Existence of global entropy solutions to the isentropic Euler equations with geometric effects

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Abstract

In the paper Lu (2011) [1], the maximum principle was used to study the uniformly bounded L^∞ estimates $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le B(x), w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le M(t)$ for the ε-viscosity and δ-flux-approximation solutions $(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon})$ of the nonhomogeneous system (1.3), where w, z are Riemann invariants of (1.3) and $M(t)$ depends on the bound of the nonlinear function $a(x)$, which excludes the class of discontinuous functions. In this short paper, we obtain the estimate $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \leq \beta$ when $a'(x) \geq 0$ for a suitable constant β depending only on the bound of $a(x)$ and prove the existence of bounded entropy solutions, for the Cauchy problem of the isentropic Euler equations with geometric effects (1.1), which extend the results of finite energy solution in [2], and weak solutions in [3] for a polytropic gas with $\gamma \in (1, \frac{5}{3}]$ to the general pressure function $P(\rho)$.

Key Words: nonhomogeneous system; isentropic Euler equations; maximum principle; compensated compactness

1 Introduction

In this paper, we are interested in the existence of global entropy solutions to the following isentropic Euler equations with geometric effects

$$\begin{cases} (\rho a(x))_t + (\rho u a(x))_x = 0, \\ (\rho u a(x))_t + (\rho u^2 a(x))_x + a(x) P(\rho)_x = 0 \end{cases} \tag{1.1}$$

with bounded initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \tag{1.2}$$

where ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure, $a(x)$ represents the cross-sectional area of a variable duct. For the polytropic gas, P takes the special form $P(\rho) = \frac{1}{\gamma}\rho^\gamma$, where $\gamma > 1$ is the adiabatic exponent.

For the convenience, we rewrite system (1.1) in the following form

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)} \rho u \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)} \rho u^2. \end{cases} \tag{1.3}$$

To study the existence of entropy solutions of the Cauchy problem (1.1) and (1.2), the main difficulty is to establish L^∞ estimate of solutions because the equations are not in conservative form and the Conley-Chuey-Smoller principle of invariant regions does not apply (See [3, 2, 4] for the details about the physical background of system (1.1) and its difficulty in analysis). A similar result about a special hyperbolic system of three equations can be also found in [5]. For the polytropic gas and the adiabatic exponent $\gamma \in (1, \frac{5}{3}]$, the definition of a finite energy solution (unbounded) is given and its existence is obtained by using the compensated compactness method in [5]. In [1], we used the maximum principle to obtain L^∞ estimate, and the compensated compactness method to prove the existence of bounded entropy solutions of the Cauchy problem (1.3) and (1.2) for general pressure function $P(\rho)$ under the uniformly bounded condition $|a'(x)| \leq M$, which excludes the class of discontinuous functions. In this paper, we study the Cauchy problem (1.1) and (1.2) for a monotonic, bounded and discontinuous function $a(x)$, and obtain the main existence result given in Theorem 1 in Section 2.

2 A priori L^∞ estimate and Existence Results

By simple calculations, two eigenvalues of system (1.3) are

$$\lambda_1 = u - \sqrt{P'(\rho)}, \quad \lambda_2 = u + \sqrt{P'(\rho)} \tag{2.1}$$

with corresponding Riemann invariants

$$z(u,\rho) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds - u, \quad w(u,\rho) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds + u, \tag{2.2}$$

where c is a constant.

For the reason to remove the boundedness condition on $a'(x)$ needed in [1], we construct a different sequence of hyperbolic systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = A^{\varepsilon_1}(x)(\rho - 2\delta)u \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x = A^{\varepsilon_1}(x)(\rho - 2\delta)u^2 \end{cases} \tag{2.3}$$

to approximate system (1.3), where $\delta > 0$ in (2.3) denotes a flux perturbation constant and the perturbation pressure is

$$P_1(\rho,\delta) = \int_{2\delta}^{\rho} \frac{t-2\delta}{t} P'(t)dt, \tag{2.4}$$

$A^{\varepsilon_1}(x) = -\frac{a^{\varepsilon_1}(x)'}{a(x)^{\varepsilon_1}}$ and $a^{\varepsilon_1}(x) = a(x) * G^{\varepsilon_1}$ is the smooth approximation of $a(x)$, G^{ε_1} a mollifier.

