L^1 ERROR ESTIMATES FOR DIFFERENCE APPROXIMATIONS OF DEGENERATE CONVECTION-DIFFUSION EQUATIONS

K. H. KARLSEN, N. H. RISEBRO, AND E. B. STORRØSTEN

Abstract. We analyze a class of semi-discrete monotone difference schemes for degenerate convection-diffusion equations in one spatial dimension. These nonlinear equations are well-posed within a class of (discontinuous) entropy solutions. We prove that the L^1 error between the approximate solutions and the unique entropy solution is $\mathcal{O}(\Delta x^{1/3})$, where Δx denotes the spatial discretization parameter. This result should be compared with the classical $\mathcal{O}(\Delta x^{1/2})$ result for conservation laws [20], and a very recent error estimate of $\mathcal{O}(\Delta x^{1/11})$ for degenerate convection-diffusion equations [18].

1. Introdcution

Nonlinear convection-dominated flow problems arise in a range of applications, such as fluid dynamics, meteorology, transport of oil and gas in porous media, electro-magnetism, as well as in many other applications. As a consequence it has become a very important undertaking to construct robust, accurate, and efficient methods for the numerical approximation of such problems. Over the years a large number of stable (convergent) numerical methods have been developed for linear and nonlinear convection-diffusion equations in which the "diffusion part" is small, or even vanishing, relative to the "convection part" of the equation. There is a large literature on this topic, and we will provide a few relevant references later.

One central but exceedingly difficult issue relating to numerical methods for convection-diffusion equations, is the derivation of (a priori) error estimates that are robust in the singular limit as the diffusion coefficient vanishes, avoiding the exponential growth of error constants. This problem has been resolved only partly in special situations, such as for linear equations or in the completely degenerate case of no diffusion (scalar conservation laws). For general nonlinear equations containing both convection and (degenerate) diffusion terms this is a long standing open problem in numerical analysis.

This paper makes a small contribution to this general problem by deriving an error estimate for a class of simple difference schemes for nonlinear and strongly degenerate convection-diffusion problems of the form

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 A(u), & (x,t) \in \Pi_T, \\ u(x,0) = u^0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Pi_T = \mathbb{R} \times (0,T)$ for some fixed final time $T > 0$, and $u(x,t)$ is the scalar unknown function that is sought. The initial function $u_0 : \mathbb{R} \to \mathbb{R}$ is a given integrable and bounded function, while the convection flux $f : \mathbb{R} \to \mathbb{R}$ and the diffusion function $A : \mathbb{R} \to \mathbb{R}$ are given functions satisfying

$$f, A \text{ locally } C^1;\ A(0) = 0;\ A \text{ nondecreasing}.$$

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The moniker *strongly degenerate* means that we allow $A'(u) = 0$ for all u in some interval $[\alpha, \beta] \subset \mathbb{R}$. Thus, the class of equations becomes very general, including purely hyperbolic equations (scalar conservation laws)

$$\partial_t u + \partial_x f(u) = 0, \tag{1.2}$$

as well as nondegenerate (uniformly parabolic) equations, such as the heat equation $\partial_t u = \partial_x^2 u$, and point-degenerate diffusion equations, such as the heat equation with a power-law nonlinearity: $\partial_t u = \partial_x(u^m \partial_x u)$, which is degenerate at $u = 0$.

Whenever the problem (1.1) is uniformly parabolic (i.e., $A' \geq \eta$ for some $\eta > 0$), it is well known that the problem admits a unique classical (smooth) solution. On the other hand, in the strongly degenerate case, (1.1) must be interpreted in the weak sense to account for possibly discontinuous (shock wave) solutions. Regarding weak solutions, it turns out that one needs an additional condition, the so-called *entropy condition*, to ensure that (1.1) is well-posed. More precisely, the following is known: For $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique solution $u \in C((0,T); L^1(\mathbb{R}^d))$, $u \in L^\infty(\Pi_T)$ of (1.1) such that $\partial_x A(u) \in L^2(\Pi_T)$ and for all convex functions $S : \mathbb{R} \to \mathbb{R}$ with $q_S' = f'S'$ and $r_S' = A'S'$,

$$\partial_t S(u) + \partial_x q_S(u) - \partial_x^2 r_S(u) \leq 0 \quad \text{in the weak sense on } [0,T) \times \mathbb{R}. \tag{1.3}$$

The satisfaction of these inequalities for all convex S is the entropy condition, and a weak solution satisfying the entropy condition is called an entropy solution. The well-posedness of entropy solutions is a famous result due to Kružkov [19] for conservation laws (1.2), and a more recent work by Carrillo [4] extends this to degenerate parabolic equations (1.1). These results are available in the multi-dimensional context, and we refer to [1, 8] for an overview of the relevant literature. For uniqueness of entropy solutions in the BV class, see [25, 27].

One traditional way of constructing entropy solutions is by the vanishing viscosity method, which starts off from classical solutions to the nondegenerate equation

$$\partial_t u_\eta + \partial_x f(u_\eta) = \partial_x^2 A(u_\eta) + \eta \partial_x^2 u_\eta, \qquad \eta > 0,$$

and establishes the strong convergence of u_η as $\eta \to 0$ by deriving BV estimates that are independent of η, see Vol′pert and Hudjaev [26].

Besides proving that u_η converges in the L^1 norm to the unique entropy solution u of (1.1), it is possible to prove the error estimate

$$\|u_\eta(t,\cdot) - u(t,\cdot)\|_{L^1} \leq C\sqrt{\eta}, \qquad (\text{whenever } u_0 \in BV), \tag{1.4}$$

see [12] (cf. also [13]). The error bound (1.4) can also be obtained as a consequence of the more general continuous dependence estimate derived in [7], see also [5, 16].

Herein we are interested in the much more difficult problem of deriving error estimates for numerical approximations of entropy solutions to convection-diffusion equations. Convergence results (without error estimates) have been obtained for finite difference schemes [10] (see also [11, 17]); finite volume schemes [14] (see also [2]); operator splitting methods [15]; and BGK approximations [3], to mention just a few references. For a posteriori estimates for finite volume schemes, see [22].

To be concrete in what follows, let us for simplicity assume $f' \geq 0$ and consider the semi-discrete difference scheme

$$\frac{d}{dt} u_j(t) + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = \frac{A(u_{j+1}) - 2A(u_j) + A(u_{j-1})}{\Delta x^2}, \tag{1.5}$$

where $u_j(t) \approx u(t, j\Delta x)$ and $\Delta x > 0$ is the spatial mesh size. Convergence of this scheme can be proved as in the works [10, 11], where explicit and implicit time discretizations are treated. Denote by $u_{\Delta x}(t,x)$ the piecewise constant interpolation of $\{u_j(t)\}_j$. The basic question we address in this paper is the following one: Does there exist a number $r \in (0,1)$ and a constant C, independent of Δx, such that

$$\|u_{\Delta x}(t,\cdot) - u(t,\cdot)\|_{L^1} \le C\,\Delta x^r, \tag{1.6}$$

where u is the unique entropy solution of (1.1). We refer to the number r as the rate of convergence.

In the purely hyperbolic case (1.2) ($A' \equiv 0$), the answer to this question is a classical result due to Kuznetsov [20], who proved that the rate of convergence is $1/2$ for viscous approximations as well as monotone difference schemes, and this is optimal for discontinuous solutions. The work of Kuznetsov [20] turned out to be extremely influential, and by now a large number of related works have been devoted to error estimation theory for conservation laws. We refer to [6] for an overview of the relevant results and literature.

Unfortunately, the situation is unclear in the degenerate parabolic case (1.1). Let us expose some reasons why adding a nonlinear diffusion term to (1.2) can make the error analysis significantly more difficult than in the streamlined Kuznetsov theory. First of all, it is well known that the purely hyperbolic difference scheme

$$\frac{d}{dt}u_j(t) + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0 \tag{1.7}$$

has as a model equation the second order viscous equation

$$\partial_t u + \partial_x f(u) = \frac{\Delta x}{2}\partial_x^2 f(u),$$

an equation that is compatible with the notion of entropy solution for (1.2). Indeed, an error estimate for this viscous equation is highly suggestive for what to expect for the upwind scheme (1.7) (this is of course what Kuznetsov proved). However, for convection-diffusion equations such as (1.1) the situation changes dramatically. The model equation for (1.5) is no longer second order but rather fourth order:

$$\partial_t u + \partial_x f(u) = \partial_x^2 A(u) + \frac{\Delta x}{2}\partial_x^2 f(u) - \frac{\Delta x^2}{12}\partial_x^4 A(u), \tag{1.8}$$

and hence the error estimate (1.4) appears no longer so relevant for numerical schemes. Fourth order equations such as (1.8) are difficult to analyze as they lack the usual maximum principle associated with first and second order equations. In general, (1.8) possesses much fewer a priori estimates than are available for the entropy solution of (1.1); indeed, conservation laws perturbed by fourth order terms are not compatible with all the entropy inequalities, see [23]. All these facts seem to render error analysis markedly more difficult than in the hyperbolic case.

Another added difficulty comes from the necessity to work with an explicit form of the parabolic dissipation term associated with (1.1). Indeed, in the analysis one needs to replace (1.3) by the following more precise entropy equation [4]

$$\begin{aligned} &\partial_t |u-c| + \partial_x\big(\operatorname{sign}(u-c)(f(u)-f(c)\big) - \partial_x^2 |A(u)-A(c)| \\ &\qquad = -\operatorname{sign}'(A(u)-A(c))\,|\partial_x A(u)|^2, \qquad c \in \mathbb{R}, \end{aligned} \tag{1.9}$$

which is formally obtained multiplying (1.1) by $\operatorname{sign}(A(u)-A(c))$, assuming for the sake of this discussion that $A'(\cdot) > 0$. The term on the right-hand side is the parabolic dissipation term, which is a finite (signed) measure and thus very singular. To illustrate why the parabolic dissipation term is needed, let $u(y,s)$ and $v(x,t)$ be two solutions satisfying (1.9). In the entropy equation for $u(y,s)$ one takes $c = v(x,t)$, while in the entropy equation for $v(x,t)$ one takes $c = u(y,s)$. Adding the two resulting equations yields

$$(\partial_t + \partial_s)\,|u-v| + (\partial_x + \partial_y)\big(\operatorname{sign}(u-v)(f(u)-f(v)\big)$$

$$-\left(\partial_x^2+\partial_y^2\right)|A(u)-A(v)| = -\mathrm{sign}'(A(u)-A(v))\left(|\partial_y A(u)|^2+|\partial_x A(v)|^2\right),$$

By adding $-2\partial_{xy}^2 |A(u)-A(v)|$ to both sides of this equation, noting that

$$-2\partial_{xy}^2 |A(u)-A(v)| = 2\mathrm{sign}'(A(u)-A(v))\partial_y A(u)\partial_x A(v),$$

we arrive at

$$\begin{aligned}(\partial_t+\partial_s)|u-v| &+ (\partial_x+\partial_y)\big(\mathrm{sign}(u-v)(f(u)-f(v)\big)\\ &\qquad -\left(\partial_x^2-2\partial_{xy}^2+\partial_y^2\right)|A(u)-A(v)|\\ &= -\mathrm{sign}'(A(u)-A(v))\big(|\partial_y A(u)|-|\partial_x A(v)|\big)^2\\ &\le 0,\end{aligned} \tag{1.10}$$

from which the contraction property $\frac{d}{dt}\|u(t,\cdot)-v(t,\cdot)\|_{L^1}\le 0$ follows [4]. Similarly, to obtain error estimates for numerical methods, it is necessary to derive a "discrete" version of (1.10) with v replaced by $u_{\Delta x}$. The main challenge is to suitably replicate at the discrete level the delicate balance between the two terms in (1.10) involving A; the difficulty stems from the lack of a chain rule for finite differences.

Despite the mentioned difficulties, we will in this paper prove that there exists a constant C, independent of Δx, such that for any $t>0$,

$$\|u_{\Delta x}(t,\cdot)-u(t,\cdot)\|_{L^1}\le C\,\Delta x^{\frac{1}{3}}.$$

The only other work we are aware of that provides L^1 error estimates for numerical approximations of (1.1) is [18]; therein (1.6) is established with $r=\frac{1}{11}$; if A is a linear function, then the convergence rate is the usual one, namely $r=\frac{1}{2}$.

