SMOOTH SOLUTIONS FOR A p-SYSTEM OF MIXED TYPE

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ABSTRACT. In this note we analyze smooth solutions of a p-system of the *mixed* type. Motivating example for this is a 2-components reduction of the Benney moments chain which appears to be connected to theory of integrable systems. We don't assume a-priory that the solutions in question are in the Hyperbolic region. Our main result states that the only smooth solutions of the system which are periodic in x are necessarily constants. As for initial value problem we prove that if the initial data is strictly hyperbolic and periodic in x then the solution can not extend to $[t_0; +\infty)$ and shocks are necessarily created.

1. MOTIVATION AND THE RESULT

Consider the p-system

$$(1) \qquad u_t = -v_x, \ v_t = (p(u))_x$$

We refer to [10] and [6] for general theory of these systems.

The general case considered in this note is somewhat different. We assume that p is a C^2-function, which is quadratic like, that is everywhere strictly convex

$$p''(u) > 0,$$

and has a minimum. We shall assume that the minimum value is zero at $u = 0$. Notice that this is not a restriction and can be achieved by changing u and $p(u)$ by constants. Thus the system is of a *mixed type* depending on the sign of u (see below), so that one needs to analyze the solutions in Hyperbolic and Elliptic regions.

Main example for me which motivates in fact the result of this paper is $p(u) = u^2/2$. In this case the system appears to be a reduction of the famous Benney chain and turns out to be related to a problem of Polynomial integrals for Hamiltonian systems with potential. This relation was first observed in [7] and smooth periodic solutions were studied in [1], [2]. More general Rich quasi-linear systems of order ≥ 2 were recently discovered in connection with other integrable Hamiltonian systems [3], [4]. The main question arising in these applications is if there exist global periodic smooth solutions of the corresponding quasi-linear systems. It is an open problem how to apply the ideas of this note to study smooth periodic solutions for these systems of order ≥ 2.

In this note we prove non-existence of classical solutions $(u(t,x), v(t,x))$ of the system, which are C^2-smooth and periodic in x.

Remark 1. Notice that for solutions u, v of the system it follows that u must satisfy the equation $u_{tt} + (p(u))_{xx} = 0$. However the requirement of periodicity in x of both u and v is a stronger assumption then just to ask u to be periodic. This can be seen on the following example. Take $p(u) = u^2/2$ and notice that $u = t$ is a periodic in x solution of $u_{tt} + (uu_x)_x = 0$ however the corresponding v for the system is $v = -x$ which is not periodic in x.

The type of the system is governed by the sign of the function $p'(u)$. It is hyperbolic for $p'(u) < 0$, that is for $u < 0$, and elliptic for $p'(u) > 0$, that is for $u > 0$. I shall denote by

$$U_+ = \{(t,x) : u(t,x) > 0\}, U_- = \{(t,x) : u(t,x) < 0\},$$

the elliptic and the hyperbolic domains on the cylinder respectively.

Our main results are given in the following two theorems below:

Theorem 1.1. *Let the pair (u,v) be a solution of the p-system of class C^2 on the semi-infinite cylinder $t \geq t_0$. If the initial data for $t = t_0$ is hyperbolic, i.e. $u(t_0, x) < 0$ for all x then it is hyperbolic everywhere and moreover the solution is constant,*

$$(u, v) = const.$$

Let me emphasize that this is important for the proof of this theorem that initially the solution has hyperbolic type and it is possible, though I don't know any single example, that there exist smooth solutions of the *mixed* type on the semi-infinite cylinder. On the other hand, we prove that if one considers the whole infinite cylinder one gets an additional rigidity and no smooth solutions of the system other than constants can exist:

Theorem 1.2. *Let (u,v) be a C^2-smooth solution of the p-system on the whole cylinder. Then both u and v are constant functions.*

One should mention that the classical Hodograph method [5] can be used to linearize the system of equations. Such a linearization leads to a second order equation of mixed type (see recent monograph [9] for extensive survey). This method works well in neighborhood of a regular points of the mapping $(x,y) \to (u,v)$. However in order to make this method global one needs to analyze different domains where this change of variables is a diffeomorphism taking into account the singularities of the mapping $(x,y) \to (u,v)$ which might be vary complicated (folds , cusps as well as singularities of rank 2).

