A NOTE ON A KINETIC FORMULATION OF THE EULER EQUATIONS

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ABSTRACT. We show that the maxwellian density $f(t,x,v)$ corresponding to a smooth solutions of the Euler equations for compressible flows of an ideal gas solves a transport-collapse-type equation. The transport part is the transport of f in x direction with a fixed velocity v and the collapse part corresponds to the transport in v direction along a vector from the subdifferential of the indicator function of the set of maxwillians in the space of regular probability measures with the 2-Wasserstein metric.

0.1. **Gradient structure.** In [3], Brenier established a kinetic formulation for scalar conservation laws in which the solution is represented by a density function $Y(t,x,v)$, non-decreasing in v, and which solved the kinetic equation

$$Y_t + f'(v)Y_x + \partial K(Y) = 0, \tag{1}$$

interaction part, ∂K, is the subdifferential of a indicator functional of set K of non-decreasing in v function in the Hilbert space $L^2_{x,v}$. The equation is the continuous version of a time discrete transport-collapse scheme introduced in Brenier[2] and Giga-Miyakawa[5]. It also, somewhat refines the kinetic formulation of scalar conservation laws given in Lions-Perthame-Tadmor[7], by defining the interaction part explicitly in terms of the solution. This difference is particular critical for the theory of measure-valued solutions given in MP[10]. The formulation (1) gives a interesting geometric way of representing the solution operator: for a time step $h > 0$ the values $Y(t_0,\cdot)$ are transported with the constant velocity and then the function $Y(t_0, x - hf'(v), v)$ is projected to the closed, convex cone K to yield the solution $Y(t_0 + h, x, v)$ with an error $o(h)$.

In the present paper, we show that for smooth solutions of the Euler equations, its maxwellian density function solves the kinetic equation that has a structure similar to (1). For this we consider the problem in the metric space of the regular probability measures with 2-Wassersten metric. The usefulness of this approach was illustrated in Carlen-Gangbo[4] where the authors apply it to kinetic Fokker-Plank equation. There, the interaction part of the kinetic equation is represented by a gradient (in the space of measures) of a relative entropy functional. The idea of formulation evolutionary PDEs using the differential structures in the space of probability measures was pioneered in Otto[9], Jordan-Kinderleher-Otto[6].

Let us descried the result. Let (ρ, u, T) denote the density, velocity and the temperature of an ideal monatomic gas, and let $E = \rho|u|^2/2 + \rho T/2$ be the gas total energy. A maxwellian density corresponding to the state (ρ, u, T) is given by

$$f(v) = \frac{\rho}{\sqrt{2\pi T}} e^{-\frac{(v-u)^2}{2T}}. \tag{2}$$

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Through the paper we assume that $(\rho(t,x),u(t,x),T(t,x))$ is a smooth solution of the Euler equations in dimension 1 for a monatomic gas $(\gamma=3)$, on $[0,T]\times\mathbb{R}_x$:

$$\left\{\begin{array}{l}\rho_t+(\rho u)_x=0,\\ (\rho u)_t+(\rho u^2+\rho T)_x=0,\\ (\frac{1}{2}\rho u^2+\frac{1}{2}\rho T)_t+((\frac{1}{2}\rho u^2+\frac{1}{2}\rho T)u+\rho Tu)_x=0.\end{array}\right. \tag{3}$$

It is formally equivalent to the system moment equations:

$$\int\left[\begin{array}{c}1\\ v\\ \frac{v^2}{2}\end{array}\right](f_t+vf_x)\,dv=0.$$

Formally, the moment equations can be written as

$$f_t+(vf)_x+(\xi f)_v=0,$$

for some function ξ such that for all (t,x),

$$\int\xi f\,dv=\int v\xi f\,dv=0. \tag{4}$$

In fact, given a smooth solution (ρ,u,T) of (3) with $\rho,T>0$, an easy computation shows that

$$\xi=-\left(1-\frac{(v-u(t,x))^2}{T(t,x)}\right)\frac{T_x(t,x)}{2}, \tag{5}$$

and in particular, for all (t,x), $\xi\in L^2_f(\mathbb{R}_v)$ – an L^2 space with the weight f. This fact, together with orthogonality conditions (4), can be seen as the inclusion $\xi\in\partial\mathscr{M}(f/\rho)$ – the subdifferential of a convex functional $\mathscr{M}$ at the (normalized maxwellian) f/ρ, in the following functional framework. We consider $\mu=f/\rho$ as an element of a metric space of the probability densities with finite second moments, $\mathscr{P}^2_{reg}$, with the metric given by the Wasserstein distance $W_2(\mu,\eta)$. Here we will be using the standard notion of Ambrosio-Gigli-Savare[1]. The set of all functions of the type (2) form a set $\mathscr{M}$, closed in the topology defined by the metric. Additionally, set $\mathscr{M}$ is displacement convex, as was shown by McCann[8] and the its subdifferential is well defined. We recall, see section 10.1 of [1], that $\xi\in L^2_{\mu(t,x,\cdot)}$ belongs to the subdifferential $\mathscr{M}$ at point $\mu(t,x,\cdot)\in\mathscr{M}$ iff

$$\int\xi(v)(t^\eta_\mu(v)-v)\mu(v)\,dv\le 0, \tag{6}$$

for all $g\in\mathscr{M}$ and $t^g_f(v)$ – the optimal transport map from $f dv$ to $g dv$. The optimal map is a linear function of v, see [8]. In fact (we will use this later), if $\mu=\frac{1}{\sqrt{2\pi T_1}}e^{-\frac{(v-u_1)^2}{2T_1}}$, $\eta=\frac{1}{\sqrt{2\pi T_2}}e^{-\frac{(v-u_2)^2}{2T_2}}$ then

$$t^\eta_\mu(v)=\sqrt{\frac{T_2}{T_1}}v-u_1\sqrt{\frac{T_2}{T_1}}+u_2, \tag{7}$$

and

$$W_2^2(\mu,\eta)=(\sqrt{T_2}-\sqrt{T_1})^2+(u_2-u_1)^2. \tag{8}$$

Then, the definition (6) is equivalent to:

$$\int \xi(v)(c_1 v + c_2)\mu(v)\,dv = 0, \quad \forall c_1, c_2.$$

Thus, we identify $T_{\mathscr{M}}(\mu) = \{c_1 v + c_2\} \subset L^2_\mu(\mathbb{R}_v)$ – the tangent plane to $\mathscr{M}$ at μ and then, $\partial\mathscr{M}(\mu)$ – its orthogonal complement. In this way the Euler equations take a form

$$f_t + (vf)_x + (\xi f)_x = 0, \quad \xi(t,x,\cdot) \in \partial\mathscr{M}(\frac{f(t,x,\cdot)}{\rho(t,x)}). \tag{9}$$

Remark 1. The structure of the equation (9) also holds for multi-dimensional Euler equations and Navier-Stokes and artificial viscosity equations, see section 0.4 for details. It is rather a charteristic of a kinetic formulation, not a particular set of equations.

Formulation (9) shows that the kinetic solutions are the solutions with the "fastest relaxation rate".

Remark 2. The equation (9) can be considered as a transport equation for a constant mass density f on $\mathbb{R}_x \times \mathbb{R}_v$.

To illustrate the transport-collapse nature of the kinetic equation (9) let us define a normalized pure transport measure

$$\eta(t,x,v) = \frac{f(0,x-tv,v)}{\int f(0,x-vt,v)\,dv}.$$

It can be easily computed that for $\mu = f/\rho$,

$$\mu_t + (\xi_1\mu)_v = 0, \quad \xi_1 = (v-u)\frac{T_t}{2T} + u_t,$$

and at $t = 0$,

$$\eta_t + (\xi_2\eta)_v = 0, \quad \xi_2 = (v-u)\frac{T_t}{2T} + u_t - \left(1 - \frac{(v-u)^2}{T}\right)\frac{T_x}{2}.$$

For each $x \in \mathbb{R}_x$ and $t = 0$, vectors ξ_1, ξ_2 are the tangent vectors to curves $\mu(t)$ and $\eta(t)$, respectively. Comparing this with the expression for ξ in (5) we find that the following relation for the tangent vectors:

$$\xi_1 = \xi_2 + \xi. \tag{10}$$

Moreover, in the above notation, $\xi, \xi_1, \xi_2 \in L^2_{\mu(0)}$, $\xi_1 \in T_{\mathscr{M}}(\mu(0))$, $\xi \in \partial\mathscr{M}(\mu(0))$. Thus, the tangent vector to the curve $\mu(t)$, representing the (normalized) solution of the Euler equations, is computed from the tangent vector ξ_2 to the pure transport curve from the minimization problem:

$$\xi_1 = \underset{\tilde{\xi} \in T_{\mathscr{M}}(\mu(0))}{\operatorname{argmin}} \|\xi_2 - \tilde{\xi}\|_{L^2_{\mu(0)}}. \tag{11}$$

The value $\min_{\tilde{\xi} \in T_{\mathscr{M}}(\mu(0))} \|\xi_2 - \tilde{\xi}\|_{L^2_{\mu(0)}}$ can be thought off as the amount of an infinitesimal interaction. Thus, at least for smooth solutions of the Euler equations, the interaction is minimized. We note, that a similar property is also shared by all solutions of (1).