Roughly speaking, the reason is two-fold for why we can significantly improve the result in [18]. First, we are herein able to provide a more faithful analog of (1.10) at the discrete level. Second, since $\mathrm{sign}'(\cdot)$ is singular, one has to work with a Lipschitz continuous approximation $\mathrm{sign}_\varepsilon(\cdot)$ of the sign function $\mathrm{sign}(\cdot)$. The use of this approximation breaks the symmetry of the corresponding entropy fluxes, and introduces new error terms that depend on the parameter ε; the process of "balancing" terms involving Δx and ε lowers the convergence rate (to $r=\frac{1}{11}$) [18]. In the present paper we are able to dispense with this balancing act. Indeed, we show that it is possible to send $\varepsilon\to 0$ independently of Δx.

The remaining part of this paper is organized as follows: In Section 2 we list some relevant a priori estimates satisfied by viscous approximations and entropy solutions, and provide a definition of entropy solutions. The semi-discrete difference scheme is defined and proved to be well-posed in Section 3. We also list several relevant a priori estimates. Section 4 is devoted to the proof of the error estimate.

2. Preliminary material

Set $A^\eta(u):=A(u)+\eta u$ for any fixed $\eta>0$, and consider the uniformly parabolic problem

$$\begin{cases} u_t^\eta + f(u^\eta)_x = A^\eta(u^\eta)_{xx}, & (x,t)\in\Pi_T,\\ u^\eta(x,0)=u^0(x), & x\in\mathbb{R}.\end{cases} \tag{2.1}$$

It is well known that (2.1) admits a unique classical (smooth) solution.

We collect some relevant (standard) a priori estimates in the next three lemmas.

Lemma 2.1. *Suppose $u^0\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})\cap BV(\mathbb{R})$, and let u^η be the unique classical solution of (2.1). Then for any $t>0$,*

$$\begin{aligned}\|u^\eta(\cdot,t)\|_{L^1(\mathbb{R})} &\le \left\|u^0\right\|_{L^1(\mathbb{R})},\\ \|u^\eta(\cdot,t)\|_{L^\infty(\mathbb{R})} &\le \|u^0\|_{L^\infty(\mathbb{R})},\\ |u^\eta(\cdot,t)|_{BV(\mathbb{R})} &\le \left|u^0\right|_{BV(\mathbb{R})}.\end{aligned}$$

For a proof of the previous and next lemmas, see for example [26].

Lemma 2.2. *Suppose* $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ *and* $f(u^0) - A(u^0)_x \in BV(\mathbb{R})$. *Let* u^η *be the unique classical solution of* (2.1). *Then for any* $t_1, t_2 > 0$,

$$\|u^\eta(\cdot, t_2) - u^\eta(\cdot, t_1)\|_{L^1(\mathbb{R})} \leq \left| f(u^0) - A(u^0)_x \right|_{BV(\mathbb{R})} |t_2 - t_1| \, .$$

Regarding the following lemma, see [24, 10].

Lemma 2.3. *Suppose* $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ *and* $f(u^0) - A(u^0)_x \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. *Let* u^η *be the unique classical solution of* (2.1). *Then for any* $t > 0$,

$$\|f(u^\eta(\cdot, t)) - A(u^\eta(\cdot, t))_x\|_{L^\infty(\mathbb{R})} \leq \left\| f(u^0) - A(u^0)_x \right\|_{L^\infty(\mathbb{R})}, \tag{2.2}$$

$$|f(u^\eta(\cdot, t)) - A(u^\eta(\cdot, t))_x|_{BV(\mathbb{R})} \leq \left| f(u^0) - A(u^0)_x \right|_{BV(\mathbb{R})}. \tag{2.3}$$

Note that $\|A(u^\eta)_x\|_{L^\infty(\Pi_T)}$ and $\|A(u^\eta)_{xx}\|_{L^1(\Pi_T)}$ are bounded independently of η provided that $A(u^0)_x$ is in $BV(\mathbb{R})$.

These results above imply that the family $\{u^\eta\}_{\eta > 0}$ is relatively compact in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$. If $u = \lim_{\eta \to 0} u^\eta$, then

$$\|u^\eta - u\|_{L^1(\Pi_T)} \leq C\eta^{1/2},$$

for some constant C which does not depend on η, see [12]. Moreover, u is an entropy solution according to the following definition.

Definition 2.1. An entropy solution of the Cauchy problem (1.1) is a measurable function $u = u(x,t)$ satisfying:

- (D.1) $u \in L^\infty(\Pi_T) \cap C((0,T); L^1(\mathbb{R}))$.
- (D.2) $A(u) \in L^2((0,T); H^1(\mathbb{R}))$.
- (D.3) For all constants $c \in \mathbb{R}$ and test functions $0 \leq \varphi \in C_0^\infty(\mathbb{R} \times [0,T))$, the following entropy inequality holds:

$$\iint_{\Pi_T} |u - c| \, \varphi_t + \operatorname{sign}(u - c)\,(f(u) - f(c))\varphi_x + |A(u) - A(c| \, \varphi_{xx} \, dtdx \\ + \int_{\mathbb{R}} |u_0 - c| \, \varphi(x, 0) \, dx \geq 0.$$

The uniqueness of entropy solutions follows from the work [4]. Actually, in view of the above a priori estimates, the relevant functional class is $BV(\Pi_T)$, in which case we can replace (D.2) by the condition $A(u)_x \in L^\infty(\Pi_T)$. For a uniqueness result in the BV class, see [27].

3. Difference scheme

We start by specifying the numerical flux to be used in the difference scheme.

Definition 3.1. (Numerical flux) We call a function $F \in C^1(\mathbb{R}^2)$ a two-point numerical flux for f if $F(u,u) = f(u)$ for $u \in \mathbb{R}$. If

$$\frac{\partial}{\partial u} F(u,v) \geq 0 \quad \text{and} \quad \frac{\partial}{\partial v} F(u,v) \leq 0$$

holds for all $u, v \in \mathbb{R}$, we call F monotone.

Let F_u and F_v denote the partial derivatives of F with respect to the first and second variable respectively. We will also assume that F is Lipschitz continuous.

Let $\Delta x > 0$ and set $x_j = j\Delta x$ for $j \in \mathbb{Z}$, and define

$$D_\pm \sigma_j = \pm \frac{\sigma_{j\pm1} - \sigma_j}{\Delta x},$$

for any sequence $\{\sigma_j\}$.

We may now define a semidiscrete approximation of the solution to (1.1) as the solution to the (infinite) system of ordinary differential equations

$$\begin{cases} \frac{d}{dt}u_j(t) + D_- F_{j+1/2} = D_- D_+ A(u_j), & t > 0, \\ u_j(0) = \frac{1}{\Delta x}\int_{I_j} u^0(x)\,dx, \end{cases} \qquad j \in \mathbb{Z}, \tag{3.1}$$

where $F_{j+1/2} = F(u_j, u_{j+1})$ is a numerical flux function and $I_j = (x_{j-1/2}, x_{j+1/2}]$.

The problem above can be viewed as an ordinary differential equation in the Banach space $\ell^1(\mathbb{Z})$ (see, e.g., [21]). To get bounds independent of Δx we define

$$\|\sigma\|_1 = \Delta x \sum_j |\sigma_j| \quad \text{and} \quad |\sigma|_{BV} = \sum_j |\sigma_{j+1} - \sigma_j| = \|D_+\sigma\|_1 .$$

If these are bounded we say that $\sigma = \{\sigma_j\}$ is in ℓ^1 and of bounded variation. Let $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$, $u^0 = \{u_j(0)\}_{j\in\mathbb{Z}}$, and define the operator $\mathcal{A} : \ell^1 \to \ell^1$ by $(\mathcal{A}(u))_j := D_-(F(u_j, u_{j+1}) - D_+A(u_j))$. Then (3.1) takes the following form

$$\frac{du}{dt} + \mathcal{A}(u) = 0, \quad t > 0, \quad u(0) = u^0. \tag{3.2}$$

This problem has a unique continuously differentiable solution since $\mathcal{A}$ is Lipschitz continuous for each fixed $\Delta x > 0$. This solution defines a strongly continuous semigroup $\mathcal{S}(t)$ on ℓ^1. If $\mathcal{S}$ also satisfies

$$\|\mathcal{S}(t)u - \mathcal{S}(t)v\|_1 \le \|u - v\|_1 \quad \text{for} \quad u, v \in \ell^1,$$

we say that it is *nonexpansive*. The next lemma sums up some important properties of the solutions to (3.1) (for a proof see [9]).

Lemma 3.1. *Suppose that F is monotone. Then there exists a unique solution $u = \{u_j\}$ to (3.1) on $[0, T]$ with the properties:*

(a) $\|u(t)\|_1 \le \|u^0\|_1$.

(b) *For every $j \in \mathbb{Z}$ and $t \in [0, T]$,*

$$\inf_k \{u_k^0\} \le u_j(t) \le \sup_k \{u_k^0\} .$$

(c) $|u(t)|_{BV} \le |u^0|_{BV}$.

(d) *If $v = \{v_j\}$ is a another solution with initial data v^0 then*

$$\|u(t) - v(t)\|_1 \le \|u^0 - v^0\|_1 .$$

Lemma 3.2. *If F is monotone, then*

$$\|F(u_j, u_{j+1}) - D_+A(u_j)\|_{\ell^\infty} \le \left\|F(u_j^0, u_{j+1}^0) - D_+A(u_j^0)\right\|_{\ell^\infty} , \tag{3.3}$$

$$|F(u_j, u_{j+1}) - D_+A(u_j)|_{BV} \le \left|F(u_j^0, u_{j+1}^0) - D_+A(u_j^0)\right|_{BV} . \tag{3.4}$$

Furthermore, $t \mapsto \{u_j(t)\}_{j\in\mathbb{Z}}$ is ℓ^1 Lipschitz continuous.

Proof. The proof follows [10]. Let $v_j = \Delta x \sum_{k\le j} \frac{du_k}{dt}$. Then v_j satisfies

$$v_j = \Delta x \sum_{k=-\infty}^{j} D_-(D_+A(u_k) - F(u_k, u_{k+1})) = D_+A(u_j) - F(u_j, u_{j+1}), \tag{3.5}$$

and we may define v_j for all $t \in [0, T]$. Note that $\{v_j(t)\}$ is in ℓ^1 for all t by Lemma 3.1. Differentiating (3.5) with respect to t we obtain

$$\begin{aligned} \frac{dv_j}{dt} &= \frac{1}{\Delta x}\Big[a(u_{j+1})\frac{du_{j+1}}{dt} - a(u_j)\frac{du_j}{dt}\Big] \\ &\quad - F_u(u_j, u_{j+1})\frac{du_j}{dt} - F_v(u_j, u_{j+1})\frac{du_{j+1}}{dt}, \end{aligned}$$

where $a(u) = A'(u)$. Note that $D_- v_j = \frac{du_j}{dt}$ and $D_+ v_j = \frac{du_{j+1}}{dt}$. Therefore

$$\frac{dv_j}{dt} = \left(\frac{1}{\Delta x} a(u_{j+1}) - F_v(u_j, u_{j+1})\right) D_+ v_j \\ - \left(\frac{1}{\Delta x} a(u_j) + F_u(u_j, u_{j+1})\right) D_- v_j. \quad (3.6)$$

Assume $v_{j_0}(t_0)$ is a local maximum in j. Then $D_+ v_{j_0}(t_0) \le 0$ and $D_- v_{j_0}(t_0) \ge 0$ so $\frac{v_{j_0}}{dt}(t_0) \le 0$ since F is monotone. Similarly, if $v_{j_0}(t_0)$ is a local minimum in j, then $\frac{v_{j_0}}{dt}(t_0) \ge 0$. Then inequality (3.3) follows by the fact that $\{v_j(t)\} \in \ell^1$. Consider (3.4). We want to show that $\frac{d}{dt}\left(|v(t)|_{BV}\right) \le 0$. Now,

$$\frac{d}{dt}\Big(\sum_j |v_{j+1} - v_j|\Big) = \sum_j \operatorname{sign}\left(v_{j+1} - v_j\right) \frac{d}{dt}\left(v_{j+1} - v_j\right),$$

so we may use (3.6). Thus

$$\begin{aligned}
&\frac{d}{dt}|v(t)|_{BV} \\
&\quad = \sum_j \left(\frac{1}{\Delta x} a(u_{j+2}) - F_v(u_{j+1}, u_{j+2})\right) (D_+ v_{j+1}) \operatorname{sign}(v_{j+1} - v_j) \\
&\quad\quad - \sum_j \left(\frac{1}{\Delta x} a(u_{j+1}) + F_u(u_{j+1}, u_{j+2})\right) |D_+ v_j| \\
&\quad\quad - \sum_j \left(\frac{1}{\Delta x} a(u_{j+1}) - F_v(u_j, u_{j+1})\right) |D_+ v_j| \\
&\quad\quad + \sum_j \left(\frac{1}{\Delta x} a(u_j) + F_u(u_j, u_{j+1})\right) ((D_- v_j) \operatorname{sign}(v_{j+1} - v_j)) \\
&\quad = \sum_j \left(\frac{1}{\Delta x} a(u_{j+1}) - F_v(u_j, u_{j+1})\right) \left[(D_+ v_j) \operatorname{sign}\left(v_j - v_{j-1}\right) - |D_+ v_j|\right] \\
&\quad\quad + \sum_j \left(\frac{1}{\Delta x} a(u_j) + F_u(u_j, u_{j+1})\right) \left[(D_- v_j) \operatorname{sign}(v_{j+1} - v_j) - |D_- v_j|\right] \\
&\quad \le 0,
\end{aligned}$$

since $a(u) > 0$ and $F_v \le 0$ and $F_u \ge 0$. Given the preceeding estimates, the ℓ^1 Lipschitz continuity is straightforward to prove. $\square$

3.1. **The numerical entropy flux.** It turns out that we need more conditions on F than mere monotonicity.

