope), the moving layer passes through without interaction with the standing layer. If the slope $p > 1$, then grains initially at rest are hit by rolling grains of the moving layer and start moving as well. Hence the moving layer gets bigger. On the other hand, if $p < 1$, grains which are rolling can be deposited on the bed. Hence the moving layer becomes smaller.

In the slow erosion limit as $\|h\|_{\mathbf{L}^\infty} \to 0$, we proved in [2] that the solution for the slope p in (1.6) provides the weak solution of the following scalar integro-differential equation

$$p_\mu + \left(\frac{p-1}{p} \cdot \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\right)_x = 0\,.$$

Here, the new time variable μ accounts for the total mass of granular material being poured downhill. Introducing $q \doteq p - 1$ and writing t for μ, we obtain the equation (1.1) with (1.5).

The result in [2] provides the existence of entropy weak solutions to the initial boundary value problem (1.1) with f given in (1.5) for finite "time" (which is actually finite total mass). However, well-posedness property was left open due to the technical difficulties caused by the non-local term in the flux. Furthermore, due to the discontinuities in q, the function $k(t,x) = K(q(t,\cdot))(x)$ is only Lipschitz continuous in its variables, therefore one can not apply directly previous results. Indeed, classical results as [19] require more smoothness on the coefficients; see also [9]. Some closer results can be found in [17,20] where the coefficient $k = k(x)$ does not depend on time.

In this paper we consider a class of more general erosion functions f that satisfy the assumptions in (1.4), and we study existence and well-posedness of BV solutions of (1.1). Assuming that the slope is always positive, i.e., $q > -1$, we seek BV solutions with bounded total mass. Therefore, we define $\mathcal{D} = \mathcal{D}_{C_0,\kappa_0}$ as the set of functions that satisfy

$$\mathcal{D}_{C_0,\kappa_0} \doteq \left\{ q(x) : \inf_{x<0} q(x) \geq \kappa_0 > -1\,,\ \mathrm{TV}\,\{q\} \leq C_0\,,\ \|q\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq C_0 \right\}. \quad (1.7)$$

Assume that the initial data satisfies $\bar{q} \in \mathcal{D}_{C_0,\kappa_0}$ for some constants $C_0 > 0, \kappa_0 > -1$. A natural definition of entropy weak solution is given below.

Definition 1.1. *Let $T > 0$. A function q is an **entropy weak solution** to* (1.1) *on $[0,T] \times \mathbb{R}_-$ with initial condition* (1.2)*, if the following holds.*

- ***(H1)*** *$q : [0,T] \to \mathbf{L}^1(\mathbb{R}_-) \cap BV(\mathbb{R}_-)$, $\inf_x q(t,x) > -1$, and the map $[0,T] \ni t \mapsto q(t)$ is Lipschitz in $\mathbf{L}^1(\mathbb{R}_-)$;*
- ***(H2)*** *q is a weak solution of the scalar conservation law*

$$\begin{cases} q_t + \left(k(t,x)\,f(q)\right)_x = 0\,, \\ q(0,x) = \bar{q}(x) \end{cases} \quad (1.8)$$

with k defined by

$$k(t,x) = K(q(t,\cdot))(x) = \exp\left\{ \int_x^0 f(q(t,\xi))\,d\xi \right\} \quad (1.9)$$

and satisfies, for all $\alpha \in \mathbb{R}$

$$\partial_t|q-\alpha| + \partial_x\left[k(x,t)|f(q)-f(\alpha)|\right] + \mathrm{sign}(q-\alpha)k_x(x,t)f(\alpha) \leq 0 \quad (1.10)$$

in the sense of distributions.

Notice that, thanks to **(H1)**, the coefficient $k(t,x)$ in (1.9) is Lipschitz continuous on $[0,T] \times \mathbb{R}_-$. Other properties of k are summarized in Proposition Appendix A.1 in the Appendix.

Now we state the main result of this paper.

Theorem 1.2. *Assume* (1.4) *and let $C_0 > 0$, $\kappa_0 > -1$ be given constants. Then for any initial data $\bar{q} \in \mathcal{D}_{C_0,\kappa_0}$ there exists an entropy weak solution $q(t,x)$ to the initial-boundary value problem* (1.1)–(1.2) *for $t \geq 0$. Moreover, consider two solutions $q_1(t,\cdot)$, $q_2(t,\cdot)$ of the integro-differential equation (1.1), corresponding to the initial data*

$$q_1(0,x) = \bar{q}_1(x)\,, \qquad q_2(0,x) = \bar{q}_2(x)\,, \qquad x < 0\,,$$

with $\bar{q}_1$, $\bar{q}_2 \in \mathcal{D}_{C_0,\kappa_0}$. Then for any $T > 0$ there exists $L = L(T, C_0, \kappa_0) > 0$ such that

$$\|q_1(t,\cdot) - q_2(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq \mathrm{e}^{Lt}\,\|\bar{q}_1 - \bar{q}_2\|_{\mathbf{L}^1(\mathbb{R}_-)}\,, \qquad t \in [0,T]\,. \quad (1.11)$$

Recalling that $q = p - 1 = u_x - 1$, the solution q established by Theorem 1.2 allows us to recover the profile u of the standing layer:

$$u(t,x) - x = \int_{-\infty}^{x} q(t,y)\,dy\,. \tag{1.12}$$

Moreover, since $K_x = -Kf(q(t,x))$, the equation (1.1) can be rewritten as

$$q_t - K_{xx} = 0\,.$$

Integrating in space on $(-\infty, x)$, using (1.12) and that $K_x(q(t,\cdot)) \in \mathbf{L}^1(\mathbb{R}_-)$, we arrive at

$$u_t - K_x = u_t + Kf\,(u_x - 1) \;=\; 0\,.$$

This nonlocal Hamilton-Jacobi equation is studied in [21], with a different class of erosion functions f. Assuming more erosion for large slope, i.e.,

$$\lim_{q\to+\infty} f'(q) = \eta_0 > 0\,,$$

the slope u_x of the standing layer would blowup, leading to jumps in the standing profile u. Notice that, in our case, only upward jumps in u_x can occur as singularities, which corresponds to convex kinks in the profile u.

About the continuous dependence notice that, when k is a prescribed coefficient, the $\mathbf{L}^1$ stability estimate (1.11) holds with $L = 0$, see (2.3). On the other hand, for the integral equation (1.1), one cannot expect $L = 0$ in general. Indeed, a small variation in the $\mathbf{L}^1$ norm of the initial data may cause a variation in the global term and then in the overall solution. However, a special case in which (1.11) holds with $L = 0$ is when $q_2 \equiv 0$, which indeed is a solution of (1.1).

Other problems involving a nonlocal term in the flux have been considered in [11,7,8]. Well-known integro-differential equations which lead to blow up of the gradients include the Camassa-Holm equation [6] and the variational wave equation [5]. The Cauchy problem for (1.1) with initial data with bounded support is studied in [3] where we use piecewise constant approximation generated by front tracing and obtain similar results.

The rest of the paper is structured in the following way. As a step toward the final result, in Section 2 we study the existence and well-posedness of the scalar equation (1.8) for a *given* coefficient $k(t,x)$. Here $k(t,x)$ is a local term, and preserves the properties of the global integral term. Such equation does not fall directly within the classical framework of [19], where more regularity on the coefficients is required (C^1). In particular, BV estimates for solutions of (1.8) are needed to obtain the continuous dependence on the initial data, see (2.21). We employ a fractional step argument to deal with the time dependence of k, and then follow an approach similar to [4,1,18,17], where the authors deal with the case of $k = k(x) \in \mathbf{L}^\infty$. See also [14]. We further refer to [9] on total variation estimates for general scalar balance laws: their result, in our context, would require more regularity (C^1) on the coefficient k.

The properties of the integral operator K, defined at (1.3), are summarized in the last Appendix.

2. Well-posedness of solutions with a given coefficient $k(t,x)$

In this section we study the well-posedness of the scalar equation (1.8) for a *given* coefficient $k(t,x)$, by reviewing some related results and completing the arguments where needed.

Throughout this section, we will use u as the unknown variable. Consider

$$u_t + \Big(k(t,x)f(u)\Big)_x = 0\,, \qquad x \le 0, \quad t \ge 0 \tag{2.1}$$

$$u(0,x) = \bar{u}(x)\,, \qquad x < 0 \tag{2.2}$$

where $k(t,x)$ satisfies the following assumptions, for some $T > 0$:

$$\textbf{(K)} \quad \begin{cases} k(t,x) \in \mathbf{L}^\infty\left([0,T]\times\mathbb{R}_-\right)\,, \text{ it is Lipschitz continuous and } \inf_{t,x} k > 0\,; \\ \mathrm{TV}\left\{k(t,\cdot)\right\}\,,\ \mathrm{TV}\left\{k_x(t,\cdot)\right\} \text{ are bounded uniformly in time}\,; \\ [0,T] \ni t \to k_x(t,\cdot) \in \mathbf{L}^1(\mathbb{R}_-) \text{ is Lipschitz continuous}\,. \end{cases}$$

The above assumptions on k are motivated by the properties of the integral operator K, see Proposition Appendix A.1 in the Appendix.