In the next section we show that at the time discrete level, (11) is equivalent to finding a maxwellian measure $\tilde{\mu}$ at the minimal distance from the transported measure $\eta(t,x,\cdot)$ to $\mathscr{M}$.

0.2. **Time discretization of the kinetic equation** (9)**.** Let $h>0$ be time step and (ρ,u,T) be a classical solution to the Euler equations. We abbreviate $\mu(t)=\mu(t,x,v)$, $\eta^h=\eta(h,x,v)$ where μ,η as above. We will prove the following theorem.

Theorem 1. *For any $h>0$, x, there is unique maxwellian $\tilde{\mu}^h=\tilde{\mu}(h,x,v)$ which minimizes*

$$\min\left\{W_2(\eta^h,\tilde{\mu})\ :\ \tilde{\mu}\in\mathscr{M}\right\}. \tag{12}$$

Let $T_{sup}=\sup_{t,x}T(t,x)$, $T_{inf}=\inf_{t,x}T(t,x)$, and $oscT=T_{sup}-T_{inf}$. There is an absolute constant $c_0>0$ such that if

$$\frac{oscT}{T_{inf}}\le c_0, \tag{13}$$

then, uniformly in x, it holds:

$$W_2(\mu(h),\tilde{\mu}^h)=O(h^2), \tag{14}$$

where $O(h^2)$ depends on C^2 norm of the solution (ρ,u,T) of (3)*. Moreover, if we define*

$$\tilde{\rho}^h=\int f(0,x-hv,v)\,dv,\quad \tilde{f}^h=\tilde{\rho}^h\tilde{\mu}^h, \tag{15}$$

then,

$$\sup_x\left|\int\begin{bmatrix}1\\ v\\ \frac{v^2}{2}\end{bmatrix}(f(h,x,v)-\tilde{f}^h(x,v))\,dv\right|=O(h^2). \tag{16}$$

The section 0.4 containes an analog fo the thorem 1 for multidimensional equations, but in somewhat greater generality; for example condition on the smallness of the oscillation (13) is not needed there. The difference in both theorems comes simply from the different approaches: in the proof of theorem 1 we use higher regularity of the optimal maps between measures defined in the scheme, while in the proof of theorem 2 we are using the estimates on the derivatives of certain moments of the optimal maps. But, certainly, the statement of theorem 2 also appies to lower dimensions.

Proof. Everywhere in the proof, for a probability density function μ, we use notation $d\mu=\mu\,dv$. Consider the minimization problem (12). Since $\mathscr{M}$ is compact and $W_2(\eta^h,\cdot)$ is lower semi-continuous, there a minimizer $\tilde{\mu}^h$. The uniqueness of the minimizer is not evident however, since the distance W_2 is not convex. To get around this difficulty, we use the fact that maxwellians are determined by their first two moments, which form minimizers can be computed explicitly. Let $t_{\mu^h}^{\eta^h}(v)$ be the optimal plan $\eta^h\,dv=t_{\mu^h}^{\eta^h}\#(\mu^h\,dv)$. It follows that $t_{\mu^h}^{\eta^h}(v)-v\in\partial\mathscr{M}(\mu^h)$, see Lemma 10.1.2 of [1]. In particular (see the previous section),

$$\int v\,d\mu^h=\int t_{\mu^h}^{\eta^h}(v)\,d\mu^h=\int v\,d\eta^h.$$

Denote the value of the minimal distance by d,

$$d = \int (t_{\mu^h}^{\eta^h}(v) - v)^2 \, d\mu^h = \int (t_{\mu^h}^{\eta^h}(v))^2 \, d\mu^h - \int v t_{\mu^h}^{\eta^h}(v) \, d\mu^h$$
$$= \int v^2 \, d\eta^h - \int v t_{\mu^h}^{\eta^h}(v) \, d\mu^h = \int v^2 \, d\eta^h - \int v^2 \, d\mu^h,$$

where the last equality holds because of the subdifferential condition for $t_{\mu^h}^{\eta^h}(v) - v$. Thus, the first and second moments of μ^h are uniquely defined by η^h, and the minimizer is unique.

The proofs of the following lemmas are technical and postponed to the appendix.

Lemma 1. *Let $\mu_0 = \mu(0,x,v)$. We abbreviate the optimal plan $t_{\mu_0}^{\eta^h}(h,x,v)$ as $t(h,v)$ or simply $t(v)$, and denote by $\xi_2(v) = \frac{t(v)-v}{h}$. Then, uniformly in x and $h \in (0,h_0]$ it holds:*

$$|t(v)| \leq C|v|. \tag{17}$$

For any $a > 0$ and sufficiently small $\frac{oscT}{T_{inf}}$,

$$\int |\partial_v t(v)| e^{-\frac{av^2}{T_{inf}}} \, dv \leq C, \quad \int |\xi_2(v)| e^{-\frac{av^2}{T_{inf}}} \, dv \leq C, \tag{18}$$

$$\int |\partial_{vv}^2 t(v)| e^{-\frac{av^2}{T_{inf}}} \, dv \leq C, \quad \int |\partial_v \xi_2(v)| e^{-\frac{av^2}{T_{inf}}} \, dv \leq C, \tag{19}$$

$$\int |\partial_h t(h,v)| e^{-\frac{av^2}{T_{inf}}} \, dv \leq C, \tag{20}$$

$$\int |\partial_{hh}^2 t(h,v)| e^{-\frac{av^2}{T_{inf}}} \, dv \leq C. \tag{21}$$

In (17), (18) *C depends on $\|(\rho_0,u_0,T_0)\|_C$ and $\inf \rho_0(x)$, $\inf T_0(x)$. In* (19), (20) *C depends on $\|(\rho_0,u_0,T_0)\|_{C^1}$ and $\inf \rho_0(x)$, $\inf T_0(x)$ and in* (21) *C depends on $\|(\rho_0,u_0,T_0)\|_{C^2}$ and $\inf \rho_0(x)$, $\inf T_0(x)$.*

Estimates in parts (19)–(21) *also hold with weights $|v|^k e^{-\frac{av^2}{T_{inf}}}$, $k > 0$, and with $|\partial_v t|, |\partial_v \xi_2|, |\xi_2|$, $|\partial_h t|, |\partial_{hh}^2 t|$, replaced by their squares.*

Lemma 2. *In the notation of Lemma 1, let*

$$\lim_{h \to 0+} \xi_2(h,v) = \lim_{h \to 0+} \frac{t(h,v) - v}{h} = \xi_{2,0}.$$

For any $a > 0$ and sufficiently small $\frac{oscT}{T_{inf}}$,

$$\int |\xi_2(h,v) - \xi_{2,0}|^2 e^{-\frac{av^2}{T_{inf}}} \, dv = O(h^2), \tag{22}$$

where $O(h^2)$ depends on $\|(\rho_0,u_0,T_0)\|_{C^2}$ and $\inf \rho_0(x)$, $\inf T_0(x)$ and is uniform in x.

For the probability densities μ_0, η^h and $\tilde\mu^h$ we denote by $t_{\mu_0}^{\eta^h}$, $t_{\tilde\mu^h}^{\eta^h}$, $t_{\tilde\mu^h}^{\mu_0}$ and $t_{\mu_0}^{\tilde\mu^h}$ the corresponding optimal plans. We also identify the following velocities $\tilde\xi = \frac{t_{\tilde\mu^h}^{\eta^h}(v)-v}{h}$, $\tilde\xi_2 = \frac{t_{\mu_0}^{\eta^h}(v)-v}{h}$, $-\hat\xi_1 = \frac{t_{\tilde\mu^h}^{\mu_0}(v)-v}{h}$ and $\tilde\xi_1^h = \frac{t_{\mu_0}^{\tilde\mu^h}(v)-v}{h}$.

