Global solutions to one-dimensional shallow water magnetohydrodynamic equations

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Abstract

In this paper, we study the Cauchy problem for the one-dimensional shallow water magnetohydrodynamic equations. The main difficulty is the case of zero depth ($h = 0$) since the nonlinear flux function $P(h)$ is singular and the definition of solution is not clear near $h = 0$. First, assuming that h has a positive and lower bound, we establish the pointwise convergence of the viscosity solutions by using the div-curl lemma from the compensated compactness theory to special pairs of functions (c, f^{ε}), and obtain a global weak entropy solution. Second, under some technical conditions on the initial data such that the Riemann invariants (w, z) are monotonic and increasing, we introduce a "variant" of the vanishing artificial viscosity to select a weak solution. Finally, we extend the results to two special cases, where $P(h)$ is for the polytropic gas or for the Chaplygin gas.

1 Introduction

The following one-dimensional, magnetohydrodynamic shallow water equations over an arbitrary boundary

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 - hG^2 + \frac{1}{2}gh^2)_x = -ghb_x(x,t), \\ (hG)_t = 0, \\ (hG)_2 = 0, \\ (hv)_t + (huv - hGF)_x = 0, \\ (hF)_t + (hFu - hGv)_x = 0 \end{cases} \tag{1.1}$$

was first derived in [Ros], where x and t are the space and the time variables, $h(x,t)$ is the fluid depth, u and v are fluid velocities, F and G are magnetic field components, g is the gravitational constant and $b(x,t)$ denotes the underlying surface.

The shallow water magnetohydrodynamic equations are important in many applications of magnetohydrodynamic to astrophysical and engineering problems, for instance, in the solar tachocline study [Gi, MG, ZCO, ZOB], in the neutron-star atmosphere dynamics study [HS, SLU], for the optimization of aluminum production process [BP, ZT] and in fusion technologies [MCR] (the details can be found in [KPT, Ros] and references cited therein).

It follows from the third and fourth equations in (1.1), $hG = const. = -C$. In [KPT], the authors studied the Riemann problem of (1.1) and showed that the simple wave solutions exist only for underlying surface that are slopes of constant inclination, i.e., $b_x(x,t) = const. = b$. Thus, we rewrite system (1.1) as follows

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{1}{2}gh^2 - \frac{C^2}{h})_x = -gbh, \\ (hv)_t + (hvu + CF)_x = 0, \\ (hF)_t + (hFu + Cv)_x = 0. \end{cases} \tag{1.2}$$

In mathematics, system (1.2) is decoupled since the first two equations are independent. We write these equations as

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{1}{2}gh^2 - \frac{C^2}{h})_x = -gbh. \end{cases} \tag{1.3}$$

System (1.3) could be also considered as the isentropic gas dynamics with a special pressure $P(h) = \frac{1}{2}gh^2 - \frac{C^2}{h}$, where h denotes the density and u is the velocity of gas [Di1]. The third and fourth equations in (1.2) are of the Temple type [Te], where the shock waves and rarefaction waves coincide.

By simple calculations, two eigenvalues (1.3) are

$$\lambda_1 = \frac{m}{h} - \sqrt{P'(h)}, \quad \lambda_2 = \frac{m}{h} + \sqrt{P'(h)}, \tag{1.4}$$

where $m = hu$ and $P(h) = \frac{1}{2}gh^2 - \frac{C^2}{h}$, with corresponding right eigenvectors

$$r_1 = (1, \lambda_1)^T, \quad r_2 = (1, \lambda_2)^T. \tag{1.5}$$

The Riemann invariants of (1.3) are functions

$$w(h,u) = \frac{m}{h} + \int_{h_0}^{h} \frac{\sqrt{P'(s)}}{s} ds, \quad z(h,u) = \frac{m}{h} - \int_{h_0}^{h} \frac{\sqrt{P'(s)}}{s} ds, \tag{1.6}$$

where $h_0 > 0$ is a constant. Then

$$\begin{aligned} &\nabla\lambda_1 \cdot r_1 = (-\tfrac{m}{h^2} - \tfrac{P''(h)}{2\sqrt{P'(h)}}, \tfrac{1}{h})(1,\lambda_1)^T \\ &= -\tfrac{hP''(h)+2P'(h)}{2h\sqrt{P'(h)}} = -\tfrac{3}{\sqrt{P'(h)}} \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &\nabla\lambda_2 \cdot r_2 = (-\tfrac{m}{h^2} + \tfrac{P''(h)}{2\sqrt{P'(h)}}, \tfrac{1}{h})(1,\lambda_2)^T \\ &= \tfrac{hP''(h)+2P'(h)}{2h\sqrt{P'(h)}} = \tfrac{3}{\sqrt{P'(h)}}. \end{aligned} \tag{1.8}$$

Therefore the system (1.3) is strictly hyperbolic from (1.4), and genuinely nonlinear from (1.7)-(1.8) in the region of $h \geq h_0 > 0$.

To study the existence of global solutions for the Cauchy problem (1.2) with bounded measurable initial data

$$(h(x,0), u(x,0), v(x,0), F(x,0)) = (h_0(x), u_0(x), v_0(x), F_0(x)), \quad h_0(x) > 0, \tag{1.9}$$

the main difficulty is the case of zero depth $h = 0$, where the function $\frac{C^2}{h}$ is singular and the definition of solution near $h = 0$ is not clear.