Remark 1. In [1], the approximate term $-4\delta B'(x)\int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho$ is added to the right-hand side of second equation in (2.3), which makes the estimate of $z \leq B(x)$ easier. However, the technical condition $|a'(x)| \leq M$, is imposed for the estimate of w and excludes the class of discontinuous functions.

We assume that $a(x)$ is monotonic and $0 < c_1 \leq a(x) \leq c_2$ for two positive constants c_1 and c_2, then $a'(x)$ exists almost everywhere, $A^{\varepsilon_1}(x)$ and $a^{\varepsilon_1}(x)$ satisfy

$$\begin{cases} \lim_{\varepsilon_1 \to 0} A^{\varepsilon_1}(x) = A(x), a.e. \text{ on } R, \\ \lim_{\varepsilon_1,\delta \to 0} \delta A^{\varepsilon_1}(x) = 0, a.e. \text{ on } R, \\ A^{\varepsilon_1}(x) \quad \text{is uniformly bounded in } L^1(R), \\ \lim_{\varepsilon_1,\varepsilon \to 0}(\varepsilon a^{\varepsilon_1}(x)', \varepsilon a^{\varepsilon_1}(x)'') = (0,0), a.e. \text{ on } R, \end{cases} \tag{2.5}$$

if ε and δ converge to zero much faster than ε_1.

So, for simplicity, we drop the index ε_1 and assume that $a(x)$ or $A(x)$ is a smooth function. The proof for the discontinuous function is completely same with the help of (2.5).

Second, we add the viscosity terms to the right-hand side of (2.3) to obtain the following parabolic system

$$\begin{cases} \rho_t + ((\rho-2\delta)u)_x = A(x)(\rho-2\delta)u + \varepsilon\rho_{xx} \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho,\delta))_x = A(x)(\rho-2\delta)u^2 + \varepsilon(\rho u)_{xx} \end{cases} \tag{2.6}$$

with initial data

$$(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) = (\rho_0(x)+2\delta, u_0(x)), \tag{2.7}$$

where $(\rho_0(x), u_0(x))$ are given in (1.2).

Using the first equation in (2.6), we can easily prove the a priori estimate $\rho^{\delta,\varepsilon} \geq 2\delta$, which is the crux to obtain the L^∞ estimate and the existence of solutions in the following Theorem 1.

Throughout this paper,we assume that $0 < c_1 \leq a(x) \leq c_2$ for two positive constants c_1, c_2 and the function $B(x)$ (or the function $B^{\varepsilon_1}(x)$ depending on $A^{\varepsilon_1}(x)$ if $a(x)$ is not smooth) belongs to the set $\in B_d^2(R)$.

Definition 1 *One function $B(x)$ is called a member in the set $\in B_d^2(R)$ if $B(x) \in C^1(R)$ and satisfies (a): there exist two positive constants β_0, β depending on the upper and lower bounds c_1, c_2 of $a(x)$ such that $0 < \beta_0 \leq B(x) \leq \beta$; (b): $B''(x) \leq 0$ or $B''(x) = B_1(x) + B_2(x)$, where $B_1(x) \leq 0$ and $|\varepsilon_2 B_2(x)| \leq |B'(x)|$ for a suitable small constant $\varepsilon_2 > 0$.*