Using the formula (7) for the optimal plan between two maxwellians and the fact that $W_2(\mu_0, \tilde\mu^h) = O(h)$, one can show that

$$\check\xi_1^h + \tilde\xi_1^h = h(c_1 + c_2 v), \tag{23}$$

for some uniformly bounded functions $c_i = c_i(h,x)$. Also, as was shown above, $\tilde\xi_1^h \in T_{\mathscr{M}}(\tilde\mu^h)$ and $\tilde\xi^h \in \partial\mathscr{M}(\tilde\mu^h)$. Clearly, $t_{\tilde\mu^h}^{\eta^h}(v) = t_{\mu_0}^{\eta^h}(t_{\tilde\mu^h}^{\mu_0}(v))$. Using this we can write

$$\tilde\xi^h = \tilde\xi_2^h + \partial_v t_{\mu_0}^{\eta^h}(v)(-\tilde\xi_1^h(v)) + \partial_v^2 t_{\mu_0}^{\eta^h}(\tilde v)\frac{(t_{\tilde\mu^h}^{\mu_0}(v)-v)^2}{2h}, \tag{24}$$

for some $\tilde v \in Int[v, t_{\mu_0}^{\eta^h}(v)]$. Since $\partial_v t_{\mu_0}^{\eta^h} + h\partial_v \tilde\xi_2^h$, and using (23) we obtain that

$$\hat\xi_2^h \equiv \tilde\xi_2^h + h\partial_v\tilde\xi_2^h(v)(-\tilde\xi_1^h(v)) + \partial_v^2 t_{\mu_0}^{\eta^h}(\tilde v)\frac{(t_{\tilde\mu^h}^{\mu_0}(v)-v)^2}{2h} + h(c_1+c_2 v) = \tilde\xi^h + \tilde\xi_1^h. \tag{25}$$

Vector $\hat\xi_2^h$ is a good approximation of ξ_2; by using Lemma 1 and Lemma 2 we can estimate

$$\begin{aligned} \int |\hat\xi_2^h - \xi_2|^2\, d\mu_0 \ \ \le \ \ & \int |\tilde\xi_2^h - \xi_2|^2\, d\mu_0 \\ & + \int \left| h\partial_v\tilde\xi_2^h(v)(-\tilde\xi_1^h(v)) + \partial_v^2 t_{\mu_0}^{\eta^h}(\tilde v)\frac{(t_{\tilde\mu^h}^{\mu_0}(v)-v)^2}{2h} + h(c_1+c_2v)\right| d\mu_0 = O(h^2), \end{aligned} \tag{26}$$

uniformly in x.

Summarizing our findings we: we have two orthogonal decompositions

$$\xi_2 = \xi + \xi_1, \quad \xi \in \partial\mathscr{M}(\mu_0),\, \xi_1 \in T_{\mathscr{M}}(\mu_0), \tag{27}$$

$$\hat\xi_2^h = \tilde\xi^h + \tilde\xi_1^h, \quad \tilde\xi^h \in \partial\mathscr{M}(\tilde\mu^h),\, \tilde\xi_1^h \in T_{\mathscr{M}}(\tilde\mu^h). \tag{28}$$

Moreover the base points μ_0, $\tilde\mu^h$ are close:

$$W_2(\mu_0, \tilde\mu^h) = O(h), \tag{29}$$

and so are the vectors ξ_2 and $\hat\xi_2^h$, see (26). We claim now that

$$\int |\xi_1 - \tilde\xi_1^h|^2\, d\mu_0,\ \int |\xi - \tilde\xi^h|^2\, d\mu_0 = O(h^2). \tag{30}$$

To show this we start by identifying ξ_1 and $\tilde\xi_1^h$ are the solutions of a minimization problem

$$\min\{\int |\xi_2 - \xi_1|^2\, d\mu_0 \ :\ \xi \in T_{\mathscr{M}}(\mu_0)\}$$

and

$$\min\{\int |\hat\xi_2 - \xi_1|^2\, d\tilde\mu^h \ :\ \xi \in T_{\mathscr{M}}(\tilde\mu^h)\}$$

respectively. The minimizer $\xi_1 = c_1 + c_2 v$, is determined from the linear system

$$\begin{bmatrix} \int d\mu_0 & \int v d\mu_0 \\ \int v d\mu_0 & \int v^2 d\mu_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int \xi d\mu_0 \\ \int \xi_2 v d\mu_0 \end{bmatrix},$$

or

$$\begin{bmatrix} 1 & u_0 \\ u_0 & |u_0|^2 + T_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int \xi_2 d\mu_0 \\ \int \xi_2 v d\mu_0 \end{bmatrix}.$$

Note, that the system is non-singular, provided that $T_0 > 0$. Similarly, for $\tilde{\xi}_1^h = \tilde{c}_1 + \tilde{c}_2 v$:

$$\begin{bmatrix} 1 & \tilde{u}^h \\ \tilde{u}^h & |\tilde{u}^h|^2 + \tilde{T}^h \end{bmatrix} \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} = \begin{bmatrix} \int \hat{\xi}_2^h d\tilde{\mu}^h \\ \int \hat{\xi}_2^h v d\tilde{\mu}^h \end{bmatrix}.$$

Because of (29),

$$(\sqrt{T_0} - \sqrt{\tilde{T}^h})^2 + (u_0 - \tilde{u}^h)^2 = W_2^2(\mu_0, \tilde{\mu}^h) = O(h^2),$$

and the coefficients of both system differ by $O(h)$. And so is the right-hand side. Indeed, by (29) and (26) we can write

$$\left| \int \xi_2 d\mu_0 - \int \hat{\xi}_2^h d\tilde{\mu}^h \right| \leq \int |\xi_2 - \tilde{\xi}_2^h| d\mu_0 + \left| \int \hat{\xi}_2^h (d\mu_0 - d\tilde{\mu}^h) \right| = O(h),$$

and similarly $\left| \int \xi_2 v d\mu_0 - \int \hat{\xi}_2^h v d\tilde{\mu}^h \right| = O(h)$. It follows that $|c_1 - \tilde{c}_1|, |c_2 - \tilde{c}_2| = O(h)$, which establishes the first part of (30). The second is immediate from (26).

To conclude the proof we define $\check{\mu}^h = (v + h\xi_1)\# d\mu_0$ and estimate

$$W_2^2(\mu(h), \tilde{\mu}^h) \leq W_2^2(\mu(h), \check{\mu}^h) + W_2^2(\check{\mu}^h, \tilde{\mu}^h). \tag{31}$$

Let us estimate the distance

$$W_2^2(\mu(h), \check{\mu}^h) \leq h^2 \int \left| \frac{t_{\mu_0}^{\mu(h)}(v) - v}{h} - \xi_1 \right|^2 d\mu_0. \tag{32}$$

The optimal plan from $\mu_0 = \mu(0)$ to $\mu(h)$ equals

$$t_{\mu_0}^{\mu(h)} = \left[\frac{T(h,x)}{T_0(x)} \right]^{\frac{1}{2}} v - \left[\frac{T(h,x)}{T_0(x)} \right]^{\frac{1}{2}} u_0(x) + u(h,x),$$

and the limit $\lim_{h\to 0} \frac{t_{\mu_0}^{\mu(h)}(v) - v}{h} = \xi_1$. Then,

$$\left| \frac{t_{\mu_0}^{\mu(h)}(v) - v}{h} - \xi_1 \right| \leq C(\|\partial_t^2 (u, T)\|_{C^0}, \inf T)(1 + |v|)h. \tag{33}$$

Combining this with (32) we see that

$$W_2(\mu(h), \check{\mu}^h) \leq O(h^2). \tag{34}$$

Now we continue (31):

$$\begin{aligned} W_2^2(\mu(h),\tilde{\mu}^h) &\leq O(h^4)+h^2\int\left|\frac{t_{\mu_0}^{\tilde{\mu}^h}(v)-v}{h}-\xi_1(v)\right|^2 d\mu_0 \\ &= O(h^4)+h^2\int|\tilde{\xi}_1^h-\xi_1|^2\,d\mu_0 = O(h^4), \end{aligned}$$

which established (14).