First, under the strong assumption $h^\varepsilon(x,t) \geq h_0 > 0$, where h_0 is a constant independent of ε, and $h^\varepsilon(x,t)$ are the viscosity approximation solutions of the parabolic system

$$\begin{cases} h_t + (hu)_x = \varepsilon h_{xx}, \\ (hu)_t + (hu^2 + \frac{1}{2}gh^2 - \frac{C^2}{h})_x = \varepsilon (hu)_{xx} - gbh, \\ (hv)_t + (hvu + CF)_x = \varepsilon (hv)_{xx}, \\ (hF)_t + (hFu + Cv)_x = \varepsilon (hF)_{xx} \end{cases} \tag{1.10}$$

with the initial data (1.9), we establish the following existence theorem

Theorem 1 *(I). Let the initial data $(h_0(x), u_0(x), v_0(x), F_0(x))$ be bounded, $h_0(x) \geq c_0 > 0$ for a positive constant c_0. Then for any fixed $\varepsilon > 0$, the viscosity solution*

$(h^\varepsilon(x,t), u^\varepsilon(x,t), v^\varepsilon(x,t), F^\varepsilon(x,t))$ *of the Cauchy problem (1.10) and (1.9) exists and satisfies*

$$0 < c(t, c_0, \varepsilon) \leq h^\varepsilon(x,t) \leq M(t), \quad |u^\varepsilon(x,t)| \leq M(t), \quad |v^\varepsilon| \leq M, \quad |F^\varepsilon| \leq M, \tag{1.11}$$

where $M(t)$ *is a positive bounded function, independent of* ε, *in any compact set* $t \in [0,T]$ *and* $c(t, c_0, \varepsilon)$ *could tend to zero as the time* t *tends to infinity or* ε *tends to zero.*

(II). If the conditions in (I) are satisfied and the total variation of $(v_0(x), F_0(x))$ *is bounded, then*

$$\int_{-\infty}^{\infty} |v_x^\varepsilon|(x,t)dx \leq M, \quad \int_{-\infty}^{\infty} |F_x^\varepsilon|(x,t)dx \leq M, \tag{1.12}$$

where M *is a positive constant depending only on the bound of the total variation of* $(v_0(x), F_0(x))$, *but independent of* ε.

(III). Assume that $h^\varepsilon(x,t)$ *have the positive, lower bound estimate* $h^\varepsilon(x,t) \geq h_0 > 0$, *where* h_0 *is a constant independent of* ε, *then there exists a subsequence (still labelled)* $(h^\varepsilon(x,t), u^\varepsilon(x,t), v^\varepsilon(x,t), F^\varepsilon(x,t))$ *such that*

$$(h^\varepsilon(x,t), u^\varepsilon(x,t), v^\varepsilon(x,t), F^\varepsilon(x,t)) \to (h(x,t), u(x,t), v(x,t), F(x,t)), \tag{1.13}$$

a.e. on any bounded and open set $\Omega \subset R \times R^+$, *and the limit functions* (h, u, v, F) *is a weak solution of the Cauchy problem (1.2) and (1.9), namely* (h, u, v, F) *satisfies (1.2) in the sense of distributions.*

Second, we choose some special initial data such that the Riemann invariants (w, z) satisfy

$$\begin{cases} l_1 \leq z_0(x) = z(h_0(x), u_0(x)) \leq l_2 < m_1 \leq w_0(x) = w(h_0(x), u_0(x)) \leq m_2, \\ z(h_0(x), u_0(x)) \quad \text{and} \quad w(h_0(x), u_0(x)) \quad \text{are monotonic increasing,} \end{cases} \tag{1.14}$$

where l_1, l_2, m_1 and m_2 are suitable constants, which ensure the a-priori positive, lower bound estimate $h(x,t) \geq h_0$. More precisely, we have

Theorem 2 *Let* $(h_0(x), u_0(x), v_0(x), F_0(x))$ *be bounded,* $(h_0(x), u_0(x))$ *satisfy (1.14) and the total variation of* $(v_0(x), F_0(x))$ *be bounded. Then the Cauchy problem (1.2) and (1.9) has a weak solution* $(h(x,t), u(x,t), v(x,t), F(x,t))$, *where the Riemann invariants* $(z(h(x,t), u(x,t)), w(h(x,t), u(x,t)))$ *are monotonic increasing with respect to* x, *and* $(v(\cdot,t)_x, F(\cdot,t)_x)$ *are bounded in* $L^1(-\infty, \infty)$.

In the sections 2 and 3, we will prove Theorems 1-2 respectively. Under the assumption $h^\varepsilon \geq h_0 > 0$, the convergence of $(h^\varepsilon, u^\varepsilon)$ can be proved by using the DiPerna's compact framework on strictly hyperbolic and genuinely nonlinear systems [Di2].

The convergence of $(v^\varepsilon, F^\varepsilon)$ is obtained by applying the Div-Curl lemma from the compensated compactness theory [Ta] to some special pairs of functions (c, v^ε) or (c, F^ε), where c is a constant. To prove Theorem 2, we introduce a "variant" of the vanishing artificial viscosity given by (3.1) and (3.14) to select the solution satisfying $h^\varepsilon \geq h_0 > 0$.

In Section 4, we will extend the results in Theorem 1 to two special cases, where $P(h)$ is for the polytropic gas $h^\gamma, \gamma > 1$ or for the Chaplygin gas $-\frac{C^2}{h}$. For the polytropic gas, the convergence of $(h^\varepsilon, u^\varepsilon)$ is proved in [LPS, LPT] and the proof of the convergence $(v^\varepsilon, F^\varepsilon)$ in Theorem 1 covers the vacuum $h = 0$. For the Chaplygin gas $P(h) = -\frac{C^2}{h}$, the main difficulty in studying the global solution is still the vacuum $h = 0$, where the definition of solution is not clear. So, we will mainly study the bounded measurable weak solution away from the vacuum.

2 Proof of Theorem 1

In this section, we shall prove (I)-(III) in Theorem 1 respectively.