Theorem 2.1. *Assume f satisfies* (1.4) *and $k(t,x)$ satisfies* ***(K)****. Let $C_0 > 0$, $\kappa_0 > -1$ be given constants. Then there exist two positive constants C_1 and κ_1, with possibly $C_1 \ge C_0$ and $-1 < \kappa_1 \le \kappa_0$, and a unique operator $P : [0,T]\times\mathcal{D}_{C_0,\kappa_0} \to \mathcal{D}_{C_1,\kappa_1}$ such that:*

1) *the function $u(t,x) = P_t(\bar{u})$ is an entropy weak solution of* (2.1) *with initial data $u(0,\cdot) = \bar{u} \in \mathcal{D}_{C_0,\kappa_0}$;*
2) *for any $\bar{u}_1$, $\bar{u}_2 \in \mathcal{D}_{C_0,\kappa_0}$ one has*

$$\|P_t(\bar{u}_1) - P_t(\bar{u}_2)\|_{\mathbf{L}^1(\mathbb{R}_-)} \le \|\bar{u}_1 - \bar{u}_2\|_{\mathbf{L}^1(\mathbb{R}_-)}\,. \tag{2.3}$$

Proof. The proof relies on introducing a small time parameter Δt and by freezing the coefficient k at the times $t_n = n\Delta t$, that will therefore depend only on x in each time interval (t_n, t_{n+1}). Then, estimates available for the case of $k = k(x)$ will lead to uniform bounds on $[0,T]$, that will allow us to pass to the limit in $\Delta t \to 0$.

Let $\bar{u} \in \mathcal{D}_{C_0,\kappa_0}$. We introduce the parameter $\Delta t > 0$ and define $t_n = n\Delta t$ for any integer $n \ge 0$. We approximate the coefficient k by

$$k_{\Delta t}(t,x) = \sum_{n\ge 0} \chi_{[t_n,t_{n+1})}(t)\, k(t_n,x)\,, \qquad (t,x) \in [0,T]\times\mathbb{R}_-\,, \tag{2.4}$$

which is constant in time on each interval $[t_n, t_{n+1})$, and consider the equation

$$u_t + \Big(k_{\Delta t}(t,x)f(u)\Big)_x = 0\,, \qquad u(0,\cdot) = \bar{u}\,. \tag{2.5}$$

The case of k independent of time is analyzed in Subsection A.2 of the Appendix. By applying that analysis to each interval $[t_n, t_{n+1})$, a unique entropy solution for (2.5) $u_{\Delta t}$ is defined, provided that $u_{\Delta t}$ is bounded from both below and above on all $[0,T]\times\mathbb{R}_-$.

We now establish the lower and upper bounds for $u_{\Delta t}$. For notation simplicity, in the following we denote by $k(t,x)$ and $u(t,x)$ the approximate coefficient and solution respectively, without causing confusion. We define the constants k_0, L, L_1 such that, recalling **(K)**, one has:

$$k_0 = \inf_{t,x} k > 0\,; \tag{2.6}$$

$$|k(t_1,x_1) - k(t_2,x_2)| \le L\left(|t_1-t_2| + |x_1-x_2|\right) \quad \forall (t_1,x_1),(t_2,x_2) \tag{2.7}$$

$$\mathrm{TV}\,\{k(t_1,\cdot) - k(t_2,\cdot)\} = \|k_x(t_1,\cdot) - k_x(t_2,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \le L_1|t_1-t_2| \tag{2.8}$$

and set $L_2 = L/k_0$.

We first give some formal arguments. The evolution of the complete flux $F = kf(u)$ along the characteristic $x(t)$ with $\dot{x} = f'(u)k$ follows the equation

$$\frac{d}{dt}F(t,x(t)) = (kf)_t + f'k(kf)_x = k_t f = \frac{k_t}{k}F\,. \tag{2.9}$$

By our assumptions (**K**), the term k_t/k is uniformly bounded. Therefore, $|F|$ grows at most at an exponential rate, and remains bounded for finite time $t \le T$. Therefore $|f(u)|$ remains bounded as well. By the 4^{th} assumption in (1.4), u never reaches -1 in finite time, leading to a lower bound on u.

The same argument leads to an upper bound for $f(u)$, if $f(u) \to +\infty$ as $u \to +\infty$. However, if $f(u) \to f_0 > 0$ as $u \to +\infty$, we need a different argument. We observe that, along a characteristic $x(t)$, one formally has

$$\dot{u} = -k_x(t,x)f(u)\,.$$

By the lower bound on u and the uniform bound on k_x, the growth of u remains uniformly bounded, yielding an upper bound.

We now make these arguments rigorous for the approximate solutions. At $t=0$ one has

$$|k(0,x)f(\bar{u}(x))| \;\le\; C_1 \tag{2.10}$$

for some $C_1 \ge 0$ that depends on the bounds for k and $\bar{u}$. We claim that, as long as the approximate solution exists, we have

$$|k(t,x)f(u(t,x))| \;\le\; C_1 \mathrm{e}^{L_2 t}\,. \tag{2.11}$$

Indeed, by (2.10) and (A.9), the inequality (2.11) is valid on $[0,t_1)$. Assume now that (2.11) is valid on $[0,t_{n+1})$, $n \ge 0$, i.e.,

$$|F(t,x)| = |k(t_n,x)f(u(t_n,x))| \le C_1\,\mathrm{e}^{L_2 t_n}\,, \qquad t \in [t_n,t_{n+1})\,. \tag{2.12}$$

At time $t = t_{n+1}$ one has

$$\begin{aligned}
|k(t_{n+1},x)f(u(t_{n+1},x))| &= \frac{k(t_{n+1},x)}{k(t_n,x)}\;|k(t_n,x)f(u(t_{n+1},x))| \\
&\le \left(1 + \frac{L}{k_0}\Delta t\right)\cdot \sup_x |k(t_n,x)f(u(t_{n+1},x))| \le \mathrm{e}^{L_2\Delta t}\cdot C_1\,\mathrm{e}^{L_2 t_n} = C_1\,\mathrm{e}^{L_2 t_{n+1}}\,.
\end{aligned}$$

By induction, this proves (2.11), which in turn gives the lower bound κ_1 for u. The upper bound also follows if $f(u) \to +\infty$ as $u \to +\infty$.

Finally, we consider the case that $f(u) \to f_0 > 0$ as $u \to +\infty$. Following the analysis in Subsection A.2, at any given point $(\bar{t}, \bar{x})$ one can trace back along an extremal backward generalized characteristic $x(t)$, which is absolutely continuous on each (t_n, t_{n+1}) and continuous up to $t = 0$. Since now the r.h.s. of (A.13) is bounded, then u grows at a linear rate, and therefore remains bounded.

We remark that the lower bound on u yields an a-priori bound on the wave speed. Indeed, since f' is a decreasing function, the characteristic speed is bounded:

$$\lambda = kf'(u) \leq \|k\|_\infty \, f'(\kappa_1) \; .$$

Bound on total variation. We estimate the total variation of $F(t,x) = k(t,x)f(u(t,x))$. On the interval (t_n, t_{n+1}) the coefficient k is constant in time and we use (A.10). On the other hand, the total variation might increase at t_n when k is updated. Then we observe that

$$F(t_n, x) = \left[1 + \frac{k(t_n,x) - k(t_{n-1},x)}{k(t_{n-1},x)}\right] F(t_n-, x)\,, \tag{2.13}$$

therefore

$$\begin{aligned} \mathrm{TV}\left\{F(t_n,\cdot)\right\} \leq & \left(1 + \frac{\|k(t_n,\cdot) - k(t_{n-1},\cdot)\|_\infty}{\inf k(t_{n-1},\cdot)}\right) \mathrm{TV}\left\{F(t_n-,\cdot)\right\} \\ & + \sup|F| \cdot \mathrm{TV}\left\{\frac{k(t_n,\cdot) - k(t_{n-1},\cdot)}{k(t_{n-1},\cdot)}\right\} . \end{aligned} \tag{2.14}$$

Thanks to (2.6)–(2.8), we have

$$\frac{\|k(t_n,\cdot) - k(t_{n-1},\cdot)\|_\infty}{\inf k(t_{n-1},\cdot)} \leq L_2 \Delta t\,, \qquad \mathrm{TV}\left\{\frac{k(t_n,\cdot) - k(t_{n-1},\cdot)}{k(t_{n-1},\cdot)}\right\} \leq L_3 \, \Delta t\,,$$

for a suitable constant L_3 independent on Δt. Moreover $F = kf$ is uniformly bounded thanks to **(K)** and the bounds on u. Hence we conclude that

$$\mathrm{TV}\left\{F(t_n,\cdot)\right\} \leq (1 + L_2\Delta t)\; \mathrm{TV}\left\{F(t_{n-1},\cdot)\right\} + L_4 \Delta t$$

for a suitable $L_4 > 0$. By induction it follows that

$$\mathrm{TV}\left\{F(t,\cdot)\right\} \leq \mathrm{e}^{L_2 t} \mathrm{TV}\left\{F(0+,\cdot)\right\} + \frac{L_4}{L_2}\left(\mathrm{e}^{L_2 t} - 1\right) .$$

Recalling that $f(u) = F/k$, one obtains the BV bound for $f(u(t))$,

$$(\inf f')\, \mathrm{TV}\left\{u(t,\cdot)\right\} \leq \mathrm{TV}\left\{f(u(t,\cdot))\right\} \leq \frac{1}{\inf k} \mathrm{TV}\left\{F(t,\cdot)\right\} + \frac{\|F\|_\infty}{(\inf k)^2} \mathrm{TV}\left\{k(t,\cdot)\right\} .$$

This gives a bound on the total variation for $u(t)$:

$$\mathrm{TV}\left\{u(t)\right\} \leq C \left[\mathrm{TV}\left\{F(t,\cdot)\right\} + \mathrm{TV}\left\{k(t,\cdot)\right\}\right] \leq C_1(t) \tag{2.15}$$

where the constant C depends on $\inf_x u$, $\sup_x u$, $\inf_x k$, $\sup_x k$. Hence the total variation of u may increase in time but it remains bounded as long as u remains bounded.

Taking the limit $\Delta t \to 0$, the coefficient $k_{\Delta t}$ converges uniformly to k. Correspondingly, the family $u_{\Delta t}$ converges (up to a subsequence) to a weak solution u of the original equation, satisfying the same upper and lower bounds and (2.15).

Finally, in the limit as $\Delta t \to 0$, the Kružkov entropy inequalities (1.10) for equation (2.1), with $q = u$ and for all $\alpha \in \mathbb{R}$, hold in the sense of distributions.