Finally, the error on the moments, (16), follows from (34) for v and v^2 moments. Consider now

$$\begin{aligned} \tilde{\rho}^h(x)-\rho(h,x) &= \int(f(0,x-hv,v)-f(h,x,v))\,dv \\ &= -h\int(f_t+vf_x)(0,x,v)\,dv+O(h^2) = h\int(\xi f)_v\,dv+O(h^2) = O(h^2). \end{aligned}$$

□

0.3. **Appendix. Proof of Lemma 1.** Optimal plan $t(v)$ is a solution to the minimal mass transport problem from measure $d\mu_0$ to $d\eta^h$. It is a non-decreasing function $\mathbb{R}_v \to \mathbb{R}_v$ and such that $\mu_0(v) = \eta^h(t(v))\partial_v t(v)$, which we consider as an ODE

$$\frac{dt}{dv} = \frac{\tilde{\rho}^h(x)}{\rho_0(x-ht)}\left[\frac{T_0(x-ht)}{T_0(x)}\right]^{\frac{1}{2}}\exp\left(-\frac{(v-u_0(x))^2}{2T_0(x)}+\frac{(t-u_0(x-ht))^2}{2T_0(x-ht)}\right), \tag{35}$$

where $\tilde{\rho}^h$ is defined in (15). For each h, the solution $t(h,\cdot)$ is a $C^1(\mathbb{R}_v)$ function if $(\rho_0,u_0,T_0) \in C(\mathbb{R}_x)^3$ and $\inf\rho_0, \inf T_0 > 0$. To prove (17) let us show first that $\forall\varepsilon > 0$, there is $V(\varepsilon) > 0$ such that

$$|t(v)| \leq \sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)|v|, \quad |v| > V(\varepsilon). \tag{36}$$

Once this is known, (17) follows since $t(v)$ is monotone. Arguing by contradiction, suppose that there is $\varepsilon > 0$, and a sequence (v_n,h_n) with $v_n > n$, such that

$$t(v_n,h_n) > \sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)v_n. \tag{37}$$

Let $[v_n, v_n+\delta_n]$ be the maximal interval on which (37) hold, i.e., either $v_n+\delta_n = +\infty$ or $t(v_n+\delta_n,h_n) = \sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)(v_n+\delta_n)$. It follows from (35) that for any $C_1 > 0$ and sufficiently large $|v| > n$, (independently of h),

$$\frac{dt}{dv} \geq C_1, \quad v \in [v_n, v_n+\delta_n],$$

and

$$\begin{aligned} t(v,h_n) &\geq C_1(v-v_n)+t(v_n,h_n) \geq C_1(v-v_n)+\sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)v_n \\ &= \left(C_1-\sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)\right)v_n+\sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)v. \end{aligned}$$

If we take $C_1 > \sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)$ in the last inequality and assume $v = v_n+\delta_n < +\infty$, we obtain that $t(v_n+\delta_n,h_n) > \sqrt{\frac{T_{sup}}{T_{inf}}}(1+\varepsilon)(v_n+\delta_n)$, contradicting the maximality of the interval $[v_n, v_n+\delta_n]$. This shows that $v_n+\delta_n = +\infty$. This implies that for all $v \in [v_n,+\infty)$,

$$\sqrt{\frac{T_0(x)}{T_0(x-h_nt)}}t(v,h_n) \geq \frac{\varepsilon}{1+\varepsilon}\sqrt{\frac{T_{sup}}{T_{inf}}}t(v,h_n)+v,$$

for large v's. It follow from (35) then that there are positive constants $c_0, c > 0$ such that

$$\frac{dt}{dv} \geq ce^{c_0t^2},$$

for all large v's. This inequality however implies that $t(v)$ blowups in finite time, contradicting the fact that it is a monotone function $\mathbb{R}_v \to \mathbb{R}_v$. Thus, (36) and (17) are proved.

In the prove of the subsequent estimates we will use (36) which shows that $t(v)$ is close to v if $oscT/T_{inf}$ is small. Consider the argument of the exponential in (35)

$$-\frac{(v-u_0(x))^2}{2T_0(x)}+\frac{(t-u_0(x-ht))^2}{2T_0(x-ht)} \leq -\frac{1}{2T_0(x)}\left(v^2\left(1-\frac{u_0(x)}{v}\right)^2-\frac{T_{sup}}{T_{inf}}t^2\left(1-\frac{u_0(x-ht)}{t}\right)^2\right).$$

For any $a>0$ we choose $\varepsilon = a/8$, and $V(\varepsilon)$ as in (36) and estimate

$$\begin{aligned} |\partial_v t|e^{-\frac{av^2}{T_{inf}}} \leq C\exp\Big\{ & -\frac{1}{2T_0(x)}\Big(a+(1-\frac{u_0(x)}{v}) \\ & -(1+\frac{oscT}{T_{inf}})^3(1+\frac{a}{8})(1-\frac{u_0(x-ht)}{t})\Big)v^2\Big\}, \end{aligned}$$

for some $C>0$ independent of (h,v). The first estimate in (18) follows provided that $\frac{oscT}{T_{inf}}$ is small enough (independently of (h,v)).

The first estimate in (19) follows by differentiating (35) in v and using the same argument as in (18).

For the second estimate in (19) consider

$$\partial_v\xi = \frac{\partial_v t-1}{h},$$

compute $\partial^2_{vh}t$ from (35) and arguing as above estimate

$$\int |\partial^2_{vh}t|e^{-\frac{av^2}{T_{inf}}}\,dv \leq C, \quad (h,x)\in[0,h]\times\mathbb{R}_x,$$

for some $C>0$ depending on C^1 norm of (ρ_0,u_0,T_0) and $\inf\rho_0$, $\inf T_0$, and the second estimate in (19) follows. To prove the second estimate in (18) we write use the change of variables

$$\int_v^{+\infty}\eta^h(t(h,s))\partial_s t(s)\,ds=\int_v^{+\infty}\mu_0(s)\,ds$$

from which we conclude

$$\int_{t(h,v)}^{v}\eta^h(s)\,ds=\int_v^{+\infty}|\eta^h(s)-\eta^0(s)|\,ds, \tag{38}$$

and

$$|t(h,v)-v|\eta^h(\tilde v)\le\int_v^{+\infty}|\eta^h(s)-\eta^0(s)|\,ds,\ \tilde v\in Int[v,t(h,v)]. \tag{39}$$

On the other hand the definition of η^h, implies

$$|\eta^h(s)-\eta^0(s)|\le C|s|h\exp\{-\frac{s^2}{2T_0(x-\tilde hs)}(1-\frac{u_0(x-\tilde hs)}{s})^2\},$$

for some $\tilde h\in[0,h]$. It is immediate from the last two estimates that for small values of v's, $|v|\le V$,

$$|\xi_2(h,v)|=|t(h,v)-v|h^{-1}\le C(V),$$

and for large values of v's and any $\varepsilon>0$,

$$|t(h,v)-v|h^{-1}\le C(\varepsilon)\exp\{\frac{(1+\varepsilon)v^2}{2T_{inf}}-\frac{(1-\varepsilon)v^2}{2T_{sup}}\}.$$

Given $a>0$ we choose a suitable small $\varepsilon>0$ and $\frac{oscT}{T_{inf}}$, we arrive at the second estimate in (18).

We will use (38) to show that $t(h,v)$ is twice differentiable in h. Indeed, for $h_1,h_2\in[0,h_0]$ we can write

$$\int_{t(h_1,v)}^{t(h_2,v)}\eta(h_1,s)\,ds=\int_{t(h_2,s)}^{+\infty}(\eta(h_2,s)-\eta(h_1,s))\,ds.$$

Dividing by h_2-h_1 and taking the limit $h_1\to h_2$ we find the $\partial_h t$ exists and equals

$$\partial_h t(h,v)=\frac{1}{\eta(h,v)}\int_{t(h,v)}^{+\infty}\partial_h\eta(h,s)\,ds. \tag{40}$$

Now, the estimate in (20) follows by the same arguments as above. If $\eta(h,s)$ has continuous second derivative in h, which happens if (ρ_0,u_0,T_0) is in $C^2(\mathbb{R})$, then $\partial^2_{hh}t(h,v)$ exists and can be computed explicitly from (40). The estimate in (21) follows then in the same manner as in (20).

The rest of the lemma can be proved using a combination of above arguments.

Proof of Lemma 2. For any fixed v the map $t(h,v)\in C^2([0,h_0])$ and there are uniform estimates on $\partial_h t$ and $\partial^2_{hh}t$, given in Lemma 1. Since $\frac{t(h,v)-v}{h}=\frac{t(h,v)-t(0,v)}{h}\to\partial_h t(0,v)=\xi_{2,0}$, and the estimate follows from (21).