Proof of (I). We multiply the first two equations in (1.10) by (w_h, w_m) and (z_h, z_m) respectively, where w, z are given by (1.6), to obtain

$$\begin{aligned} & w_t + \lambda_2 w_x \\ & = \varepsilon w_{xx} + \tfrac{2\varepsilon}{h} h_x w_x - \tfrac{\varepsilon}{2h^2\sqrt{P'(h)}}(2P' + hP'')h_x^2 - gb \leq \varepsilon w_{xx} + \tfrac{2\varepsilon}{h} h_x w_x - gb \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} & z_t + \lambda_1 z_x \\ & = \varepsilon z_{xx} + \tfrac{2\varepsilon}{h} h_x z_x + \tfrac{\varepsilon}{2h^2\sqrt{P'(h)}}(2P' + hP'')h_x^2 - gb \geq \varepsilon w_{xx} + \tfrac{2\varepsilon}{h} h_x z_x - gb. \end{aligned} \tag{2.2}$$

Let $\bar{w} = w + gbt$ and $\bar{z} = z + gbt$. Then we have from (2.1) and (2.2) that

$$\begin{cases} \bar{w}_t + \lambda_2 \bar{w}_x \leq \varepsilon \bar{w}_{xx} + \frac{2\varepsilon}{h} h_x \bar{w}_x, \\ \bar{z}_t + \lambda_1 \bar{z}_x \geq \varepsilon \bar{z}_{xx} + \frac{2\varepsilon}{h} h_x \bar{z}_x. \end{cases} \tag{2.3}$$

If we consider (2.3) as inequalities about the variables $\bar{w}$ and $\bar{z}$, then we can get the estimates $\bar{w}(h^\varepsilon, m^\varepsilon) \le \bar{w}(h^\varepsilon, m^\varepsilon)|_{t=0} = w(h^\varepsilon, m^\varepsilon)|_{t=0} \le M, \bar{z}(h^\varepsilon, m^\varepsilon) \ge -M$ by applying the maximum principle to (2.3). Thus we obtain the estimates $0 < h^\varepsilon \le M(t)$ and $|u^\varepsilon| \le M(t)$ for a suitable function $M(t)$, which is bounded in any compact set $t \in [0, T]$ and independent of ε. Using the estimate $|u^\varepsilon| \le M(t)$ and Theorem 1.0.2 given in [Lu1], we have the estimate $h^\varepsilon \ge c(t, c_0, \varepsilon) > 0$.

To prove the L^∞ bounds of v^ε and F^ε, we substitute the first equations in (1.10) into the third and fourth equations to obtain

$$\begin{cases} v_t + uv_x + \frac{C}{h}F_x = \varepsilon v_{xx} + 2\varepsilon \frac{h_x}{h} v_x, \\ \\ F_t + uF_x + \frac{C}{h}v_x = \varepsilon F_{xx} + 2\varepsilon \frac{h_x}{h} F_x. \end{cases} \tag{2.4}$$

Let $w_1 = v + F, z_1 = v - F$. We have from (2.4) that

$$\begin{cases} w_{1t} + (u + \frac{C}{h})w_{1x} = \varepsilon w_{1xx} + 2\varepsilon \frac{h_x}{h} w_{1x}, \\ \\ z_{1t} + (u - \frac{C}{h})z_{1x} = \varepsilon z_{1xx} + 2\varepsilon \frac{h_x}{h} z_{1x}. \end{cases} \tag{2.5}$$

If we consider (2.5) as equations about the variables w_1 and z_1, then we have the estimates $|w_1| \le |w_1|_{t=0}| \le M, |z_1| \le |z_1|_{t=0}| \le M$ by applying the maximum principle to (2.5). Thus we obtain the estimates $|v^\varepsilon| \le M$ and $|F^\varepsilon| \le M$ for a suitable positive constant M, which is independent of ε.

So, the proof of (I) about the existence of the viscosity solution for the Cauchy problem (1.10)-(1.9) is completed by the standard theory of semilinear parabolic systems, namely the local existence and the above a priori bounded estimates.

Proof of (II). To prove the BV estimate in (1.12), we use the ideal given in [Se]. Let $\theta = w_{1x}$ and choose one sequence of smooth functions $g(\theta, \alpha)$ such that $g''(\theta, \alpha) \ge 0, g'(\theta, \alpha) \to sign\theta, g(\theta, \alpha) \to |\theta|$ as $\alpha \to 0$.

Differentiating the first equation in (2.5) with respect to x and then multiplying the sequence of smooth functions $g'(\theta, \alpha)$ to the result, we have

$$\theta_t + ((u + \frac{C}{h})\theta)_x = \varepsilon \theta_{xx} + (2\varepsilon \frac{h_x}{h} \theta)_x \tag{2.6}$$

and

$$\begin{aligned} & g(\theta, \alpha)_t + ((u + \tfrac{C}{h})g(\theta, \alpha))_x + (g'(\theta, \alpha)\theta - g(\theta, \alpha))(u + \tfrac{C}{h})_x \\ & = \varepsilon g(\theta, \alpha)_{xx} - \varepsilon g''(\theta, \alpha)\theta_x^2 \\ & +(2\varepsilon \tfrac{h_x}{h} g(\theta, \alpha))_x + (2\varepsilon \tfrac{h_x}{h})_x (g'(\theta, \alpha)\theta - g(\theta, \alpha)). \end{aligned} \tag{2.7}$$

Now we let $\alpha \to 0$ in (2.7) to get

$$|\theta|_t + ((u + \frac{C}{h})|\theta|)_x \leq \varepsilon |\theta|_{xx} + (2\varepsilon \frac{h_x}{h}|\theta|)_x \tag{2.8}$$

in the sense of distributions. Integrating (2.8) in $R \times [0, t]$, we have

$$\int_{-\infty}^{\infty} |w_{1x}|(x,t)dx = \int_{-\infty}^{\infty} |\theta|(x,t)dx \leq \int_{-\infty}^{\infty} |\theta|(x,0)dx \leq M. \tag{2.9}$$

Similarly we can prove that

$$\int_{-\infty}^{\infty} |z_{1x}|(x,t)dx \leq \int_{-\infty}^{\infty} |z_{1x}|(x,0)dx \leq M \tag{2.10}$$

and so the estimates

$$\int_{-\infty}^{\infty} |v_x^\varepsilon|(x,t)dx \leq M, \quad \int_{-\infty}^{\infty} |F_x^\varepsilon|(x,t)dx \leq M. \tag{2.11}$$

