# Global Solutions to the One-dimensional Compressible Navier-Stokes-Poisson Equations with Large Data

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**Abstract**

This paper is concerned with the construction of global smooth solutions away from vacuum to the Cauchy problem of the one-dimensional compressible Navier-Stokes-Poisson system with large data and density dependent viscosity coefficient and density and temperature dependent heat conductivity coefficient. The proof is based on some detailed analysis on the bounds on the density and temperature functions.

**Key Words and Phrases:** Navier-Stokes-Poisson equations, global solutions with large data, viscosity and heat conductivity coefficients.

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## 1 Introduction

The compressible Navier-Stokes-Poisson (denoted by NSP) system consisting of the compressible Navier-Stokes equations coupled with Poisson equation models the viscous fluid under the influence of the self-induced electric force:

$$\begin{cases} \rho_\tau + \nabla_\xi \cdot (\rho u) = 0, \\ (\rho u)_\tau + \nabla_\xi \cdot (\rho u \otimes u) + \nabla_\xi p = \rho \nabla_\xi \Phi + \nabla_\xi \cdot \mathbf{T}, \\ (\rho \mathbf{E})_\tau + \nabla_\xi \cdot (\rho u \mathbf{E} + up) = \rho u \cdot \nabla_\xi \Phi + \nabla_\xi \cdot (u\mathbf{T}) + \nabla_\xi \cdot (\kappa(v,\theta)\nabla_\xi \theta), \\ \Delta_\xi \Phi = \rho - \overline{\rho}(\xi), \quad \lim_{|\xi|\to+\infty} \Phi(\tau,\xi) = 0. \end{cases}$$

Here, $\rho > 0$, $u = (u^1, u^2, u^3)$, $\theta > 0$, $p = p(\rho, \theta)$, $e$ and $\Phi$ denote the density, velocity, absolute temperature, pressure, internal energy and the electrostatic potential function, respectively. And $\mathbf{E} = \frac{1}{2}|u|^2 + e$ is the specific total energy, $\mathbf{T} = \mu(\rho, \theta)\left(\nabla_\xi u + (\nabla_\xi u)^t\right) + \nu(\rho, \theta)(\nabla_\xi \cdot u)\mathbf{I}$ is the stress tensor with $\mathbf{I}$ being the identity matrix. The viscosity coefficients $\mu(\rho, \theta) > 0$ and $\nu(\rho, \theta)$ satisfy $\mu(\rho, \theta) + \frac{2}{3}\nu(\rho, \theta) > 0$. The thermodynamic variables $p$, $\rho$, and $e$ are related by Gibbs equation $de = \theta ds - pd\rho^{-1}$ with $s$ being the specific entropy. $\kappa(\rho, \theta) > 0$ denotes the heat conductivity coefficient, and $\overline{\rho}(\xi) > 0$ is the background doping profile, cf. [30].

To explain the purpose of this paper, we firstly give the following remarks on the viscosity and heat conductivity coefficients:

- When the viscosity coefficients $\mu(\rho, \theta) > 0$, $\nu(\rho, \theta)$ and the heat-conductivity coefficient $\kappa(\rho, \theta) > 0$ are constants, (1.1) is used in semiconductor theory to model the transport of charged particles under the influence of self-induced electric field, cf. [30].

- In the kinetic theory, the time evolution of the particle distribution function for the charged particles in a dilute gas can be modelled by the Vlasov-Poisson-Boltzmann system, cf. [4], [3], [34]. When we derive the NSP (1.1) from the Vlasov-Poisson-Boltzmann system by using the Chapman-Enskog expansion, cf. [4], [12], [34], the viscosity coefficients $\mu$, $\nu$ and the heat-conductivity coefficient $\kappa$ depend on the absolute temperature $\theta$ and $\nu = -\frac{2}{3}\mu$ for the monatomic gas. If the inter-molecular potential is proportional to $r^{-\alpha}$ with $\alpha > 1$, where $r$ represents the intermolecular distance, then $\mu$, $\nu$ and $\kappa$ are proportional to the temperature to some power:

  $$\mu,\ -\nu,\ \kappa \propto \theta^{\frac{\alpha+4}{2\alpha}}.$$

  In particular, for the Maxwellian molecule ($\alpha = 4$), such dependence is linear, while for the hard sphere model and also the case when $\alpha \to +\infty$, the dependence is in the form of $\sqrt{\theta}$.