0.4. **3D-case.** We consider the classical solutions to the Euler equations descirbing the motion of a monatomic gas in all of the space $\mathbb{R}^3_x$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla(\rho T) = 0, \\ \partial_t(\rho T) + \operatorname{div}(\rho T) + (\gamma - 1)\rho T \operatorname{div} u = 0, \end{cases} \tag{41}$$

with $\gamma = \frac{5}{3}$. Setting

$$f(t,x,v) = \frac{\rho(t,x)}{(2\pi T(t,x))^{\frac{3}{2}}} e^{-\frac{|v-u(t,x)|^2}{2T(t,x)}}, \tag{42}$$

the maxwellian corresponding to the solution (ρ, uT), the following kinetic representation holds:

$$f_t + v \cdot \nabla_x f + \operatorname{div}_v(\xi f) = 0, \tag{43}$$

where

$$\xi = -\left(3 - \frac{|v-u|^2}{T}\right)\frac{\nabla T}{2} + (v-u)^t \left(\mathbb{D} - \frac{1}{3} \operatorname{tr}(\mathbb{D})\mathbb{I}\right), \tag{44}$$

and $\mathbb{D} = (\nabla_x u + \nabla_x^t u)/2$, and $\operatorname{tr}(\mathbb{D})$ stands for its trace. This representation is verifeid by a direct computation, given a smooth solution (ρ, u, T).

By $\mathscr{M} \subset \mathscr{P}^2_{reg}(\mathbb{R}^3_v)$ we denote the set of maxwellians on $\mathbb{R}^3_v$:

$$\mathscr{M} = \left\{ \frac{f}{\rho} = \frac{1}{(2\pi T)^{\frac{3}{2}}} e^{-\frac{|v-u|^2}{2T}} : T \in \mathbb{R}^1_+, u \in \mathbb{R}^3 \right\}.$$

$\mathscr{M}$ is closed, displacement convex subset of $\mathscr{P}^2_{reg}(\mathbb{R}^3_v)$, with the metric defined by the quadratic Wasserstein distance $W_2(\mu, \nu)$. The subdifferential to its indicator function at point $\mu \in \mathscr{M}$, $\partial\mathscr{M}(\mu)$, is defined as all $\xi \in L^2(\mathbb{R}^3; \mu)$ such that

$$\int \xi \cdot (\alpha v + \beta)\mu \, dv = 0, \quad \alpha \in \mathbb{R}, \beta \in \mathbb{R}^3, \tag{45}$$

see for example [1], Section 10.1. In this notation one verifies easily that for ξ from (44), for all (t,x), $\xi(t,x,\cdot) \in \partial\mathscr{M}(f(t,x,\cdot)/\rho(t,x))$. The tangent plane to $\mathscr{M}$ at μ is defined as

$$T_{\mathscr{M}}(\mu) = \{\alpha v + \beta : \alpha \in \mathbb{R}, \beta \in \mathbb{R}^3.\} \tag{46}$$

In fact, if $\mu = \frac{1}{(2\pi T_1)^{\frac{3}{2}}} e^{-\frac{|v-u_1|^2}{2T_1}}$, $\eta = \frac{1}{(2\pi T_2)^{\frac{3}{2}}} e^{-\frac{|v-u_2|^2}{2T_2}}$ then

$$t^\eta_\mu(v) = \sqrt{\frac{T_2}{T_1}} v - u_1 \sqrt{\frac{T_2}{T_1}} + u_2, \tag{47}$$

and

$$W_2^2(\mu, \eta) = (\sqrt{T_2} - \sqrt{T_1})^2 + (u_2 - u_1)^2. \tag{48}$$

This space consists of all tangent vectors to curves in $\mathscr{M}$ passing through μ.

Given $f(t,x,v)$ in (42) define $\mu(t)=\mu(t,x,v)=f/\rho(t,x)\in\mathscr{M}$. A direct computation shows that

$$\partial_t\mu+\operatorname{div}_v(\xi_1\mu)=0,\quad \xi_1=(v-u)\frac{T_t}{2T}+\frac{u_t}{T}. \tag{49}$$

Denote the kinetic function representing the normalized transport in x-direction by

$$\eta^h=\eta(h,x,v)=\frac{f(0,x-hv,v)}{\tilde\rho},\ \tilde\rho^h=\int f(0,x-hv,v)\,dv. \tag{50}$$

Then at $h=0$,

$$\partial_h\eta^h+\operatorname{div}_v(\xi_2\eta^h),\quad \xi_2=\xi_1-\xi. \tag{51}$$

Note that $\xi,\xi_1,\xi_2\in L^2(\mathbb{R}^3;\mu(0))$ and ξ is orthogonal to ξ_1 in that space, i.e.

$$\xi_1=\operatorname{argmin}\{\|\xi_2-\tilde\xi\|_{L^2(\mathbb{R}^3;\mu(0))}\,:\,\tilde\xi\in T_{\mathscr{M}}(\mu(0))\}. \tag{52}$$

Remark 1. *The kinetic function $f(t,x,v)$ corrsponding to the Navier-Stokes equations verifies,*

$$f_t+v\cdot\nabla_x f-\Delta_x(f/\rho)+\,div_v(\xi f)=0,$$

and for the artificial viscosity equations:

$$f_t+v\cdot\nabla_x f-\Delta_x(f)+\,div_v(\xi f)=0,$$

with $\xi\in\partial\mathscr{M}(f/\rho)$, in both cases.

Our main result gives an infinitesimal characterisation of (52).

Theorem 2. *Let $(\rho,u,T)\in C^2_{t,x}([0,T_0]\times\mathbb{R}^3)$ be a solution of the Euler equations with*

$$\|(\rho,u,T)\|_{C^2_{t,x}}<+\infty \tag{53}$$

$$\inf\rho=\inf_{[0,T_0]\times\mathbb{R}^3}\rho(t,x)>0, \tag{54}$$

$$\inf T=\inf_{[0,T_0]\times\mathbb{R}^3}T(t,x)>0. \tag{55}$$

Let $f(t,x,v)$, be as in (42)*, $\mu^h=f(h,x,v)/\rho(h,v)$, and η^h be defind by* (50)*. For any $(h,x)\in[0,T_0]\times\mathbb{R}^3$ there is a unique minimizer $\tilde\mu^h$ of*

$$\min\left\{W_2(\eta^h,\tilde\mu)\,:\,\tilde\mu\in\mathscr{M}\right\}, \tag{56}$$

and uniformly in $x\in\mathbb{R}^3$,

$$W_2(\mu^h,\tilde\mu^h)=O(h^2). \tag{57}$$

Additionally, for $\tilde\rho^h$ defined by (50) *and $\tilde f(h,x,v)=\tilde\rho^h\tilde\mu^h$, uniformly in $x\in\mathbb{R}^3$,*

$$\int\begin{bmatrix}1\\ v_i\\ |v|^2\end{bmatrix}(\tilde f(h,x,v)-f(h,x,v))\,dv=O(h^2),\ i=1..3. \tag{58}$$

In the above theorem the statement $\phi(h,x) = O(h^k)$ uniformly in x, is understood as

$$\limsup_{h\to 0+} \sup_{x\in\mathbb{R}^3} \frac{|\phi(h,x)|}{h^k} < +\infty.$$

Proof. Thoughout the proof we use the convention that for the probability density μ, $\int \phi(v)\,d\mu = \int \phi(v)\mu(v)\,dv$.