Remark 1. The above BV estimates of $(v^\varepsilon(x,t), F^\varepsilon(x,t))$ with respect to the variable x are true when we have the lower, positive estimate depending on ε, $h^\varepsilon(x,t) \geq c(\varepsilon, t) > 0$. However these estimates are not enough to guarantee the pointwise convergence of $(v^\varepsilon(x,t), F^\varepsilon(x,t))$ as ε goes to zero. If one could also prove the uniformly bounded estimates of $(w_{1t}^\varepsilon, z_{1t}^\varepsilon)$ in $L_{loc}^1(R \times R^+)$, then one would have the pointwise convergence of $(v^\varepsilon(x,t), F^\varepsilon(x,t))$. Unfortunately, the terms $2\varepsilon \frac{h_x}{h} w_{1x}$ and $2\varepsilon \frac{h_x}{h} z_{1x}$ in the right-hand side of (2.5) make the L^1 estimates of w_{1t}^ε and z_{1t}^ε very difficult, or even impossible near $h = 0$. In [Lu4, Lu5], the author used the standard method from the compensated compactness theory to study the convergence of $(v^\varepsilon(x,t), F^\varepsilon(x,t))$. Here we introduce a different method.

Proof of (III). Since the assumption $h^\varepsilon(x,t) \geq h_0 > 0$ in Theorem 1, system (1.3) is strictly hyperbolic from (1.4) and genuinely nonlinear from (1.7)-(1.8) in the sense of Lax [Lax], the following DiPerna's compact framework gives us the pointwise convergence of $(h^\varepsilon(x,t), u^\varepsilon(x,t))$ immediately.

Lemma 3 *For the 2×2 strictly hyperbolic and genuinely nonlinear system*

$$\begin{cases} a_t + f(a,b)_x = 0, \\ b_t + g(a,b)_x = 0, \end{cases} \tag{2.12}$$

if the viscosity solutions $(a^\varepsilon, b^\varepsilon)$ of the Cauchy problem for the inhomogeneous system

$$\begin{cases} a_t + f(a,b)_x + f_1(a,b) = \varepsilon a_{xx}, \\ b_t + g(a,b)_x + g_1(a,b) = \varepsilon b_{xx} \end{cases} \tag{2.13}$$

are uniformly bounded, then there exists a subsequence $(a^{\varepsilon_l}(x,t), b^{\varepsilon_l}(x,t))$ *such that*

$$(a^{\varepsilon_l}(x,t), b^{\varepsilon_l}(x,t)) \to (a(x,t), b(x,t)), \tag{2.14}$$

pointwisely on any bounded and open set $\Omega \subset R \times R^+$ *as* ε_l *goes to zero.*

To prove the pointwise convergence of $(v^\varepsilon(x,t), F^\varepsilon(x,t))$, we first prove the following lemma

Lemma 4

$$c_t + v_x^\varepsilon, \quad h_t^\varepsilon + (h^\varepsilon u^\varepsilon)_x, \quad (h^\varepsilon v^\varepsilon)_t + (h^\varepsilon v^\varepsilon u^\varepsilon)_x \tag{2.15}$$

are all compact in $H_{loc}^{-1}(R \times R^+)$ *, where c is a constant.*

Proof of Lemma 4. Since v_x^ε are bounded in $L_{loc}^1(R \times R^+)$, then they are compact in $W_{loc}^{-1,\alpha}(R \times R^+)$ for a suitable $\alpha \in (1,2)$ by the Sobolev's embedding theorem. Furthermore since $v_x^\varepsilon(x,t)$ are bounded in $W^{-1,\infty}(R \times R^+)$, so Murat's theorem [Mu] gives that $v_x^\varepsilon(x,t)$ or $c_t + v_x^\varepsilon(x,t)$ are compact in $H_{loc}^{-1}(R \times R^+)$.

$(h^\varepsilon, h^\varepsilon u^\varepsilon)$ is a weak entropy-entropy flux pair corresponding to the system of isentropic gas dynamics (1.3). System (1.3) has a convex entropy $\eta^* = \frac{hu^2}{2} + h\int_0^h \frac{P(s)}{s^2} ds$, which reduces that ([Di1, LPT, LPS, Lu3])

$$\varepsilon(h\int_0^h \frac{P(s)}{s^2} ds)_{hh}(h_x^\varepsilon)^2 = \varepsilon(g + \frac{C^2}{h^3})(h_x^\varepsilon)^2 \text{ are bounded in } L_{loc}^1(R \times R^+) \tag{2.16}$$

and so

$$\varepsilon(h_x^\varepsilon)^2 \text{ are bounded in } L_{loc}^1(R \times R^+). \tag{2.17}$$

Thus, we have from (1.10) and (2.17) that $h_t^\varepsilon + (h^\varepsilon u^\varepsilon)_x$ are compact in $H_{loc}^{-1}(R \times R^+)$.

Finally we choose $\eta(h,v) = hf(v)$, where f is a strictly convex function of v and multiply the first equation in (1.10) by $f(v)$, the first equation in (2.4) by $hf'(v)$, and add the results to obtain

$$\begin{aligned} &(hf(v))_t + (hf(v)u)_x + Cf'(v)F_x \\ &= \varepsilon(hf(v))_{xx} - \varepsilon f''(v)hv_x^2 \le \varepsilon(hf(v))_{xx} - \varepsilon c_0 hv_x^2 \end{aligned} \tag{2.18}$$

for a suitable positive constant c_0. Since F_x is bounded in $L_{loc}^1(R\times R^+)$, we obtain from (2.18) that

$$\varepsilon hv_x^2 \quad \text{are bounded in} \quad L_{loc}^1(R \times R^+). \tag{2.19}$$

We may rewrite the right-hand side of the third equation in (1.10) as $\varepsilon(hv_x+vh_x)_x$, which is clearly compact in $H^{-1}_{loc}(R\times R_+)$ since the estimates (2.19) and (2.17).