Definition 3.2. Given an entropy pair (ψ, q) and a numerical flux F, we define $Q \in C^1(\mathbb{R}^2)$ by

$$\begin{aligned}
Q(u, u) &= q(u), \\
\frac{\partial}{\partial v} Q(v, w) &= \psi'(v) \frac{\partial}{\partial v} F(v, w), \\
\frac{\partial}{\partial w} Q(v, w) &= \psi'(w) \frac{\partial}{\partial w} F(v, w).
\end{aligned}$$

We call Q a numerical entropy flux.

The next lemma gives a sufficient condition on the numerical flux to ensure that there exists a numerical entropy flux.

Lemma 3.3. *Given a two-point numerical flux F, assume that there exist C^1 functions F_1, F_2 such that*

$$F(u,v) = F_1(u) + F_2(v), \qquad F_1'(u) + F_2'(u) = f'(u), \tag{3.7}$$

for all relevant u and v. Then there exists a numerical entropy flux Q for any entropy flux pair (ψ, q).

Proof. Let (ψ, q) be an entropy pair. Then q has the form

$$q(u) = \int_c^u \psi'(z) f'(z)\, dz + C,$$

for some constant C. Define Q by

$$Q(u,v) = \int_c^u \psi'(z) F_1'(z)\, dz + \int_c^v \psi'(z) F_2'(z)\, dz + C. \tag{3.8}$$

It is easily verified that Q is a numerical entropy flux. □

Let us list a few numerical flux functions to which Lemma 3.3 applies.

Example 3.1 (Engquist-Osher flux)**.** Let

$$f_+'(s) = \max(f'(s), 0) \qquad \text{and} \qquad f_-'(s) = \min(f'(s), 0).$$

Then, in the terminology of Lemma 3.3, let $F(u,v) = F_1(u) + F_1(v)$ with

$$F_1(u) = f(0) + \int_0^u f_+'(s)\, ds \qquad \text{and} \qquad F_2(v) = \int_0^v f_-'(s)\, ds.$$

It is easily seen that the criteria given in Lemma 3.3 are satisfied, and F is also clearly monotone.

Example 3.2. Let $a, b \in \mathbb{R}$ and define

$$F_1(u) = af(u) + bu \qquad \text{and} \qquad F_2(v) = (1-a)f(v) - bv.$$

Note that $F(u,v) = F_1(u) + F_2(v)$ is monotone if

$$a \inf_x \{f'(x)\} \geq -b \quad \text{and} \quad (1-a)\sup_x \{f'(x)\} \leq b.$$

This example includes both the upwind scheme and the Lax-Friedrichs scheme.

From a more general point of view we may consider any flux splitting. That is $f(u) = f^+(u) + f^-(u)$ where $(f^+(u))' \geq 0$ and $(f^-(u))' \leq 0$ for all $u \in \mathbb{R}$. Then the numerical flux F defined by

$$F(u,v) = f^+(u) + f^-(v)$$

satisfies the assumptions of Lemma 3.3. Note also that any convex combination of numerical flux functions which satisfy the hypothesis of Lemma 3.3, itself satisfies the assumptions of the lemma.

If (3.7) holds, then we have a representation of Q given by (3.8). It follows that

$$Q(u,v) = q(u) + \int_u^v \psi'(z) F_2'(z)\, dz.$$

Note that we may obtain another representation depending on F_1 by splitting up the first integral.

Lemma 3.4. *Let Q be a numerical entropy flux associated with the entropy pair (ψ, q) and the monotone numerical flux F. Then*

$$\psi'(u)(F(u,w) - F(v,u)) \geq Q(u,w) - Q(v,u),$$

for all relevant u, v and w.

Proof. Fix u and define $p(v,w) = p_1(w) + p_2(v)$, where

$$p_1(w) = -\psi'(u)F(u,w) + Q(u,w) + \psi'(u)f(u) - q(u),$$
$$p_2(v) = \psi'(u)F(v,u) - Q(v,u) - \psi'(u)f(u) + q(u).$$

Then we have

$$p(v,w) = -\psi'(u)(F(u,w) - F(v,u)) + (Q(u,w) - Q(v,u)),$$

and so the lemma is proved if we can show that $p(v,w) \le 0$. Now

$$\begin{aligned} p_1'(w) &= -\psi'(u)\frac{\partial}{\partial w}F(u,w) + \psi'(w)\frac{\partial}{\partial w}F(u,w) \\ &= \psi''(\xi_1)\frac{\partial}{\partial w}F(u,w)(w-u) \end{aligned}$$

for some $\xi_1 \in \mathrm{int}(u,w)$[1]. Similarly

$$\begin{aligned} p_2'(v) &= \psi'(u)\frac{\partial}{\partial v}F(v,u) - \psi'(v)\frac{\partial}{\partial v}F(v,u) \\ &= \psi''(\xi_2)\frac{\partial}{\partial v}F(v,u)(u-v), \end{aligned}$$

for some $\xi_2 \in \mathrm{int}(u,v)$. Since F is monotone and ψ is convex, $p_i'(z)(z-u) \le 0$ for $i = 1,2$. It remains to observe that $p_1(u) = p_2(u) = 0$ and so

$$p_i(z) = \int_u^z p_i'(\xi)\,d\xi \le 0, \qquad \text{for } i = 1,2.$$

Hence $p(v,w) \le 0$. □

4. Error estimate

Let $\{u_j\}_{j\in\mathbb{Z}}$ be the solution to (3.1). To any sequence $\{u_j(t)\}_{j\in\mathbb{Z}}$ we associate the piecewise constant function

$$u_{\Delta x}(x,t) = u_j(t) \quad \text{for } x \in I_j. \tag{4.1}$$

To derive the error estimate we need many of the uniform bounds from Sections 2 and 3. For these estimates to hold independently of Δx, we make the following assumptions on the initial data u^0:

(i) $u^0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$.
(ii) $A(u^0)_x \in BV(\mathbb{R})$.

We may now state the theorem.

Theorem 4.1. *Let u be the entropy solution to* (1.1) *and $\{u_j(t)\}_{j\in\mathbb{Z}}$ solve the semidiscrete difference scheme* (3.1). *If u^0 satisfies* (i) *and* (ii) *above, then for all sufficiently small Δx,*

$$\|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} \le \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} + C_T \Delta x^{\frac{1}{3}}, \qquad t \in [0,T], \tag{4.2}$$

where the constant C_T depends on A, f, u^0, and T, but not on Δx.

From now on we suppose $A(\sigma) = \hat{A}(\sigma) + \eta\sigma$, where $\hat{A}' \ge 0$ is the original degenerate diffusion function. In fact, we will prove (4.2) in this case, and then note that the right-hand side is independent of η, so we can send $\eta \to 0$.

Let us define some of the functions we are going to work with. First, let the approximation of the sign function be given by

$$\mathrm{sign}_\varepsilon(\sigma) = \begin{cases} \sin(\frac{\pi\sigma}{2\varepsilon}) & \text{for } |\sigma| < \varepsilon, \\ \mathrm{sign}\,(\sigma) & \text{otherwise,} \end{cases}$$

[1]By $\mathrm{int}(a,b)$ we mean the closed interval between a and b.

where $\varepsilon > 0$. Note that $\mathrm{sign}_\varepsilon$ is continuously differentiable and non-decreasing. Then we define

$$|u|_\varepsilon = \int_0^u \mathrm{sign}_\varepsilon(z)\,dz.$$

The next lemma is a replacement for the chain rule when working with sequences and finite differences.

Lemma 4.1. *Let $\{u_j\}_{j\in\mathbb{Z}}$ be some sequence in $\mathbb{R}$ and $A:\mathbb{R}\to\mathbb{R}$ a strictly increasing continuously differentiable function. For any $u\in\mathbb{R}$ there exist sequences $\{\tau_j\}_{j\in\mathbb{Z}}, \{\theta_j\}_{j\in\mathbb{Z}}$ such that for each $j\in\mathbb{Z}$ both τ_j and θ_j are in $\mathrm{int}(u_j,u_{j+1})$ and*

$$\begin{aligned} D_+\mathrm{sign}_\varepsilon(A(u_j)-A(u)) &= \mathrm{sign}'_\varepsilon(A(\tau_j)-A(u))D_+A(u_j),\\ D_+|A(u_j)-A(u)|_\varepsilon &= \mathrm{sign}_\varepsilon(A(\theta_j)-A(u))D_+A(u_j). \end{aligned}$$

If u is a differentiable function of y then for each $j\in\mathbb{Z}$

$$\mathrm{sign}'_\varepsilon(A(\tau_j)-A(u))A(u)_y = -(\mathrm{sign}_\varepsilon(A(\theta_j)-A(u)))_y. \tag{4.3}$$

Both $\{\tau_j\}_{j\in\mathbb{Z}}$ and $\{\theta_j\}_{j\in\mathbb{Z}}$ depend on u and ε.

Proof. The first statement is a direct consequence of the mean value theorem. Consider (4.3). If $u_j = u_{j+1}$ then $\theta_j = \tau_j$ is independent of u and hence of y, so (4.3) follows by the chain rule. In general

$$\begin{aligned} \mathrm{sign}'_\varepsilon(A(\tau_j)-A(u))A(u)_yD_+A(u_j) &= D_+\mathrm{sign}_\varepsilon(A(u_j)-A(u))A(u)_y\\ &= -D_+(|A(u_j)-A(u)|_\varepsilon)_y\\ &= -\mathrm{sign}_\varepsilon(A(\theta_j)-A(u))_yD_+A(u_j). \end{aligned}$$

In the case $u_j\neq u_{j+1}$ we have $D_+A(u_j)\neq 0$ and (4.3) follows. □

4.1. **Doubling of the variables.** We let (x,t,y,s) denote a point in Π_T^2, where x and y are the spatial variables and s and t are the time variables. Moreover, we let $u_{\Delta x} = u_{\Delta x}(x,t)$ be defined by (4.1), and let $u = u(y,s)$ be the classical solution of (1.1) with $A(\sigma) = \hat{A}(\sigma)+\eta\sigma$. Although both u and $u_{\Delta x}$ depend on η, we do not indicate this dependence in our notation. To avoid writing four integral signs we will in general write one for each domain Π_T and let $dX = dxdtdyds$. For a function $\varphi:\Pi_T^2\to\mathbb{R}$, we let $\varphi^\sigma(x,t,y,s) = \varphi(x+\sigma,t,y,s)$.