Theorem 1 . *Let $a'(x) \geq 0$. (1). Assume $P(\rho) = \frac{1}{\gamma}\rho^\gamma$. If $1 < \gamma \leq 3$ and for any $B(x) \in B_d^2(R)$ satisfying*

$$(\theta - 1)^2 B'^2(x) - 2\theta(\theta + 1)A(x)B(x)B'(x) + \theta^2(A(x)B(x))^2 < 0, \tag{2.8}$$

where $\theta = \frac{\gamma-1}{2}$ or if $\gamma \geq 3$ and for any $B(x) \in B_d^2(R)$ satisfying $B'(x) \leq A(x)B(x)$, the Riemann invariant z of system (1.3) with respect to the approximated solutions of the Cauchy problem (2.6)-(2.7) satisfies the estimate

$$z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) = \frac{1}{\theta}(\rho^{\delta,\varepsilon})^\theta - u^{\delta,\varepsilon} \leq B(x) \tag{2.9}$$

when $z(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) \leq B(x) \leq \beta$ and the Riemann invariant w satisfies the estimate

$$w(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)) = \frac{1}{\theta}(\rho^{\delta,\varepsilon})^\theta + u^{\delta,\varepsilon} \leq \beta \tag{2.10}$$

when $w(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) \leq \beta$; (2). Assume that there exists a small constant $\rho_0 > 0$ such that $P(\rho)$ has the same principal singularity as the γ-law when $\rho \in (0, \rho_0)$. For simplicity, let

$$P(\rho) = \frac{1}{\gamma}\rho^\gamma, \quad \text{when} \quad \rho \in (0, \rho_0), \tag{2.11}$$

where $\gamma \in (1, 3]$, and $P'(\rho)) > 0, P''(\rho)) > 0$ when $\rho \geq \rho_0$. Then for any function $B(x) \in B_d^2(-\infty, \infty)$ satisfying (2.8) and

$$B(x) \leq \frac{1}{2}\int_0^{\rho_0} \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \beta, \quad B'(x) \geq \frac{1}{2}A(x)\sqrt{P'(\rho_0)}, \tag{2.12}$$

the Riemann invariants z and w satisfy the estimates (2.9)-(2.10); (3). *If the function $a(x)$ and the initial data satisfy the conditions in Parts (1) and (2), then there exists a subsequence of $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$, which converges pointwisely to a pair of bounded functions $(\rho(x,t), u(x,t))$ as δ, ε tend a zero, and the limit is a weak entropy solution of the Cauchy problem (1.1)-(1.2)*

Definition 2 *For integrable function $a'(x) \in L^1(R)$, a pair of bounded functions $(\rho(x,t), u(x,t))$ is called a weak entropy solution of the Cauchy problem (1.1)-(1.2) if*

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty \rho a\phi_t + \rho u a\phi_x dxdt + \int_{-\infty}^\infty \rho_0 a\phi(x,0)dx = 0, \\ \int_0^\infty \int_{-\infty}^\infty \rho u a\phi_t + (\rho u^2 + P(\rho))a\phi_x + a'P(\rho)\phi dxdt + \int_{-\infty}^\infty \rho_0 u_0 a\phi(x,0)dx = 0 \end{cases} \tag{2.13}$$

hold for all test function $\phi \in C_0^1(R \times R^+)$ and the entropy inequalities

$$\int_0^\infty \int_{-\infty}^\infty \eta(\rho,m)a(x)\phi_t + q(\rho,m)a(x)\phi_x + a'(x)(q - \eta_\rho \rho u - \eta_m \rho u^2)\phi dxdt \geq 0 \tag{2.14}$$

hold for any non-negative test function $\phi \in C_0^\infty(R \times R^+ - \{t=0\})$, where $m = \rho u$ and $(\eta(\rho,m), q(\rho,m))$ is a pair of convex entropy-entropy flux of system (1.3).

Remark 2. When $a'(x) \leq 0$, a similar L^∞ solution of the Cauchy problem (1.1)-(1.2) can be also obtained.

Remark 3. The estimate depending on the space variable $w(\rho,m) \leq B(x)$ is first obtained in [3] for a polytropic gas with $1 < \gamma < \frac{5}{3}$ by using a modified Godunov scheme.

Now we are going to prove Theorem 1 in the next section.