Since F_x is bounded both in $L^1_{loc}(R\times R^+)$ and in $W^{-1,\infty}(R\times R^+)$, we have from the Murat's theorem ([Mu]) and the third equation in (1.10) that

$$(h^\varepsilon v^\varepsilon)_t+(h^\varepsilon v^\varepsilon u^\varepsilon)_x=\varepsilon(h^\varepsilon v^\varepsilon)_{xx}-CF^\varepsilon_x \quad \text{are compact in} \quad H^{-1}_{loc}(R\times R^+), \tag{2.20}$$

which ends the proof of Lemma 4.

Now we apply for the div-curl lemma in the compensated compactness theory [Ta, Lu2] to the following special pairs of functions

$$(c,v^\varepsilon),\quad (h^\varepsilon,h^\varepsilon u^\varepsilon) \tag{2.21}$$

and

$$(c,v^\varepsilon),\quad (h^\varepsilon v^\varepsilon,h^\varepsilon v^\varepsilon u^\varepsilon), \tag{2.22}$$

respectively for any constant c, to obtain

$$\overline{h^\varepsilon}\cdot\overline{v^\varepsilon}=\overline{h^\varepsilon v^\varepsilon} \quad \text{and} \quad \overline{h^\varepsilon v^\varepsilon}\cdot\overline{v^\varepsilon}=\overline{h^\varepsilon(v^\varepsilon)^2}, \tag{2.23}$$

where $\overline{f(\theta^\varepsilon)}$ denotes the weak-star limit of $f(\theta^\varepsilon)$.

Let $(\overline{h^\varepsilon},\overline{v^\varepsilon})=(h,v)$. By simple calculations, we have

$$\overline{h^\varepsilon(v^\varepsilon-v)^2}=\overline{h^\varepsilon(v^\varepsilon)^2}-2v\overline{h^\varepsilon v^\varepsilon}+hv^2=0 \tag{2.24}$$

due to (2.23), which implies the pointwise convergence of v^ε on any compact, support set $\{(x,t):h(x,t)>0\}$. Similarly we can prove the pointwise convergence of F^ε on any compact, support set $\{(x,t):h(x,t)>0\}$. Therefore, we complete the proof of Theorem 1.

3 Proof of Theorem 2

We introduce a "variant" of the vanishing artificial viscosity to study system (1.3). Consider the following parabolic system

$$\begin{cases} w_t+\lambda_2 w_x=\varepsilon w_{xx}-gb, \\ z_t+\lambda_1 z_x=\varepsilon z_{xx}-gb \end{cases} \tag{3.1}$$

with initial data

$$(w^{\varepsilon_1}(x,0), z^{\varepsilon_1}(x,0)) = (w(h_0(x), u_0(x)), z(h_0(x), u_0(x)) * G^{\varepsilon_1}, \tag{3.2}$$

where w, z are Riemann invariants given in (1.6) and G^{ε_1} is a mollifier such that $w^{\varepsilon_1}(x,0)$ and $z^{\varepsilon_1}(x,0)$ are smooth, monotonic, increasing functions satisfying

$$\begin{cases} \lim_{\varepsilon_1 \to 0}(w^{\varepsilon_1}(x,0), z^{\varepsilon_1}(x,0)) = (w_0(x), z_0(x)), a.e. \text{ on } R, \\ l_1 \le z^{\varepsilon_1}(x,0) \le l_2 < m_1 \le w^{\varepsilon_1}(x,0) \le m_2 \\ 0 \le \varepsilon_1 z_x^{\varepsilon_1}(x,0) \le M, \quad 0 \le \varepsilon_1 w_x^{\varepsilon_1}(x,0) \le M, \quad \varepsilon = o((\varepsilon_1)^2). \end{cases} \tag{3.3}$$

We rewrite (3.1) as

$$\begin{cases} \bar{w}_t + \lambda_2 \bar{w}_x = \varepsilon \bar{w}_{xx}, \\ \bar{z}_t + \lambda_1 \bar{z}_x = \varepsilon \bar{z}_{xx}, \end{cases} \tag{3.4}$$

where $\bar{z} = z + bgt, \bar{w} = w + bgt$. We differentiate (3.4) with respect to x and let $\bar{w}_x = r, \bar{z}_x = s$; then

$$\begin{cases} r_t + \lambda_2 r_x + (\lambda_{2w} r + \lambda_{2z} s) r = \varepsilon r_{xx}, \\ s_t + \lambda_1 s_x + (\lambda_{1w} r + \lambda_{1z} s) s = \varepsilon s_{xx}. \end{cases} \tag{3.5}$$

A simple calculation yields

$$u_w = \frac{1}{2}, \quad h_w = \frac{h}{2\sqrt{P'(h)}}, \quad u_z = \frac{1}{2}, \quad h_z = -\frac{h}{2\sqrt{P'(h)}} \tag{3.6}$$

and

$$\lambda_{1w} = \lambda_{2z} = \frac{2P'(h) - hP''(h)}{4P'(h)} > 0, \quad \lambda_{1z} = \lambda_{2w} = \frac{2P'(h) + hP''(h)}{4P'(h)} > 0. \tag{3.7}$$

By applying the maximum principle to (3.4) and (3.5), we can get the following estimates immediately for the solutions $(w^{\varepsilon,\varepsilon_1}(x,t), z^{\varepsilon,\varepsilon_1}(x,t))$ of the Cauchy problem (3.1) and (3.2) ([Lu1])

$$\begin{cases} l_1 - gbt \le z^{\varepsilon,\varepsilon_1}(x,t) \le l_2 - gbt < m_1 - gbt \le w^{\varepsilon,\varepsilon_1}(x,t) \le m_2 - gbt \\ 0 \le \varepsilon_1 z_x^{\varepsilon,\varepsilon_1}(x,t) \le M, \quad 0 \le \varepsilon_1 w_x^{\varepsilon,\varepsilon_1}(x,t) \le M \end{cases} \tag{3.8}$$

which imply the following estimates on $(h^{\varepsilon,\varepsilon_1}(x,t), u^{\varepsilon,\varepsilon_1}(x,t))$

$$h_0 \le h^{\varepsilon,\varepsilon_1} \le M(t), \quad |u^{\varepsilon,\varepsilon_1}| \le M(t), \quad 0 \le \varepsilon_1 u_x^{\varepsilon,\varepsilon_1} \le M, \quad \varepsilon_1 |h_x^{\varepsilon,\varepsilon_1}| \le M, \tag{3.9}$$

where M is a positive constant, $M(t)$ a positive bounded function on any compact set $t \in [0, T]$ and both are independent of $\varepsilon, \varepsilon_1$.