Indeed, fix $\alpha \in \mathbb{R}$ and a sequence $\eta = \eta^\varepsilon(u)$ of convex, smooth approximations to $\bar{\eta}(u) = |u - \alpha|$ (choose as in [1, p.257]). The corresponding entropy flux is

$$q = q^\varepsilon(k, u) = k \int_\alpha^u \eta'(u) f'(v)\, dv\,.$$

In place of (A.11) on each (t_n, t_{n+1}), we consider the inequality

$$\partial_t\, \eta(u_{\Delta t}) + \partial_x\, q(k_{\Delta t}, u_{\Delta t}) + \eta'(u_{\Delta t})\, [k_{\Delta t}]_x\, (x) f(\alpha) \leq 0\,, \tag{2.16}$$

which holds in the sense of distributions. The term $(k_{\Delta t})_x$ converges strongly in $\mathbf{L}^1$ to k_x, as $\Delta t \to 0$:

$$\begin{aligned}\iint_{\mathbb{R}_- \times [0,T]} |(k_{\Delta t})_x - k_x|\; dxdt &= \sum_n \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}_-} |k_x(t,x) - k_x(t_n, x)|\; dxdt \\ &\leq L_1 \sum_n \int_{t_n}^{t_{n+1}} (t - t_n) dt \;=\; L_1 \frac{\Delta t}{2} T \to 0\,.\end{aligned}$$

Hence, thanks to the L^1_{loc} convergence of $u_{\Delta t}$ to u, the inequality (2.16) is valid in the limit:

$$\partial_t\, \eta^\varepsilon(u) + \partial_x\, q^\varepsilon(k, u) + (\eta^\varepsilon)'(u) k_x(x) f(\alpha) \leq 0\,. \tag{2.17}$$

The last step consists in passing to the limit in $\varepsilon \to 0$. By setting $\mathrm{sign}(v) = 0$ if $v = 0$, and since

$$\eta^\varepsilon(u) \to |u - \alpha|\,, \qquad (\eta^\varepsilon)'(u) \to \mathrm{sign}(u - \alpha)\,, \qquad q^\varepsilon(k, u) \to k|f(u) - f(\alpha)|$$

pointwise, by dominated convergence theorem we obtain the desired inequality (1.10). Uniqueness and $\mathbf{L}^1$-contraction follow by extending the argument in [1, Theorem 7.1, p.261] to the case $a(x,t) = k(t,x)$, being Lipschitz continuous also in time (with $g = 0 = \gamma$). □

Next we establish the continuous dependence on the coefficient function. We rely on a result in [17] (Corollary 3.2) that applies to Cauchy problems and to the case of $k = k(x)$, that is, the coefficient does not depend on time.

For convenience of the reader we report that statement of [17] adapted to our situation. Consider the two equations

$$u_t + \Big(k f(u)\Big)_x = 0\,, \qquad t \geq 0\,, \tag{2.18}$$

$$u_t + \Big(\tilde{k} f(u)\Big)_x = 0\,, \qquad t \geq 0\,. \tag{2.19}$$

Proposition 2.2. *For $x \in \mathbb{R}$, let $k(x)$, $\tilde{k}(x) \in BV(\mathbb{R})$ satisfy*

$$k_x\,,\; \tilde{k}_x \in BV(\mathbb{R})\,; \qquad \inf k\,,\; \inf \tilde{k} \geq \alpha > 0$$

for some positive α. Consider the initial data u_0, $\tilde u_0 \in BV(\mathbb{R})$ for the two equations (2.18), (2.19) *respectively and let $u(t,x)$, $\tilde u(t,x)$ be the corresponding solutions, assuming that they are bounded from above and bounded away from -1. Let C_1 be a bound on $|f|$ over the range of the solutions. Then*

$$\begin{aligned}\|u(t,\cdot)-\tilde u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} &\le \|u_0-\tilde u_0\|_{\mathbf{L}^1(\mathbb{R})} \\ &+ t\left\{C_1 \mathrm{TV}\,\{k-\tilde k\} + C_2\left(1+\mathrm{TV}\,u_0+\mathrm{TV}\,\tilde u_0\right)\|k-\tilde k\|_\infty\right\} \qquad (2.20)\end{aligned}$$

where C_2 depends on the bounds on u, k, $\mathrm{TV}\,\{k\}$ and on $\tilde u$, $\tilde k$, $\mathrm{TV}\,\{\tilde k\}$.

Comparing this statement with the one of Corollary 3.2 in [17], one assumption is missing. Indeed in [17] the authors assume $f(0) = f(b) = 0$, for some $b > 0$ and that the initial data take value in $[0, b]$. This assumption is then used to obtain an invariant region for the solution, i.e., $0 \le u \le b$, which is essential for their proof. For our equation, the upper and lower bounds were achieved by other means in the proof of Theorem 2.1.

The continuous dependence property for our problem follows from Proposition 2.2, by properly extending the IBVP into Cauchy problems.

Theorem 2.3. *For $x < 0$, let $k(t,x)$, $\tilde k(t,x)$ satisfy the assumption **(K)**, and assume that the initial data $\bar u$ belongs to $\mathcal{D}_{C_0,\kappa_0}$ (defined at* (1.7)*). Let $u(t,\cdot)$, $\tilde u(t,\cdot)$ be the solutions of the conservation laws* (2.18), (2.19) *respectively, with the same initial data $\bar u$, for some time interval $[0,T]$ ($T > 0$).*

Then, the following estimate holds

$$\begin{aligned}\frac{1}{t}\|u(t,\cdot)-\tilde u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} &\le \widehat{C}_1 \sup_{t\in[0,T]} \mathrm{TV}\,\left\{k(t,\cdot)-\tilde k(t,\cdot)\right\} \\ &+ \widehat{C}_2\left(1+\sup_\tau \mathrm{TV}\,u(\tau,\cdot)+\sup_\tau \mathrm{TV}\,\tilde u(\tau,\cdot)\right)\|k-\tilde k\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}_-)}\,, \quad (2.21)\end{aligned}$$

where $\widehat{C}_1$ is a bound on $|f|$ over the range of the solutions and $\widehat{C}_2$ depends on the bounds on the solutions, the coefficients and their total variation $\mathrm{TV}\,\{k(t,\cdot\}$, $\mathrm{TV}\,\{\tilde k(t,\cdot\}$.

Proof. The IBVP (2.1)–(2.2) can be extended to the following Cauchy problem

$$u_t + \Big(k(t,x)f(u)\Big)_x = 0\,, \qquad x\in\mathbb{R}, \quad t\ge 0\,, \qquad (2.22)$$

with extended initial data

$$u(0,x) = \begin{cases}\bar u(x) & \text{for } x \le 0\,,\\ \bar u(0-) & \text{for } x > 0\end{cases} \qquad (2.23)$$

and the extended coefficient function $k(t,x)$

$$k(t,x) = \lim_{y\to 0-} k(t,y) \qquad \text{for } x > 0\,.$$

Due to the fact that the characteristic speed is positive, the solution for the Cauchy problem (2.22)–(2.23) restricted on $x \leq 0$ will match the solution for the IBVP (2.1).

In a same way, the IBVP (2.19) is extended to the Cauchy problem for

$$u_t + \left(\tilde{k}(t,x) f(u)\right)_x = 0\,, \qquad x \in \mathbb{R}\,, \quad t \geq 0 \tag{2.24}$$

with data (2.23). Without causing confusion, let's still denote $u(t,x)$ and $\tilde{u}(t,x)$ the solutions for (2.22) and (2.24), respectively, and let $u_\Delta(t,x)$ and $\tilde{u}_\Delta(t,x)$ be the corresponding approximate solutions, constructed in the same way as in the proof of Theorem 2.1, with approximate coefficients $k_{\Delta t}$ and $\tilde{k}_{\Delta t}$ as in (2.4).

Denote the distance between these two solutions by

$$e_\Delta(t) \doteq \|u_\Delta(t,\cdot) - \tilde{u}_\Delta(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \,.$$

Notice that $e_\Delta(0) = 0$ and that $e_\Delta(t) \geq \|u_\Delta(t,\cdot) - \tilde{u}_\Delta(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)}$.

On each time interval $[t_n, t_{n+1})$ the coefficient is constant in time and the assumptions of Proposition 2.2 are satisfied. Hence, from (2.20), we have the following estimate

$$\begin{aligned} e_\Delta(t_{n+1}) - e_\Delta(t_n) \;\leq\; & \Delta t\, \widehat{C}_1 \mathrm{TV}_{\mathbb{R}} \left\{ k_{\Delta t}(t_n,\cdot) - \tilde{k}_{\Delta t}(t_n,\cdot) \right\} \\ & + \Delta t\, \widehat{C}_2 \left(1 + \mathrm{TV}_{\mathbb{R}_-} u(t_n,\cdot) + \mathrm{TV}_{\mathbb{R}_-} \tilde{u}(t_n,\cdot)\right) \left\| k_{\Delta t}(t_n,\cdot) - \tilde{k}_{\Delta t}(t_n,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R})} \end{aligned} \tag{2.25}$$

for some constants $\widehat{C}_1$ and $\widehat{C}_2$ that are uniform on $[0,T]$. Notice that, in the above lines, $\mathrm{TV}_{\mathbb{R}} \left\{ k_{\Delta t} - \tilde{k}_{\Delta t} \right\}$ coincides with $\mathrm{TV}_{\mathbb{R}_-}$ of the same quantity and, similarly, the $\mathbf{L}^\infty$-norm on $\mathbb{R}$ coincides with the $\mathbf{L}^\infty$-norm on $\mathbb{R}_-$. Concerning $\mathrm{TV}_{\mathbb{R}} u$ (similarly for $\mathrm{TV}_{\mathbb{R}} \tilde{u}$), we replaced it with $\mathrm{TV}_{\mathbb{R}_-} u$ with an error that is bounded and possibly depending on T.