Therefore our approach is different and is based on the method of characteristics by P. Lax [8] in the hyperbolic region together with certain convexity argument in the elliptic region. Special attention is payed to the interface between them.

In the sequel we treat the Hyperbolic domain first and then the Elliptic one. We use the following main ingredients: Near the boundary of the Hyperbolic region the genuine non-linearity increases and becomes "infinite"; Analysis of Hyperbolic region is in fact non-local and uses in a strong way periodicity assumption. For the Elliptic region we replace maximum principle which works well for bounded domains by a simple convexity argument.

2. Hyperbolic region

In the Hyperbolic region u and $p'(u)$ are negative and the boundary of the Hyperbolic region lies in $U_0 = \{u = 0\}$. We shall denote by $\lambda_{1,2} = \pm\sqrt{-p'(u)}$ (as usual subindex 1 corresponds to the upper sign here and in the sequel) the eigenvalues of the matrix

$$(2) \qquad A = \begin{pmatrix} 0 & 1 \\ -p'(u) & 0 \end{pmatrix},$$

with the Riemann invariants

$$r_{1,2} = v \mp \int_u^0 \sqrt{-p'(u)}\,du$$

and so u, v can be recovered from the Riemann invariants by the formulas:

$$(3) \qquad v = \frac{r_1 + r_2}{2}; \ u = q^{-1}\left(\frac{r_2 - r_1}{2}\right),$$

where by definition

$$q(u) := \int_u^0 \sqrt{-p'(s)}\,ds,$$

is a positive monotone decreasing function for $u < 0$ with

$$q(0) = 0, \ q'(u) = -\sqrt{-p'(u)}, \ q''(u) = \frac{p''(u)}{2\sqrt{-p'(u)}}.$$

It is crucial fact that both eigenvalues are genuinely non-linear by the formulas:

$$(\lambda_1)_{r_1} = (\lambda_2)_{r_2} = \frac{p''(u)}{4p'(u)}.$$

Notice that near the boundary ∂U_- the non-linearity becomes infinite. Moreover, verifying literarily the Lax method [8] for our p-system one arrives to the following Ricatti equations along characteristics of the first and the second eigenvalues:

$$(4) \qquad L_{v_1}(\beta_1) + k\beta_1^2 = 0, \ \ L_{v_2}(\beta_2) + k\beta_2^2 = 0$$

where

$$\beta_1 := (r_1)_x(-p'(u))^{\frac{1}{4}}; \ \beta_2 := (r_2)_x(-p'(u))^{\frac{1}{4}}; \quad k := -\frac{p''(u)}{4(-p'(u))^{\frac{5}{4}}},$$

and L_{v_1}, L_{v_2} stands for derivatives along the first and the second characteristic fields respectively. The following two lemmas will be very useful for the proofs.

Lemma 2.1. *(1) If a characteristic curve of the first or of the second eigenvalue starting from the initial time t_0 reaches the boundary ∂U_- in a finite time $t_+ > t_0$ (respectively $t_- < t_0$), then the derivative of the corresponding Riemann invariant r_x must be non-positive (resp. non-negative) along this characteristic.*

(2) If a characteristic curve of the first or of the second eigenvalue extends to a semi-infinite interval $(t_0, +\infty)$ (resp.$(-\infty, t_0)$), then for the corresponding Riemann invariant either $(r)_x \le 0$ (resp.$r_x \ge 0$) or otherwise $-u$ tends to $+\infty$ along this characteristic curve when $t \to +\infty$ (resp.$t \to -\infty$).