3 Proofs of Theorem 1.

As done in [1], we multiply (2.4) by (w_ρ, w_m) and (z_ρ, z_m) respectively, where

$$z(u,v) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds - \frac{m}{\rho}, \quad w(u,v) = \int_c^\rho \frac{\sqrt{P'(s)}}{s} ds + \frac{m}{\rho}, \tag{3.1}$$

are the Riemann invariants of system (1.3), and obtain

$$
\begin{aligned}
& w_t+\lambda_2^\delta w_x \\
& =\varepsilon w_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x w_x-\tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2 \\
& +A(x)(\rho-2\delta)u(-\tfrac{m}{\rho^2}+\tfrac{\sqrt{P'(\rho)}}{\rho})+\tfrac{1}{\rho}A(x)(\rho-2\delta)u^2 \\
& =\varepsilon w_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x w_x-\tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2 \\
& +A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho}
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
& z_t+\lambda_1^\delta z_x \\
& =\varepsilon z_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x z_x-\tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2 \\
& +A(x)(\rho-2\delta)u(\tfrac{m}{\rho^2}+\tfrac{\sqrt{P'(\rho)}}{\rho})-\tfrac{1}{\rho}A(x)(\rho-2\delta)u^2 \\
& =\varepsilon z_{xx}+\tfrac{2\varepsilon}{\rho}\rho_x z_x-\tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2 \\
& +A(x)(\rho-2\delta)u\tfrac{\sqrt{P'(\rho)}}{\rho},
\end{aligned}
\tag{3.3}
$$

where

$$
\lambda_1^\delta=\frac{m}{\rho}-\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}, \quad \lambda_2^\delta=\frac{m}{\rho}+\frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}. \tag{3.4}
$$

Let $z=B(x)+v$. Using the same technique given in [1] (in our case, the term $\frac{4\delta}{\rho}B'(x)\int_0^\rho\frac{\sqrt{P'(\rho)}}{\rho}d\rho$ is not imposed), we have

$$
\begin{aligned}
& v_t+(u-\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)})(v_x+B'(x)) \\
& =\varepsilon v_{xx}+\varepsilon B''(x)+\tfrac{2\varepsilon}{\rho}\rho_x v_x+\tfrac{2\varepsilon}{\rho}\rho_x B'(x)-\tfrac{\varepsilon}{2\rho^2\sqrt{P'(\rho)}}(2P'+\rho P'')\rho_x^2 \\
& -A(x)\tfrac{\rho-2\delta}{\rho}\sqrt{P'(\rho)}(B(x)+v-\int_c^\rho\tfrac{\sqrt{P'(\rho)}}{\rho}d\rho)
\end{aligned}
\tag{3.5}
$$

or

$$
\begin{aligned}
&v_t + a(x,t)v_x + b(x,t)v + [-\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 - \varepsilon B''(x) - \varepsilon_3 B(x)B'(x)] \\
&+ \textstyle\int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho B'(x) - (1-\varepsilon_3)B(x)B'(x) \\
&+[A(x)B(x) - B'(x)](\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \le \varepsilon v_{xx}
\end{aligned}
\tag{3.6}
$$

where $\varepsilon_3 > 0$ is a suitable constant, $a(x,t) = u - \frac{\rho-2\delta}{\rho}\sqrt{P'(\rho)} - \frac{2\varepsilon}{\rho}\rho_x$ and $b(x,t) = -B'(x) + A(x)(\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho}$.

Proof of Theorem 1: If $P(\rho) = \frac{1}{\gamma}\rho^\gamma$ and $\gamma > 3$, from the first equation in System (2.4), we have the a priori estimate $\rho \ge 2\delta$. Since $B(x) \in B_d^2, -\beta^{\varepsilon_1} \le B'(x) \le 0$ in Theorem 1, where β^{ε_1} is a constant depending on the bound of A^{ε_1}, we can choose $\varepsilon = o(\delta)$ and suitable relation between ε and ε_3 such that the following three terms in the left-hand side of (3.6)

$$
-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P' + \rho P''}B'(x)^2 - \varepsilon B''(x) - \varepsilon_3 B(x)B'(x) \ge 0. \tag{3.7}
$$