4.1.1. *Rewriting the continuous equation.* Define an entropy pair $(\psi_\varepsilon, q_\varepsilon)$ by

$$\begin{aligned} \psi_\varepsilon(u,c) &= \int_c^u \mathrm{sign}_\varepsilon(A(z)-A(c))\,dz,\\ q_\varepsilon(u,c) &= \int_c^u \psi'_\varepsilon(z,c)f'(z)\,dz = \int_c^u \mathrm{sign}_\varepsilon(A(z)-A(c))f'(z)\,dz, \end{aligned}$$

where ψ'_ε is the derivative with respect to the first variable. Let $\varphi=\varphi(x,t,y,s)$ be a non-negative smooth function such that for each (y,s), the function $(x,t)\mapsto\varphi(x,t,y,s)\in C_0^\infty(\mathbb{R}\times(0,T))$, and for each (x,t), the function $(y,t)\mapsto\varphi(x,t,y,s)\in C_0^\infty(\mathbb{R}\times(0,T))$. Multiply equation (2.1) by $\psi'_\varepsilon(u,u_j)\varphi$ and integrate in both space and time to get

$$\int_{\Pi_T} \psi_\varepsilon(u,u_j)_s\varphi + \psi'_\varepsilon(u,u_j)(f(u)-f(u_j))_y\varphi\,dyds = \int_{\Pi_T}(\psi'_\varepsilon(u,u_j)\varphi)A(u)_{yy}\,dyds.$$

Integration by parts and the chain rule gives

$$\begin{aligned} &\int_{\Pi_T} \psi_\varepsilon(u,u_j)\varphi_s - q_\varepsilon(u,u_j)_y\varphi\,dyds\\ &\quad= \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u)-A(u_j))A(u)_y\varphi_y + \mathrm{sign}'_\varepsilon(A(u)-A(u_j))(A(u)_y)^2\varphi\,dyds. \end{aligned}$$

Using the chain rule and integration by parts we get

$$\int_{\Pi_T} \operatorname{sign}_\varepsilon(A(u)-A(u_j))A(u)_y\varphi_y\,dyds = -\int_{\Pi_T} \left|A(u)-A(u_j)\right|_\varepsilon \varphi_{yy}\,dyds.$$

so

$$\begin{aligned}\int_{\Pi_T} &\psi_\varepsilon(u,u_j)\varphi_s - q_\varepsilon(u,u_j)_y\varphi\,dyds \\ &= \int_{\Pi_T} -\left|A(u)-A(c)\right|_\varepsilon \varphi_{yy} + \operatorname{sign}'_\varepsilon(A(u)-A(u_j))(A(u)_y)^2\varphi\,dyds. \quad (4.4)\end{aligned}$$

Note that since $u_{\Delta x}$ is piecewise constant in x,

$$\int_{\mathbb{R}} h\left(u_{\Delta x}(x,t)\right)\,dx = \Delta x \sum_j h\left(u_j(t)\right),$$

for any function h taking pointwise values. We thus multiply (4.4) by Δx and sum over j. This yields

$$\begin{aligned}\iint_{\Pi_T^2} &\psi_\varepsilon(u,u_{\Delta x})\varphi_s + q_\varepsilon(u,u_{\Delta x})\varphi_y\,dX \\ &= \iint_{\Pi_T^2} -\left|A(u)-A(u_{\Delta x})\right|_\varepsilon\varphi_{yy} + \operatorname{sign}'_\varepsilon(A(u)-A(u_{\Delta x}))(A(u)_y)^2\varphi\,dX\end{aligned}$$

after integrating the resulting expression over $[0,T]$ in the variable t. Let us take the limit as $\varepsilon \downarrow 0$. By dominated convergence

$$\begin{aligned}\iint_{\Pi_T^2} &\left|u-u_{\Delta x}\right|\varphi_s + \operatorname{sign}\left(u-u_{\Delta x}\right)\left(f(u)-f(u_{\Delta x})\right)\varphi_y\,dX \\ &= -\iint_{\Pi_T^2} \left|A(u)-A(u_{\Delta x})\right|\varphi_{yy}\,dX \qquad (4.5)\\ &\quad + \lim_{\varepsilon\downarrow 0}\iint_{\Pi_T^2} \operatorname{sign}'_\varepsilon(A(u_{\Delta x})-A(u))(A(u)_y)^2\varphi\,dX,\end{aligned}$$

since

$$\lim_{\varepsilon\downarrow 0} q_\varepsilon(u,u_{\Delta x}) = \int_{u_{\Delta x}}^{u} \operatorname{sign}\left(z-u_{\Delta x}\right)f'(z)\,dz = \operatorname{sign}\left(u-u_{\Delta x}\right)\left(f(u)-f(u_{\Delta x})\right),$$

and

$$\lim_{\varepsilon\downarrow 0}\psi_\varepsilon(u,u_{\Delta x}) = \lim_{\varepsilon\downarrow 0}\int_{u_{\Delta x}}^{u} \operatorname{sign}_\varepsilon(A(z)-A(u_{\Delta x})) = |u-u_{\Delta x}|.$$

4.1.2. *Rewriting the semidiscrete equation.* Next we will obtain an analogous expression for $u_{\Delta x}$. For a function $\sigma = \sigma(x,t,y,s)$ we let

$$D_\pm\sigma = \pm\frac{\sigma^{\pm\Delta x}-\sigma}{\Delta x},$$

where $\sigma^{\Delta x}(x,t,y,s) = \sigma(x+\Delta x,t,y,s)$. From (3.1) it follows that

$$\frac{d}{dt}u_{\Delta x}(x,t) + D_-F(u_{\Delta x}(x,t),u_{\Delta x}(x+\Delta x,t)) = D_-D_+A(u_{\Delta x}(x,t))$$

holds for all $(x,t)\in\Pi_T$. Multiply by $\psi'_\varepsilon(u_{\Delta x},u)\varphi$ and integrate in both time and space to obtain

$$\begin{aligned}\int_{\Pi_T} &\psi_\varepsilon(u_{\Delta x},u)_t\varphi + \psi'_\varepsilon(u_{\Delta x},u)D_-F\left(u_{\Delta x},(u_{\Delta x})^{\Delta x}\right)\varphi\,dxdt \\ &= \int_{\Pi_T} \operatorname{sign}_\varepsilon(A(u_{\Delta x})-A(u))\left(D_-D_+A(u_{\Delta x})\right)\varphi\,dxdt.\end{aligned}$$

Note that for any two functions u, v of x we have $D_+(uv) = u^{\Delta x} D_+ v + (D_+u)v$. It follows that

$$\int_{\mathbb{R}} (D_+u)v \, dx = -\int_{\mathbb{R}} uD_-v \, dx,$$

given that uv lies in $C_0(\mathbb{R})$. We will refer to these identities respectively as Leibniz rule for difference quotients and integration by parts for difference quotients.

By the above,

$$\begin{aligned}\int_{\Pi_T} \psi_\varepsilon(u_{\Delta x}, u)\varphi_t - \psi'_\varepsilon(u_{\Delta x}, u) D_- F(u_{\Delta x}, (u_{\Delta x})^{\Delta x})\varphi \, dxdt \\ = \int_{\Pi_T} D_+ \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+ A(u_{\Delta x}) \varphi^{\Delta x} \, dxdt \\ + \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+ A(u_{\Delta x}) D_+ \varphi \, dxdt.\end{aligned} \tag{4.6}$$

Let us introduce the sequences from Lemma 4.1 and form the piecewise constant functions $\tau_{\Delta x}$ and $\theta_{\Delta x}$ as in (4.1). Note that since both τ_j and θ_j depends on t, y, s and ε the same is true for $\tau_{\Delta x}$ and $\theta_{\Delta x}$. By Lemma 4.1

$$\begin{aligned}\int_{\Pi_T} D_+ \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+ A(u_{\Delta x}) \varphi^{\Delta x} \, dxdt \\ = \int_{\Pi_T} \mathrm{sign}'_\varepsilon(A(\tau_{\Delta x}) - A(u)) [D_+ A(u_{\Delta x})]^2 \varphi^{\Delta x} \, dxdt.\end{aligned} \tag{4.7}$$

Concerning the second term on the right of (4.6) we add and subtract to apply Lemma 4.1 again. Using integration by parts for difference quotients

$$\begin{aligned}&\int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+ A(u_{\Delta x}) D_+ \varphi \, dxdt \\ &\quad = \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(\theta_{\Delta x}) - A(u)) D_+ A(u_{\Delta x}) D_+ \varphi \, dxdt \\ &\qquad + \int_{\Pi_T} \left[\mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) - \mathrm{sign}_\varepsilon(A(\theta_{\Delta x}) - A(u))\right] D_+ A(u_{\Delta x}) D_+ \varphi \, dxdt \\ &\quad = -\int_{\Pi_T} \left|A(u_{\Delta x}) - A(u)\right|_\varepsilon D_- D_+ \varphi \, dxdt \\ &\qquad + \int_{\Pi_T} \left[\mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) - \mathrm{sign}_\varepsilon(A(\theta_{\Delta x}) - A(u))\right] D_+ A(u_{\Delta x}) D_+ \varphi \, dxdt.\end{aligned}$$

Integrate in y and s, and apply (4.7) and the above equation to turn equation (4.6) into

$$\begin{aligned}&\iint_{\Pi_T^2} \psi_\varepsilon(u_{\Delta x}, u)\varphi_t - \psi'_\varepsilon(u_{\Delta x}, u) D_- F(u_{\Delta x}, (u_{\Delta x})^{\Delta x})\varphi \, dX \\ &= \iint_{\Pi_T^2} \mathrm{sign}'_\varepsilon(A(\tau_{\Delta x}) - A(u))(D_+ A(u_{\Delta x}))^2 \varphi^{\Delta x} \, dX \\ &\quad - \iint_{\Pi_T^2} \left|A(u_{\Delta x}) - A(u)\right|_\varepsilon (D_- D_+ \varphi) \, dX \\ &\quad + \iint_{\Pi_T^2} \left[\mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) - \mathrm{sign}_\varepsilon(A(\theta_{\Delta x}) - A(u))\right] D_+ A(u_{\Delta x}) D_+ \varphi \, dX.\end{aligned}$$

Let $\varepsilon \downarrow 0$ and apply dominated convergence to obtain

$$\iint_{\Pi_T^2} |u_{\Delta x} - u| \, \varphi_t - \mathrm{sign}\,(u_{\Delta x} - u) \, D_- F(u_{\Delta x}, (u_{\Delta x})^{\Delta x})\varphi \, dX \tag{4.8}$$

$$
\begin{aligned}
&= -\iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\,(D_- D_+ \varphi)\ dX \\
&\quad + \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon'(A(\tau_{\Delta x}) - A(u))(D_+ A(u_{\Delta x}))^2 \varphi^{\Delta x}\, dX \\
&\quad + \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} [\operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_{\Delta x}) - A(u))]\, D_+ A(u_{\Delta x}) D_+ \varphi\, dX.
\end{aligned}
$$

Since $A(u_{\Delta x})$ is of bounded variation in x we may apply the dominated convergence theorem and Lemma 4.3 to compute an explicit expression for the last limit (details will be given later). It follows by the above equation that the first limit exists.

4.1.3. *Adding the equations.* By Lemma 4.1,

$$
\begin{aligned}
0 &= \iint_{\Pi_T^2} D_+ \left(\operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) A(u)_y \varphi\right)\ dX \\
&= \iint_{\Pi_T^2} D_+ (\operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))) A(u)_y \varphi^{\Delta x} \\
&\qquad\qquad + \operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) A(u)_y D_+ \varphi\, dX \\
&= \iint_{\Pi_T^2} D_+ (\operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u))) A(u)_y \varphi^{\Delta x} \\
&\qquad\qquad - \left(|A(u_{\Delta x}) - A(u)|_\varepsilon\right)_y D_+ \varphi\, dX \\
&= \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon'(A(\tau_{\Delta x}) - A(u)) D_+ A(u_{\Delta x}) A(u)_y \varphi^{\Delta x} \\
&\qquad\qquad + |A(u_{\Delta x}) - A(u)|_\varepsilon D_+ \varphi_y\, dX.
\end{aligned}
$$

Taking the limit as $\varepsilon \downarrow 0$,

$$
\begin{aligned}
0 = \lim_{\varepsilon\to 0} \iint_{\Pi_T^2} & \operatorname{sign}_\varepsilon'(A(\tau_{\Delta x}) - A(u)) D_+ A(u_{\Delta x}) A(u)_y \varphi^{\Delta x}\, dX \\
& + \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\, D_+ \varphi_y\, dX.
\end{aligned} \tag{4.9}
$$