Proof. The proof easily follows from the exact formulas for the solutions of (4):

$$\beta(t) = \frac{\beta(t_0)}{1 + \beta(t_0) \int_{t_0}^{t} k(s)ds}.$$

Indeed in order to prove (1) suppose that characteristic extends to the maximal interval $(t_0; t_+)$. Recall that the characteristics are solutions of the ODE's

$$\dot{x} = \pm\sqrt{-p'(u)},$$

so when characteristic curve approaches the boundary of Hyperbolic domain, so that u tends to zero, then the characteristic curve must converge to a limit point say (t_+, x_+) on the boundary. Moreover, it follows then that the integral

$$\int_{t_0}^{t_+} k(s)ds = -\int_{t_0}^{t_+} \frac{p''(u)}{4(-p'(u))^{\frac{5}{4}}}ds$$

diverges to $-\infty$. Indeed, for $t \to t_+$ the function $u(t, x(t)) \to 0$ and can be estimated from above by

$$|u(t, x(t))| \le C_1 |t - t_+|,$$

also for u close to zero one can estimate:

$$\left|p'(u)\right| = \left|\int_{u}^{0} p''(u)du\right| \le C_2 |u|, \ where \ C_2 = \max_{u\in[-1,0]} p''(u).$$

So the nominator in of the integrand is bounded away from zero and the denominator is less or equal then $C_1 C_2 |t - t_+|^{\frac{5}{4}}$, thus the integral diverges. This proves the first part of the lemma.

The second part is analogous. Indeed for an infinite characteristic there are two possibilities. The first is when the integral diverges to $-\infty$, in this case $r_x \le 0$ exactly as in the previous case. In the second case the integral converges, then in particular the integrand must tend to zero along characteristic. Then since p is strictly convex it follows that $p'(u)$ tends to $-\infty$ and so u also. Lemma is proved $\qquad\square$

This lemma enables to divide between two types of characteristics which start at t_0 in a positive or negative direction of time as follows.

Definition 2.2. Let γ be a characteristic curve defined on a *maximal* interval $[t_0, t_+)$ (or respectively $(t_-, t_0]$). We shall say that γ is of type B_+ (res. B_-) if $t_+ = +\infty$ (resp. $t_- = -\infty$) and $-u \to +\infty$ when $t \to +\infty$ (resp. $t \to -\infty$).

We shall say that γ is of type A_+ (resp. A_-) in the opposite case. That is if either t_+ (resp. t_-) is finite, or $t_+ = +\infty$ (resp. $t_- = -\infty$) and $-u$ does not tend to $+\infty$ when $t \to +\infty$ (resp. $t \to -\infty$).

Lemma 2.3. *There cannot exist two semi-infinite characteristics in the same direction*

$$\gamma_1 = (t, x_1(t)), \ \gamma_2 = (t, x_2(t))$$

of the first and the second eigenvalue belonging both to the same class B_+ or B_-.

Proof. Assume on the contrary that there exist such $\gamma_1 = (t, x_1(t))$, $\gamma_2 = (t, x_2(t))$ belonging to the same class, say B_+, so that $-u|_{\gamma_{1,2}} \to +\infty$ when $t \to +\infty$.

Then by periodicity we can shift the characteristics to get

$$\gamma_1^{(k)} = (t, x_1(t) + k), \ \gamma_2^{(l)} = (t, x_2(t) + l)$$

for all $k, l \in \mathbf{Z}$ are characteristics of class B_+ also. Since the functions x_1, x_2 are solutions of the ODEs

$$\dot{x} = \pm\sqrt{-p'(u)}$$

respectively, it follows that x_1 (respectively x_2) are strictly monotone increasing (resp.decreasing) function with the derivative bounded away from zero. Therefore for sufficiently large k the characteristics γ_1 and $\gamma_2^{(k)}$ must intersect in a unique point, call it P_k. Denote by t_k the t-coordinates of P_k. One can see that t_k is monotone increasing and must tend to $+\infty$. Indeed in the opposite case there exist limits $t_k \nearrow t_*$ and $P_k \to P_*$ so that by intermediate value theorem $p'(u)$ is unbounded on the compact segment $[t_0; t_*]$, contradiction.

