PROPER ENTROPY CONDITIONS FOR SCALAR CONSERVATION LAWS WITH DISCONTINUOUS FLUX

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Abstract. We introduce entropy admissibility conditions for scalar conservation laws with discontinuous flux which precisely describe behavior of solutions at the interface. The conditions provide well-posedness of appropriate Cauchy problem. We assume that the flux is such that the maximum principle holds, but we allow multiple flux crossings and we do not need any kind of genuine nonlinearity conditions. Proposed concept is a proper generalization to the standard Kruzhkov entropy conditions and it does not involve transformation of the equation or use of adapted entropies.

The subject of the paper is the following Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \left(H(x)f(u) + H(-x)g(u) \right) = 0, & (t,x) \in I\!R^+ \times I\!R \\ u|_{t=0} = u_0(x) \in L^\infty(I\!R), \end{cases} \tag{1}$$

where u is the scalar unknown function; u_0 is a function such that $a \leq u_0 \leq b$, $a, b \in I\!R$; H is the Heaviside function; and $f, g \in C^1(\mathbf{R})$ are such that $f(a) = g(a) = c_1$, $f(b) = g(b) = c_2$ for some constants c_1 and c_2.

Problems such as (1) describe many physical phenomena related to flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, gas flow in a variable duct... Therefore, they are under intensive investigations since its introduction in [23], but specially in the last twenty years.

As usual in conservation laws, Cauchy problem (1) in general does not possess classical solution, and it can have several weak solutions. Since it is not possible to directly generalize the standard theory of entropy admissible solutions [18], in order to choose a proper weak solution to (1) many admissibility conditions were proposed. We mention minimal jump condition [14], minimal variation condition and Γ condition [9, 10], entropy conditions [16, 2], vanishing capillary pressure limit [15], admissibility conditions via adapted entropies [7, 8] or via conditions at the interface [3, 4, 11]. Excellent overview on the subject as well as a kind of unification of the mentioned approaches can be found in [6].

However, in every of the mentioned approaches, in order to prove existence or uniqueness of a weak solution to the considered problem, some structural hypothesis on the flux (such as convexity, genuine nonlinearity, the crossing condition) or on the form of the solution (see [3, 4]) were assumed. An exception is paper [19] where none of mentioned assumptions has been used in order to prove existence and stability of several stable semi-groups of admissible solutions to (1). The proof was based on a transformation of the equation which provides a kind of the crossing conditions. As it comes to the crossing conditions, they are introduced in [16] where

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degenerate parabolic equation with discontinuous flux is considered:

$$\begin{cases} \partial_t u + \partial_x \left(H(x)f(u) + H(-x)g(u)\right) = \partial_{xx}A(u), & (t,x) \in (0,T) \times I\!R \\ u|_{t=0} = u_0(x) \in BV(I\!R) \cap L^1(I\!R), & x \in I\!R, \end{cases}$$

where A is non-decreasing with $A(0) = 0$. Assuming that $A \equiv 0$ we obtain the problem of type (1). In order to obtain uniqueness of a weak solution to the problem, the following reminiscent of the Kruzhkov admissibility condition [18] is used:

Definition 1. [16] Let u be a weak solution to problem (1). We say that u is an entropy admissible weak solution to (1) if the following entropy condition is satisfied for every fixed $\xi \in \mathbf{R}$:

$$\partial_t |u - \xi| + \partial_x \Big\{ \mathrm{sgn}(u-\xi) \Big[H(x)(f(u) - f(\xi)) + H(-x)(g(u) - g(\xi)) \Big] \Big\} \\ - |f(\xi) - g(\xi)| \delta(x) \le 0 \;\; \text{in} \;\; \mathcal{D}'(I\!R^+ \times I\!R).$$

The entropy condition from Definition 1 rely on a rough estimate of behavior around the interface $x = 0$ of solutions to equations regularized with vanishing visosity and flux regularization. Therefore, the latter concept provides the well posedness only under the additional assumptions: the crossing conditions and existence of traces of entropy solutions at $x = 0$.

Let us first recall the notion of traces.

Definition 2. Let $W : I\!R \times I\!R^+ \to I\!R$ be a function that belongs to $L^\infty(I\!R \times I\!R^+)$. By the right and left traces of $W(\cdot, t)$ at the point $x = 0$ we call the functions $t \mapsto W(0\pm, t) \in L^\infty_{loc}(I\!R^+)$ that satisfy for every $\varphi \in C_c(I\!R^+)$:

$$\lim_{x \to 0} \int_{I\!R^+} |W(t,x) - W(t, 0+)| \varphi(t) dt = 0, \;\; \lim_{x \to 0} \int_{I\!R^+} |W(t,x) - W(t, 0-)| \varphi(t) dt = 0.$$

As we have shown in [19], results on existence of traces from [20] allow us to assume that the traces always exist. We shall now recall the results.

Definition 3. We say that the function $u \in L^\infty(I\!R^d)$ is a quasi-solution to the scalar conservation law

$$\mathrm{div}_x F(u) = 0, \;\; x \in I\!R^d,$$

where $F = (F_1, \dots, F_d) \in C(I\!R^d; I\!R)$ if it satisfies for almost every $\xi \in I\!R$:

$$\mathrm{div}_x \mathrm{sgn}(u - \xi)(F(u) - F(\xi)) = \gamma_k \;\; \text{in} \;\; \mathcal{D}'(I\!R^d),$$

where γ_k is a locally bounded Borel measure.

Theorem 4. [20] *Let $h, f \in C(I\!R)$. Suppose that the function u is a quasi-solution to*

$$\partial_t h(u) + \partial_x f(u) = 0, \;\; (t,x) \in I\!R^+ \times I\!R,$$

where the vector (h, f) is such that the mappings $\xi \mapsto h(\xi)$ and $\xi \mapsto f(\xi)$ are not constant on any non-degenerate interval.

Then, the function u admits right and left strong traces at $x = 0$.