Summing up (2.25) in n, we get

$$\begin{aligned} & e_\Delta(t_N) - e_\Delta(0) \\ = & \sum_{n=0}^{N-1} e_\Delta(t_{n+1}) - e_\Delta(t_n) \;\leq\; t_N \widehat{C}_1 \sup_{t \in [0,t_N]} \mathrm{TV}_{\mathbb{R}_-} \left\{ k_{\Delta t} - \tilde{k}_{\Delta t} \right\} \\ & + t_N \widehat{C}_2 \left(1 + \sup_t \mathrm{TV}_{\mathbb{R}_-} u(t,\cdot) + \sup_t \mathrm{TV}_{\mathbb{R}_-} \tilde{u}(t,\cdot) \right) \left\| k_{\Delta t} - \tilde{k}_{\Delta t} \right\|_{\mathbf{L}^\infty([0,t_N] \times \mathbb{R}_-)} \,. \end{aligned}$$

Now taking the limit $\Delta t \to 0$, we get (2.21), completing the proof. □

3. Well-posedness of the integro-differential equation

In this section we prove the main Theorem 1.2. In Subsection 3.1 we define a family of approximate solutions to (1.1)–(1.2) and show their compactness, locally in time. Then in Subsection 3.2 we show that the limit solution can be prolonged beyond the existence time, by improving the estimates on upper and lower bound for the exact solution of (1.1)–(1.2). Finally, in Subsection 3.3 we show that the flow generated by the integro-differential equation (1.1) is Lipschitz continuous, restricted to any domain $\mathcal{D}$ given at (1.7).

3.1. *Local in time existence of BV solutions*

In this Subsection we prove the following existence theorem.

Theorem 3.1. *Let C_0, κ_0 be given constants and let $\bar q(x) \in \mathbf{L}^1(\mathbb{R}_-) \cap BV(\mathbb{R}_-)$ such that*

(a) $\inf_{x<0} \bar q(x) \ \geq\ \kappa_0 > -1$ *;*
(b) $\mathrm{TV}\,\{\bar q(\cdot)\} \leq C_0$ *;*
(c) $\|\bar q\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq C_0$ *.*

Then there exist $T > 0$, $\kappa_1 > -1$ and $C_1 > 0$ such that

$$\begin{cases} q_t + \left(\exp\left\{\int_x^0 f(q(t,\xi))\,d\xi\right\}\ f(q)\right)_x = 0\,, \\ q(0,x) = \bar q(x)\,, \end{cases} \tag{3.1}$$

admits an entropy weak solution $q(t,x)$ on $[0,T] \times \mathbb{R}_-$ that satisfies

(a)' $\inf_{x<0} q(t,x) \ \geq\ \kappa_1 > -1$ *;*
(b)' $\mathrm{TV}\,\{q(t,\cdot)\} \leq C_1$ *;*
(c)' $\|q(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq \|\bar q\|_{\mathbf{L}^1(\mathbb{R}_-)}$ *.*

Proof. We define a sequence of approximate solution to the scalar equation (1.1)–(1.3). We fix $\Delta t > 0$ and set $t_n = n\Delta t$, $n \in \mathbb{N}$. The approximation is generated recursively, as n starts from 0 and increases by 1 after each step. For each step with $n \geq 0$, let $q(t,x)$ be defined on $[0,t_n) \times \mathbb{R}_-$ and set

$$k_n(x) \doteq \exp\left\{\int_x^0 f(q(t_n,\xi))\,d\xi\right\}\,.$$

Then we define q on $[t_n,t_{n+1}) \times \mathbb{R}_-$ as the solution of the problem

$$\begin{cases} q_t + (k_n(x)\,f(q))_x = 0\,, & t \in [t_n,t_{n+1}) \\ q(t_n,x) = q(t_n-,x)\,. \end{cases}$$

This procedure leads to a solution operator $t \mapsto S_t^{\Delta t}\bar q = q^{\Delta t}(t,\cdot)$, defined up to a certain time $T = T(\Delta t,\bar q) > 0$, of the problem

$$\begin{cases} q_t + \left(k^{\Delta t}(t,x)\,f(q)\right)_x = 0\,, & t > 0 \\ q(0,x) = \bar q(x)\,, \end{cases} \tag{3.2}$$

where $k = k^{\Delta t}$ is defined by

$$k^{\Delta t}(t,x) = \sum_{n\geq 0} \chi_{[t_n,t_{n+1})}(t) \cdot k_n(x)\,. \tag{3.3}$$

Notice that the operator $S_t^{\Delta t}$ has the semigroup property $S_{\tau_1+\tau_2}^{\Delta t} = S_{\tau_1}^{\Delta t} S_{\tau_2}^{\Delta t}$ for τ_1, $\tau_2 \in (\Delta t)\mathbb{N}$. Now we prove uniform bounds, independent of Δt, on the family of approximate solutions.

The L^1 bound. This follows by the application of (2.3) in Theorem 2.1, at each time step $[t_n, t_{n+1})$, and the fact that $t \mapsto q(t,\cdot)$ is continuous in $\mathbf{L}^1$. Until the solution is defined, we have

$$\|q(t,\cdot)\|_{\mathbf{L}^1} \le \|q(0,\cdot)\|_{\mathbf{L}^1} . \tag{3.4}$$

Lower and upper bound on q. Define

$$z(t) \;=\; \inf_x \, q(t,x)\,, \qquad w(t) \;=\; \sup_x \, q(t,x)\,.$$

We observe that, by comparison with the equilibrium solution $u \equiv 0$, (i) if $z(0) \ge 0$ then $z(t) \ge 0$; and (ii) if $w(0) \le 0$ then $w(t) \le 0$ for all $t > 0$.

Now consider $-1 < z(0) < 0$ and $w(t) > 0$. Choose δ and M such that $z(0) \ge -1+2\delta$ and $w(0) \le M/2$. For example, one can take $\delta = (\kappa_0+1)/2$ and $M = 2w(0)$. Let $T = T(\delta, M) > 0$ be the first time that one of the following bounds fails,

$$z(t) \ge -1+\delta\,, \qquad w(t) \le M\,. \tag{3.5}$$

Then, for $t \le T$, from the analysis of equation (3.2) (see (A.13)), we find that z and w are continuous and satisfy

$$z(t) \ge z(0) + \sup_x \left|k_x^{\Delta t}(t,x)\right| \int_0^t f(z(\tau))\,d\tau\,, \qquad z<0\,, \tag{3.6}$$

$$w(t) \le w(0) + \sup_x \left|k_x^{\Delta t}(t,x)\right| \int_0^t f(w(\tau))\,d\tau\,, \qquad w>0\,. \tag{3.7}$$

Note that in (3.6) we have $f(z) \le 0$, and in (3.7) we have $f(w) \ge 0$. For $\left|k_x^{\Delta t}\right|$, we have the estimate

$$\begin{aligned}\left|k_x^{\Delta t}(t,x)\right| = \left|k^{\Delta t}(x) f(q(t_n,x))\right| &\le \exp\left\{\int_x^0 |f(q(t_n,\xi))|\,d\xi\right\} f(M)\\ &\le f(M)\exp\{f'(-1+\delta)\,\|\bar q\|_{\mathbf{L}^1}\} \le C(\delta,M)\,.\end{aligned}$$

This gives us

$$z(t) \ge z(0) + C(\delta,M)\int_0^t f(z(\tau))\,d\tau \ge z(0) - C(\delta,M)t\,|f'(-1+\delta)|\,,$$

$$w(t) \le w(0) + C(\delta,M)\int_0^t f(w(\tau))\,d\tau \le w(0) + C(\delta,M)tf(M)\,.$$

We conclude that the bounds in (3.5) hold for $t \le T$ with

$$T(\delta,M) = \min\{T_1, T_2\}\,,$$

where

$$T_1(\delta,M) = \frac{\delta}{C(\delta,M)\,|f'(-1+\delta)|}\,, \qquad T_2(\delta,M) = \frac{M/2}{C(\delta,M)f(M)}\,,$$

yielding the lower and upper bounds.

Finally, if $z(0) \geq 0$ and $w(0) > 0$, or if $z(0) < 0$ and $w(0) \leq 0$, then we would only need to establish one of the bounds in (3.5), and the result follows.

Bounds on f, f', k. Once we have a lower, upper bound on q and the bound on $\|q\|_{\mathbf{L}^1}$, we immediately find that

$$f(q(t,x)), \quad f'(q(t,x)), \quad \int_x^0 f(q(t,\xi))\, d\xi \quad \in \quad \mathbf{L}^\infty([0,T] \times \mathbb{R}_-) \tag{3.8}$$

uniformly w.r.t. Δt. By definition (3.3) of k, we can easily verify that the following properties hold uniformly w.r.t. Δt:

(i) $k \in \mathbf{L}^\infty([0,T] \times \mathbb{R}_-)$, $\inf_{t,x} k > 0$;
(ii) $k_x \in \mathbf{L}^\infty([0,T] \times \mathbb{R}_-)$;
(iii) $\mathrm{TV}\, k(t,\cdot)$ is bounded uniformly in time.

Indeed, (i) follows from the definition of k and (3.8). About (ii), at each time t we have $k(t,\cdot) = k_n(\cdot)$ for some n, and $k_x = -k_n f(q(t_n,\cdot))$. Then $k_x \in \mathbf{L}^\infty$ because of (i) and (3.8). Finally

$$\mathrm{TV}\, k(t,\cdot) = \|k_x\|_{\mathbf{L}^1} = \|k_n f(q(t_n,\cdot))\|_{\mathbf{L}^1} \leq M\, \|k\|_\infty\, \|q(t_n,\cdot)\|_{\mathbf{L}^1} \leq M\, \|k\|_\infty \|\bar{q}\|_{\mathbf{L}^1}$$

where $M = \sup f'$, that depends on the lower bound on q.

Lastly, from (i) and (3.8) one obtains a uniform bound on the characteristic speed $kf'(q)$.