Therefore we have,

$$-u(t_k, x(t_k)) \to +\infty, \ k \to +\infty,$$

and then by formula (3) also

$$r_2(P_k) - r_1(P_k) \to +\infty, \ k \to +\infty.$$

But this is not possible since by periodicity in x of (u, v) one has that $r_1(t_0, x)$ and $r_2(t_0, x)$ are bounded, and so by conservation of r_1, r_2 along characteristics $r_2(P_k) - r_1(P_k)$ must be bounded also. This contradiction proves the lemma. $\square$

Let me prove now Theorem 1.1.

Proof. Introduce

$$t' = \sup\{t : [t_0, t] \times \mathbf{S}^1 \subseteq U_-\},$$

this means that t' is the first moment where non-Hyperbolic type appears. In other words u must vanish at some point on $\{t'\} \times \mathbf{S}^1$. Write $U'_- = [t_0, t') \times \mathbf{S}^1$. We prove that t' equals in fact to $+\infty$. Indeed, it follows from the second lemma that all characteristics of at least one of the eigenvalues are of class A_+. Without loss of generality let it be the family of the second eigenvalue with this property. Then it follows from the lemma 2.1 that

$$(r_2)_x(t_0, x) \leq 0,$$

for every x. But by periodicity this is possible only when $r_2(t_0, x)$ is in fact constant for the initial moment and so also everywhere on the whole U'_-. This means that within the domain U'_- only r_1 can vary. But then u is a function of r_1 only and therefore has constant values along characteristics

of the first eigenvalue. By the construction there exists a point on $\{t'\} \times \mathbf{S}^1$ where u vanishes. Since there exists a characteristic of the first family terminating at this point, then u vanishes also on the whole characteristic. But this is a contradiction, since u must be strictly negative on U_-. This implies that the hyperbolic domain U_- coincides with the whole semi-infinite cylinder $t \geq t_0$.

Furthermore since we know that r_2 is a constant on the whole half cylinder then u depends only on r_1 and has constant values along characteristics of the first family (in particular $-u$ does not tend to infinity) so by Lemma 2.1 this implies

$$(r_1)_x \leq 0.$$

Using periodicity again we conclude that r_1 is constant also everywhere on the half-cylinder. Thus (u, v) is a constant solution. We are done. $\square$

The idea of the proof of the second theorem is somewhat similar. The difference is that one should take into account both directions of characteristics of the Hyperbolic domain. Also in this case elliptic domains cannot be excluded as before, one needs an additional tool. In the next section we treat the Elliptic region. These two steps provide the proof of theorem 1.2. The first step is the following:

Theorem 2.4. *Let (u, v) be a C^2-solution of the system on the infinite cylinder. Then either U_- coincides with the whole cylinder and u, v are constants everywhere, or U_- is empty, i.e $u \geq 0$ everywhere.*

Proof. To give a proof assume that U_- is not the whole cylinder (otherwise Theorem 1.1 yields the result). We need to show that $u \geq 0$, everywhere on the cylinder. Fix a connected component, denote it U'_-, and take any initial moment t_0 with the property that the intersection of $\{t = t_0\}$ with component U'_- is not empty. It consists of the disjoint union of open intervals-intervals of Hyperbolicity (the case when it is the whole circle is covered already by theorem 1.1).

Consider the following 2 complementary cases:

Case1. *For any initial moment t_0 with the property $\{t = t_0\} \cap U_- \neq \emptyset$, for at least one of the eigenvalues (say for the second one) all characteristics started from t_0 in positive and negative direction belong to the classes A_+, A_-.*

In this case by the first lemma $r_2(t_0, x)$ is constant. Since t_0 is arbitrary this implies that r_2 is constant on every connected component of U_-. This implies that only r_1 varies on every component and so $u, v, \lambda_1, \lambda_2$ are functions of r_1 only. This means that u keeps constant values along characteristics of the first eigenvalue. Therefore every such characteristic can be extended infinitely in both directions because if it reaches the boundary of the Hyperbolic region u must have value zero on the whole characteristic, which contradicts Hyperbolicity. Moreover, since λ_1 is a function of r_1 only so that it is constant along λ_1-characteristics, then these characteristics are parallel horizontal straight lines, $\{x = const\}$. But then λ_1 must be zero and so again $u = 0$, contradiction.