Adding (4.5) and (4.8) and subtracting twice (4.9) we get:

$$
\begin{aligned}
&\iint_{\Pi_T^2} |u - u_{\Delta x}|\,(\varphi_s + \varphi_t) + \operatorname{sign}(u - u_{\Delta x})\,(f(u) - f(u_{\Delta x}))\,\varphi_y\, dX \\
&\qquad - \iint_{\Pi_T^2} \operatorname{sign}(u_{\Delta x} - u)\, D_- F(u_{\Delta x}, (u_{\Delta x})^{\Delta x}) \varphi\, dX \\
&\quad = -\iint_{\Pi_T^2} |A(u) - A(u_{\Delta x})|\,(D_- D_+ \varphi + 2 D_+ \varphi_y + \varphi_{yy})\ dX && (4.10)\\
&\quad\quad + \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon'(A(\tau_{\Delta x}) - A(u))(A(u)_y - D_+ A(u_{\Delta x}))^2 \varphi^{\Delta x}\, dX && (4.11)\\
&\quad\quad + \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon'(A(u_{\Delta x}) - A(u))\varphi \\
&\qquad\qquad - \operatorname{sign}_\varepsilon'(A(\tau_{\Delta x}) - A(u))\varphi^{\Delta x}\big] (A(u)_y)^2\, dX && (4.12)\\
&\quad\quad + \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) \\
&\qquad\qquad - \operatorname{sign}_\varepsilon(A(\theta_{\Delta x}) - A(u))\big] D_+ A(u_{\Delta x}) D_+ \varphi\, dX. && (4.13)
\end{aligned}
$$

4.2. **The main inequality.** Following Lemma 3.3, we define the numerical entropy flux $Q^u(u_j, u_{j+1})$ by

$$Q^u(u_j, u_{j+1}) = \mathrm{sign}\,(u_j - u)\,(f(u_j) - f(u)) + \int_{u_j}^{u_{j+1}} \mathrm{sign}\,(z-u)\,F_2'(z)\,dz. \tag{4.14}$$

By Lemma 3.4

$$\mathrm{sign}(u_j - u) D_- F(u_j, u_{j+1}) \geq D_- Q^u(u_j, u_{j+1}).$$

The term (4.11) is positive and so

$$\begin{aligned}\iint_{\Pi_T^2} &|u_{\Delta x} - u|\,(\varphi_t + \varphi_s)\ dX \\ &+ \iint_{\Pi_T^2} \mathrm{sign}\,(u - u_{\Delta x})\,(f(u) - f(u_{\Delta x}))\,\varphi_y + Q^u(u_{\Delta x}, (u_{\Delta x})^{\Delta x}) D_+\varphi\,dX \\ &\qquad + \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\,(D_- D_+ \varphi + 2D_+\varphi_y + \varphi_{yy})\ dX \geq \Re,\end{aligned} \tag{4.15}$$

where

$$\Re := (4.13) + (4.12). \tag{4.16}$$

By (4.14)

$$\begin{aligned}\iint_{\Pi_T^2} &\mathrm{sign}\,(u - u_{\Delta x})\,(f(u) - f(u_{\Delta x}))\,\varphi_y + Q^u(u_{\Delta x}, (u_{\Delta x})^{\Delta x}) D_+\varphi\,dX \\ &= \iint_{\Pi_T^2} \mathrm{sign}\,(u - u_{\Delta x})\,(f(u) - f(u_{\Delta x}))\,(\varphi_y + D_+\varphi)\ dX \\ &\qquad + \iint_{\Pi_T^2} \int_{u_{\Delta x}}^{(u_{\Delta x})^{\Delta x}} \mathrm{sign}\,(z-u)\,F_2'(z)\,dz\,D_+\varphi\,dX.\end{aligned}$$

Let

$$\gamma := \iint_{\Pi_T^2} \int_{u_{\Delta x}}^{(u_{\Delta x})^{\Delta x}} \mathrm{sign}\,(z-u)\,F_2'(z)\,dz\,D_+\varphi\,dX. \tag{4.17}$$

We obtain from (4.15) the inequality

$$\begin{aligned}\iint_{\Pi_T^2} &|u_{\Delta x} - u|\,(\varphi_t + \varphi_s)\ dX \\ &+ \iint_{\Pi_T^2} \mathrm{sign}\,(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(D_+\varphi + \varphi_y)\ dX \\ &+ \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\,(D_- D_+ \varphi + 2D_+\varphi_y + \varphi_{yy})\ dX \\ &\qquad \geq -\gamma + \Re.\end{aligned} \tag{4.18}$$

Let us specify the test function φ. Let $\rho \in C_0^\infty(\mathbb{R})$ satisfy

$$\mathrm{supp}(\rho) \subset [-1,1], \quad \rho(-\sigma) = \rho(\sigma), \quad \rho(\sigma) \geq 0, \quad \int_{\mathbb{R}} \rho(\sigma)\,d\sigma = 1,$$

and set

$$\omega_r(x) = \frac{1}{r}\rho\left(\frac{x}{r}\right), \quad \rho_\alpha(\xi) = \frac{1}{\alpha}\rho\left(\frac{\xi}{\alpha}\right), \quad \rho_{r_0}(t) = \frac{1}{r_0}\rho\left(\frac{r}{\rho_0}\right),$$

for positive (small) r, α and r_0. Let ν and τ be such that $0 < \nu < \tau < T$ and define

$$\psi^\alpha(t) := H_\alpha(t - \nu) - H_\alpha(t - \tau), \quad H_\alpha(t) = \int_{-\infty}^t \rho_\alpha(\xi)\,d\xi.$$

Let
$$\varphi(x,t,y,s) = \psi^\alpha(t)\omega_r(x-y)\rho_{r_0}(t-s).$$
To ensure $\varphi_{|t=0} \equiv 0$, $\varphi_{|s=0} \equiv 0$, we choose ν and τ such that $0 < r_0 < \min(\nu, T-\tau)$ and $0 < \alpha < \min(\nu - r_0, T-\tau-r_0)$. Note that
$$\begin{aligned}\varphi_t + \varphi_s &= \psi_t^\alpha \omega_r \rho_{r_0},\\ \varphi_x + \varphi_y &= 0,\\ \varphi_{xx} + 2\varphi_{xy} + \varphi_{yy} &= 0.\end{aligned}$$
In equation (4.18) these expressions appear with difference quotients instead of x-derivatives. It should then be expected that these equalities turns into good approximations as long as Δx tends relatively fast to zero compared to r. This will be seen in what follows. Consider the first term on the left in (4.18),
$$\begin{aligned}\iint_{\Pi_T^2} |u_{\Delta x} - u| (\varphi_t + \varphi_s)\, dX = \iint_{\Pi_T^2} &|u_{\Delta x} - u|\, \rho_\alpha(t-\nu)\phi\omega_r\rho_{r_0}\, dX \\ &- \iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\tau)\phi\omega_r\rho_{r_0}\, dX.\end{aligned}$$
Hence
$$\begin{aligned}&\iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\nu)\phi\omega_r\rho_{r_0}\, dX \\ &\qquad + \iint_{\Pi_T^2} \operatorname{sign}(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(D_+\varphi + \varphi_y)\, dX \\ &\qquad + \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\,(D_-D_+\varphi + 2D_+\varphi_y + \varphi_{yy})\, dX + \gamma \\ &\qquad\qquad \geq \iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\tau)\phi\omega_r\rho_{r_0}\, dX + \Re. \qquad (4.19)\end{aligned}$$

4.3. **Estimates.** The subject of this section is to find bounds on the "unwanted" terms in (4.19). In these computations we let C denote a generic constant. By constant it is meant that it does not depend on the "small" variables but it might depend on T and the initial conditions. Similarly we let $\Gamma = \Gamma(\Delta x, \eta, \alpha, r, r_0)$ denote a generic function (taking a variable number of arguments) with the property that it is locally bounded, positive and increasing in each argument. Note the maximum of two such functions is itself of this type.

We first write down some standard computations for future reference.

Lemma 4.2. *Let* $D^k = \frac{\partial^k}{\partial x^k}$. *Then*
$$\left|D^k\varphi(x,t,y,s)\right| \leq \psi(t)\frac{\left\|\rho^{(k)}\right\|_{L^\infty}}{r^{k+1}} \mathbb{1}_{\{|x-y|\leq r\}}(x,y)\rho_{r_0}(t-s).$$
Let $\varphi^\sigma(x,t,y,s) = \varphi(x+\sigma,t,y,s)$. *If* $|\sigma| \leq \Delta x$ *then*
$$\left|D^k\varphi^\sigma(x,t,y,s)\right| \leq \psi(t)\frac{\left\|\rho^{(k)}\right\|_{L^\infty}}{r^{k+1}} \mathbb{1}_{\{|x-y|\leq r+\Delta x\}}(x,y)\rho_{r_0}(t-s).$$
Considering the difference quotient applied to ω_r *we have*
$$|D_+\omega_r(x-y)|| \leq \frac{\|\rho'\|_{L^\infty}}{r^2}\mathbb{1}_{\{|x-y|\leq r+\Delta x\}}(x,y).$$

Proof. Note that
$$D^k\omega_r(x) = \frac{1}{r^{k+1}}\rho^{(k)}\left(\frac{x}{r}\right).$$

Since $\mathrm{supp}(\rho) \subset [-1,1]$ we have

$$\left|D^k \omega_r(x)\right| \leq \frac{\left\|\rho^{(k)}\right\|_{L^\infty}}{r^{k+1}} \mathbb{1}_{\{|x|\leq r\}}(x),$$

which proves the first statement. Consider the second statement. If $|x-y| \geq r+\Delta x$, then

$$|x+\sigma-y| \geq |x-y| - |\sigma| \geq r + \Delta x - \Delta x = r,$$

so it follows that $\mathbb{1}_{\{|x+\sigma-y|\leq r\}}(x,y) \leq \mathbb{1}_{\{|x-y|\leq r+\Delta x\}}(x,y)$; this proves the second statement.

To prove the last statement, recall that

$$D_+\omega_r(x) = \frac{\omega_r(x+\Delta x) - \omega_r(x)}{\Delta x}.$$

If $|x| \geq r+\Delta x$ then $\omega_r(x+\Delta x) = \omega_r(x) = 0$, so $\mathrm{supp}(D_+(\omega_r)) \subset [-r-\Delta x, r+\Delta x]$. By the mean value theorem and the fact that $\|\omega_r'\|_{L^\infty} = \|\rho'\|_{L^\infty} r^{-2}$ we get

$$|\omega_r(x+\Delta x) - \omega_r(x)| \leq \frac{\|\rho'\|_{L^\infty}}{r^2}\Delta x.$$

The last statement follows from this. □

Now observe that

$$D_+\varphi + \varphi_y = D_+\varphi - \varphi_x,$$

and so

$$\begin{aligned}
&\iint_{\Pi_T^2} \mathrm{sign}\left(u_{\Delta x} - u\right)\left(f(u_{\Delta x}) - f(u)\right)\left(D_+\varphi + \varphi_y\right)\, dX \\
&\qquad = \iint_{\Pi_T^2} \mathrm{sign}\left(u_{\Delta x} - u\right)\left(f(u_{\Delta x}) - f(u)\right)\left(D_+\varphi - \varphi_x\right)\, dX =: \beta. \qquad (4.20)
\end{aligned}$$

Estimate 4.1. Let β be defined by (4.20), then

$$|\beta| \leq C\frac{\Delta x}{r}\left(1 + \frac{\Delta x}{r}\right).$$

Proof. We claim that

$$\left(D_+\varphi - \varphi_x\right)(x,t,y,s) = \frac{1}{\Delta x}\int_0^{\Delta x} (\Delta x - \sigma)\varphi_{xx}(x+\sigma,t,y,s)\, d\sigma. \qquad (4.21)$$

Hence

$$\beta = \frac{1}{\Delta x}\iint_{\Pi_T^2}\int_0^{\Delta x} \mathrm{sign}_\varepsilon\left(A(u_{\Delta x}) - A(u)\right)\left(f(u_{\Delta x}) - f(u)\right)(\Delta x - \sigma)\varphi_{xx}^\sigma\, d\sigma\, dX.$$

We can write

$$\begin{aligned}
&\mathrm{sign}\left(u_{\Delta x} - u\right)\left(f(u_{\Delta x}) - f(u)\right)(x,t,y,s) \\
&\qquad = \sum_j \underbrace{\mathrm{sign}\left(u_j - u\right)\left(f(u_j) - f(u)\right)(t,y,s)}_{\Theta_j} \mathbb{1}_{\{I_j\}}(x).
\end{aligned}$$