Bound on the total variation of q. By definition of the total variation

$$\mathrm{TV}\,\{q(t,\cdot)\} \doteq \lim_{h\to 0+} \frac{1}{h} \int_{-\infty}^0 |q(t,x) - q(t,x-h)|\, dx\,,$$

we have, for any $h > 0$

$$\frac{1}{h} \int_{-\infty}^0 |q(t,x) - q(t,x-h)|\, dx \leq \mathrm{TV}\,\{q(t,\cdot)\}\,. \tag{3.9}$$

The total variation of q does not change at time t_n when k is updated. Now consider a time interval $t \in [t_n, t_{n+1})$, and we estimate the change of the total variation of q in this time interval. We have

$$\begin{aligned} &\int_{-\infty}^0 |q(t_{n+1},x) - q(t_{n+1},x-h)|\, dx \\ &\leq \int_{-\infty}^0 |q(t_n,x) - q(t_n,x-h)|\, dx + \int_{t_n}^{t_{n+1}} \mathcal{E}(\tau)\, d\tau \end{aligned} \tag{3.10}$$

where

$$\mathcal{E}(\tau) = \limsup_{\theta\to 0+} \frac{\int_{-\infty}^0 |q(\tau+\theta, x-h) - \hat{q}(\tau+\theta,x)|\, dx}{\theta}\,.$$

Here $\hat{q}$ is the entropy solution to

$$\begin{cases} u_t + (k_n(x)f(u))_x = 0\,, & t \geq \tau\,,\ x < 0 \\ u(\tau, x) = q(\tau, x-h)\,. \end{cases}$$

On the other hand, $q(t, x-h)$ is a solution of

$$\begin{cases} u_t + (k_n(x-h)f(u))_x = 0\,, & t \geq \tau\,,\ x < 0 \\ u(\tau, x) = q(\tau, x-h)\,. \end{cases}$$

Using the estimate (2.20) we find

$$\mathcal{E}(\tau) \leq \|f\|_\infty \mathrm{TV}\left\{k_n(\cdot - h) - k_n(\cdot)\right\} + C\left(1 + \mathrm{TV}\left\{q(\tau, \cdot)\right\}\right)\|k_n(\cdot - h) - k_n(\cdot)\|_\infty$$

for a suitable constant C. Notice that

$$|k_n(x-h) - k_n(x)| = \left|\int_{x-h}^{x} (k_n)_x(\tau, y)\,dy\right| \leq h\|k_n f\|_\infty$$

and that

$$\begin{aligned} &\mathrm{TV}\left\{k_n(\cdot - h) - k_n(\cdot)\right\} = \|(k_n)_x(\cdot - h) - (k_n)_x(\cdot)\|_{\mathbf{L}^1} \\ &\leq \|(k_n(\cdot - h) - k_n(\cdot))\, f\left(q(\tau, \cdot)\right)\|_{\mathbf{L}^1} + \|k_n(\cdot - h)\cdot\left(f\left(q(\tau, \cdot - h)\right) - f\left(q(\tau, \cdot)\right)\right)\|_{\mathbf{L}^1} \\ &\leq h\|k_n f\|_\infty \cdot \|f\left(q(\tau, \cdot)\right)\|_{\mathbf{L}^1} + \|k_n\|_{\mathbf{L}^\infty}\|f'\|_\infty\ \|q(\tau, \cdot) - q(\tau, \cdot - h)\|_{\mathbf{L}^1}\ . \end{aligned}$$

In conclusion, using also (3.9), we obtain

$$\begin{aligned} \mathcal{E}(\tau) &\leq h\left\{M_1 + M_2 \mathrm{TV}\left\{q(\tau, \cdot)\right\} + M_3 \frac{1}{h}\|q(\tau, \cdot) - q(\tau, \cdot - h)\|_{\mathbf{L}^1}\right\} \\ &\leq h\left\{M_1 + (M_2 + M_3)\,\mathrm{TV}\left\{q(\tau, \cdot)\right\}\right\} \end{aligned}$$

where M_i depend only on a-priori bounded quantities. Now from (3.10) we obtain

$$\mathrm{TV}\left\{q(t_{n+1}, \cdot)\right\} \leq \mathrm{TV}\left\{q(t_n, \cdot)\right\} + \int_{t_n}^{t_{n+1}} \Big[M_1 + (M_2 + M_3)\,\mathrm{TV}\left\{q(\tau, \cdot)\right\}\Big]\,d\tau\,. \quad (3.11)$$

We conclude that the total variation of q may grow exponentially in t on each interval (t_n, t_{n+1}), but it remains bounded for any bounded time t.

Convergence to weak solutions; Existence of BV solutions. Now, without causing confusion, we will use $q^\Delta(t, x)$ for the approximate solution, where $\Delta = \Delta t$ is the step size. Let k^Δ be the approximated coefficient of the equation, defined in (3.3).

By compactness, there exists a subsequence of $\{q^\Delta(t, x)\}$, as $\Delta \to 0$, that converges to a limit function $q(t, x)$ in $\mathbf{L}^1_{loc}$. Let $k(t, x)$ be the integral term, (1.9), corresponding to q, which is uniformly bounded as well as the k^Δ. We have

$$\begin{aligned} &k^\Delta(t, x) - k(t, x) \\ &= \mathcal{O}(1)\left\{\int_x^0 f(q^\Delta(t_n, \xi))\,d\xi - \int_x^0 f(q(t, \xi))\,d\xi\right\} \\ &= \mathcal{O}(1)\left\{\sup_\tau \mathrm{TV}\left\{f(q^\Delta(\tau, \cdot))\right\}\sup \dot{x}\,\Delta + \int_x^0 \left[f(q^\Delta(t, \xi)) - f(q(t, \xi))\right]d\xi\right\} \end{aligned}$$

that vanishes as $\Delta \to 0$. Therefore we can pass to the limit in the weak formulation. On the interval $[t_n, t_{n+1}]$, $q^\Delta(t,x)$ satisfies

$$\int_{t_n}^{t_{n+1}} \int_{-\infty}^{0} (q^\Delta \phi_t + k^\Delta f(q^\Delta)\phi_x)\, dx\, dt = \int_{-\infty}^{0} \left[q^\Delta \phi(t_{n+1}, x) - q^\Delta \phi(t_n, x)\right]\, dx$$

for some test function ϕ with compact support inside $[0,T] \times \mathbb{R}_-$. Summing this up over n, we get

$$\int_0^T \int_{-\infty}^{0} (q^\Delta \phi_t + k^\Delta f(q^\Delta)\phi_x)\, dx\, dt = \int_{-\infty}^{0} \left[q^\Delta \phi(T, x) - q^\Delta \phi(0, x)\right]\, dx\,. \tag{3.12}$$

Since

$$q^\Delta \to q \text{ in } \mathbf{L}^1_{loc}\,, \quad f(q^\Delta) \to f(q) \text{ in } \mathbf{L}^1_{loc}\,, \quad k^\Delta \to k \text{ pointwise}$$

and k^Δ, k are uniformly bounded, by dominated convergence we can take the limit as $\Delta \to 0$ and have the convergence of (3.12) to

$$\begin{aligned} \int_0^T \int_{-\infty}^{0} & \left[q(t,x)\phi_t(t,x) + k(t,x) f(q(t,x))\phi_x(t,x)\right]\, dx\, dt \\ &= \int_{-\infty}^{0} \left[q\phi(T,x) - q\phi(0,x)\right]\, dx\,. \end{aligned}$$

The "entropy" part of the statement follows as in the proof of Theorem 2.1.

This completes the proof of existence of BV solutions for (1.1). □

3.2. *Global existence of BV solutions*

Once the BV solutions exist locally in time, we can further show that they enjoy better properties than the ones deduced from the approximate solutions. In particular we show that the lower and upper bounds on q can be taken independently of time t, leading to global in time existence of BV solutions.

Let q be an entropy weak solution of (3.1) on $[0,T] \times \mathbb{R}_-$. We will now improve the lower and upper bounds obtained in Theorem 3.1; to do this, we again employ the tool of generalized characteristics introduced [10].

The theory presented in [10] applies to the scalar conservation law

$$u_t + [f(u,x,t)]_x \;=\; 0\,,$$

with f of class C^2 in all variables, and f strictly convex in u. Here, for equation (3.1), we still have the convexity property of f (being concave down), while the dependence on x, t in the flux is only Lipschitz continuous. However, denoting k the integral term as in (1.9) and recalling that $k_x = -kf(u)$, the discontinuities of k_x propagate along the discontinuities of u. Hence sufficient regularity is still available along genuine characteristics.

More precisely, one can define the notion of generalized characteristic as in [10, Def. 3.1], whose existence follows from [13]. Since $f'' \neq 0$, the conclusion of

Theorem 3.2 in [10] is still valid: for any $(\bar{t}, \bar{x}) \in (0,T) \times \mathbb{R}_-$, the minimal and maximal backward characteristics issued at $(\bar{t}, \bar{x})$ are genuine.

To validate the characteristic system

$$\begin{cases} x'(t) = k(t,x) f'(q(t)) \,, \\ q'(t) = -k_x(t, x(t)) f(q) = k f(q)^2 \geq 0 \end{cases}$$

along genuine characteristics, one follows the proof of Theorem 3.3 in [10] to obtain the following.