Now we consider the matrix

$$A = \begin{pmatrix} w_h & w_m \\ \\ z_h & z_m \end{pmatrix}^{-1} = \begin{pmatrix} \frac{h}{2\sqrt{P'(h)}} & -\frac{h}{2\sqrt{P'(h)}} \\ \\ \frac{1}{2}(h + \frac{m}{\sqrt{P'(h)}}) & \frac{1}{2}(h - \frac{m}{\sqrt{P'(h)}}) \end{pmatrix},$$

where $m = hu$. Multiply (3.1) by A, then (3.1) can be rewritten as follows:

$$\begin{cases} h_t + (hu)_x = \frac{\varepsilon h}{2\sqrt{P'(h)}}(w_{xx} - z_{xx}) = \varepsilon h_{xx} + \varepsilon \frac{h}{\sqrt{P'(h)}}(\frac{\sqrt{P'(h)}}{h})' h_x^2, \\ \\ (hu)_t + (hu^2 + P(h))_x = \frac{\varepsilon h}{2}(w_{xx} + z_{xx}) + \frac{\varepsilon m}{2\sqrt{P'(h)}}(w_{xx} - z_{xx}) - bgh. \end{cases} \tag{3.10}$$

Then for any smooth entropy-entropy flux pair $(\eta(h,m), q(h,m))$ of system (1.3), we multiply (3.10) by $(\eta_h(h,m), \eta_m(h,m))$ to obatin

$$\begin{aligned} &\eta_t(h^{\varepsilon,\varepsilon_1}, m^{\varepsilon,\varepsilon_1}) + q_x(h^{\varepsilon,\varepsilon_1}, m^{\varepsilon,\varepsilon_1}) \\ &= \eta_h(h^{\varepsilon,\varepsilon_1}, m^{\varepsilon,\varepsilon_1})(\frac{\varepsilon h}{2\sqrt{P'(h)}}(w_{xx} - z_{xx})) \\ &+ \eta_m(h^{\varepsilon,\varepsilon_1}, m^{\varepsilon,\varepsilon_1})(\frac{\varepsilon h}{2}(w_{xx} + z_{xx}) + \frac{\varepsilon m}{2\sqrt{P'(h)}}(w_{xx} - z_{xx}) - bgh). \end{aligned} \tag{3.11}$$

Since the estimates given in (3.9) and $\varepsilon = o((\varepsilon_1)^2)$, we can easily prove that

$$\eta_t(h^{\varepsilon,\varepsilon_1}, m^{\varepsilon,\varepsilon_1}) + q_x(h^{\varepsilon,\varepsilon_1}, m^{\varepsilon,\varepsilon_1}) \quad \text{are compact in} \quad H^{-1}_{loc}(R \times R^+) \tag{3.12}$$

as $\varepsilon, \varepsilon_1$ go to zero. Thus DiPerna's compactness framework (Lemma 3) reduces again the pointwise convergence of $(h^{\varepsilon,\varepsilon_1}, m^{\varepsilon,\varepsilon_1})$ as $\varepsilon, \varepsilon_1$ go to zero. The limit (h, m) satisfies system (1.3) in the sense of distributions

$$\begin{cases} \int_0^\infty \int_R h\phi_t + hu\phi_x dxdt + \int_{t=0} h_0 \phi dx = 0, \\ \\ \int_0^\infty \int_R hu\phi_t + (hu^2 + P(h))\phi_x - gbh\phi dxdt + \int_{t=0} h_0 u_0 \phi dx = 0 \end{cases} \tag{3.13}$$

for all $\phi \in C_0^1(R \times R^+)$.

Now let $(v^{\varepsilon,\varepsilon_1}, F^{\varepsilon,\varepsilon_1})$ be the solutions of the parabolic system

$$\begin{cases} (hv)_t + (hvu + CF)_x = \varepsilon(hv)_{xx} + \varepsilon \frac{h}{\sqrt{P'(h)}}(\frac{\sqrt{P'(h)}}{h})' h_x^2 v, \\ \\ (hF)_t + (hFu + Cv)_x = \varepsilon(hF)_{xx} + \varepsilon \frac{h}{\sqrt{P'(h)}}(\frac{\sqrt{P'(h)}}{h})' h_x^2 F \end{cases} \tag{3.14}$$

with the initial data

$$(v(x,0), F(x,0)) = (v_0(x), F_0(x)). \tag{3.15}$$

Substituting the first equation in (3.10) into (3.14), then (v, F) satisfies (2.4). Since the estimates in (3.9) and $\varepsilon = o((\varepsilon_1)^2)$, then both terms

$$\varepsilon \frac{h}{\sqrt{P'(h)}}(\frac{\sqrt{P'(h)}}{h})' h_x^2 v \quad \text{and} \quad \varepsilon \frac{h}{\sqrt{P'(h)}}(\frac{\sqrt{P'(h)}}{h})' h_x^2 F \tag{3.16}$$

in (3.14) converge to zero as $\varepsilon, \varepsilon_1$ go to zero. Using the same technique in the proof of (III) in Theorem 1, we may prove the pointwise convergence of $(v^{\varepsilon,\varepsilon_1}(x,t), F^{\varepsilon,\varepsilon_1}(x,t)) \to (v(x,t), F(x,t))$, as $\varepsilon, \varepsilon_1$ go to zero, and the limit (v, F) satisfies the third and the fourth equations in system (1.2) in the sense of distributions. Therefore we complete the proof of Theorem 2.