The existence of minimizer in (56) follows from the compactness of $\mathscr{M}$ in $\mathscr{P}^2_{reg}(\mathbb{R}^3_v)$ with the metric $W_2(\mu,\nu)$. Let us show that the minimizer in unique. An element μ of $\mathscr{M}$ is uniquelly determined by its moments $\int v_i\mu\,dv$, $i = 1..3$, and $\int |v^2|\mu\,dv$. Let $\mu \in \mathscr{M}$ and donte by $t^{\eta^h}_{\tilde{\mu}^h}(v)$, $t^{\mu}_{\tilde{\mu}^h}(v)$ the optimal maps from $\tilde{\mu}^h$ to η^h and from $\tilde{\mu}^h$ to μ, respectivelly. Also choose a map $t(v)$ that takes μ to η^h and such that

$$t\circ t^{\mu}_{\tilde{\mu}^h}(v) = t^{\eta^h}_{\tilde{\mu}^h}(v),\, v\in\mathbb{R}^3.$$

The map t with this property exists because $t^{\mu}_{\tilde{\mu}^h}(v) = \alpha v+\beta$, for some $\alpha\in\mathbb{R}_+$ and $\beta\in\mathbb{R}^3$, see (47). Since $W_2^2(\tilde{\mu}^h,\eta^h) \le W_2^2(\mu,\eta)$ we obtain:

$$W_2^2(\tilde{\mu}^h,\eta^h) = \int |t^{\eta^h}_{\tilde{\mu}^h}(v)-v|^2\,d\tilde{\mu}^h \le \int |t(v)-v|^2\,d\mu = \int |t^{\eta^h}_{\tilde{\mu}^h}(v) - t^{\mu}_{\tilde{\mu}^h}(v)|^2\,d\tilde{\mu}^h,$$

from which we conclude that

$$\int |t^{\mu}_{\tilde{\mu}^h}(v)-v|^2\,d\tilde{\mu}^h + 2\int (t^{\eta^h}_{\tilde{\mu}^h}(v)-v)\cdot(v-t^{\mu}_{\tilde{\mu}^h}(v))\,d\tilde{\mu}^h \ge 0.$$

Using again the fact that $t^{\mu}_{\tilde{\mu}^h}(v) = \alpha v+\beta$, for $\alpha\in\mathbb{R}_+$ and $\beta\in\mathbb{R}^3$, we obtain the following orthogonality condition:

$$\int (t^{\eta^h}_{\tilde{\mu}^h}(v)-v)\cdot(\alpha v+\beta)\,d\tilde{\mu}^h = 0, \quad \forall\alpha\in\mathbb{R},\, \beta\in\mathbb{R}^3. \tag{59}$$

From this we conlude that

$$\int v_i\,d\eta^h = \int v_i\,d\tilde{\mu}^h,\, i = 1..3, \tag{60}$$

and

$$\begin{aligned} d^2 = W_2^2(\eta^h,\tilde{\mu}^h) &= \int |t^{\eta^h}_{\tilde{\mu}^h}(v)-v|^2\,d\tilde{\mu}^h = \int |v|^2\,d\eta^h + \int |v|^2\,d\tilde{\mu}^h - 2\int t^{\eta^h}_{\tilde{\mu}^h}(v)\cdot v\,d\tilde{\mu}^h \\ &= \int |v|^2\,d\eta^h + \int |v|^2\,d\tilde{\mu}^h - 2\int (t^{\eta^h}_{\tilde{\mu}^h}(v)\pm v)\cdot v\,d\tilde{\mu}^h = \int |v|^2\,d\eta^h - \int |v|^2\,d\tilde{\mu}^h, \end{aligned}$$

That is

$$\int |v|^2\,d\tilde{\mu}^h = \int |v|^2\,d\eta^h - d^2. \tag{61}$$

This and (60) determines $\tilde{\mu}^h$ uniquelly.

Now we will prove (57). For that consider a map $t(v) = t^{\eta^h}_{\eta^0}\circ t^{\eta^0}_{\tilde{\mu}^h}(v)$ that takes $\tilde{\mu}^h$ to η^h.

Claim 1. *t is the optimal transport map from $\tilde{\mu}^h$ to η^h.*

Proof. Indeed, since $t^{\eta^h}_{\eta^0}$ is optimal, there is a convex function $\psi_1 \in W^{1,2}(\mathbb{R}^3;\mu_0)$ such that $t^{\eta^h}_{\eta^0}(v) = \nabla\psi_1(v)$. Also, the optimal map $t^{\eta^0}_{\tilde\mu^h} = \alpha v+\beta$, for some $\alpha\in\mathbb{R}_+$ and $\beta\in\mathbb{R}^3$. Thus, $t(v) = \nabla\psi_1(\alpha v+\beta) = \nabla(\alpha^{-1}\psi_1(\alpha v+\beta))$ – the gradient of a convex function. Using the uniqueness of the Brenier's polar decomposition, we conclude that $t(v)$ is optimal. □

The claim allows us to write

$$t^{\eta^h}_{\tilde\mu^h}(v) = t^{\eta^h}_{\eta^0}\circ t^{\eta^0}_{\tilde\mu^h}(v). \tag{62}$$

Lets make the notation

$$\xi^h = -\frac{t^{\eta^h}_{\tilde\mu^h}(v)-v}{h},\ \xi^h_2 = \frac{t^{\eta^h}_{\eta^0}(v)-v}{h},\ \xi^h_1 = -\frac{t^{\eta^0}_{\tilde\mu^h}(v)-v}{h}. \tag{63}$$

We write (62) as

$$-\xi^h(v) = \xi^h_2(t^{\eta^0}_{\tilde\mu^h}) - \xi^h_1(v). \tag{64}$$

Since ξ_1 is linear in v, the orthogonality condition (59) implies that

$$\int \xi^h\cdot\xi^h_1\,d\tilde\mu^h = 0. \tag{65}$$

Let $\xi_2(x,v)$ be defined through (51) that is

$$\mathrm{div}_v(\xi_2\eta^0) = -\partial_h\eta^h\Big|_{h=0}.$$

Lemma 3. *Let $\phi(v)$ be a smooth function with a polynomial growth at infinity. Then,*

$$\int\phi(v)(\eta^h-\eta^0)\,dv = O(h), \tag{66}$$

and

$$\int(\phi(v)\eta^h-\phi(v)\eta^0-h\nabla\phi\cdot\xi_2\eta^0)\,dv = O(h^2), \tag{67}$$

uniformly in $x\in\mathbb{R}^3$.

Proof. Expression on the left in (67) can be written as

$$\int\phi(v)(\eta^h-\eta^0-\partial_h\eta^h\Big|_{h=0}h)\,dv.$$

Since η^h is given by (50) and (ρ,u,T) as in the statement of the theorem, the both estimates follows immidiately. □

The next lemma is most technically involved and we postpone its proof to the end of the proof.

Lemma 4. *Uniformly in $x\in\mathbb{R}^3$,*

$$W_2(\eta^h,\eta^0) = O(h). \tag{68}$$

Lemma 5. *Denote by $T_{\tilde\mu^h}$, T_{η^0} the variancies $\int |v|^2 d\tilde\mu^h - |\int v\, d\tilde\mu^h|^2$ and $\int |v|^2 d\eta^0 - |\int v\, d\eta^0|^2$. Then,*

$$W_2(\tilde\mu^h,\eta^0) = \left[(\sqrt{T_{\eta^0}} - \sqrt{T_{\tilde\mu^h}})^2 + |\int v\, d\eta^0 - \int v\, d\tilde\mu^h|^2\right]^{\frac{1}{2}} = O(h). \tag{69}$$

For all sufficiently small h, but independent of x,

$$\inf T_{\tilde\mu^h} \ge \frac{1}{2}\inf T, \quad \inf \tilde\rho^h \ge \frac{1}{2}\inf\rho, \tag{70}$$

$$\sup|\int v\, d\tilde\mu^h| \le \frac{3}{2}\sup|u|. \tag{71}$$

Proof. The first equality in (69) follows from the formula (48) of the optimal map from $\tilde\mu^h$ to η^0, since both of them are in $\mathscr{M}$. The first moment $\int v\, d\tilde\mu^h = \int v\, d\eta^h$, by (60), and by the estimate (66) we conclude that

$$|\int v\, d\tilde\mu^h - \int v\, d\eta^0| = O(h).$$

On the other hand, by (61) $\int |v|^2 d\tilde\mu^h - \int |v|^2 d\eta^0 = \int |v|^2 d\eta^h - \int |v|^2 d\eta^0 - d^2$ and by (66) and (68)

$$\int |v|^2 d\eta^h - \int |v|^2 d\eta^0 - d^2 = O(h).$$

The estimate (69) then follows, as well as (70), (71). □

Set

$$\tilde\xi_2^h(v) = \xi_2^h(t_{\tilde\mu^h}^{\eta^0}(v)).$$

Lemma 6. *Uniformly in $x \in \mathbb{R}^3$,*

$$\int \tilde\xi_2^h\, d\tilde\mu^h - \int \xi_2\, d\eta^0 = O(h), \tag{72}$$

$$\int v\cdot\tilde\xi_2^h\, d\tilde\mu^h - \int v\cdot\xi_2\, d\eta^0 = O(h). \tag{73}$$