Let $t \to \xi(t)$, $t \in [a,b]$ be a genuine characteristic and let $J \subset [a,b]$ such that

$$q(t, \xi(t)-) = q(t, \xi(t)+) \qquad \text{for } t \in J \,, \tag{3.13}$$

with full Lebesgue measure on $[a,b]$ $(m_1(J) = b-a)$. Then there exists a measurable function $v(t)$ defined on J such that

$$q(t, \xi(t)-) = v(t) = q(t, \xi(t)+) \qquad \text{for } t \in J$$

and

$$v(\tau) = v(\sigma) - \int_\sigma^\tau k_x(t, \xi(t)) f(v(t)) \, dt$$

for $\sigma, \tau \in J$. To reach this, one applies suitably the Green theorem to equation $(3.1)_1$ and uses the fact that, for $t \in J$, the integral term

$$x \mapsto \int_x^0 f(q(t, \xi)) \, d\xi$$

is continuously differentiable at $x = \xi(t)$. In this case one has

$$k_x(t, \xi(t)) = -k(t, \xi(t)) f(q(t, \xi(t)))$$

and hence

$$v(\tau) = v(\sigma) + \int_\sigma^\tau k(t, \xi(t)) f^2(v(t)) \, dt \,.$$

Thus v can be extended to an absolutely continuous function on $[a,b]$, which indeed is C^1; then also $t \to \xi(t)$ is continuously differentiable and the system

$$\begin{cases} \xi'(t) = k(t,x) f'(v(t)) \,, \\ v'(t) = -k_x(t, \xi(t)) f(v) = k f^2(v) \end{cases}$$

is satisfied in the classical sense on (a,b).

Lower bound on q. Given any point $(\bar{t}, \bar{x}) \in (0,T) \times \mathbb{R}_-$, let $t \to x(t)$ be the minimal backward characteristic, defined for $t \in [0, \bar{t}]$. By setting $q(t) = q(t, x(t))$, we have

$$\begin{cases} x'(t) = k(t,x) f'(q(t)) \,, & x(\bar{t}) = \bar{x} \,, \\ q'(t) = -k_x(t, x(t)) f(q) = k f^2(q) \geq 0 \,, & q(\bar{t}) = q(\bar{t}, \bar{x}-) \,. \end{cases} \tag{3.14}$$

We see that the solution q is non-decreasing along any characteristics. Therefore, we have $\inf_x q(t,x) \geq \inf \bar{q}(x) \geq \kappa_0 > -1$ for all $t \geq 0$.

Upper bound on q. Again, consider a point $(\bar{t}, \bar{x})$ and let $t \to x(t)$ be the minimal backward characteristic through it. From the second equation in (3.14) we see that if $q(0, t(0)) \le 0$, then $q \to 0$ as $t \to +\infty$. Now consider $q(0, x(0)) > 0$, and we have $q(t, x(t)) \ge 0$ for all t. Define

$$W(t,x) = \int_{-\infty}^{x} |q(t,y)|\, dy\,, \qquad x < 0\,, \tag{3.15}$$

that satisfies

$$0 \le W(t,x) \le \|q(0,\cdot)\|_{L^1(\mathbb{R}_-)}\,.$$

Using (1.10) with $\alpha = 0$, we have

$$W_t = \int_{-\infty}^{x} |q(t,y)|_t\, dy \le -\int_{-\infty}^{x} \Big(k(t,x)\,|f(q)|\Big)_x\, dy = -k|f(q)|\,.$$

The variation of W along the characteristic is

$$\begin{aligned} \frac{d}{dt} W(t,x(t)) &= W_t + x' W_x \le -k|f| + |q| k f' = k\left(-|f| + |q| f'(q)\right) \\ &= k\left(-f + q f'(q)\right) = -f^2 k \frac{f - q f'(q)}{f^2} = -\frac{d}{dt}\left(\frac{q(t,x(t))}{f(q(t,x(t)))}\right). \end{aligned} \tag{3.16}$$

Here we remove the absolute value signs because $q > 0$. Then, (3.16) implies that

$$W(t,x(t)) + \frac{q(t,x(t))}{f(q(t,x(t)))} \equiv C$$

along characteristics. This gives the bound

$$\frac{q(t,x(t))}{f(q(t,x(t)))} = \frac{q(0,x(0))}{f(q(0,x(0)))} + W(0,x(0)) - W(t,x(t)) \le C_1\,, \tag{3.17}$$

where C_1 can be chosen independently of $(\bar{t}, \bar{x})$. Recalling (1.4), we have

$$\lim_{q \to +\infty} \frac{q}{f(q)} = +\infty\,.$$

Therefore, (3.17) implies an upper bound for q for all t. The uniform bound on the total variation follows because the constants M_i in (3.11) are now bounded uniformly in time.

3.3. *Continuous dependence on the data for the integro-differential equation*

In this section we prove the last part of Theorem 1.2, showing that the flow generated by the integro-differential equation (1.1) is Lipschitz continuous, restricted to any domain $\mathcal{D} \subset \mathbf{L}^1(\mathbb{R}_-)$ of functions $q(\cdot)$ satisfying the following uniform bounds in (1.7), for some constants C_0, κ_0.

Consider two solutions $q_1(t,\cdot)$, $q_2(t,\cdot)$ of the integro-differential equation (1.1), say with initial data

$$q_1(0,x) = \bar{q}_1(x)\,, \qquad q_2(0,x) = \bar{q}_2(x) \qquad x < 0\,,$$

and satisfying the conditions in (1.7) for $t \in [0,T]$. We are going to prove that

$$\begin{aligned} &\|q_1(t,\cdot) - q_2(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \\ &\le \|\bar q_1 - \bar q_2\|_{\mathbf{L}^1(\mathbb{R}_-)} + L \cdot \int_0^t \|q_1(s,\cdot) - q_2(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)}\, ds\,, \end{aligned} \tag{3.18}$$

for a suitable constant L. By Gronwall lemma, this yields (1.11), hence the Lipschitz continuous dependence of solutions of (1.1) on the initial data.

Define the functions $k_1(t,x)$, $k_2(t,x)$ as in (1.9), corresponding to $q_1(t,x)$, $q_2(t,x)$ respectively. Now set

$$k^\theta(t,x) \doteq \begin{cases} k_1(t,x) & \text{if } \ t \in [0,\theta]\,, \\ k_2(t,x) & \text{if } \ t > \theta\,. \end{cases}$$

Finally, for any given $\theta \in [0,T]$, let $q^\theta = q^\theta(t,x)$ be the solution of the conservation law

$$q_t + \big(k^\theta(t,x)\, f(q)\big)_x = 0\,, \qquad q^\theta(0,x) = \bar q_2(x)\,. \tag{3.19}$$

Observe that, for each fixed θ, the distance between any two entropy-admissible solutions of the conservation law (3.19) is non-increasing in time. In particular, for $\theta = T$, call $\hat q$ the solution of

$$q_t + (k_1(t,x)\, f(q))_x = 0\,,$$

with initial data $\hat q(0,x) = \bar q_2(x)$ (see Figure 1). We have

$$\|q_1(t,\cdot) - \hat q(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \;\le\; \|\bar q_1 - \bar q_2\|_{\mathbf{L}^1(\mathbb{R}_-)} \qquad \forall t \in [0,T]\,. \tag{3.20}$$

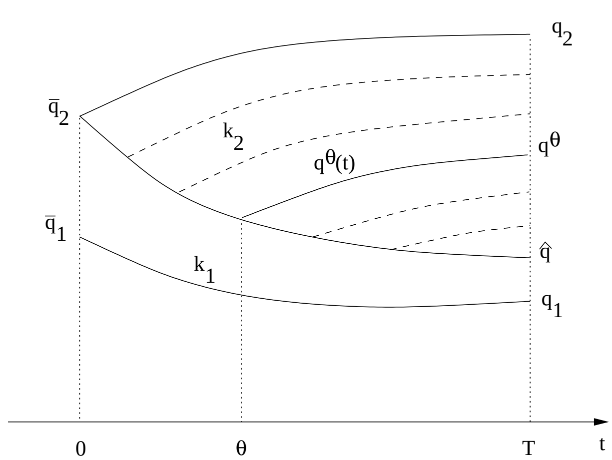

Fig. 1. The flow of solutions $q_1, \hat q, q^\theta, q_2$ for the integro-differential equation.

Moreover we can use the Lipschitz property of the solution operator for (1.1) with $k = k_2$ fixed, and get the distance estimate

$$\|\hat{q}(T,\cdot) - q_2(T,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \;\leq\; \int_0^T E(\tau)\,d\tau\,, \tag{3.21}$$

where

$$E(\tau) \;\doteq\; \limsup_{h\to 0+} \frac{\|q^\tau(\tau+h,\cdot) - \hat{q}(\tau+h,\cdot)\|_{\mathbf{L}^1}}{h}\,.$$

Indeed, observe that $\hat{q}(\tau,\cdot) = q^\theta(\tau,\cdot)$ whenever $\tau \leq \theta$, for any $\tau \in [0,T]$.