4 Applications on Polytropic and Chaplygin gas

In this section, we study the following system

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + P(h))_x = -gbh, \\ (hv)_t + (hvu + CF)_x = 0, \\ (hF)_t + (hFu + Cv)_x = 0 \end{cases} \tag{4.1}$$

for the polytropic gas $P(h) = h^\gamma, \gamma > 1$ and for the Chaplygin gas $P(h) = -\frac{1}{h}$. Since the global weak solution for the polytropic gas is well studied in [LPS, LPT], we have the following existence theorem immediately

<u>Theorem</u> 5 *Let $P(h) = h^\gamma, \gamma > 1$ and the initial data $(h_0(x), u_0(x), v_0(x), F_0(x))$ be bounded, $h_0(x) \geq 0$, the total variation of $(v_0(x), F_0(x))$ be bounded. Then the Cauchy problem (4.1) and (1.9) has a global bounded weak solution (h, u, v, F) satisfying (4.1) in the sense of distributions, where $h(x,t) \geq 0$, $v_x(\cdot, t)$ and $F_x(\cdot, t))$ are bounded in L^1.*

Remark 2. In Theorem 5, the initial date $h_0(x) \geq 0$. We may first add a small ε to $h_0(x)$ and consider system (4.1) with the initial date $h_0(x) + \varepsilon > 0$. The proof of Theorem 5 is similar to that of Theorem 1.

Now we consider the case of Chaplygin gas and rewrite the first two equations in (4.1) as

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + (hu^2 - h^{-1})_x = -gbh. \end{cases} \tag{4.2}$$

By simple calculations, two eigenvalues of system (4.2) are

$$\lambda_1 = u - h^{-1}, \quad \lambda_2 = u + h^{-1} \tag{4.3}$$

with corresponding right eigenvectors

$$r_1 = (1, u - h^{-1})^T, \quad r_2 = (1, u + h^{-1})^T; \tag{4.4}$$

the two corresponding Riemann invariants are

$$z = u + h^{-1}, \quad w = u - h^{-1}; \tag{4.5}$$

and

$$\begin{cases} \nabla\lambda_1 \cdot r_1 = (-uh^{-1} + h^{-2}, h^{-1})(1, u - h^{-1})^T = 0, \\ \\ \nabla\lambda_2 \cdot r_2 = (-uh^{-1} - h^{-2}, h^{-1})(1, u + h^{-1})^T = 0. \end{cases} \tag{4.6}$$

Therefore, it follows from (4.3) that $\lambda_1 < \lambda_2$ for all $0 < h < \infty$, and from (4.6) that both characteristic fields are linearly degenerate on all the points (h, hu) (Temple's type also [Te]).

Any entropy-entropy flux pair $(\eta(h, m), q(h, m))$ of system (4.2) satisfies the additional system

$$q_h = u\eta_h + h^{-3}\eta_u, \quad q_u = h\eta_h + u\eta_u. \tag{4.7}$$

Eliminating the q from (4.7), we have

$$\eta_{hh} = h^{-4}\eta_{uu}. \tag{4.8}$$

Let $\eta = hH(h, u)$, then $\eta_h = H + hH_h, \eta_{hh} = 2H_h + hH_{hh}, \eta_u = hH_u, \eta_{uu} = hH_{uu}$. Thus using the entropy equation (4.8), we have the following equation on the function H

$$2H_h + hH_{hh} = h^{-3}H_{uu}. \tag{4.9}$$

Let $H(h, u) = L(w, z)$, then

$$H_h = h^{-2}(L_w - L_z), \quad H_u = L_w + L_z, \quad H_{uu} = L_{ww} + L_{zz} + 2L_{wz},$$

$$H_{hh} = h^{-4}(L_{ww} + L_{zz}) - 2h^{-4}L_{wz} - 2h^{-3}(L_w - L_z).$$

Substituting all above equalities into (4.9), we have $L_{wz} = 0$ or $L(w,z) = f(w) + g(z)$ and so the following lemma.

Lemma 6 *System (4.2) has two special families of entropy-entropy flux pairs:*

$$(A).\quad \eta_1(h,u) = hf(u - h^{-1}), \quad q_1(h,u) = \lambda_2\eta_1(h,u) = (hu+1)f(u-h^{-1});$$

$$(B).\quad \eta_2(h,u) = hg(u + h^{-1}), \quad q_2(h,u) = \lambda_1\eta_2(h,u) = (hu-1)g(u+h^{-1}),$$

where f, g are two arbitrary differentiable functions.

Proof of Lemma 6: We need to prove that the entropy-entropy flux pairs $(\eta_1, q_1), (\eta_2, q_2))$ satisfy (4.7). By simple calculations,

$$q_{1h} = uf + h^{-2}(uh+1)f', \quad q_{1u} = hf + (uh+1)f',$$

$$\eta_{1h} = f + h^{-1}f', \quad \eta_{1u} = hf'.$$

So (η_1, q_1) satisfy (4.7). Similarly we can prove that $(\eta_2, q_2))$ satisfy (4.7) also.