We recall next the crossing conditions. We were not able to cope with them in a natural way in [19]:

Crossing condition: For any states u, v the following condition must hold:

$$f(u) - g(u) < 0 < f(v) - g(v) \Rightarrow u < v.$$

The latter condition can be fulfilled only in the case when the functions f and g have a single intersection point between a and b. One of the ways to overcome this obstacle (proposed in [19]) is to introduce a transformation of the unknown function u:

$$v = \tilde{\alpha}(u)H(x) + \tilde{\beta}(u)H(-x) \Rightarrow u = \alpha(v)H(x) + \beta(v)H(-x),$$

and denoting $f_\alpha = f \circ \alpha$ and $g_\beta = g \circ \beta$, we have from (1):

$$\begin{cases} \partial_t(\alpha(v)H(x) + \beta(v)H(-x)) + \partial_x \left(H(x)f_\alpha(v) + H(-x)g_\beta(v)\right) = 0, \\ v|_{t=0} = \tilde{\alpha}(u_0)H(x) + \tilde{\beta}(u_0)H(-x). \end{cases} \tag{2}$$

So, instead of dealing with the flux $H(x)f(u) + H(-x)g(u)$, we deal with the new flux $H(x)f_\alpha(v) + H(-x)g_\beta(v)$. By an appropriate choice of the functions α and β, the functions f_α and g_β will satisfy the crossing conditions and we shall have well posedness for (2). If the functions α and β are monotonic, the latter implies well-posedness of (1).

However, this is not completely satisfactory. What we want to find are entropy conditions which provide well posedness without (more less) artificial transformation of the equation. To be more succinct, let us consider the usual vanishing viscosity approximation to (1)

$$\partial_t u_\varepsilon + \partial_x(f(u_\varepsilon)H(x) + g(u_\varepsilon)H(-x))] = \varepsilon \partial_{xx} u_\varepsilon. \tag{3}$$

and assume that (u_ε) is L^1_{loc}-strongly precompact. An L^1_{loc}-limit along a subsequence u of (u_ε) will represent a weak solution to (1). Remark, in passing, that solutions to (1) corresponding to the limit of (u_ε) defined above represent the vanishing viscosity germ from [6].

Assume now that we have a weak solution u to (1). We would like to know what conditions should u satisfy so that it represents a subsequential L^1_{loc}-limit to (u_ε). It appears that they can be obtained by an analysis of the vanishing viscosity approximation to (1) at the interface. The latter is described by the following admissibility concept.

Definition 5. Let u be a weak solution to problem (1).

We say that u is an entropy admissible solution to (1) if

(D.1.) $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ and $u(t,x) \in [a,b]$ for almost every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$;

(D.2) there exists a function $p : \mathbb{R}^+ \to [a,b]$ such that for every $\xi \in \mathbb{R}$:

$$\partial_t u + \partial_x \Big\{ \mathrm{sgn}(u-\xi) \Big[H(x)(f(u) - f(\xi)) + H(-x)(g(u) - g(\xi)) \Big] \Big\} \tag{4}$$
$$+ \mathrm{sgn}(p(t) - \xi)(f(\xi) - g(\xi))\delta(x) \leq 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}).$$

The paper is organized as follows.

In Section 2, we shall prove L^1_{loc}-well-posedness to (1) under the genuine nonlinearity assumption on the flux.

In Section 3, we shall prove the general well-posedness result.

1. Well posedness under the genuine nonlinearity assumptions

In this section we shall assume that the flux satisfies the genuine nonlinearity conditions. This is necessary since the existence proof reduces to a convergence of a family of approximate solutions to (1). The latter convergence is, in turn, provided by the genuine nonlinearity conditions. More precisely, we can also assume a little bit less [5], but we shall use this in the next section.

Definition 6. We say that the flux from the equation in (1) is genuinely nonlinear if the mappings

$$\xi \mapsto f'(\xi) \quad \text{and} \quad \xi \mapsto g'(\xi),$$

are not identically equal to a constant on non-degenerate subintervals of (a, b).

We shall start with the existence proof.

Theorem 7. *Assume that the flux from the equation from* (1) *is genuinely nonlinear and that* $u_0 \in C_c^\infty(I\!R)$. *Then, there exists a weak solution* u *to* (1) *satisfying the conditions from Definition 5.*

Proof: Consider the following approximation to the equation given in (1):

$$\partial_t u_\varepsilon + \partial_x \left(f(u_\varepsilon)H(x) + g(u_\varepsilon)H(-x)\right) = \varepsilon u_{xx}. \tag{5}$$

Remark that, due to the energy inequality, there exists $u_\varepsilon \in L^1_{loc}(I\!R^+; H^1_{loc}(I\!R))$ which solves (5) with the initial data $u_\varepsilon|_{t=0} = u_0$, $a \leq u_0 \leq b$.

If we denote $p_\varepsilon(t) = u_\varepsilon(t, 0)$, it is not difficult to conclude that u_ε satisfies the following entropy inequality for every $\xi \neq p_\varepsilon(t)$:

$$\begin{aligned} &\partial_t |u_\varepsilon - \xi| + \partial_x \left(\mathrm{sgn}(u_\varepsilon - \xi)((f(u_\varepsilon) - f(\xi))H(x) + (g(u_\varepsilon) - g(\xi))H(-x)\right) \\ &\quad + \mathrm{sgn}(p_\varepsilon(t) - \xi)(f(\xi) - g(\xi))\delta(x) = \varepsilon \mathrm{sgn}(u_\varepsilon - \xi)\partial_{xx} u_\varepsilon \leq \varepsilon \partial_{xx}|u_\varepsilon - \xi|. \end{aligned} \tag{6}$$

Since we have assumed that the flux is genuinely nonlinear, according to results from [21], we conclude that there exists $u \in L^1_{loc}(I\!R^+ \times I\!R)$ such that for a zero sequence (ε_m), it holds

$$L^1_{loc} - \lim_{m\to\infty} u_{\varepsilon_m} = u.$$

Remark that the latter implies that for almost every $t \in I\!R^+$ and every relatively compact $K \subset\subset I\!R$:

$$\lim_{m\to\infty} \int_K |u_{\varepsilon_m}(t, x) - u(t, x)| dx = 0. \tag{7}$$

Next, for every $t \in I\!R^+$, let (ε_n^t) be a subsequence of the sequence (ε_m) such that

$$p(t) = \lim_{n\to\infty} p_{\varepsilon_n^t}(t) = \limsup_{m\to\infty} p_{\varepsilon_m}(t). \tag{8}$$