To compute the integrand in (3.21), observe that the functions $h \mapsto q^\tau(\tau+h,\cdot)$ and $h \mapsto \hat{q}(\tau+h,\cdot)$ take the same value $\hat{q}(\tau,\cdot)$ when $h = 0$, and $h \mapsto q^\tau(\tau+h,x)$ satisfies the conservation law

$$q_h + \left(k_2(\tau+h,x)\,f(q)\right)_x = 0\,, \tag{3.22}$$

while $h \mapsto \hat{q}(\tau+h,x)$ solves

$$q_h + \left(k_1(\tau+h,x)\,f(q)\right)_x = 0\,, \tag{3.23}$$

for $h \geq 0$. By using (2.21) in Theorem 2.3, we can measure the error term $E(\tau)$. By the facts that $\|q^\tau(\tau,\cdot)\|_{\mathbf{L}^\infty}$, $\|\hat{q}(\tau,\cdot)\|_{\mathbf{L}^\infty}$, $\mathrm{TV}\{q^\tau(\tau,\cdot)\}$, $\mathrm{TV}\{\hat{q}(\tau,\cdot)\}$, $\mathrm{TV}\{k_1(\tau,\cdot)\}$ and $\mathrm{TV}\{k_2(\tau,\cdot)\}$ are all bounded, the coefficients $\widehat{C}_1$ and $\widehat{C}_2$ in (2.21) are all bounded constants. Let M be a generic bounded constant, we get

$$\begin{aligned}&\|q^\tau(\tau+h,\cdot) - \hat{q}(\tau+h,\cdot)\|_{\mathbf{L}^1}\\ &\leq Mh\left[\sup_{\tau\leq t\leq\tau+h} \mathrm{TV}\left(k_1(t,\cdot) - k_2(t,\cdot)\right) + \|k_1 - k_2\|_{\mathbf{L}^\infty([\tau,\tau+h]\times\mathbb{R}_-)}\right].\end{aligned}$$

Therefore, we have

$$E(\tau) = M\cdot \mathrm{TV}\left\{k_1(\tau,\cdot) - k_2(\tau,\cdot)\right\} + M\cdot\|k_1(\tau,\cdot) - k_2(\tau,\cdot)\|_{\mathbf{L}^\infty}\,. \tag{3.24}$$

Recalling the definitions of k_1, k_2 we deduce that

$$\begin{aligned}\|k_1(\tau,\cdot) - k_2(\tau,\cdot)\|_{\mathbf{L}^\infty} &= M\cdot\sup_{x<0}\left|\int_x^0 f\left(q_1(\tau,y)\right)\,dy - \int_x^0 f\left(q_2(\tau,y)\right)\,dy\right|\\ &= M\cdot\|q_1(\tau,\cdot) - q_2(\tau,\cdot)\|_{\mathbf{L}^1}\,,\end{aligned} \tag{3.25}$$

and, using also (3.25),

$$\begin{aligned}&\mathrm{TV}\left\{k_1(\tau,\cdot) - k_2(\tau,\cdot)\right\} \;=\; \|(k_1)_x(\tau,\cdot) - (k_2)_x(\tau,\cdot)\|_{\mathbf{L}^1}\\ &= \|k_1(\tau,\cdot)\,f\left(q_1(\tau,\cdot)\right) - k_2(\tau,\cdot)\,f\left(q_2(\tau,\cdot)\right)\|_{\mathbf{L}^1}\\ &\leq \|(k_1(\tau,\cdot) - k_2(\tau,\cdot))\,f\left(q_1(\tau,\cdot)\right)\|_{\mathbf{L}^1} + \|k_2(\tau,\cdot)\cdot\left(f\left(q_1(\tau,\cdot)\right) - f\left(q_2(\tau,\cdot)\right)\right)\|_{\mathbf{L}^1}\\ &= \|k_1(\tau,\cdot) - k_2(\tau,\cdot)\|_{\mathbf{L}^\infty}\cdot\|f\left(q_1(\tau,\cdot)\right)\|_{\mathbf{L}^1} + \|k_2(\tau,\cdot)\|_{\mathbf{L}^\infty}\,\|q_1(\tau,\cdot) - q_2(\tau,\cdot)\|_{\mathbf{L}^1}\\ &= M\,\|q_1(\tau,\cdot) - q_2(\tau,\cdot)\|_{\mathbf{L}^1}\,.\end{aligned} \tag{3.26}$$

Putting the estimates (3.25) and (3.26) into (3.24), we get

$$E(\tau) \;\leq\; L \cdot \|q_1(\tau,\cdot) - q_2(\tau,\cdot)\|_{\mathbf{L}^1}$$

for a suitable constant L. Inserting this estimate in (3.21) and using (3.20) one finally obtains (3.18).

Appendix A. Technicalities

A.1. *Properties of the integral operator*

Now we prove some properties of the integral term k in terms of a Lipschitz flow $t \mapsto q(t,\cdot)$. The operator K, see (1.3), is defined on the set

$$\left\{ q \in \mathbf{L}^1(\mathbb{R}_-) \cap BV(\mathbb{R}_-)\,; \quad \inf_{x<0} q(x) > -1 \right\}$$

and valued in $Lip(\mathbb{R}_-)$. Its properties are summarized in the following Proposition.

Proposition Appendix A.1. *Let C_0, κ_0, T be given positive constants. Assume that the map $q : [0,T] \to \mathcal{D}_{C_0,\kappa_0}$ is Lipschitz continuous as a function in $\mathbf{L}^1(\mathbb{R}_-)$.*

Define k as in (1.9). *Then*

$$\textbf{\textit{(K)}} \quad \begin{cases} k(t,x) : [0,T] \times \mathbb{R}_- \to \mathbb{R}_+ \text{ is bounded and Lipschitz continuous, with} \\ \qquad\qquad \inf_{t,x} k > 0\,; \\ \mathrm{TV}\, k(t,\cdot)\,,\ \mathrm{TV}\, k_x(t,\cdot) \text{ are bounded uniformly in time;} \\ [0,T] \ni t \to k_x(t,\cdot) \in \mathbf{L}^1(\mathbb{R}_-) \text{ is Lipschitz continuous.} \end{cases}$$

Proof. To begin, notice that the quantity k is well-defined and is Lipschitz continuous on $[0,T] \times \mathbb{R}_-$.

Let L be a Lipschitz constant of the map $[0,T] \ni t \mapsto q(t) \in \mathbf{L}^1(\mathbb{R}_-)$. From the bounds (1.7) one easily deduces that

$$\|q(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_-)} \leq C_0\,, \tag{A.1}$$

$$\|f(q(t,\cdot))\|_{\mathbf{L}^\infty(\mathbb{R}_-)} \leq \max\left\{|f(C_0)|\,, |f(\kappa_0)|\right\}\,, \tag{A.2}$$

$$\|f(q(t,\cdot))\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq |f'(\kappa_0)| \cdot \|q(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq C_0|f'(\kappa_0)|\,, \tag{A.3}$$

$$\|f(q(t_1,\cdot)) - f(q(t_2,\cdot))\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq L|f'(\kappa_0)| \cdot |t_1 - t_2|\,. \tag{A.4}$$

By the assumptions on q we find that

$$\left| \int_x^0 f(q(t,\xi))\, d\xi \right| \;\leq\; \|f(q(t,\cdot))\|_{\mathbf{L}^1(\mathbb{R}_-)} \;\leq\; C_0|f'(\kappa_0)|\,.$$

Hence the integral term k is bounded and satisfies

$$0 < \exp\left(-C_0|f'(\kappa_0)|\right) \leq k(t,x) \leq \exp\left(C_0|f'(\kappa_0)|\right)\,.$$

Moreover, for all $0 \leq t_1 < t_2$ we have

$$\left| \int_x^0 [f(q(t_1,\xi)) - f(q(t_2,\xi))] \, d\xi \right| \leq \|f(q(t_1,\cdot)) - f(q(t_2,\cdot))\|_{\mathbf{L}^1(\mathbb{R}_-)} \leq L|f'(\kappa_0)| \cdot |t_1 - t_2| \,.$$

This leads to the Lipschitz continuity in t for $k(t,x)$. Namely, for all x we have

$$|k(t_1,x) - k(t_2,x)| = \mathcal{O}(1) \left| \int_x^0 [f(q(t_1,\xi)) - f(q(t_2,\xi))] \, d\xi \right| \leq \widehat{L} \quad |t_1 - t_2| \,. \tag{A.5}$$

Here the Lipschitz constant $\widehat{L}$ depends on the parameters L, C_0, κ_0.

From the definition of k, the derivative function k_x satisfies

$$k_x = -kf(q) \ \in \mathbf{L}^1 \cap \mathbf{L}^\infty \,. \tag{A.6}$$

This immediately shows three facts: (i) $k(t,x)$ is Lipschitz in space variable x, (ii) $k(t,\cdot) \in BV(\mathbb{R}_-)$ where the BV bounds are uniform in t, and (iii) $k_x(t,\cdot) \in BV(\mathbb{R}_-)$.

From (A.6) we get the estimate on the total variation of k_x

$$\mathrm{TV}\,(k_x) \leq \mathrm{TV}\,(k) \cdot \|f(q)\|_{\mathbf{L}^\infty(\mathbb{R}_-)} \ + \ \|k\|_{\mathbf{L}^\infty(\mathbb{R}_-)} \mathrm{TV}\,(f(q(t,\cdot))) \leq M \, \mathrm{TV}\,(q) \,,$$

with M depending on the parameters L, C_0, κ_0.

Finally, we show that $[0,T] \ni t \to k_x(t,\cdot) \in \mathbf{L}^1(\mathbb{R}_-)$ is Lipschitz continuous. By using (A.5), (A.3) and (A.4), one has

$$\begin{aligned}
& \|k_x(t_1,\cdot) - k_x(t_2,\cdot)\|_{\mathbf{L}^1(\mathbb{R}_-)} \\
&= \mathrm{TV}\,\{k(t_1,\cdot) - k(t_2,\cdot)\} \\
&= \|k(t_1,\cdot) f(q(t_1,\cdot)) - k(t_2,\cdot) f(q(t_2,\cdot))\|_{\mathbf{L}^1(\mathbb{R}_-)} \\
&\leq \|k(t_1,\cdot) - k(t_2,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_-)} \|f(q(t_1,\cdot))\|_{\mathbf{L}^1(\mathbb{R}_-)} \\
&\quad + \|k(t_2,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R}_-)} \|f(q(t_1,\cdot)) - f(q(t_2,\cdot))\|_{\mathbf{L}^1(\mathbb{R}_-)} \\
&\leq \widehat{M} |t_1 - t_2|
\end{aligned}$$

with $\widehat{M}$ depending on the parameters L, C_0, κ_0. □

A.2. *Proof of Theorem 2.1 for $k = k(x)$*

Let $k(x)$ satisfy the condition **(K)**, stated in Sect. 2, that here reduces to assume that

$$k \in BV\,(\mathbb{R}_-) \cap Lip\,(\mathbb{R}_-) \,, \qquad k_x \in BV\,(\mathbb{R}_-) \,, \qquad \inf_x k(x) > 0 \,. \tag{A.7}$$

Let $\bar{u} \in \mathcal{D}_{C_0,\kappa_0}$, see (1.7). By adapting the analysis in [4,1,18], the entropy solution for

$$u_t + \Big(k(x) f(u)\Big)_x = 0 \,, \quad x \leq 0, \ t \geq 0 \,; \qquad u(0,x) = \bar{u}(x) \,, \quad x < 0 \,, \tag{A.8}$$

exists and is unique on $[0,T]\times\mathbb{R}_-$ *provided that* u is shown to be bounded from above and below on this set (see below for details). As long as the solution is defined, it satisfies the following properties:

(a) the sup norm of the *complete flux*

$$F(t,x) \doteq k(x) f(u(t,x))\,,$$

as well as its total variation, cannot increase in time:

$$|F(t,x)| \leq \sup |F(0,\cdot)|\,, \qquad t\geq 0 \tag{A.9}$$

$$\mathrm{TV}\,\{F(t,\cdot)\} \leq \mathrm{TV}\,\{F(0,\cdot)\}\,, \qquad t\geq 0\,; \tag{A.10}$$

(b) the Kružkov entropy inequalities

$$\partial_t|u-\alpha| + \partial_x\left[k(x)|f(u)-f(\alpha)|\right] + \mathrm{sign}(u-\alpha)k_x(x)f(\alpha) \leq 0\,, \tag{A.11}$$

for all $\alpha\in\mathbb{R}$, hold in the sense of distributions;

(c) the corresponding operator $(t,\bar u)\mapsto u(t,\cdot)$ is contractive in $\mathbf{L}^1(\mathbb{R}_-)$.