Using summation by parts

$$\begin{aligned}
&\frac{1}{\Delta x}\int_{\mathbb{R}}\int_0^{\Delta x} \mathrm{sign}\left(u_{\Delta x} - u\right)\left(f(u_{\Delta x}) - f(u)\right)(\Delta x - \sigma)\varphi_{xx}^\sigma\, d\sigma dx \\
&\qquad = \frac{1}{\Delta x}\int_0^{\Delta x}\sum_j \Theta_j \int_{\mathbb{R}} \mathbb{1}_{\{I_j\}}(x)(\Delta x - \sigma)\varphi_{xx}^\sigma\, dx d\sigma \\
&\qquad = \frac{1}{\Delta x}\int_0^{\Delta x}\sum_j \Theta_j \int_{I_j} \varphi_{xx}(x+\sigma,t,y,s)\, dx(\Delta x - \sigma)\, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\Delta x} \sum_j \Theta_j \left(D_- \varphi^{\sigma}_{x,j+1/2} \right) (\Delta x - \sigma)\, d\sigma \\
&= -\sum_j D_+ \Theta_j \int_0^{\Delta x} \varphi^{\sigma}_{x,j+1/2} (\Delta x - \sigma)\, d\sigma,
\end{aligned}
$$

where $\varphi^{\sigma}_{x,j+1/2} = \varphi_x(x_{j+1/2} + \sigma, t, y, s)$. By Lemma 4.2 we have

$$
|\varphi_x(x+\sigma, t, y, s)| \le C \frac{1}{r^2} \mathbb{1}_{\{|x-y| \le r + \Delta x\}}(x,y) \rho_{r_0}(t-s).
$$

Hence

$$
\left| \int_0^{\Delta x} \varphi^{\sigma}_{x,j+1/2}(\Delta x - \sigma)\, d\sigma \right| \le C \Delta x^2 \frac{1}{r^2} \mathbb{1}_{\left\{|x_{j+1/2} - y| \le r + \Delta x\right\}}(y) \rho_{r_0}(t-s).
$$

Now

$$
|D_+ \Theta_j| \le \|f\|_{\mathrm{Lip}} |D_+ u_j|\,.
$$

Therefore

$$
\begin{aligned}
&\Big| \frac{1}{\Delta x} \int_{\mathbb{R}} \int_0^{\Delta x} \operatorname{sign}(u_{\Delta x} - u)\,(f(u_{\Delta x}) - f(u))\,(\Delta x - \sigma) \varphi^{\sigma}_{xx}\, d\sigma dx \Big| \\
&\qquad \le \sum_j |D_+ \Theta_j| \left| \int_0^{\Delta x} \varphi^{\sigma}_{x,j+1/2}(\Delta x - \sigma)\, d\sigma \right| \\
&\qquad \le C \|f\|_{\mathrm{Lip}} \Delta x^2 \sum_j |D_+ u_j| \frac{1}{r^2} \mathbb{1}_{\left\{|x_{j+1/2} - y| \le r + \Delta x\right\}}(y) \rho_{r_0}(t-s).
\end{aligned}
$$

It follows by the above and Lemma 3.1 that

$$
\begin{aligned}
|\beta| &\le C \Delta x^2 \frac{r + \Delta x}{r^2} \int_0^T \sum_j |D_+ u_j|\, dt \\
&= C \frac{r + \Delta x}{r^2} \int_{\Pi_T} |u_{\Delta x}(x + \Delta x, t) - u_{\Delta x}(x,t)|\, dxdt \\
&= CT \frac{1}{r} \left(1 + \frac{\Delta x}{r} \right) \Delta x \left| u^0_{\Delta x} \right|_{BV(\mathbb{R})}.
\end{aligned}
$$

This conclude the proof. □

Next, let us consider the term (4.10). First observe that

$$
D_- D_+ \varphi + 2 D_+ \varphi_y + \varphi_{yy} = (D_- D_+ \varphi - \varphi_{xx}) + 2\left(D_+ \varphi - \varphi_x\right)_y.
$$

Thus (4.10) can be rewritten

$$
\begin{aligned}
&\iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\, (D_- D_+ \varphi + 2 D_+ \varphi_y + \varphi_{yy})\, dX \\
&\qquad = \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\, (D_- D_+ \varphi - \varphi_{xx})\, dX \\
&\qquad\qquad + 2 \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)|\, \left(D_+ \varphi - \varphi_x\right)_y\, dX \\
&\qquad =: \zeta_1 + \zeta_2.
\end{aligned}
$$

Estimate 4.2.

$$
|\zeta_1 + \zeta_2| \le C \frac{\Delta x}{r^2} \left(1 + \frac{\Delta x}{r} \right)^2.
$$

Proof. Consider the term ζ_1. We use the same strategy as for the β term. Write $\mu(\sigma) = \varphi(x+\sigma, t, y, s)$. By a Taylor expansion

$$\mu(z) - \mu(0) = z\mu'(0) + \frac{1}{2}z^2\mu''(0) + \frac{1}{6}z^3\mu^{(3)}(0) - \frac{1}{6}\int_0^z (\sigma - z)^3 \mu^{(4)}(\sigma)\, d\sigma.$$

Using this, we get

$$\begin{aligned}\mu(\Delta x) &- 2\mu(0) + \mu(-\Delta x) - \Delta x^2 \mu''(0) \\ &= -\frac{1}{6}\int_0^{\Delta x} (\sigma - \Delta x)^3 \mu^{(4)}(\sigma)\, d\sigma + \frac{1}{6}\int_{-\Delta x}^0 (\sigma + \Delta x)^3 \mu^{(4)}(\sigma)\, d\sigma.\end{aligned}$$

It follows that

$$\begin{aligned}D_+D_-\varphi - \varphi_{xx} = &-\frac{1}{6\Delta x^2}\int_0^{\Delta x} (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\, d\sigma \\ &+ \frac{1}{6\Delta x^2}\int_{-\Delta x}^0 (\sigma + \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\, d\sigma.\end{aligned}$$

Splitting ζ_1 according to this equality we get

$$\begin{aligned}\zeta_1 &= \iint_{\Pi_T^2} |A(u_{\Delta x}) - A(u)| \left(D_-D_+\varphi - \varphi_{xx}\right)\, dX \\ &= -\frac{1}{6\Delta x^2}\iint_{\Pi_T^2}\int_0^{\Delta x} |A(u_{\Delta x}) - A(u)| (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\, d\sigma\, dX \\ &\quad + \frac{1}{6\Delta x^2}\iint_{\Pi_T^2}\int_{-\Delta x}^0 |A(u_{\Delta x}) - A(u)| (\sigma + \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\, d\sigma\, dX \\ &=: T_1 + T_2.\end{aligned}$$

We also have that

$$|A(u_{\Delta x}) - A(u)|(x, t, y, s) = \sum_j \underbrace{|A(u_j) - A(u)|(t, y, s)}_{\Phi_j} \mathbb{1}_{\{I_j\}}(x).$$

Now consider T_1,

$$\begin{aligned}&-\int_0^{\Delta x}\int_{\mathbb{R}} |A(u_{\Delta x}) - A(u)| (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi^\sigma\, dx d\sigma \\ &\quad = -\sum_j |A(u_j) - A(u)|(t, y, s) \int_0^{\Delta x} (\sigma - \Delta x)^3 \int_{\mathbb{R}} \mathbb{1}_{\{I_j\}}(x) \frac{\partial^4}{\partial x^4}\varphi^\sigma\, dx d\sigma \\ &\quad = -\Delta x \int_0^{\Delta x} (\sigma - \Delta x)^3 \sum_j \Phi_j D_-\varphi^\sigma_{xxx, j+1/2}\, d\sigma \\ &\quad = \Delta x \sum_j D_+\Phi_j \int_0^{\Delta x} (\sigma - \Delta x)^3 \varphi^\sigma_{xxx, j+1/2}\, d\sigma\end{aligned}$$

where

$$\varphi^\sigma_{xxx, j+1/2}(t, y, s) = \frac{\partial^3}{\partial x^3}\varphi(x_{j+1/2} + \sigma, t, y, s).$$

Now we use Lemma 4.2 to estimate this term,

$$\begin{aligned}|T_1| &= \Big|\frac{1}{6\Delta x^2}\iint_{\Pi_T^2}\int_0^{\Delta x} |A(u_{\Delta x}) - A(u)| (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4}\varphi(x+\sigma, t, y, s)\, d\sigma\, dX\Big| \\ &= \Big|\frac{1}{6\Delta x}\int_{\Pi_T}\int_0^T \sum_j D_+\Phi_j \int_0^{\Delta x} (\sigma - \Delta x)^3 \varphi^\sigma_{xxx, j+1/2}\, d\sigma\ dt\, dy ds\Big|\end{aligned}$$

$$
\begin{aligned}
&\leq C\frac{r+\Delta x}{\Delta x^2 r^4}\int_{\Pi_T} |D_+A(u_{\Delta x})| \Big(\int_0^{\Delta x} (\sigma-\Delta x)^3\, d\sigma\Big)\, dxdt \\
&\leq C\Delta x^2 \frac{r+\Delta x}{r^4} \\
&= C\frac{\Delta x^2}{r^3}\left(1+\frac{\Delta x}{r}\right),
\end{aligned}
$$

where we have used that $|A(u_{\Delta x}(\cdot,t))|_{BV(\mathbb{R})}$ is bounded independently of $\Delta x, t, \eta$ by Lemma 3.1. The term T_2 is estimated in a similar way. Now consider ζ_2. Integration by parts and the dominated convergence theorem imply

$$
\begin{aligned}
\zeta_2 &= \lim_{\varepsilon\downarrow 0} 2\iint_{\Pi_T^2} |A(u_{\Delta x})-A(u)|_\varepsilon \,(D_+\varphi-\varphi_x)_y\; dX \\
&= -2\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_{\Delta x})-A(u))A(u)_y\,(D_+\varphi-\varphi_x)\; dX \\
&= -2\iint_{\Pi_T^2} \operatorname{sign}(u_{\Delta x}-u)\, A(u)_y\,(D_+\varphi-\varphi_x)\; dX.
\end{aligned}
$$

By (4.21)

$$
\begin{aligned}
|\zeta_2| &= 2\frac{1}{\Delta x}\Big|\iint_{\Pi_T^2}\int_0^{\Delta x} \operatorname{sign}(u_{\Delta x}-u)A(u)_y(\Delta x-\sigma)\varphi_{xx}^\sigma\, d\sigma\, dX\Big| \\
&\leq C\frac{r+\Delta x}{(\Delta x)r^3}\int_{\Pi_T} |A(u)_y| \int_0^{\Delta x}(\Delta x-\sigma)\, d\sigma\, dyds \\
&\leq C\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right),
\end{aligned}
$$

since $|A(u(\cdot,s))|_{BV(\mathbb{R})}$ is bounded independently of s and η by Lemma 2.1. This concludes the proof of the estimate. □

Estimate 4.3. Let γ be defined by (4.17), then

$$
|\gamma| \leq C\frac{\Delta x}{r}\left(1+\frac{\Delta x}{r}\right).
$$

Proof. By definition F_2' is bounded, hence

$$
\Big|\int_{u_j}^{u_{j+1}} \operatorname{sign}(z-u)\, F_2'(z)\, dz\Big| \leq \|F_2\|_{\text{Lip}}\, \Delta x\, |D_+u_j|\,.
$$

Note that $|u_{\Delta x}(\cdot,t)|_{BV(\mathbb{R})}$ is bounded independently of $\Delta x, t$ and η by Lemma 3.1 so we may apply Lemma 4.2 to obtain the result. □

Estimate 4.4. Let $\Re$ be defined by (4.16), then

$$
|\Re| \leq \Gamma(r)\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right)\left(1+\left(\frac{\Delta x}{r}\right)^3\right)+C\frac{\Delta x}{r_0}.
$$

Proof. Consider the term (4.13), i.e.,

$$
\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T^2}\big[\operatorname{sign}_\varepsilon(A(u_{\Delta x})-A(u))-\operatorname{sign}_\varepsilon(A(\theta_{\Delta x})-A(u))\big]D_+A(u_{\Delta x})D_+\varphi\, dX.
$$

As stated before we may apply dominated convergence and Lemma 4.3 to show that this limit exists. First observe that

$$
\begin{aligned}
0 &\leq [\operatorname{sign}_\varepsilon(A(\theta_j)-A(u))-\operatorname{sign}_\varepsilon(A(u_j)-A(u))]\, D_+A(u_j) \\
&\leq \Delta x D_+\operatorname{sign}_\varepsilon(A(u_j)-A(u))D_+A(u_j).
\end{aligned}
$$

Hence

$$|(4.13)| \leq \Delta x \lim_{\varepsilon\downarrow 0} \iint_{\Pi_T^2} D_+\mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+A(u_{\Delta x}) \left|D_+\varphi\right| \, dX$$