Consider now (6) for an arbitrary fixed t for which (7) holds. We have for an arbitrary $\phi \in C_c^2(I\!R)$:

$$\begin{aligned} &-\int_{I\!R} \partial_t |u_\varepsilon(t, \cdot) - \xi| \phi dx \\ &\geq -\int_{I\!R} \left(\mathrm{sgn}(u_\varepsilon(t, \cdot) - \xi)((f(u_\varepsilon(t, \cdot)) - f(\xi))H(x) + (g(u_\varepsilon(t, \cdot)) - g(\xi))H(-x)\right) \partial_x \phi dx \\ &- \mathrm{sgn}(p_\varepsilon(t) - \xi)(f(\xi) - g(\xi))\phi(0) + \varepsilon \int_{I\!R} |u_\varepsilon(t, \cdot) - \xi| \partial_{xx}\phi dx. \end{aligned}$$

We let here $\varepsilon \to 0$ along the subsequence from (8). From (7) and (8) we conclude that for all $\xi \neq p(t)$:

$$
\begin{aligned}
&\lim_{n\to\infty}\Big(-\int_{I\!R}\partial_t|u_{\varepsilon_n^t}(t,\cdot)-\xi|\phi dx\Big)\\
&=-\int_{I\!R}(\mathrm{sgn}(u(t,\cdot)-\xi)((f(u(t,\cdot))-f(\xi))H(x)+(g(u(t,\cdot))-g(\xi))H(-x))\,\partial_x\phi dx\\
&-\mathrm{sgn}(p(t)-\xi)(f(\xi)-g(\xi))\phi(0).
\end{aligned}
$$

We multiply the latter expression by an arbitrary $\varphi \in C_c^1(I\!R^+)$ and integrate over $I\!R^+$. From here, we have according to the Fatou lemma for almost every $\xi \in I\!R$:

$$
\begin{aligned}
&\int_{I\!R^+\times I\!R}|u-\xi|\phi\partial_t\varphi dxdt\\
&=\liminf_{n\to\infty}\Big(-\int_{I\!R^+\times I\!R}\partial_t|u_{\varepsilon_n^t}(t,x)-\xi|\phi(x)\varphi(t)dxdt\Big)\\
&\geq\int_{I\!R^+}\liminf_{n\to\infty}\Big(-\int_{I\!R}\partial_t|u_{\varepsilon_n^t}(t,x)-\xi|\phi(x)\varphi(t)dx\Big)\,dt\\
&\geq-\int_{I\!R^+\times I\!R}(\mathrm{sgn}(u-\xi)((f(u)-f(\xi))H(x)+(g(u)-g(\xi))H(-x))\,\varphi\partial_x\phi dxdt\\
&-\phi(0)\int_{I\!R^+}\mathrm{sgn}(p(t)-\xi)(f(\xi)-g(\xi))\varphi dt.
\end{aligned}
$$

From here, we conclude that the function u is a weak solution to (1) which satisfies conditions from Definition 5.

❑

Now, we pass to harder part of the well posedness proof – uniqueness. Let us first single out admissible shock waves lying at the interface $x = 0$. Since there are many possibilities, we shall not formulate a statement, but we shall split analysis on several cases which will contain necessary information.

First, assume that the shock wave of the form

$$
u(t,x)=\begin{cases} u^-, & x<0\\ u^+, & x>0,\end{cases}
$$

represents a weak solution to (1). Being a weak solution, the constants u^+ and u^- must satisfy the Rankine-Hugoniot conditions

$$
g(u^-)=f(u^+). \tag{9}
$$

Now, we shall analyze admissibility of the shock depending on the relation between u^+ and u^-, and positions of the points $p = p(t) \in [a, b]$ defined in (4).

In order to inspect admissibility of the shock wave u, we simply insert it in (4), and conclude that it must be for any $\xi \in [a, b]$

$$
\begin{aligned}
\Big(&\mathrm{sgn}(u^+-\xi)(f(u^+)-f(\xi))-\mathrm{sgn}(u^--\xi)(g(u^-)-g(\xi))\\
&\qquad+\mathrm{sgn}(p(t)-\xi)(f(\xi)-g(\xi))\Big)\delta(x)\leq 0.
\end{aligned}\tag{10}
$$

Remark that if $\xi \geq \max\{u^+, u^-, p(t)\}$ or $\xi \leq \min\{u^+, u^-, p(t)\}$ the the left-hand side in (10) is equal to zero according to (9). Now, we investigate other possible cases.

Case 1: $u^+ \leq u^-$

- $u^+ \leq u^- \leq p(t) = p$ (assume that the time is fixed);

  (i) If
$$\begin{cases} u^+ \leq u^- \leq \xi \leq p & \Longrightarrow \\ f(\xi) \leq g(\xi). \end{cases} \tag{11}$$
  (ii) If
$$\begin{cases} u^+ \leq \xi \leq u^- \leq p & \Longrightarrow \\ f(\xi) \leq f(u^+). \end{cases} \tag{12}$$

- $u^+ \leq p \leq u^-$

  (iii) If
$$\begin{cases} u^+ \leq p \leq \xi \leq u^- & \Longrightarrow \\ g(\xi) \leq g(u^-). \end{cases} \tag{13}$$
  (iv) If
$$\begin{cases} u^+ \leq \xi \leq p \leq u^- & \Longrightarrow \\ f(\xi) \leq f(u^+). \end{cases} \tag{14}$$

- $p \leq u^+ \leq u^-$

  (v) If
$$\begin{cases} p \leq u^+ \leq \xi \leq u^- & \Longrightarrow \\ f(\xi) \leq f(u^+). \end{cases} \tag{15}$$
  (vi) If
$$\begin{cases} p \leq \xi \leq u^+ \leq u^- & \Longrightarrow \\ g(\xi) \leq f(\xi). \end{cases} \tag{16}$$

Case 2: $u^- \leq u^+$

- $u^- \leq u^+ \leq p(t) = p$;

  (i) If
$$\begin{cases} u^- \leq u^+ \leq \xi \leq p & \Longrightarrow \\ f(\xi) \leq g(\xi). \end{cases} \tag{17}$$
  (ii) If

$$\begin{cases} u^- \leq \xi \leq u^+ \leq p & \Longrightarrow \\ g(u^-) \leq g(\xi). \end{cases} \tag{18}$$

- $u^- \leq p \leq u^+$

(iii) If

$$\begin{cases} u^- \leq p \leq \xi \leq u^+ & \Longrightarrow \\ f(u^+) \leq f(\xi). \end{cases} \tag{19}$$

(iv) If

$$\begin{cases} u^- \leq \xi \leq p \leq u^+ & \Longrightarrow \\ g(u^-) \leq g(\xi). \end{cases} \tag{20}$$

- $p \leq u^- \leq u^+$

(v) If

$$\begin{cases} p \leq u^- \leq \xi \leq u^+ & \Longrightarrow \\ f(u^+) \leq f(\xi). \end{cases} \tag{21}$$

(vi) If

$$\begin{cases} p \leq \xi \leq u^- \leq u^+ & \Longrightarrow \\ g(\xi) \leq f(\xi). \end{cases} \tag{22}$$

Now, we can prove the uniqueness result.