Case2. *There exists t_0 such that for each eigenvalue λ_1, λ_2 there exists a characteristic of class B started at t_0 in some direction.*

It follows from the second lemma that the directions of these characteristics must be opposite. So assume without loss of generality that γ_1, γ_2 are λ_1, λ_2-characteristics in the classes B_+, B_- respectively. Then it follows from the lemmas that the characteristics γ_1, γ_2 being extended beyond t_0 in the negative and positive direction respectively belong to classes A_-, A_+ respectively. And thus by the first lemma

$$(r_1)_x(t_0, x) \geq 0, \ (r_2)_x(t_0, x) \leq 0,$$

for all x in the intervals of Hyperbolicity. So in this case $(r_1 - r_2)(t_0, x)$ is a monotone function in x. Then also $u(t_0, x)$ is monotone by the formula (3) and since u vanishes at the ends of the intervals of Hyperbolicity, u must vanish identically. This contradiction completes the proof of the theorem. $\square$

3. Elliptic case

Theorem 3.1. *Suppose (u, v) is a C^2-solution of the system on the whole cylinder satisfying $u \geq 0$ everywhere. Then u, v are constants.*

If the elliptic domain is bounded one could use the maximum principle for the proof. But in general the following tool can be used:

Proof. Take any function $f : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$f(0) = 0, \ f(u) > 0 \ for \ all \ u > 0 \ and \ f''(u) < 0 \ for \ all \ u \geq 0.$$

Introduce

$$E(t) = \int_{S^1} f(u(t, x)) dx,$$

which by the construction is a positive function of t unless u is identically zero. Compute the second derivative of E using the p-system and integration by parts. We have

$$\ddot{E} = \int_{S_1} f''(u) \left((v_x)^2 + p'(u)(u_x)^2 \right) dx.$$

Since u is non-negative then $p'(u)$ is non-negative also, since p is quadratic like by the assumption. So we get that E is a positive concave function and thus must be constant. Then obviously u, v are constants everywhere. $\square$

References

[1] Bialy, M. Polynomial integrals for a Hamiltonian system and breakdown of smooth solutions for quasi-linear equations. Nonlinearity 7 (1994), no. 4, 1169-1174.

[2] M. Bialy. On periodic solutions for a reduction of Benney chain // Nonlinear Differ. Equ. Appl. 2009. V. 16. P. 731–743.

[3] M. Bialy, A. Mironov. Rich quasi-linear system for integrable geodesic flows on 2-torus // Discrete and Continuous Dynamical Systems - Series A. 2011. V. 29. N. 1. P. 81–90.

[4] M. Bialy, A. Mironov. Qubic and Quartic integrals for geodesic flow on 2-torus via system of Hydrodynamic type. Preprint arXiv:1101.3449, 2011

[5] R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience Publishers, Inc., New York, N. Y., 1948.

[6] H. Fan and M. Slemrod, Dynamic flows with liquid/vapor phase transitions, pp 373-420. in S. Friedlander , D. Serre, editors, Handbook of Mathematical Fluid Dynamics v. 1, Elsevier Science, 2002.

[7] V.V. Kozlov Polynomial integrals of dynamical systems with one-and-a-half degrees of freedom. Math. Notes (translation) 45 (1989), no. 3-4, 296-300

[8] Lax, P. D. Hyperbolic systems of conservation laws and the mathematical theory of shock waves. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973.

[9] Otway, T. H. The Dirichlet problem for elliptic-hyperbolic equations of Keldysh type. Lecture Notes in Math. 2043. Berlin: Springer. ix, 214 p.

[10] D. Serre. Systems of Conservation Laws. Vol.1,2.Cambridge University Press, 1999.

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