Now we consider the other terms

$$
\begin{aligned}
&L = \textstyle\int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho B'(x) - (1-\varepsilon_3)B(x)B'(x) \\
&+[A(x)B(x) - B'(x)](\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_c^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho
\end{aligned}
\tag{3.8}
$$

in the left-hand side of (3.6). Let $c = 2\delta$. Since $\gamma > 3$, then $\int_{2\delta}^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho \le (\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho}$. We first let $B'(x) \le A(x)B(x) \le 0$, then $B'(x) \le \frac{A(x)B(x)}{4(1-\varepsilon_3)}$ for a small ε_3, and

$$
\begin{aligned}
&L \ge \textstyle\int_{2\delta}^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho B'(x) - (1-\varepsilon_3)B(x)B'(x) \\
&+[A(x)B(x) - B'(x)]\textstyle\int_{2\delta}^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - A(x)(\int_{2\delta}^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho)^2 \\
&= -A(x)[\textstyle\int_{2\delta}^\rho \frac{\sqrt{P'(\rho)}}{\rho} d\rho - \frac{B(x)}{2}]^2 - (1-\varepsilon_3)B(x)B'(x) + \frac{A(x)B^2(x)}{4} \ge 0.
\end{aligned}
\tag{3.9}
$$

If $1 < \gamma \le 3$, we let $c = 0$ and rewrite (3.6) as

$$\begin{aligned}
&v_t + a(x,t)v_x + b(x,t)v \\
&-[\tfrac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 + \varepsilon B''(x) - \tfrac{2\delta}{\rho}B'(x)\textstyle\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho + \varepsilon_3 B(x)B'(x)] \\
&+[\textstyle\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x) - \tfrac{2\delta}{\rho}B'(x)\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho] - (1-\varepsilon_3)B(x)B'(x) \\
&+[A(x)B(x) - B'(x)](\rho - 2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho \le \varepsilon v_{xx}.
\end{aligned} \tag{3.10}$$

Since

$$\frac{2\delta}{\rho}\int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \le \frac{2}{\gamma-1}(2\delta)^\theta, \tag{3.11}$$

we can also choose suitable constant ε_3 and $\varepsilon = o(\delta)$ such that the following four terms in the left-hand side of (3.10)

$$-\frac{2\varepsilon\sqrt{P'(\rho)}}{2P'+\rho P''}B'(x)^2 - \varepsilon B''(x) + \frac{2\delta}{\rho}B'(x)\int_0^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho - \varepsilon_3 B(x)B'(x) \ge 0. \tag{3.12}$$

Thus the other terms in the left-hand side of (3.10)

$$\begin{aligned}
&L_1 = [\textstyle\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho B'(x) - \tfrac{2\delta}{\rho}B'(x)\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho] - (1-\varepsilon_3)B(x)B'(x) \\
&+[A(x)B(x) - B'(x)](\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho} - A(x)(\rho-2\delta)\tfrac{\sqrt{P'(\rho)}}{\rho}\textstyle\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho}d\rho \\
&= -\tfrac{1}{\theta}A(x)(\rho-2\delta)^2\rho^{2\theta-2} + (A(x)B(x) - B'(x) + \tfrac{B'(x)}{\theta})(\rho-2\delta)\rho^{\theta-1} \\
&-(1-\varepsilon_3)B(x)B'(x) - \tfrac{2\delta}{\theta}A(x)(\rho-2\delta)\rho^{2\theta-2}.
\end{aligned} \tag{3.13}$$

Since

$$-\frac{2\delta}{\theta}A(x)(\rho-2\delta)\rho^{2\theta-2} \ge 0, \tag{3.14}$$

we have

$$\begin{aligned}
&L_1 \ge -\tfrac{1}{\theta}A(x)(\rho-2\delta)^2\rho^{2\theta-2} + (A(x)B(x) - B'(x) + \tfrac{B'(x)}{\theta})(\rho-2\delta)\rho^{\theta-1} \\
&-(1-\varepsilon_3)B(x)B'(x) \\
&= -\tfrac{1}{\theta}A(x)[(\rho-2\delta)^2\rho^{2\theta-2} + \tfrac{\theta B'(x) - B'(x) - \theta A(x)B(x)}{A(x)}(\rho-2\delta)\rho^{\theta-1} \\
&+(\tfrac{\theta B'(x) - B'(x) - \theta A(x)B(x)}{2A(x)})^2] - (1-\varepsilon_3)B(x)B'(x) + \tfrac{(\theta B'(x) - B'(x) - \theta A(x)B(x))^2}{4\theta A(x)}.
\end{aligned} \tag{3.15}$$