Proof. Let $\phi(v)$ a function v_i, $i = 1..3$, or $|v|^2$. Using the fact that $(t_{\tilde\mu^h}^{\eta^0})^{-1} = t_{\eta^0}^{\tilde\mu^h}$, since transformations are linear, the followoing indentity hold

$$\begin{aligned}\int \nabla\phi\cdot\tilde\xi_2^h\, d\tilde\mu^h - \int\nabla\phi\cdot\xi_2\, d\eta^0 &= \int\nabla\phi(t_{\eta^0}^{\tilde\mu^h}(v))\cdot\xi_h^2\, d\eta^0 - \int\nabla\phi\cdot\xi_2\, d\eta^0\\ &= \int\nabla\phi(v)\cdot\xi_h^2\, d\eta^0 - \int\nabla\phi\cdot\xi_2\, d\eta^0 + \int(\nabla\phi(t_{\eta^0}^{\tilde\mu^h}) - \nabla\phi(v))\cdot\xi_h^2\, d\eta^0 = I_1 + I_2.\end{aligned} \tag{74}$$

Using the definition of $\xi_2^h = h^{-1}(t_{\eta^0}^{\eta^h}(v) - v)$, the first term can be written as

$$\begin{aligned}I_1 = h^{-1}(\int\phi(v)\, d\eta^h - \int h\phi(v)\, d\eta^0 - \int\nabla\phi(v)\cdot\xi_2(v)\, d\eta^0)&\\ - h^{-1}\int(t_{\eta^0}^{\eta^h}(v) - v)^t\nabla^2\phi(t_{\eta^0}^{\eta^h}(v) - v)\, d\eta^0,&\end{aligned}$$

where $\nabla^2\phi$ is the Hessian of ϕ evaluated at some point on the segment from v to $t_{\eta^0}^{\eta^h}(v)$ (in fact it is constant, since ϕ is at most quadratic). Using (67) and (68) we see that $I_1 = O(h)$.

To estimate I_2, we notice that for linear or quadratic ϕ's, $|\nabla\phi(t_{\eta^0}^{\tilde\mu^h}(v)) - \nabla\phi(v))| \le 2|t_{\eta^0}^{\tilde\mu^h}(v) - v|$ and

$$
(75)\quad |I_2| \le 2h^{-1}\int |t_{\eta^0}^{\tilde\mu^h}(v) - v|^2\,d\eta^0 + 2h^{-1}\int |t_{\eta^0}^{\eta^h}(v) - v|^2\,d\eta^0 \\
= 2h^{-1}\int |t_{\tilde\mu^h}^{\eta^0}(v) - v|^2\,d\tilde\mu^h + 2h^{-1}\int |t_{\eta^0}^{\eta^h}(v) - v|^2\,d\eta^0.
$$

From (69) and (68) we obtain that $I_2 = O(h)$. □

Our final lemma shows that the difference between the tangent vector, ξ_1 to the curve $\mu(t)$ representing the actual solution of the Euler equations and the tangent vector ξ_1^h to the aproximation is small.

Lemma 7. *Uniformly in* $x \in \mathbb{R}^3$,

$$
(76)\qquad \int |\xi_1 - \xi_1^h|^2\,d\eta^0,\ \int |\xi - \xi^h|^2\,d\eta^0,\ \int |\xi_1 - \xi_1^h|^2\,d\tilde\mu^h = O(h^2).
$$

Proof. Two pairs of vectors (ξ_1, ξ) and (ξ_1^h, ξ^h) come from the orthogonal decomposition of vectors ξ_2 and $\tilde\xi_2^h$, respectively:

$$
\xi_2 = \xi_1 - \xi, \quad \xi_1 \perp \xi \text{ in } L^2(\mathbb{R}^3;\eta^0), \\
\tilde\xi_2^h = \xi_1^h - \xi^h, \quad \xi_1^h \perp \xi^h \text{ in } L^2(\mathbb{R}^3;\tilde\mu^h),
$$

Because, the decomposition is determined by the zero and first order moments only, bounds (72), (76) on ξ_2, $\tilde\xi_2^h$, and bound (69) on the difference between the base measures $\tilde\mu^h$, η^0, imply the statement of the lemma, as the argument following (27), (28) in 1-d case. □

Now we're in the position to prove (57). We can estimate

$$
(77)\quad W_2^2(\mu^h, \tilde\mu^h) \le \int |t_{\eta^0}^{\mu^h}(t_{\tilde\mu^h}^{\eta^0}(v)) - v|^2\,d\tilde\mu^h = h^2\int \left| \frac{t_{\eta^0}^{\mu^h}(t_{\tilde\mu^h}^{\eta^0}(v)) - t_{\tilde\mu^h}^{\eta^0}(v)}{h} - \xi_1^h(v) \right|^2 d\tilde\mu^h \\
\le h^2\int \left| \frac{t_{\eta^0}^{\mu^h}(t_{\tilde\mu^h}^{\eta^0}(v)) - t_{\tilde\mu^h}^{\eta^0}(v)}{h} - \xi_1(t_{\tilde\mu^h}^{\mu^h}(v)) \right|^2 d\tilde\mu^h + h^2\int \left|\xi_1(t_{\tilde\mu^h}^{\mu^h}(v)) - \xi_1(v)\right|^2 d\tilde\mu^h \\
+ h^2\int \left|\xi_1(v) - \xi_1^h(v)\right|^2 d\tilde\mu^h.
$$

The optimal plan from $\eta^0 (= \mu^0)$ to μ^h equals

$$
t_{\mu^0}^{\mu^h}(v) = \left[\frac{T(h,x)}{T(0,x)}\right]^{\frac12} v - \left[\frac{T(h,x)}{T(0,x)}\right]^{\frac12} u(0,x) + u(h,x),
$$

and the limit $\lim_{h\to 0}\frac{t_{\mu^0}^{\mu^h}(v)-v}{h}=\xi_1$. Then by a strighforward computation

$$\left|\frac{t_{\mu^0}^{\mu^h}(v)-v}{h}-\xi_1\right|\le C(\|\partial_t^2(u,T)\|_{C^0_{t,x}},\inf T)(1+|v|)h. \tag{78}$$

Since the map $t_{\tilde\mu^h}^{\eta^0}(v)$ is linear in v, we also get

$$\left|\frac{t_{\eta^0}^{\mu^h}(t_{\tilde\mu^h}^{\eta^0}(v))-t_{\tilde\mu^h}^{\eta^0}(v)}{h}-\xi_1^h(v)\right|\le C(\|(\rho,u,T)\|_{C^2_{t,x}},\inf T,\inf\rho)(1+|v|)h \tag{79}$$

Since the function $\xi_1(v)$ is linear in v, see (49), we have $|\xi_1(t_{\tilde\mu^h}^{\eta^0}(v))-\xi_1(v)|\le C|t_{\tilde\mu^h}^{\eta^0}(v)-v|$, and

$$\int\left|\xi_1(t_{\tilde\mu^h}^{\mu^h}(v))-\xi_1(v)\right|^2 d\tilde\mu^h\le C\int|t_{\tilde\mu^h}^{\eta^0}(v)-v|^2\,d\tilde\mu^h=CW_2^2(\tilde\mu^h,\eta^0)=O(h^2). \tag{80}$$

Combining (79), (80) and (76) in (77) we get $W_2^2(\mu^h,\tilde\mu^h)=O(h^4)$.
Estimate (58) then also follows. □

0.5. **Proof of (68).**

Proof. We use Brenier-Benamou formula for

$$W_2^2(\eta^h,\eta^0)=\inf\left\{h\int_0^h\int|v_s(v)|^2\eta^s\,dvds\right\}$$

where the inf is taken over of velocity fileds $v_h(v)$ such that

$$\partial_h\eta^h+\operatorname{div}_v(v_h(v)\eta^h)=0,\quad\text{in }\mathscr{D}'((0,T_0)\times\mathbb{R}^3).$$

We will show that there is a veclosity field $v_h=\nabla w_h$ such that $\int|\nabla w_h|^2\eta^h\,dv$ is uniformly bounded in (h,x), wich will imply that $W_2(\eta^h,\eta^0)=O(h)$, as required.