Theorem 8. *Let $u, v \in L^\infty(I\!R^+ \times I\!R)$ be two admissible solutions to* (1) *with initial data u_0 and v_0, respectively, which admit left and right traces at $x = 0$. Then, for every $R, T > 0$ there exists $C > 0$ depending only on f and g such that it holds*

$$\int_0^T \int_{B(0,R)} |u(t,x) - v(t,x)| dxdt \leq T \int_{B(0,R+CT)} |u_0(x) - v_0(x)| dx, \tag{23}$$

Proof: Our aim is to derive the Kato inequality, i.e. to prove that for every $\varphi \in C_c^1(I\!R^+ \times I\!R)$.

$$\int_{I\!R^+ \times I\!R} \Big(|u - v| \varphi_t \tag{24}$$

$$+ \operatorname{sgn}(u - v) \left((f(u) - f(v)) H(x) + (g(u) - g(v)) H(-x) \right) \varphi_x \Big) dxdt \geq 0.$$

It is well known that (24) holds for $\varphi \in C_c^1(I\!R^+ \times (I\!R \backslash \{0\}))$ (see e.g. [16]). In order to prove that it holds for any $\varphi \in C_c^1(I\!R^+ \times I\!R)$ we introduce the function

Introduce the function

$$\mu_h(x) = \begin{cases} \frac{1}{h}(x + 2h), & x \in [-2h, -h] \\ 1, & x \in [-h, h] \\ \frac{1}{h}(2h - x), & x \in [h, 2h] \\ 0, & |x| > 2h \end{cases} \tag{25}$$

and for an arbitrary $\psi \in C_0^1(I\!R^+ \times I\!R)$, put $\varphi = (1 - \mu_h)\psi$ in (24). After letting $h \to 0$, we get

$$\int_{\mathbb{R}^+\times\mathbb{R}}\Big(|u-v|\psi_t+\mathrm{sgn}(u-v)\left((f(u)-f(v))H(x)+(g(u)-g(v))H(-x)\right)\psi_x\Big)dxdt \tag{26}$$
$$\geq \int_{\mathbb{R}^+}\left(-\mathrm{sgn}(u^+-v^+)(f(u^+)-f(v^+))+\mathrm{sgn}(u^- -v^-)(g(u^-)-g(v^-))\right)\varphi(t,0)dt$$
$$= \int_{\mathbb{R}^+} S(u^\pm,v^\pm)\varphi(t,0)dt.$$

Now, we shall prove that the right-hand side of the latter expression is greater or equal to zero. The proof is tedious and it is accomplished by considering numerous different possibilities depending on relations between $u^\pm$, $v^\pm$ and $p(t)$.

Concerning the relation between $u^\pm$ and $v^\pm$, we see that, according to the Rankine-Hugoniot conditions, only when

$$\begin{cases} u^- > v^- \ \text{ and } \ u^+ < v^+ \ \text{ or}\\ u^- < v^- \ \text{ and } \ u^+ > v^+, \end{cases} \tag{27}$$

the quantity $S(u^\pm,v^\pm)$ is not zero (see Cases 1–5 in the proof of [16, Theorem 2.1]). On the other hand, the two cases from (27) are symmetric (since $S(u^\pm,v^\pm) = S(v^\pm,u^\pm)$) and their analysis is thus the same. Thus, it is enough to prove that $S(u^\pm,v^\pm)\geq 0$ if the first relation from (27) is satisfied.

We shall consider the following possible cases (for almost every fixed t).

$$\begin{array}{ll} \text{Case 1:} & u^+ < v^+ < v^- < u^- \\ \text{Case 2:} & u^+ < v^- < v^+ < u^- \\ \text{Case 3:} & v^- < u^+ < v^+ < u^- \\ \text{Case 4:} & v^- < u^+ < u^- < v^+ \\ \text{Case 5:} & v^- < u^- < u^+ < v^+. \end{array}$$

Before that, notice that, according to the disposition of $u^\pm$ and $v^\pm$ and the Rankine-Hugoniot conditions:

$$S(u^\pm,v^\pm) = -\mathrm{sgn}(u^+-v^+)(f(u^+)-f(v^+))+\mathrm{sgn}(u^- -v^-)(g(u^-)-g(v^-))$$
$$= f(u^+)-f(v^+)+g(u^-)-g(v^-) = 2(f(u^+)-f(v^+)) = 2(g(u^-)-g(v^-)).$$

Thus, we aim to prove that for almost every $t\in\mathbb{R}^+$ it holds

$$f(u^+)-f(v^+)\geq 0 \ \text{ or } \ g(u^-)-g(v^-)\geq 0. \tag{28}$$

Since $u,v\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ are two admissible solutions to (1), we consider two functions $p_u = p_u(t)$ and $p_v = p_v(t)$, defined in Definition 5, corresponding to u and v, respectively.

Case 1 For almost every fixed $t\in\mathbb{R}^+$, we have the following possibilities.

- $u^+ < v^+ < v^- < u^- < p_u$
  The conclusion follows by taking $\xi = v^+$ in (12).
- $u^+ < v^+ \leq p_u \leq u^-$
  The conclusion follows by taking $\xi = v^+$ in (14).

- $u^+ \leq p_u \leq v^- < u^-$
  The conclusion follows by taking $\xi = v^-$ in (13).
- $p_u < u^+ < v^+ < v^- < u^-$
  The conclusion follows by taking $\xi = v^-$ in (15).

Case 5 This case is symmetric with the previous one. We need to simply consider position of p_v instead of p_u and to apply (17)–(22) instead of (11)–(16).

Case 2 For almost every fixed $t \in I\!R^+$, we have the following possibilities.