We have now to comment on the applicability of the results in [4,1,18], since our assumptions (1.4) on the flux do not match precisely the ones in these papers. Indeed they require (various) sufficient conditions on the flux in order to have uniform L^∞ bounds on the solution, that mainly consist in the existence of bounded stationary solutions and the use of comparison arguments.

Recalling (1.4), our assumptions differ for two reasons: (i) f is singular at -1, (ii) in the case $f(u)\to f_0>0$ as $u\to+\infty$, there is a possible loss of bounded stationary solutions.

First case: $f(u)\to+\infty$ as $u\to+\infty$. Following for instance [4], a solution is defined locally in time and an immediate property is (A.9). Since f diverges to $-\infty$ as $u\to-1$, then a uniform lower bound follows. This also implies that the propagation speed kf' is uniformly bounded.

The same argument leads to an upper bound on $f(u)$ and hence on u. In other words the existence of global bounded stationary solutions (that satisfy $k(x)f(u(x))=C$) lead, by comparison, to uniformly a priori bounds.

Second case: $f(u)\to f_0>0$ as $u\to+\infty$. Here we cannot apply directly the mentioned references, due to the lack of coercitivity for large u.

We first give a formal argument. The lower bound for u would follow as before. About the upper bound, the characteristic analysis gives

$$\dot x = k(x)f'(u)\,, \tag{A.12}$$

$$\dot u = -k_x(x)f(u)\,. \tag{A.13}$$

Because of the assumption on f and lower bound on u, then $f(u)$ is bounded from above and below. Since also k_x is bounded, it turns out that the r.h.s. of (A.13) is bounded, leading to a uniformly bounded growth of u in finite time.

To render this argument rigorous, one can proceed in two steps:

(1) show that the solution exists locally in time.

(2) show that, given any $T > 0$, the solution existing in the time interval $[0, T]$ is uniformly bounded, independently on T.

Proof of (1). For each $\Delta x > 0$ define

$$k_{\Delta x}(x) = \sum_{j \in \mathbb{Z}} \chi_{I_j}(x)\, k(j\Delta x)\,, \qquad I_j = (j\Delta x, (j+1)\Delta x)$$

and consider the equation

$$u_t + (k_{\Delta x} f(u))_x = 0\,, \qquad u(0, \cdot) = \bar{u}\,. \tag{A.14}$$

Notice that on each I_j the flux is independent on x. At each interface $x_j = j\Delta x$ the relation

$$k_l f\,(u(t, x_j-)) = k_r f\,(u(t, x_j+)) \tag{A.15}$$

must be satisfied for a.e. t, with $k_l = k((j-1)\Delta x)$ and $k_r = k(j\Delta x)$.

We remark that, given k_l, k_r and $u_l = u(t, x_j-)$, the solvability of (A.15) requires

$$f\,(u(t, x_j+)) = \frac{k_l}{k_r} f\,(u_l) < f_0\,. \tag{A.16}$$

Denote by L, k_0, L_2 the constant values

$$L = \|k_x\|_\infty\,, \quad k_0 = \inf_x k(x)\,, \quad L_2 = L/k_0\,.$$

If Δx is small enough, the property (A.16) is certainly true whenever $u_l \le 0$ (since also the l.h.s. would be negative), and also when $0 < u_l \le M$ because

$$\frac{k_l}{k_r} f\,(u(t, x_j-)) \le (1 + L_2\Delta x)\, f(M) < f_0$$

for Δx sufficiently small, depending on M. Moreover notice that, since $k_{\Delta x}$ is constant on each I_j, then the maximum (positive) value of u do not increase inside each strip. In conclusion, it may increase only at interfaces.

According to the previous remarks, a weak entropy solution of (A.14) is defined locally in time, following for instance [4], since the upper bound

$$\sup_x f(u(t, x)) \le (1 + L_2\Delta x)\, f(\sup \bar{u}) < f_0 \tag{A.17}$$

is satisfied for small times, provided that Δx is small enough.

Indeed, this can be proved at the level of piecewise constant approximate solutions of (A.14).

A lower bound on u is available by estimate (A.9), as already used before, say $u \ge u_{min} > -1$ where u_{min} is a global bound depending only on k and on the initial data.

As remarked before, this provides an upper bound on the characteristic speed kf': one has

$$k(x) f'(u) \;\le\; \|k\|_\infty f'(u_{min}) \doteq \lambda_{max}\,.$$

Each front that travels from some x_j to x_{j+1} takes a time t_1 which is larger than $\Delta x/\lambda_{max}$. Therefore, for $t < \Delta x/\lambda_{max}$, the estimate (A.17) is valid.

Now we need to prove that there exists $T_1 > 0$ and $M_1 > 0$, independent on small Δx, such that

$$\sup_x f(u(t,x)) \leq f(M_1) < f_0 \,, \qquad 0 \leq t \leq T_1 \,.$$

An increase of u occurs only at interactions of a u-front with a k-front. Hence the increase of u at some time t is estimated by

$$\sup_x f(u(t,x)) \leq (1 + L_2 \Delta x)^{n+1} f(\sup \bar{u})$$

where n is the largest integer such that $t \geq n\Delta x/\lambda_{max}$, so that

$$\begin{aligned} \sup_x f(u(t,x)) &\leq (1 + L_2\Delta x)^{1+t\lambda_{max}/\Delta x} f(\sup \bar{u}) \\ &\leq (1 + L_2\Delta x) \cdot \mathrm{e}^{tL_2\lambda_{max}} f(\sup \bar{u}) \end{aligned}$$

which is $< f_0$ for Δx sufficiently small and $t \leq T_1$, where $T_1 > 0$ is chosen to satisfy

$$\mathrm{e}^{T_1 L_2 \lambda_{max}} f(\sup \bar{u}) \;<\; f_0 \,.$$

By using this argument, one can deduce an upper bound on piecewise constant approximate solutions of (A.14), independent on Δx. Passing to the limit as $\Delta x \to 0$, one then obtains the local existence of the solution (A.8) for $t \leq T_1$.

Proof of (2). One can justify the formal analysis done through equations (A.12)–(A.13) by using the properties of generalized characteristics [12, Ch. 10-11].

Indeed, the scalar equation in (A.8) can be recasted as a 2×2 system

$$k_t = 0 \,, \qquad u_t + \Big(kf(u)\Big)_x = 0 \,. \tag{A.18}$$

The characteristic speeds are $\lambda_1 = 0$ and $\lambda_2 = kf'(u) > 0$. Hence the system is strictly hyperbolic whenever u belongs to a bounded set (recalling that $f'' \neq 0$ and that $f(u) \to f_0$ and as $u \to \infty$, one has $f'(u) \to 0$ as $u \to \infty$).

Moreover, the 2^{nd} characteristic field is genuinely nonlinear, due to the strict convexity of u.

Let $(k(x), u(t,x))$ be a weak, entropic and bounded solution of (A.18) defined on $(0,T_1) \times \mathbb{R}_-$, for some $T_1 > 0$, whose existence follows from (1). We assume that u, being of locally bounded variation in the two variables (t,x), is normalized as described in [12, Sect. 1.7].

Following [12, Th.10.3.2], minimal and maximal generalized characteristics emanating from any point $(\bar{t},\bar{x})$ are shock-free (see [12, Def. 10.2.4]). Since characteristic curves of the 2^{nd} family are globally defined, the assumption of small oscillation of (k,u) is not needed. Hence, given any $(\bar{t},\bar{x})$, a minimal or maximal generalized characteristic $[0,\bar{t}] \ni t \mapsto x(t)$ satisfies

$$\lim_{x \to x(t)} u(t,x) = u(t,x(t)) \,, \qquad \dot{x} = k(x)f'(u) \,, \qquad \text{for a.e. } t \in [0,\bar{t}] \,.$$

Notice that $t \to x(t)$ is Lipschitz continuous and strictly increasing. Then one can adapt the proofs of [12], Th. 11.1.1 and 11.1.3, to obtain the following: if $x(t)$ is a minimal backward characteristic, one has

$$u(t, x(t)) = u(\bar{t}, \bar{x}-) - \int_{\bar{t}}^{t} k_x(x(\tau)) f(u(\tau, x(\tau)))\, d\tau\,, \qquad t \in [0, \bar{t}]\,.$$

Hence $t \mapsto u(t, x(t))$ turns out to be absolutely continuous. A similar property holds for the maximal backward characteristics. Therefore, the formal argument based on (A.12)–(A.13) is validated.

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