Using both integration by parts for difference quotients and Leibniz rule for difference quotients we obtain

$$\begin{aligned}
\iint_{\Pi_T^2} & D_+\mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+A(u_{\Delta x}) \left|D_+\varphi\right| \, dX \\
&= -\iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+A(u_{\Delta x}) D_+ \left|D_-\varphi\right| \, dX \\
&\quad - \iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_-D_+A(u_{\Delta x}) \left|D_-\varphi\right| \, dX \\
&=: S_1 + S_2.
\end{aligned}$$

To estimate S_2 we first observe that $D_+\left|D_-\varphi\right| \leq \left|D_+D_-\varphi\right|$. Furthermore, when proving Estimate 4.2, we established that

$$\begin{aligned}
D_+&D_-\varphi(x,t,y,s) \\
&= \varphi_{xx}(x,t,y,s) - \frac{1}{6\Delta x^2} \int_0^{\Delta x} (\sigma - \Delta x)^3 \frac{\partial^4}{\partial x^4} \varphi(x+\sigma,t,y,s) \, d\sigma \\
&\quad + \frac{1}{6\Delta x^2} \int_{-\Delta x}^0 (\sigma + \Delta x)^3 \frac{\partial^4}{\partial x^4} \varphi(x+\sigma,t,y,s) \, d\sigma.
\end{aligned}$$

By Lemma 4.2,

$$\begin{aligned}
\Big| \int_0^{\pm\Delta x} & (\sigma \mp \Delta x)^3 \frac{\partial^4}{\partial x^4} \varphi(x+\sigma,t,y,s) \, d\sigma \Big| \\
&\leq C \frac{(\Delta x)^4}{r^5} \mathbb{1}_{\{|x-y| \leq r + \Delta x\}}(x,y) \rho_{r_0}(t-s).
\end{aligned}$$

Using Lemma 4.2 once more, the above implies that

$$\begin{aligned}
\int_{\Pi_T} \left|D_+D_-\varphi\right| \, dyds &\leq \int_{\Pi_T} \left|\varphi_{xx}\right| \, dyds + C\frac{(\Delta x)^2}{r^4}\left(1 + \frac{\Delta x}{r}\right) \\
&\leq C\left(\frac{1}{r^2} + \frac{(\Delta x)^2}{r^4}\right)\left(1 + \frac{\Delta x}{r}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
|S_1| &= \Big| \iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) D_+A(u_{\Delta x}) D_+ \left|D_-\varphi\right| \, dX \Big| \\
&\leq \int_{\Pi_T} \left|D_+A(u_{\Delta x})\right| \Big(\int_{\Pi_T} \left|D_+D_-\varphi\right| \, dyds \Big) \, dxdt \\
&\leq C\left(\frac{1}{r^2} + \frac{(\Delta x)^2}{r^4}\right)\left(1 + \frac{\Delta x}{r}\right) \int_{\Pi_T} \left|D_+A(u_{\Delta x})\right| \, dxdt.
\end{aligned}$$

Recall that $\left|A(u_{\Delta x}(\cdot,t))\right|_{BV(\mathbb{R})}$ is bounded independently of $\Delta x, t, \eta$ by Lemma 3.1. Considering S_2 we have

$$\begin{aligned}
|S_2| &= \Big| \iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u)) \left(D_-D_+A(u_{\Delta x})\right) \left|D_-\varphi\right| \, dX \Big| \\
&\leq \iint_{\Pi_T^2} \left|D_-D_+A(u_{\Delta x})\right| \left|D_-\varphi\right| \, dX \\
&\leq C\frac{r + \Delta x}{r^2} \int_{\Pi_T} \left|D_-D_+A(u_{\Delta x})\right| \, dxdt.
\end{aligned}$$

Note that it follows by (3.1) and Lemma 3.1 that $\|D_-D_+A(u_{\Delta x}(\cdot,t))\|_{L^1(\mathbb{R})}$ is bounded independently of $\Delta x, t, \eta$. Hence

$$\begin{aligned}\Delta x \iint_{\Pi_T^2} D_+\mathrm{sign}_\varepsilon(A(u_{\Delta x}) - A(u))D_+A(u_{\Delta x})\,|D_+\varphi|\;dX&\\ &\le \Delta x\,(|S_1|+|S_2|)\\ &\le \Gamma(r)\left(\frac{\Delta x}{r^2}+\frac{(\Delta x)^3}{r^4}\right)\left(1+\frac{\Delta x}{r}\right).\end{aligned}$$

Next we estimate (4.12). We first split the term.

$$\begin{aligned}&\iint_{\Pi_T^2}\left(\mathrm{sign}_\varepsilon'(A(u_{\Delta x})-A(u))\varphi - \mathrm{sign}_\varepsilon'(A(\tau_{\Delta x})-A(u))\varphi^{\Delta x}\right)\left(A(u)_y\right)^2\;dX\\ &\quad= \iint_{\Pi_T^2}\left(\mathrm{sign}_\varepsilon'(A(u_{\Delta x})-A(u)) - \mathrm{sign}_\varepsilon'(A(\tau_{\Delta x})-A(u))\right)\varphi^{\Delta x}\left(A(u)_y\right)^2\;dX\\ &\qquad+\iint_{\Pi_T^2}\mathrm{sign}_\varepsilon'(A(u_{\Delta x})-A(u))\left(\varphi-\varphi^{\Delta x}\right)\left(A(u)_y\right)^2 dX\\ &\quad=: W_1+W_2.\end{aligned}$$

We start with W_1. By (2.1) we have

$$\psi_\varepsilon'(u,u_j)u_s\varphi^{\Delta x}+\psi_\varepsilon'(u,u_j)f(u)_y\varphi^{\Delta x}=\psi_\varepsilon'(u,u_j)A(u)_{yy}\varphi^{\Delta x}$$

Now

$$\begin{aligned}\left(\psi_\varepsilon'(u,u_j)A(u)_y\right)_y &= \left(\mathrm{sign}_\varepsilon(A(u)-A(u_j))A(u)_y\right)_y\\ &= \mathrm{sign}_\varepsilon'\left(A(u)-A(u_j)\right)\left(A(u)_y\right)^2\\ &\qquad+\mathrm{sign}_\varepsilon(A(u)-A(u_j))A(u)_{yy}.\end{aligned}\tag{4.22}$$

Hence

$$\begin{aligned}&\int_{\Pi_T}\mathrm{sign}_\varepsilon'(A(u)-A(u_j))\left(A(u)_y\right)^2\varphi^{\Delta x}\,dyds\\ &\qquad=\int_{\Pi_T}(\mathrm{sign}_\varepsilon(A(u)-A(u_j))A(u)_y)_y\varphi^{\Delta x}\,dyds\\ &\qquad\quad-\int_{\Pi_T}\mathrm{sign}_\varepsilon(A(u)-A(u_j))A(u)_{yy}\varphi^{\Delta x}\,dyds,\end{aligned}$$

and so

$$\begin{aligned}&\int_{\Pi_T}\mathrm{sign}_\varepsilon'(A(u)-A(u_j))\left(A(u)_y\right)^2\varphi^{\Delta x}\,dyds\\ &\quad=\int_{\Pi_T}\left[\mathrm{sign}_\varepsilon(A(u)-A(u_j))A(u)_y\right]_y\varphi^{\Delta x}\,dyds-\int_{\Pi_T}\psi_\varepsilon'(u,u_j)u_s\varphi^{\Delta x}\,dyds\\ &\qquad-\int_{\Pi_T}\mathrm{sign}_\varepsilon(A(u)-A(u_j))f(u)_y\varphi^{\Delta x}\,dyds.\end{aligned}$$

We shall obtain a similar expression for

$$\int_{\Pi_T}\mathrm{sign}_\varepsilon'(A(u)-A(\tau_j))(A(u)_y)^2\varphi^{\Delta x}\,dyds.$$

We have that

$$\psi_\varepsilon'(u,\theta_j)u_s\varphi^{\Delta x}+\psi_\varepsilon'(u,\theta_j)f(u)_y\varphi^{\Delta x}=\psi_\varepsilon'(u,\theta_j)A(u)_{yy}\varphi^{\Delta x}.$$

Using Lemma 4.1,

$$\begin{aligned}\left(\psi_\varepsilon'(u,\theta_j)A(u)_y\right)_y &= \left(\mathrm{sign}_\varepsilon(A(u)-A(\theta_j))A(u)_y\right)_y\\ &= \mathrm{sign}_\varepsilon'(A(u)-A(\tau_j))\left(A(u)_y\right)^2\end{aligned}$$

$$+\operatorname{sign}_\varepsilon(A(u)-A(\theta_j))A(u)_{yy},$$

and so

$$\begin{aligned}\int_{\Pi_T} &\operatorname{sign}'_\varepsilon(A(u)-A(\tau_j))\left(A(u)_y\right)^2\varphi^{\Delta x}\,dyds\\ &= \int_{\Pi_T}(\operatorname{sign}_\varepsilon(A(u)-A(\theta_j))A(u)_y)_y\varphi^{\Delta x}\,dyds\\ &\quad - \int_{\Pi_T}\operatorname{sign}_\varepsilon(A(u)-A(\theta_j))A(u)_{yy}\varphi^{\Delta x}\,dyds.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\Pi_T}&\operatorname{sign}'_\varepsilon(A(u)-A(\tau_j))(A(u)_y)^2\varphi^{\Delta x}\,dyds\\ &= \int_{\Pi_T}(\operatorname{sign}_\varepsilon(A(u)-A(\theta_j))A(u)_y)_y\varphi^{\Delta x}\,dyds - \int_{\Pi_T}\psi'_\varepsilon(u,\theta_j)u_s\varphi^{\Delta x}\,dyds\\ &\quad - \int_{\Pi_T}\psi'_\varepsilon(u,\theta_j)f(u)_y\varphi^{\Delta x}\,dyds.\end{aligned}$$

From this it follows

$$W_1 = \int_{\Pi_T}\left(\operatorname{sign}'_\varepsilon(A(u)-A(u_j)) - \operatorname{sign}'_\varepsilon(A(u)-A(\tau_j))\right)\left(A(u)_y\right)^2\varphi^{\Delta x}\,dyds$$

$$= \int_{\Pi_T}\left(\left(\operatorname{sign}_\varepsilon(A(u)-A(u_j)) - \operatorname{sign}_\varepsilon(A(u)-A(\theta_j))\right)A(u)_y\right)_y\varphi^{\Delta x}\,dyds \tag{4.23}$$

$$- \int_{\Pi_T}\left(\operatorname{sign}_\varepsilon(A(u)-A(u_j)) - \operatorname{sign}_\varepsilon(A(u)-A(\theta_j))\right)u_s\varphi^{\Delta x}\,dyds \tag{4.24}$$

$$- \int_{\Pi_T}\left(\operatorname{sign}_\varepsilon(A(u)-A(u_j)) - \operatorname{sign}_\varepsilon(A(u)-A(\theta_j))\right)f(u)_y\varphi^{\Delta x}\,dyds \tag{4.25}$$

Next we want to send ε to zero and then estimate the terms (4.23), (4.24) and (4.25). The next two lemmas will enable us to do this.

Lemma 4.3. *For real numbers u, a and b define*

$$g(u,a,b) = \begin{cases}\frac{|A(b)-A(u)|-|A(a)-A(u)|}{A(b)-A(a)} & \text{if } a\neq b,\\ \operatorname{sign}(A(a)-A(u)) & \text{if } a=b,\ u\neq b,\\ 0 & \text{if } a=b=u.\end{cases}$$

Under the same assumptions as in Lemma 4.1

$$\lim_{\varepsilon\downarrow 0}\operatorname{sign}_\varepsilon(A(\theta_j)-A(u)) = g(u,u_j,u_{j+1}).$$

Proof. Recall the definition of θ_j:

$$\operatorname{sign}_\varepsilon\left(A(\theta_j)-A(u)\right)\left(A(u_{j+1})-A(u_j)\right) = \left|A(u_{j+1})-A(u)\right|_\varepsilon - \left|A(u_j)-A(u)\right|_\varepsilon.$$

If $u_{j+1}=u_j$, then $\theta_j=u_j$ for all u and ε, since $\theta_j\in\operatorname{int}(u_j,u_{j+1})$. Thus in this case

$$\lim_{\varepsilon\downarrow 0}\operatorname{sign}_\varepsilon\left(A(\theta_j)-A(u)\right) = \begin{cases}0 & \text{if } u=u_j,\\ \operatorname{sign}\left(A(u_j)-A(u)\right) & \text{otherwise.}\end{cases}$$

Now assume that $D_+A(u_j)\neq 0$. Then

$$\operatorname{sign}_\varepsilon(A(\theta_j)-A(u)) = \frac{\left|A(u_{j+1})-A(u)\right|_\varepsilon - \left|A(u_j)-A(u)\right|_\varepsilon}{A(u_{j+1})-A(u_j)},$$

and the result follows by letting $\varepsilon\downarrow 0$. $\square$

Regarding the function g.