- $u^+ < v^- < v^+ < u^- < p_u$
  The conclusion follows by taking $\xi = v^+$ in (12).
- $u^+ < v^- < v^+ \leq p_u \leq u^-$
  The conclusion follows by taking $\xi = v^+$ in (14).
- $u^+ < v^- \leq p_u \leq v^+ < u^-$
  Here, we must involve the position of p_v. Before that, recall that from (13) and (14)

$$\begin{aligned} &g(\xi) \leq g(u^-), \quad \xi \in [p_u, u^-] \\ &f(\xi) \leq f(u^+), \quad \xi \in [u^+, p_u]. \end{aligned} \tag{29}$$

Now, we have the following possibilities.

1. $u^+ < v^- \leq p_u \leq v^+ \leq p_v$
From (18) (applied on v) and (29), we have respectively $g(v^-) \leq g(p_u) \leq g(u^-)$ which is (28).

2. $u^+ < v^- \leq p_u \leq p_v \leq v^+ < u^-$
From (20) and (29), we have respectively $g(v^-) \leq g(p_u) \leq g(u^-)$.

3. $u^+ < v^- \leq p_v \leq p_u \leq v^+ < u^-$
From (19) and (29), we have respectively $f(v^+) \leq f(p_u) \leq f(u^+)$.

4. $p_v \leq v^- \leq p_u \leq v^+ < u^-$
From (21) and (29), we have respectively $f(v^+) \leq f(p_u) \leq f(u^+)$.

- $u^+ \leq p_u \leq v^- < v^+ < u^-$
  The conclusion follows by taking $\xi = v^-$ in (13).
- $p_u < u^+ < v^+ < v^- < u^-$
  The conclusion follows by taking $\xi = v^-$ in (15).

Case 4 This case is symmetric with the previous one. We need to simply consider position of p_v instead of p_u and to apply (17)–(22) instead of (11)–(16) or vice verse when needed.

Case 3 For almost every fixed $t \in I\!R^+$, we have the following possibilities.

- $v^- < u^+ < v^+ < u^- \leq p_u$
  In this case, the first relation in (28) follows by taking $\xi = v^+$ in (12).
- $v^- < u^+ < v^+ \leq p_u \leq u^-$
  In this case, (28) follows from (14) by taking $\xi = v^+$ there.
- $v^- < u^+ \leq p_u \leq v^+ < u^-$
  We must involve the position of p_v again. We have the following possibilities

1. $v^- < u^+ \leq p_u \leq v^+ < u^- \leq p_v$

From (18), on v, it follows $g(v^-) \leq g(p_u)$ while from (29), $g(p_u) \leq g(u^-)$. Thus, (28) follows.

2. $v^- < u^+ \leq p_u \leq v^+ \leq p_v \leq u^-$

The situation is the same as the previous one.

3. $v^- < u^+ \leq p_u \leq p_v \leq v^+ \leq u^-$

From (20), on v, and (29), it follows respectively $g(v^-) \leq g(p_u) \leq g(u^-)$ which is (28).

4. $v^- < u^+ \leq p_v \leq p_u \leq v^+ \leq u^-$

From (19) and (29), it follows respectively $f(v^+) \leq f(p_u) \leq f(u^+)$ which is (28).

5. $v^- \leq p_v \leq u^+ \leq p_u \leq v^+ \leq u^-$

Relation (28) follows as in the previous case.

6. $p_v \leq v^- < u^+ \leq p_u \leq v^+ \leq u^-$

From (21) and (29), it follows respectively $f(v^+) \leq f(p_u) \leq f(u^+)$.

- $v^- \leq p_u \leq u^+ < v^+ < u^-$

  Relation (28) follows from (15).
- $p_u \leq v^- < u^+ < v^+ < u^-$

  The conclusion is the same as in the previous item.

From the given considerations, we conclude that $S(u^\pm, v^\pm) \geq 0$, i.e. that the Kato inequality (relation (24)) holds. From here, the proof of the theorem follows in the standard way [18].

❏

2. Well posedness in the general situation

In this section, we only assume that $f, g \in C^1(\mathbf{R})$ are such that $f(a) = g(a) = c_1$ and $f(b) = g(b) = c_2$ (so that we have the maximum principle), and that there exists a finite number of intervals (a_{r_j}, a_{r_j+1}), $j = 1, \dots, k_r$, and $(b_{l_i}, b_{l_i}+1)$, $i = 1, \dots, k_l$, $k_l, k_r \in I\!N$, such that the mappings

$$\begin{aligned}&\xi \mapsto g(\xi) \text{ and } \xi \mapsto f(\xi) \text{ are constant on the intervals}\\ &(b_{l_i}, b_{l_i+1}),\ i = 1, \dots, k_l, \text{ and } (a_{r_j}, a_{r_j+1}),\ j = 1, \dots, k_r, \text{ respectively.}\end{aligned} \tag{30}$$

For a convenience, assume that $[a, b] = \cup_{i=1}^{n_r} [a_i, a_{i+1})$ and $[a, b] = \cup_{i=1}^{n_l} [b_i, b_{i+1})$, where $n_l, n_r \in I\!N$, and $a_1 = b_1 = a$, and $a_{n_r} = b_{n_l} = b$. The methodology that we are using is adapted from [19].

We shall need the notion of Young measures and remind that a typical use of the notion in the field of conservation laws can be found in [12]. We shall rely on a procedure from there.

Theorem 9. [22] *Assume that the sequence (u_{ε_k}) is uniformly bounded in $L^p_{loc}(I\!R^+ \times I\!R^d))$, $p \geq 1$. Then, there exists a subsequence (not relabeled) (u_{ε_k}) and a family of probability measures*

$$\nu_{t,x} \in \mathcal{M}(I\!R), \quad (t, x) \in I\!R^+ \times I\!R^d$$

such that the limit

$$\bar{g}(t, x) := \lim_{k \to \infty} g(u_{\varepsilon_k}(t, x))$$

exists in the distributional sense for all $g \in C(I\!R)$. *The limit is represented by the expectation value*

$$\bar{g}(t,x) = \int_{I\!R^+ \times I\!R^d} g(\xi) d\nu_{t,x}(\xi),$$

for almost all points $(t,x) \in I\!R^+ \times I\!R^d$.

We refer to such a family of measures $\nu = (\nu_{(t,x)})_{(t,x)\in I\!R^+ \times I\!R}$ *as the Young measure associated to the sequence* $(u_{\varepsilon_k})_{k\in\mathbf{N}}$.