Since the first three terms in the right-hand side of (3.15) is non-negative, $A(x) \leq 0$ and $B(x) > 0$, then $L_1 \geq 0$ if we let $B'(x) \leq 0$ satisfy

$$-(1-\varepsilon_3)B(x)B'(x) + \frac{(\theta B'(x) - B'(x) - \theta A(x)B(x))^2}{4\theta A(x)} \geq 0, \tag{3.16}$$

which is equivalent to

$$(\theta B'(x) - B'(x) - \theta A(x)B(x))^2 \leq 4\theta A(x)B(x)B'(x) - 4\theta\varepsilon_3 A(x)B(x)B'(x) \tag{3.17}$$

or

$$(\theta-1)^2 B'^2(x) - 2\theta(\theta+1)A(x)B(x)B'(x) + 4\theta\varepsilon_3 A(x)B(x)B'(x) + \theta^2(A(x)B(x))^2 \leq 0. \tag{3.18}$$

(3.18) is true for a suitably small $\varepsilon_3 > 0$ since the condition (2.8) given in Theorem 1. Thus under the conditions of Theorem 1, (3.6) is reduced to

$$v_t + a(x,t)v_x + b(x,t)v \leq \varepsilon v_{xx}, \tag{3.19}$$

and we can prove that $v \leq 0$ or $z \leq B(x)$ if applying for the maximum principle to (3.19)

Second, we have from (3.2) that

$$w_t + a_1(x,t)w_x \leq \varepsilon w_{xx} + A(x)(\rho - 2\delta)u\frac{\sqrt{P'(\rho)}}{\rho}, \tag{3.20}$$

where $a_1(x,t) = u + (\rho - 2\delta)\frac{\sqrt{P'(\rho)}}{\rho} - \frac{2\varepsilon}{\rho}\rho_x$.

Let

$$w = (X + \beta + \frac{N(x^2 + Lhe^{3t})}{L^2}), \tag{3.21}$$

where $|B(x)| \leq \beta$ is the upper bound of $w_0(x)$ and N, h are the bounds of $|w|, |a_1(x,t)|$ obtained from the local solution. Then we can prove the estimate $w \leq \beta$ by using the maximum principle (cf. [1]).

About the Part (2) in Theorem 1, we can prove $L_1 \geq 0$ for $\rho \in (0, \rho_0)$ if $B(x)$ satisfies (2.8).

If $\rho \geq \rho_0$, then L_1 given in (3.13) satisfies

$$
\begin{aligned}
&L_1 = [\int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho B'(x) - \tfrac{2\delta}{\rho} B'(x) \int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho] - (1-\varepsilon_3) B(x) B'(x) \\
&+[A(x)B(x) - B'(x)](\rho - 2\delta) \tfrac{\sqrt{P'(\rho)}}{\rho} - A(x)(\rho - 2\delta) \tfrac{\sqrt{P'(\rho)}}{\rho} \int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho \\
&\geq B'(x) \tfrac{\rho - 2\delta}{\rho} \int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho - \tfrac{1}{2} A(x)(\rho - 2\delta) \tfrac{\sqrt{P'(\rho)}}{\rho} \int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho \\
&+[A(x)B(x) - B'(x)](\rho - 2\delta) \tfrac{\sqrt{P'(\rho)}}{\rho} - \tfrac{1}{2} A(x)(\rho - 2\delta) \tfrac{\sqrt{P'(\rho)}}{\rho} \int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho \\
&\geq [B'(x) - \tfrac{1}{2} A(x) \sqrt{P'(\rho_0)}] \tfrac{\rho - 2\delta}{\rho} \int_0^\rho \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho \\
&+[A(x)B(x) - B'(x) - \tfrac{1}{2} A(x) \int_0^{\rho_0} \tfrac{\sqrt{P'(\rho)}}{\rho} d\rho] \tfrac{\rho - 2\delta}{\rho} \sqrt{P'(\rho)} \geq 0
\end{aligned}
\tag{3.22}
$$

if we let $B(x)$ satisfy (2.12) or

$$
B'(x) - \frac{1}{2} A(x) \sqrt{P'(\rho_0)} \geq 0, \quad -\frac{1}{2} A(x) \int_0^{\rho_0} \frac{\sqrt{P'(\rho)}}{\rho} d\rho + A(x)B(x) - B'(x) \geq 0. \tag{3.23}
$$