Lemma 4.4. *Suppose $a \neq b$. Then $u \mapsto g(u,a,b)$ is non-increasing function and*

$$g(u,a,b) = \begin{cases} 1 & \text{if } u \leq \min\{a,b\}, \\ -1 & \text{if } \max\{a,b\} \leq u. \end{cases}$$

Proof. First observe that $g(u,a,b) = g(u,b,a)$ so we can assume that $a < b$.

$$\begin{aligned} (A(b)-A(a))g(u,a,b) &= |A(b)-A(u)| - |A(a)-A(u)| \\ &= \operatorname{sign}(b-u)\,(A(b)-A(u)) - \operatorname{sign}(a-u)\,(A(a)-A(u)) \\ &= \begin{cases} \operatorname{sign}(b-u)\,(A(b)-A(a)) & \text{if } u \notin (a,b), \\ A(b)+A(a)-2A(u) & \text{if } u \in (a,b). \end{cases} \end{aligned}$$

Since A is increasing, this proves the lemma. □

Let

$$H_j(u) = \int_{-\infty}^{u} \operatorname{sign}(z-u_j) + g(z,u_j,u_{j+1})\,dz.$$

By the above lemma, the support of the integrand belongs to $\operatorname{int}(u_j,u_{j+1})$. Besides, its absolute value is bounded by 2, so H_j is Lipschitz continuous, and $|H_j(u)| \leq 2\,|u_{j+1}-u_j|$. Therefore

$$\Big|\int_0^T H_j(u)\frac{\partial}{\partial s}\rho_{r_0}(t-s)\,ds\Big| \leq \frac{C}{r_0}\,|u_{j+1}-u_j|\,.$$

Regarding (4.24), we proceed as follows:

$$\begin{aligned} \Big|\lim_{\varepsilon\downarrow 0}\int_{\Pi_T}&\big[\operatorname{sign}_\varepsilon(A(u)-A(u_j)) - \operatorname{sign}_\varepsilon(A(u)-A(\theta_j))\big]u_s\varphi^{\Delta x}\,dyds\Big| \\ &= \Big|\int_{\Pi_T} H_j'(u)u_s\varphi^{\Delta x}\,dyds\Big| \\ &= \Big|\int_{\Pi_T} \frac{\partial}{\partial s}\left(H_j(u)\right)\varphi^{\Delta x}\,dyds\Big| \\ &= \Big|\int_{\Pi_T} H_j(u)\varphi_s^{\Delta x}\,dyds\Big| \\ &\leq \frac{C}{r_0}|u_{j+1}-u_j| = C\frac{\Delta x}{r_0}|D_+u_j|. \end{aligned}$$

Now we estimate (4.25). To this end, let

$$Q_j(u) = \int_0^u H_j'(z)f'(z)\,dz.$$

Then

$$|Q_j(u)| = \Big|\int_0^u H_j'(z)f'(z)\,dz\Big| \leq C\int_0^u \mathbb{1}_{\{\operatorname{int}(u_j,u_{j+1})\}}(z)\,dz \leq C|u_{j+1}-u_j|.$$

Hence

$$\begin{aligned} \Big|\lim_{\varepsilon\downarrow 0}\int_{\Pi_T}&\big(\operatorname{sign}_\varepsilon(A(u)-A(u_j)) - \operatorname{sign}_\varepsilon(A(u)-A(\theta_j))\big)f'(u)u_y\varphi^{\Delta x}\,dyds\Big| \\ &= \Big|\int_{\Pi_T} Q_j(u)\varphi_y^{\Delta x}\,dyds\Big| \leq C\frac{\Delta x}{r}\,|D_+u_j| \end{aligned}$$

For the term (4.23) we let

$$P_j(u) = \int_0^u H_j'(z)A'(z)\,dz,$$

and perform the same trick.

$$\Big|\lim_{\varepsilon\downarrow 0}\int_{\Pi_T}\Big(\big(\mathrm{sign}_\varepsilon(A(u)-A(u_j))-\mathrm{sign}_\varepsilon(A(u)-A(\theta_j))\big)A(u)_y\Big)_y\varphi^{\Delta x}\,dyds\Big|$$
$$=\Big|\int_{\Pi_T}P_j(u)\varphi^{\Delta x}_{yy}\,dyds\Big|\le C\frac{\Delta x}{r^2}\left|D_+u_j\right|.$$

It follows that

$$\begin{aligned}|W_1|&=\Big|\lim_{\varepsilon\downarrow 0}\iint_{\Pi_T^2}\big[\mathrm{sign}'_\varepsilon(A(u)-A(u_j))-\mathrm{sign}'_\varepsilon(A(u)-A(\tau_j))\big]\left(A(u)_y\right)^2\varphi^{\Delta x}\,dX\Big|\\&\le\Gamma(r)\frac{\Delta x}{r^2}.\end{aligned}$$

Finally we need to bound W_2 independently of ε. By equation (4.22)

$$\begin{aligned}W_2&=\iint_{\Pi_T^2}\mathrm{sign}'_\varepsilon(A(u_{\Delta x})-A(u))\left(\varphi-\varphi^{\Delta x}\right)\left(A(u)_y\right)^2\,dX\\&=-\Delta x\iint_{\Pi_T^2}\mathrm{sign}'_\varepsilon(A(u)-A(u_{\Delta x}))\left(A(u)_y\right)^2D_+\varphi\,dX\\&=-\Delta x\iint_{\Pi_T^2}\left(\mathrm{sign}_\varepsilon(A(u)-A(u_{\Delta x})A(u)_y\right)_yD_+\varphi\,dX\\&\quad+\Delta x\iint_{\Pi_T^2}\mathrm{sign}_\varepsilon(A(u)-A(u_{\Delta x}))A(u)_{yy}D_+\varphi\,dX\\&=:W_2^1+W_2^2.\end{aligned}$$

Now, by Lemma 2.1, $|A(u(\cdot,s))|_{BV(\mathbb{R})}$ is bounded independently of s and η so

$$\begin{aligned}\left|W_2^1\right|&\le\Delta x\iint_{\Pi_T^2}|A(u)_y|\,|D_+\varphi_y|\,dX\\&=\Delta x\int_{\Pi_T}|A(u)_y|\Big(\int_{\Pi_T}|D_+\varphi_y|\,dxdt\Big)\,dyds\\&\le\Delta x\int_{\Pi_T}|A(u)_y|\Big(\int_{\Pi_T}|\psi(D_+\omega_r)_y\rho_{r_0}|\,dxdt\Big)\,dyds\\&\le\left(C\frac{\Delta x}{r}+C\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right)\right)\int_{\Pi_T}|A(u)_y|\,dyds\\&\le\Gamma(r)\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right).\end{aligned}$$

Also, using the uniform bound on $\|A(u)_{yy}\|_{L^1(\Pi_T)}$ from Lemma 2.3,

$$\begin{aligned}\left|W_2^2\right|&\le\Delta x\iint_{\Pi_T^2}|A(u)_{yy}|\,|D_+\varphi|\,dX\\&\le\Delta x\int_{\Pi_T}|A(u)_{yy}|\Big(\int_{\Pi_T}|D_+\varphi|\,dxdt\Big)\,dyds\\&\le\Delta xC\left(1+\frac{1}{r}\left(1+\frac{\Delta x}{r}\right)\right)\int_{\Pi_T}|A(u)_{yy}|\,dyds\\&\le\Gamma(r)\frac{\Delta x}{r}\left(1+\frac{\Delta x}{r}\right).\end{aligned}$$

It follows that

$$|W_2|\le\Gamma(r)\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right).$$

□

Now we return to inequality (4.19), and define

$$\Xi(\Delta x, r, \eta, \varepsilon, r_0) = \beta + \zeta_1 + \zeta_2 + |\Re| + \gamma.$$

Using this, (4.19) reads

$$\iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\tau)\omega_r(x-y)\rho_{r_0}(t-s)\, dX \\ \leq \iint_{\Pi_T^2} |u_{\Delta x} - u|\, \rho_\alpha(t-\nu)\omega_r(x-y)\rho_{r_0}(t-s)\, dX + \Xi(\Delta x, r, r_0).$$

Combining Estimate 4.1, Estimate 4.2, Estimate 4.3, and Estimate 4.4 we find that

$$|\Xi| \leq \Gamma\left(r, \frac{\Delta x}{r}\right)\frac{\Delta x}{r^2} + C\frac{\Delta x}{r_0},$$

independently of α. Hence we can send α to zero and get

$$\kappa(\tau) \leq \kappa(\nu) + \Xi(\Delta x, r, r_0),$$

where

$$\kappa(t) := \int_{\mathbb{R}} \iint_{\Pi_T} |u_{\Delta x}(x,t) - u(y,s)|\, \omega_r(x-y)\rho_{r_0}(t-s)\, dydsdx.$$

Lemma 4.5. *Let $t \geq r_0$, and L^c be the Lipschitz constant of $t \mapsto |u(\cdot,t)|_{L^1(\mathbb{R})}$. Then*

$$\left|\kappa(t) - \|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})}\right| \leq |u(\cdot,t)|_{BV(\mathbb{R})}\, r + L^c r_0.$$

Proof. By the triangle inequality,

$$\left|\kappa(t) - \|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})}\right| \\ \leq \int_{\mathbb{R}} \int_{\Pi_T} |u(y,s) - u(x,t)|\, \omega_r(x-y)\rho_{r_0}(t-s)\, dydsdx \\ \leq \int_0^T \Big(\int_{\mathbb{R}} |u(y,s) - u(y,t)|\, dy\Big)\rho_{r_0}(t-s)\, ds \\ + \int_{\mathbb{R}}\int_{\mathbb{R}} |u(y,t) - u(x,t)|\omega_r(x-y) dydx \\ \leq L^c r_0 + |u(\cdot,t)|_{BV(\mathbb{R})}\, r.$$

$\square$

Recall that we had to pick $\nu > r_0$. Let L^d be the L^1-Lipschitz constant of $t \mapsto u_{\Delta x}(\cdot,t)$. By the triangle inequality

$$\|u_{\Delta x}(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbb{R})} \\ \leq \left\|u_{\Delta x}(\cdot,\nu) - u^0_{\Delta x}\right\|_{L^1(\mathbb{R})} + \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} + \left\|u^0 - u(\cdot,\nu)\right\|_{L^1(\mathbb{R})} \\ \leq L^d\nu + \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} + L^c\nu.$$

This means that

$$\|u_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \left\|u^0_{\Delta x} - u^0\right\|_{L^1(\mathbb{R})} \\ + \left(L^c + L^d\right)\nu + 2\left(L^c r_0 + \left|u^0\right|_{BV(\mathbb{R})} r\right) + \Gamma\Big(r, \frac{\Delta x}{r}\Big)\frac{\Delta x}{r^2} + C\frac{\Delta x}{r_0}.$$

Now choose $r^3 = r_0^2 = \Delta x$ and $\nu = 2r_0$. Then there exist a constant C such that

$$\|u_{\Delta x}(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(\mathbb{R})} \leq \left\|u^0_{\Delta x} - u^0\right\| + C\Delta x^{\frac{1}{3}}.$$

Now recall that $A(\sigma) = \hat{A}(\sigma) + \eta\sigma$, with $\hat{A}'(\sigma) \geq 0$, and so it remains to send η to zero to conclude the proof. If v is the entropy solution of the non-regularized equation, then $u(\cdot, t) \to v(\cdot, t)$ in $L^1(\mathbb{R})$ as $\eta \to 0$ (cf. Section 2). Concerning the difference scheme, one can prove continuous dependence in ℓ^1 on the parameter η using Grönwall's inequality. Hence, we can also send η to zero in the scheme. This finishes the proof of Theorem 4.1.

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(Kenneth H. Karlsen)
Center of Mathematics for Applications (CMA)
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
E-mail address: kennethk@math.uio.no

(Nils Henrik Risebro)
Center of Mathematics for Applications (CMA)
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
E-mail address: nilshr@math.uio.no

(Erlend Briseid Storrøsten)
Center of Mathematics for Applications (CMA)
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
E-mail address: erlenbs@math.uio.no