Furthermore,

$$u_{\varepsilon_k} \to u \ \text{ in } \ L^r_{\rm loc}(I\!R^+ \times I\!R^d), \ 1 \le r < p$$

if and only if

$$\nu_{t,x}(\xi) = \delta(\xi - u(t,x)) \ \ a.e. \ \ (t,x) \in I\!R^+ \times I\!R,$$

where δ *is the Dirac distribution.*

We shall avoid intervals on which the functions f and g lose genuine nonlinearity via the truncation operator $s_{l,r}(u) = \max\{l, \min\{r,u\}\}$, $l < r$, $l, r \in I\!R$. In order to apply it, we shall need to (slightly) adapt ideas from [5] on the following family of problems:

$$\begin{cases} \partial_t u_\varepsilon + \partial_x \left(f(u_\varepsilon)H(x) + g(u_\varepsilon)H(-x)\right) = \varepsilon \partial_{xx} u_\varepsilon \\ u_\varepsilon\Big|_{t=0} = u_0 \in C_c^\infty(I\!R). \end{cases} \tag{31}$$

Roughly speaking, we shall split the interval (a,b) on subintervals where the genuine nonlinearity conditions are fulfilled (and apply results from [21]), on intervals where the flux is linear but not constant (and apply ideas from [5]), and on intervals where the flux is constant (easy to deal with). In order to formalize the ideas, we need the following three lemmas whose proofs are omitted since they are the same as the corresponding proofs from [17]. Actually, they hold if merely $u_0 \in BV(I\!R)$.

Lemma 10. [17, Lemma 4.1] *[L^∞-bound] There exists constant* $c_0 > 0$ *such that for all* $t \in (0,T)$ *the solutions* u_ε *to* (31) *satisfy,*

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(I\!R)} \le c_0.$$

More precisely,

$$a \le u_\varepsilon \le b.$$

Lemma 11. [17, Lemma 4.2] *[Lipschitz regularity in time] Then, there exists constant* c_1, *independent of* ε, *such that for all* $t > 0$ *the solutions* u_ε *to* (31) *satisfy,*

$$\int_{I\!R} |\partial_t u_\varepsilon(\cdot,t)|\, dx \le c_1.$$

Lemma 12. [17, Lemma 4.3] *[Entropy dissipation bound] There exists a constant* c_2 *independent from* ε *such that the solutions* u_ε *to* (31) *satisfy*

$$\varepsilon \int_{I\!R} \left(\partial_x u_\varepsilon(t,x)\right)^2 dx \le c_2,$$

for all $t > 0$.

We also need Murat's lemma:

Lemma 13. [13] *Assume that the family* (Q_ε) *is bounded in* $L^p(\Omega)$, $\Omega \subset I\!R^d$, $p>2$. *Then,*

$$(\operatorname{div} Q_\varepsilon)_\varepsilon \in W^{-1,2}_{\rm c,loc} \quad \textit{if} \quad \operatorname{div} Q_\varepsilon = p_\varepsilon + q_\varepsilon,$$

with $(q_\varepsilon)_\varepsilon \in W^{-1,2}_{\rm c,loc}(\Omega)$ *and* $(p_\varepsilon)_\varepsilon \in \mathcal{M}_{b,loc}(\Omega)$.

The proof of the next lemma is almost the same as the corresponding one from [19]. It is based on Lemmas 10–13.

Lemma 14. *Denote for a fixed* $\xi \in I\!R$:

$$\begin{aligned} q(x,\lambda) &= H(\lambda-\xi)\Big(H(x)(f(\xi)-f(\xi))+H(-x)(g(\xi)-g(\xi))\Big),\\ \bar q(x,\xi) &= H(\lambda-\xi)\Big(H(x)(f^2(\xi)-f^2(\xi))+H(-x)(g^2(\xi)-g^2(\xi))\Big). \end{aligned} \tag{32}$$

The family

$$\partial_t \bar q(x,u_\varepsilon) + \partial_x q(x,u_\varepsilon), \quad \varepsilon>0, \tag{33}$$

is precompact in $W^{-1,2}_{\rm loc}(I\!R^+\times I\!R)$.

Proof: Denote $\eta'(\lambda)=H(\lambda-\xi)$. By multiplying (31) by $\eta'(u_\varepsilon)$, we conclude

$$\partial_t|u_\varepsilon-\xi| + \partial_x q(x,u_\varepsilon) \le |f(\xi)-g(\xi)|\delta(x) + \varepsilon\partial_x(u_{\varepsilon x}H(u_\varepsilon-\xi)).$$

We rewrite the latter expression in the form

$$\begin{aligned} &\partial_t\bar q(x,u_\varepsilon) + \partial_x q(x,u_\varepsilon)\\ &= (\partial_t \bar q(x,u_\varepsilon) - \partial_t|u_\varepsilon-\xi|) + |f(\xi)-g(\xi)|\delta(x) + \varepsilon\partial_x(u_{\varepsilon x}H(u_\varepsilon-\xi)) + \mu_\varepsilon(t,x,\xi), \end{aligned}$$

where (μ_ε) is a family of Radon measures which exists according to the Schwarz lemma on non-negative distributions.

The conclusion of the lemma now follows from the Murat lemma after taking Lemmas 10-12 into account. For details, see [19, Lemma 1.8].

❑

Now, we can prove the following lemma.

Lemma 15. *Denote by* (u_ε) *family of solutions to* (31) *and assume that* f *and* g *satisfy* (30). *Assume that the mapping* $\xi\mapsto f(\xi)$ *is not constant on any subinterval of an interval* (l,r). *Then, the sequence* $(H(x)s_{l,r}(u_\varepsilon))$ *is strongly precompact in* $L^1_{loc}(I\!R^+\times I\!R)$.

Similarly, if the mapping $\xi\mapsto g(\xi)$ *is not constant on any subinterval of an interval* (l,r), *then the sequence* $(H(-x)s_{l,r}(u_\varepsilon))$ *is strongly precompact in* $L^1_{loc}(I\!R^+\times I\!R)$.