So, we get the proof of the part (2) in Theorem 1.

As to the third part in Theorem 1, since system (1.3) and its flux perturbation system (2.3) have the same entropies (cf. [6]), for the homogeneous case $A(x) = 0$, the H^{-1}_{loc} compactness of $\eta_t(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon}) + q_x(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon})$ was proved in [6] for any pair of weak entropy-entropy flux of system (1.3) constructed in [7, 8, 9, 10]. Because $A(x)$ is integrable in$R$, after we have the uniformly bounded estimates of $(\rho^{\delta,\varepsilon}, m^{\delta,\varepsilon})$ given in (2.9) and (2.10), we can also prove the H^{-1}_{loc} compactness for the nonhomogenerous case. Thus the convergence of $(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \to (\rho, u)$ as δ, ε tend to zero in the Part (3) of Theorem 1 can be proved by using the compensated compactness theory and the already existed compact framework given in [7, 8, 9, 10, 11].

Now we shall prove the limit (ρ, u) is an entropy solution of the Cauchy problem (1.1) and (1.2).

Multiplying $a(x)$ to the second equation of system (2.6), we have

$$
\begin{aligned}
&(\rho u a)_t + ((\rho u^2 + P(\rho))a)_x - a' P(\rho) + \delta(a f(\rho, u))_x + \delta a' g(\rho, u)_x \\
&= \varepsilon a(x)(\rho u)_{xx} = \varepsilon[(a(x)\rho)_{xx} - 2(a'(x)\rho)_x + a''(x)\rho]
\end{aligned}
\tag{3.24}
$$

for two suitable continuous functions $f(\rho, u)$ and $g(\rho, u)$. Thus the second equation of (2.13) is true when we let ε, δ in (3.24) go to zero in the sense of distributions. Similarly, we can prove that (ρ, u) satisfies the first equation in (2.13).

Since systems (1.3) and (2.3) have the same entropies, for any convex entropy-entropy flux pair (η, q) of system (1.3), we multiply (η_ρ, η_m) to system (2.6) to obtain

$$\begin{aligned}
&\eta_t + q_x + \delta Q(\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta})_x \\
&= \varepsilon\eta_{xx} - \varepsilon(\rho_x^{\varepsilon,\delta}, m_x^{\varepsilon,\delta}) \cdot \nabla^2\eta(\rho^{\varepsilon,\delta}, m^{\varepsilon,\delta}) \cdot (\rho_x^{\varepsilon,\delta}, m_x^{\varepsilon,\delta})^T \\
&+\eta_\rho A(x)(\rho - 2\delta)u + \eta_m A(x)(\rho - 2\delta)u^2 \\
&\leq \varepsilon\eta_{xx} + \eta_\rho A(x)(\rho - 2\delta)u + \eta_m A(x)(\rho - 2\delta)u^2
\end{aligned} \tag{3.25}$$

for a suitable function $Q(\rho, m)$ depending on η.

Multiplying $a(x)$ to (3.25), we have

$$\begin{aligned}
&(a(x)\eta)_t + (a(x)q)_x - a'(x)q + a'(x)(\eta_\rho\rho u + \eta_m\rho u^2) \\
&\leq \varepsilon a(x)\eta_{xx} - \delta a(x)Q_x + 2\delta a'(x)(\eta_\rho u + \eta_m u^2).
\end{aligned} \tag{3.26}$$

Thus (2.14) is true when we multiply a nonnegative test function ϕ to (3.26) and let ε, δ go to zero. So we complete the proof of Theorem 1.

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