We consider the Cauchy problem for the related parabolic system

$$\begin{cases} h_t + (hu)_x = \varepsilon h_{xx} \\ (hu)_t + (hu^2 - h^{-1})_x = \varepsilon(hu)_{xx} - gbh \end{cases} \tag{4.10}$$

with bounded initial data

$$(h(x,0), u(x,0)) = (h_0(x), u_0(x)), \quad h_0(x) > 0, \tag{4.11}$$

and multiply (4.10) by (w_h, w_m) and $(z_h, z_m), m = hu$ respectively to obtain

$$\begin{aligned} w_t + \lambda_2 w_x &= \varepsilon w_{xx} - \varepsilon(w_{hh}h_x^2 + 2w_{hm}h_x m_x + w_{mm}m_x^2) - gb \\ &= \varepsilon w_{xx} - \varepsilon(-2h^{-2}h_x m_x + 2(m-1)h^{-3}h_x^2) - gb \\ &= \varepsilon w_{xx} + 2\varepsilon h^{-1}h_x w_x - gb, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} z_t + \lambda_1 z_x &= \varepsilon z_{xx} - \varepsilon(z_{hh}h_x^2 + 2z_{hm}h_x m_x + z_{mm}m_x^2) - gb \\ &= \varepsilon z_{xx} - \varepsilon(-2h^{-2}h_x m_x + 2(m+1)h^{-3}h_x^2) - gb \\ &= \varepsilon z_{xx} + 2\varepsilon h^{-1}h_x z_x - gb. \end{aligned} \tag{4.13}$$

Lemma 7 *(A). Let the initial data* $(h_0(x), u_0(x))$ *be bounded measurable and* $h_0(x) \geq c_0 > 0$ *such that*

$$c_1 \leq w(h_0(x), u_0(x)) \leq c_2 < d_1 \leq z(h_0(x), u_0(x)) \leq d_2 \tag{4.14}$$

for some constants $c_i, d_i, i = 1, 2$. *Then the viscosity solutions of the Cauchy problem (4.10) and (4.11) exist and have the estimates*

$$c_1 - gbt \leq w(h^\varepsilon, u^\varepsilon) \leq c_2 - gbt < d_1 - gbt \leq z(h^\varepsilon, u^\varepsilon) \leq d_2 - gbt. \tag{4.15}$$

(B). If the total variations of $w(h_0(x), u_0(x))$ *and* $z(h_0(x), u_0(x))$ *are bounded in* $(-\infty, \infty)$, *then* $z^\varepsilon_x(\cdot, t)$ *and* $w^\varepsilon_x(\cdot, t))$ *are uniformly bounded in* L^1.

(C). If the conditions in (A) and (B) are satisfied, we may select a subsequence (still labelled) $(h^\varepsilon(x,t), u^\varepsilon(x,t))$ *such that*

$$(h^\varepsilon(x,t), u^\varepsilon(x,t)) \to (h(x,t), u(x,t)), \tag{4.16}$$

a.e. on any bounded and open set $\Omega \subset R \times R^+$.

Proof of Lemma 7. By using the maximum principle to (4.12) and (4.13), we can easily prove the estimates in (4.15), which coupled with (4.5) and (4.15) reduce the following L^∞ estimates about $(h^\varepsilon, u^\varepsilon)$

$$0 < \frac{2}{d_2 - c_1} \leq h^\varepsilon \leq \frac{2}{d_1 - c_2}, \quad \frac{c_1 + d_1}{2} - gbt \leq u^\varepsilon \leq \frac{c_1 + d_1}{2} - gbt, \tag{4.17}$$

which ensure the existence of the viscosity solutions for the Cauchy problem (4.10) and (4.11).

Applying for the technique in the proof of Theorem 1, we can also prove that $z^\varepsilon_x(\cdot, t)$ and $w^\varepsilon_x(\cdot, t))$ are uniformly bounded in $L^1(R)$.

Finally, from the conclusions in (B), we have that $h^\varepsilon_x(\cdot, t)$ and $u^\varepsilon_x(\cdot, t)$ are bounded in $L^1(R)$, and so compact in $H^{-1}_{loc}(R \times R^+)$. Using the same technique in the proof of Theroem 1, we can easily prove that $h^\varepsilon_t + (h^\varepsilon u^\varepsilon)_x$ and $(h^\varepsilon u^\varepsilon)_t + (h^\varepsilon (u^\varepsilon)^2 - \frac{1}{h^\varepsilon})_x$ are compact in $H^{-1}_{loc}(R \times R^+)$. So we may apply for the div-curl lemma to the following pairs of functions

$$(c, h^\varepsilon), \quad (h^\varepsilon, h^\varepsilon u^\varepsilon) \tag{4.18}$$

to obtain

$$\overline{h^\varepsilon} \cdot \overline{h^\varepsilon} = \overline{(h^\varepsilon)^2} \tag{4.19}$$

which implies the pointwise convergence of h^ε. Then we apply for the div-curl lemma to the pairs of functions

$$(c, u^\varepsilon), \quad (h^\varepsilon u^\varepsilon, h^\varepsilon (u^\varepsilon)^2 - \frac{1}{h^\varepsilon}) \tag{4.20}$$

to obtain

$$\overline{u^\varepsilon} \cdot \overline{h^\varepsilon u^\varepsilon} = \overline{h^\varepsilon (u^\varepsilon)^2} \tag{4.21}$$

which implies the pointwise convergence of u^ε since the pointwise convergence of h^ε . Thus we complete the proof of Lemma 7.

Now we end this paper by the following theorem.

<u>Theorem</u> 8 *Let $P(h) = -h^{-1}$. If the initial data $(h_0(x), u_0(x), v_0(x), F_0(x))$ are bounded in L^∞; $(h_{0x}(x), u_{0x}(x), v_{0x}(x), F_{0x}(x))$ are bounded in $L^1(R)$; $h_0(x) \geq c_0 > 0$ such that*

$$c_1 \leq w(h_0(x), u_0(x)) \leq c_2 < d_1 \leq z(h_0(x), u_0(x)) \leq d_2 \tag{4.22}$$

for some constants $c_i, d_i.i = 1, 2$, then the Cauchy problem (4.1) and (1.9) has a weak solution (h, u, v, F). Moreover, $(h_x(\cdot, t), u_x(\cdot, t), v_x(\cdot, t), F_x(\cdot, t))$ are uniformly bounded in $L^1(R)$.

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