Proof: Notice that from Lemma 14, it follows that for the family of functions u_ε and any $r,l\in I\!R$, the families

$$\begin{aligned} &\partial_t\bar q(x,H(x)s_{l,r}(u_\varepsilon)) + \partial_x q(x,H(x)s_{l,r}(u_\varepsilon)) \ \text{ and}\\ &\partial_t\bar q(x,H(-x)s_{l,r}(u_\varepsilon)) + \partial_x q(x,H(-x)s_{l,r}(u_\varepsilon)), \end{aligned} \tag{34}$$

where the functions $\bar q, q$ given by (32), are strongly precompact in $W^{-1,2}_{loc}(I\!R^+\times I\!R)$. Indeed, notice that

$$\begin{aligned} q(x,H(x)s_{l,r}(u_\varepsilon)) &= H(x)q(x,s_{l,r}(u_\varepsilon)) - H(-\xi)H(-x)(g(0)-g(\xi))\\ \bar q(x,H(x)s_{l,r}(u_\varepsilon)) &= H(x)\bar q(x,s_{l,r}(u_\varepsilon)) - H(-\xi)H(-x)(g^2(0)-g^2(\xi)) \end{aligned}. \tag{35}$$

Since $\partial_t\bar{q}(x,s_{l,r}(u_\varepsilon))+\partial_x q(x,s_{l,r}(u_\varepsilon))$ is strongly precompact in $W^{-1,2}_{loc}(\mathbb{R}^+\times\mathbb{R})$ if $\partial_t\bar{q}(x,u_\varepsilon)+\partial_x q(x,u_\varepsilon)$ is (see [21, Theorem 6]), we conclude from (35) that (34) holds.

Furthermore, notice that if the mapping $\xi\mapsto f(\xi)$ is not constant on any subinterval of an interval (l,r) then the vector $(\bar{q}(x,\xi),q(x,\xi))$ from (32) is genuinely nonlinear on the interval (l,r) for $x>0$. Indeed, for $x>0$ the vector reduces to $(f^2(\xi),f(\xi))$ and this is obviously genuinely nonlinear vector since, due to the assumptions of the lemma, for any $\xi_0,\xi_1\in\mathbb{R}$, it holds $\xi_0 f^2(\xi)\neq\xi_1 f(\xi)$ for a.e. $\xi\in(l,r)$. Now, from [21] and Lemma 14, we conclude that the family $(H(x)s_{l,r}(u_\varepsilon))$ is strongly precompact in $L^1_{loc}(\mathbb{R}^+\times\mathbb{R})$.

In the completely same way, we conclude that the family $(H(-x)s_{l,r}(u_\varepsilon))$ is strongly precompact in $L^1_{loc}(\mathbb{R}^+\times\mathbb{R})$ if the mapping $\xi\mapsto g(\xi)$ is different from a constant on every subinterval of the interval (l,r). ❑

Lemma 16. *Assume that the flux functions f and g from* (1) *satisfy* (30)*. Then, there exists a function $u\in L^\infty(\mathbb{R})$ such that*

$$f(u_\varepsilon)H(x)+g(u_\varepsilon)H(-x)\to f(u)H(x)+g(u)H(-x) \tag{36}$$

strongly in $L^1_{loc}(\mathbb{R}^+\times\mathbb{R})$. Moreover, the function u admits left and right traces at the interface $x=0$.

Proof: Denote

$$\tilde{u}_\varepsilon(t,x)=\begin{cases} u_\varepsilon(t,x), & u_\varepsilon(t,x)\notin\cup_{i=1}^{k_r}[a_{r_i},a_{r_i+1}),\quad x>0\\ u_\varepsilon(t,x), & u_\varepsilon(t,x)\notin\cup_{i=1}^{k_l}[b_{l_i},b_{l_i+1}),\quad x\leq 0,\\ a_{r_j}, & u_\varepsilon(t,x)\in[a_{r_j},a_{r_j+1}],\quad x>0,\\ b_{l_j}, & u_\varepsilon(t,x)\in[b_{l_j},b_{l_j+1}],\quad x\leq 0.\end{cases} \tag{37}$$

Notice that $f(u_\varepsilon)H(x)+g(u_\varepsilon)H(-x)=f(\tilde{u}_\varepsilon)H(x)+g(\tilde{u}_\varepsilon)H(x)$ according to assumptions (30). Then, notice that

$$\tilde{u}_\varepsilon=H(x)\left(\sum_{i=1}^{n_r}s_{a_i,a_{i+1}}(\tilde{u}_\varepsilon)-\sum_{i=2}^{n_r-1}a_i\right)+H(-x)\left(\sum_{i=1}^{n_l}s_{b_i,b_{i+1}}(\tilde{u}_\varepsilon)-\sum_{i=2}^{n_l-1}b_i\right). \tag{38}$$

According to Lemma 15 and the definition of the function $\tilde{u}_\varepsilon$, we see that $(\tilde{u}_\varepsilon)$ is strongly precompact in $L^1_{loc}(\mathbb{R}^+\times\mathbb{R})$ (since this property has each of the summands on the right-hand side of (38)). Denote an accumulation point of the family $(\tilde{u}_\varepsilon)$ by u. Clearly, the function u satisfies (36).

In order to prove that the function u admits traces at the interface, denote by $H(x)u^{a_ia_{i+1}}$, $i=1,\dots,n_r$, and $H(-x)u^{b_ib_{i+1}}$, $i=1,\dots,n_l$, strong L^1_{loc}-limits along subsequences of the families $(s_{a_i,a_{i+1}}(\tilde{u}_\varepsilon))$, $i=1,\dots,n_r$, and $(s_{b_i,b_{i+1}}(\tilde{u}_\varepsilon))$, $i=1,\dots,n_l$, respectively. From (38), it follows:

$$u=H(x)\left(\sum_{i=1}^{n_r}u^{a_i,a_{i+1}}-\sum_{i=2}^{n_r-1}a_i\right)+H(-x)\left(\sum_{i=1}^{n_l}u^{b_i,b_{i+1}}-\sum_{i=2}^{n_l-1}b_i\right). \tag{39}$$

Also, notice that $H(x)u^{a_ia_{i+1}}$, $i=1,\dots,n_r$, and $H(-x)u^{b_ib_{i+1}}$, $i=1,\dots,n_l$, are quasi-solutions to (1). Therefore, according to Theorem 4, they admit strong traces at $x=0$. From (39), we see that u admits strong traces as well. ❑

We are finally ready to prove the main theorem of the paper.

Theorem 17. *There exists a unique entropy admissible weak solution to* (1).