Claim 2. *For any $(h,x)\in[0,T_0]\times\mathbb{R}^3$, there is $w_h\in L^2_{loc}(\mathbb{R}^3)$, with $\nabla w_h\in L^2_{loc}(\mathbb{R}^3)$, such that for any pointwisely (h,x,v) :*

$$div_v(\nabla w_h\eta^h)=-\partial_h\eta^h, \tag{81}$$

and there is C, independent of $(h,x)\in[0,T_0]\times\mathbb{R}^3$, such that

$$\int|\nabla w_h|^2\eta^h\,dv\le C.$$

Proof. We consider an elliptic equation (81) and construct a solution form a sequence of solutions to an approximate boundary value problem (for a fixed (h,x)):

$$\begin{cases}\operatorname{div}_v(\nabla w^R\eta^h)=-\partial\eta^h,\ v\in B_R,\\ w^R(v)=0,\ v\in\partial B_R.\end{cases} \tag{82}$$

Here B_R is the ball of radius R centered at 0. The equation can also be wrtten as

$$\Delta w^R+\nabla\ln\eta^h\cdot\nabla w^R=-\partial_h\ln\eta^h. \tag{83}$$

In the condition of our theorem $\nabla \ln \eta^h$, $\partial_h \ln \eta^h \in C^1(\overline{B_R})$, for any R and thus there is a unique classical solution of (82). Let $\bar{w} = \frac{1}{|B_1|}\int_{B_1} w^R\,dv$. We multiply the equation by w^R and integrate over the domain:

$$
\begin{aligned}
(84)\quad \int_{B_R} |\nabla w^R|^2 \eta^h\,dv &= \int_{B_R} (w^R - \bar{w})\partial_h \eta^h\,dv + \bar{w}\int_{B_R} \partial_h \eta^h\,dv \\
&= \int_{B_R} (w^R - \bar{w})\partial_h \eta^h\,dv - \bar{w}\int_{B_R^c} \partial_h \eta^h\,dv = I_1 + I_2,
\end{aligned}
$$

where in the last line we used the fact that η^h is a unit measure on $\mathbb{R}^3$. Let us estimate

$$
\begin{aligned}
\int_{B_R} |w-\bar{w}||\partial_h \eta^h|\,dv &\le \int_{B_R} dv \int_0^1 d\lambda \int_{B_1} dy |\nabla w^R(\lambda v + (1-\lambda)y)||v-y||\partial_h \eta^h| \\
&\le \int_0^1 d\lambda \int_{B_1} dy \int_{B_R \cap \{\lambda|v|>R_0\}} |\nabla w^R(\lambda v + (1-\lambda)y)||v-y||\partial_h \eta^h|\,dv \\
&+ \int_0^1 d\lambda \int_{B_1} dy \int_{B_R \cap \{\lambda|v|\le R_0\}} |\nabla w^R(\lambda v + (1-\lambda)y)||v-y||\partial_h \eta^h|\,dv \\
&\le \int_0^1 d\lambda \int_{B_1} dy \int_{B_R \cap \{|v|>R_0\}} |\nabla w^R(v + (1-\lambda)y)||v/\lambda - y||\partial_h \eta^h(v/\lambda)|\lambda^{-3}\,dv \\
&+ \int_0^1 d\lambda \int_{B_1} dy \int_{B_R \cap \{\lambda|v|\le R_0\}} |\nabla w^R(\lambda v + (1-\lambda)y)||v-y||\partial_h \eta^h|\,dv \\
&= J_1 + J_2,
\end{aligned}
$$

where R_0 will be chosen later. We estimate

$$
|J_2| \le C \sup_{B_{R_0+1}} |\nabla w^R|,
$$

where C is independent of (h,x,R). Now we will show that there are numbers C_1, C_2 independent of (h,x,R) such that

$$
(85)\qquad |J_2| \le C \sup_{B_{R_0+1}} |\nabla w^R| \le C_1 \|\nabla w^R\|_{L^2(B_{4R_0+4})} + C_2.
$$

Indeed, the classical estimate for the equation (83), Theorem 8.8 of[11], implies that there are $C_i = C_i(\|\ln \eta^h\|_{C^2_{h,v}([0,T_0]\times B_{2R_0+2})})$ (this is controlled by $\|\rho,u,T\|_{C^2_{t,x}}$) such that

$$
(86)\qquad \|\nabla w^R\|_{W^{2,2}(B_{2R_0+2})} \le C\|\nabla w^R\|_{L^2(B_{4R_0+4})}.
$$

Moreover, the ebedding theorems imply that

$$
\sup_{B_{R_0+1}} |\nabla w^R| \le \left| |B_{R_0+1}|^{-1} \int_{B_{R_0+1}} \nabla w^R\,dv \right| + C\|D^2 w^R\|_{L^6(B_{2R_0+2})}
$$

and

$$
\|D^2 w^R\|_{L^6(B_{2R_0+2}))} \le C\|D^2 w^R\|_{L^2(B_{2R_0+2}))} + C\|D^3 w^R\|_{L^2(B_{2R_0+2}))}.
$$

Combining the last two estimates with (86), we obtain (85). From that we also have

$$|J_2| \le C_1 \|\nabla w^R \sqrt{\eta^h}\|_{L^2(B_R)} + C_2, \tag{87}$$

with a possibly different C_1, because $\eta^h(v)$ has a non-zero lower bound, independent of (h,x,R) for $v \in B_{4R_0+4}$.

To estimate J_1, we write

$$\partial_h \eta^h(v) = (\partial_h \ln \eta^h) e^{-\frac{|v-u(0,x-hv)|^2}{2T(0,x-hv)}},$$

and use the estimates $|\partial_h \ln \eta^h(v)| \le C(1+|v|^2)$ and

$$(\eta^h(v+(1-\lambda)y))^{-1} \le Ce^{\frac{|v-u(0,x-h(v-(1-\lambda)y))|^2}{2T(0,x-h(v-(1-\lambda)y))}},$$

to obtain

$$\begin{aligned}
|J_1| \;\le\;& \int_0^1 d\lambda \int_{B_1} dy \left[\int_{B_R} |\nabla w^R(v)|^2 \eta^h(v)\, dv\right]^{\frac{1}{2}} \\
&\times \left[\int_{B_1 \cap \{|v|>R_0\}} \lambda^{-6} |(v/\lambda)^2 - y|^2 |\partial_h \eta^h(v/\lambda)|^2 (\eta^h(v+(1-\lambda)y))^{-1}\, dv\right]^{\frac{1}{2}} \\
\le\;& C\left[\int_{B_R} |\nabla w^R(v)|^2 \eta^h(v)\, dv\right]^{\frac{1}{2}} \\
&\times \int_0^1 d\lambda \int_{B_1} dy \left[\int_{B_1 \cap \{|v|>R_0\}} \lambda^{-6} |(v/\lambda)^2 + 1|^6 e^{-\frac{|v/\lambda - u(0,x-hv)|^2}{T(0,x-hv)}} e^{\frac{|v-u(0,x-h(v-(1-\lambda)y))|^2}{2T(0,x-h(v-(1-\lambda)y))}}\, dv\right]^{\frac{1}{2}}
\end{aligned}$$

Choosing

$$R_0 > \frac{\sup T}{\inf T} + \sup |u|,$$

we have

$$e^{-\frac{|v/\lambda - u(0,x-hv)|^2}{T(0,x-hv)}} e^{\frac{|v-u(0,x-h(v-(1-\lambda)y))|^2}{2T(0,x-h(v-(1-\lambda)y))}} \le Ce^{-\frac{|v|^2}{4\lambda^2 \sup T}}, \tag{88}$$

for all $|v| > R_0$. With this estimate,

$$|J_1| \le C\left[\int_{B_R} |\nabla w^R(v)|^2 \eta^h(v)\, dv\right]^{\frac{1}{2}}, \tag{89}$$

where C is independent of (h,x,R).

To estimate the average $\bar{w}$ we write $|\bar{w}| \le C_0 \|w^R\|_{L^6(B_1)} \le C_0 \|\nabla w^R\|_{L^2(B_R)}$, for some absolute constant C_0. Considering the coefficients in the equation and the right-hand side (83), there is C, independent of (h,x,R) such that

$$\sup_{v \in B_R} |\nabla \ln \eta^h|, \sup_{v \in B_R} |\partial_h \ln \eta^h| \le CR^2, \quad \forall R > 1.$$

A basic energy estimate then produces

$$\|\nabla w^R\|_{L^2(B_R)} \le CR^{n_0},$$

for some $C, n_0 > 0$, independent of (h,x,R). Thue we have

$$|\bar{w}| \leq CR^{n_0}. \tag{90}$$

With this, I_2 from (84) can be estimated by

$$|I_2| \leq CR^{n_0} e^{-\frac{R^2}{4\sup T}} \leq C,$$

because $\eta^h(v) \geq e^{-\frac{R^2}{2\sup T}}$, for $|v| > R$.

Collecting the estimates on $I_1 = J_1 + J_2$, and I_2, we arrive at

$$\int |\nabla w^R|^2 \eta^h \, dv \leq C, \tag{91}$$

with C independent of (h,x,R). By taking the limit on a suitable subsequence of ∇w^R as $R \to +\infty$ we obtain an function ∇w, wich verifies the statement of the claim. □

The claim implies the estimates of the lemma, as was explained above.

□

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