Proof: At the beginning, assume that $u_0 \in C_c^\infty(\mathbb{R};[a,b])$ and, as usual, denote by (u_ε) the family of solutions to (31). By applying the procedure from the proof of Theorem 7, we conclude that u_ε satisfies for every $\xi \in \mathbb{R}$:

$$\begin{aligned}\partial_t|u_\varepsilon-\xi| + \partial_x\left(\operatorname{sgn}(u_\varepsilon-\xi)((f(u_\varepsilon)-f(\xi))H(x)+(g(u_\varepsilon)-g(\xi))H(-x)\right)\\ + \operatorname{sgn}(p_\varepsilon(t)-\xi)(f(\xi)-g(\xi))\delta(x) \leq \mathcal{O}_{\mathcal{D}'}(\varepsilon),\end{aligned} \tag{40}$$

where $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ is a family of distributions tending to zero in the sense of distributions as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in (40) and taking Lemma 16 and Theorem 9 into account, we obtain in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$:

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \Big(|\lambda-\xi| d\nu_{t,x}(\lambda) + \partial_x\left(\operatorname{sgn}(u-\xi)((f(u)-f(\xi))H(x)+(g(u)-g(\xi))H(-x)\right)\\ + \operatorname{sgn}(p(t)-\xi)(f(\xi)-g(\xi))\delta(x) \leq 0,\end{aligned} \tag{41}$$

where $\nu_{t,x}$ is a Young measure corresponding to the sequence (u_ε), and u is the function satisfying (36). The Young measure $\nu_{t,x}$ and the function u (admitting strong traces at $x = 0$), we shall call an entropy admissible measure valued solution to (1).

Denote by $\sigma_{t,x}$ a Young measure and by v a function representing an entropy admissible measure valued solution to (1) corresponding to initial data $v_0 \in C_c^\infty(\mathbb{R};[a,b])$.

Using the classical arguments by DiPerna [12], we conclude that for any test function $\varphi \in C_0^1(\mathbb{R}^+ \times (\mathbb{R}\backslash\{0\}))$ it holds:

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{R}}\int_{\mathbb{R}^2} |\lambda-\eta|\partial_t\varphi d\nu_{t,x}(\lambda)d\sigma_{t,x}(\eta)dxdt\\ &+\int_{\mathbb{R}^+\times\mathbb{R}} \operatorname{sgn}(u-v)\left((f(u)-f(v))H(x)+(g(u)-g(v))H(-x)\right)\partial_x\varphi dxdt \geq 0.\end{aligned} \tag{42}$$

Now we take the function μ_h from (25) again and, for an arbitrary $\psi \in C_0^1(\mathbb{R}^+\times\mathbb{R})$, put $\varphi = (1-\mu_h)\psi$ in (42). We obtain:

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{R}}\int_{\mathbb{R}^2} |\lambda-\eta|\partial_t\psi d\nu_{t,x}(\lambda)d\sigma_{t,x}(\eta)dxdt\\ &+\int_{\mathbb{R}^+\times\mathbb{R}} \operatorname{sgn}(u-v)\left((f(u)-f(v))H(x)+(g(u)-g(v))H(-x)\right)\partial_x\psi dxdt\\ &\geq -J(h)+\mathcal{O}(h),\end{aligned} \tag{43}$$

where $J(h) = \int_{\mathbb{R}^+\times\mathbb{R}} (((f(u)-f(v))H(x)+(g(u)-g(v))H(-x)))\,\mu_h'\psi dxdt$, while $\mathcal{O}(h)$ is the standard Landau symbol. Since v and w admit strong traces at $x = 0$, as in the proof of Theorem 7, we conclude $\lim\limits_{h\to 0} J(h) \geq 0$. From here, after letting $h \to 0$ in (43), we conclude:

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{R}}\int_{\mathbb{R}^2} |\lambda-\eta|\partial_t\psi d\nu_{t,x}(\lambda)d\sigma_{t,x}(\eta)dxdt\\ &+\int_{\mathbb{R}^+\times\mathbb{R}} \operatorname{sgn}(u-v)\left((f(u)-f(v))H(x)+(g(u)-g(v))H(-x)\right)\partial_x\psi dxdt \geq 0,\end{aligned}$$

and from here, using well known procedure [18], we conclude that for any $T, R > 0$ and appropriate $C > 0$:

$$\int_0^T \int_{B(0,R)} \int_{I\!R^2} |\lambda - \eta| d\nu_{t,x}(\lambda) d\sigma_{t,x}(\eta) dx dt \leq T \int_{B(0,R+CT)} |u_0 - v_0| dx. \tag{44}$$

Taking $u_0 = v_0$, we see from (44) that for almost every $(t,x) \in [0,T] \times I\!R$ the Young measures $\nu_{t,x}$ and $\sigma_{t,x}$ are the same and they are supported at the same point. This actually means that $\sigma_{t,x}(\xi) = \nu_{t,x}(\xi) = \delta(\xi - u(t,x))$ for a function u. From Theorem 9, we conclude that $v_\varepsilon \to u$ strongly in $L^1_{loc}(I\!R^+ \times I\!R)$ along a subsequence. The function u will obviously represent the entropy admissible solution to (1).

Since we have just concluded that for any $u_0 \in C^\infty_c(I\!R)$, the family (u_ε) of solutions to (31) is strongly L^1_{loc}-precompact, from (44) we get (23).

Now, we consider the case $u_0 \in L^\infty(I\!R)$. First, we take a sequence $(u_{0\varepsilon})$ of smooth compactly supported functions such that $u_{0\varepsilon} \to u_0$ in $L^1_{loc}(I\!R)$. Then, we take the sequence (u_ε) of entropy admissible solutions to (1) with $u_0 = u_{0\varepsilon}$. The sequence (u_ε) satisfy:

$$\int_0^T \int_{B(0,R)} |u_{\varepsilon_1} - u_{\varepsilon_2}| dx dt \leq T \int_{B(0,R+CT)} |u_{0\varepsilon_1} - u_{0\varepsilon_2}| dx.$$

This readily implies that the sequence (u_ε) is convergent in $L^1_{loc}(I\!R^+ \times I\!R)$. Its limit is clearly an entropy admissible solution to (1). Uniqueness of such entropy admissible solution is proved in the completely same way as when $u_0 \in BV(I\!R; [a,b])$ (since the existence of traces on $x = 0$ does not depend on the properties of initial data). ❑

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