This paper is concerned with the global existence of large data solutions when the viscosity coefficients $\mu$, $\nu$ and the heat conductivity coefficient $\kappa$ depend on $\rho$ and $\theta$. Unlike the small perturbation solutions, such dependence has strong influence on the solution behavior and thus leads to difficulties in analysis not for the case of constant coefficients. In fact, for the one-dimensional compressible Navier-Stokes equations, recently there are a lot of works on the construction of non-vacuum solutions to the one-dimensional compressible Navier-Stokes equations with density and temperature dependent transportation coefficients in various forms, cf. [1] [5], [18], [19], [21], [22], [23], [24], [25] and the references therein. However, there is a gap between the physical models and the satisfactory existence theory.

The main purpose in this paper is devoted to the construction of globally smooth, non-vacuum solutions to the one-dimensional non-isentropic compressible NSP with degenerate density dependent viscous coefficient and degenerate density and temperature dependent heat conductivity coefficient for arbitrarily large data. We hope that the analysis here can shed some light on the construction of global classical solutions to the fluid model derived from the Vlasov-Poisson-Boltzmann system with large data.

Let $x$ be the Lagrangian space variable, $t$ be the time variable, and $v = \frac{1}{\rho}$ denote the specific volume. Then the one-dimensional compressible NSP system (1.1) with viscous coefficient $\mu(v)$

and heat conductivity coefficient $\kappa(v,\theta)$ becomes

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\theta)_x = \left(\frac{\mu(v)u_x}{v}\right)_x + \frac{\Phi_x}{v}, \\ e_t + p(v,\theta)u_x = \frac{\mu(v)u_x^2}{v} + \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x, \\ \left(\frac{\Phi_x}{v}\right)_x = 1 - v, \quad \lim\limits_{|x|\to+\infty} \Phi(t,x) = 0. \end{cases} \tag{1.1}$$

Throughout this paper, we will concentrate on the ideal, polytropic gases:

$$p(v,\theta) = \frac{R\theta}{v} = Av^{-\gamma}\exp\left(\frac{\gamma-1}{R}s\right), \quad e = C_v\theta = \frac{R\theta}{\gamma-1}, \tag{1.2}$$

where the specific gas constant $R$ and the specific heat at constant volume $C_v$ are positive constants and $\gamma > 1$ is the adiabatic constant. Moreover, to simplify the presentation, we will only consider the case when the background doping profile $\overline{\rho}$ is a positive constant which is normalized to 1 as in $(1.1)_4$.

Take the initial data

$$(v(0,x),u(0,x),\theta(0,x)) = (v_0(x),u_0(x),\theta_0(x)), \quad \lim_{x\to\pm\infty}(v_0(x),u_0(x),\theta_0(x)) = (v_\pm,u_\pm,\theta_\pm) \tag{1.3}$$

satisfying $v_- = v_+, u_- = u_+, \theta_- = \theta_+$. Without loss of generality, we assume $v_- = v_+ = 1, u_- = u_+ = 0, \theta_- = \theta_+ = 1$.

The first result is concerned with the case

$$\mu(v) = v^{-a}, \quad \kappa(v,\theta) = \theta^b, \tag{1.4}$$

which is stated as follows.

**Theorem 1.1** *Suppose*

- $(v_0(x)-1, u_0(x), \theta_0(x)-1, \Phi_{0x}(x)) \in H^1(\mathbf{R})$, *and there exist positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *such that*

$$\underline{V} \le v_0(x) \le \overline{V}, \quad \underline{\Theta} \le \theta_0(x) \le \overline{\Theta}; \tag{1.5}$$

- $\frac{1}{3} < a < \frac{1}{2}$;
- $b$ *satisfies one of the following conditions*
  - *(i)* $1 \le b < \frac{2a}{1-a}$,
  - *(ii)* $0 < b < 1$, $\frac{2-b}{2} + \frac{(a^2-a+2)(1-b)}{(1-2a)(3a-1)} < 1$, $\frac{(1-b)(3+a-2a^2)}{(3a-1)(1-2a)} < 1$.

*Then the Cauchy problem (1.1), (1.3) with $\mu(v)$ and $\kappa(v,\theta)$ given by (1.4) admits a unique global solution $(v(t,x),u(t,x),\theta(t,x))$ satisfying*

$$\begin{gathered} \Big(v(t,x)-1, u(t,x), \theta(t,x)-1\Big) \in C^0\left(0,T;H^1(\mathbf{R})\right), \\ \Big(u_x(t,x), \theta_x(t,x)\Big) \in L^2\left(0,T;H^2(\mathbf{R})\right), \\ \Phi(t,x) \in C^0\left(0,T;H^3(\mathbf{R})\right), \\ 0 < V_0^{-1} \le v(t,x) \le V_0, \quad 0 < \Theta_0^{-1} \le \theta(t,x) \le \Theta_0, \quad \forall (t,x) \in [0,T]\times\mathbf{R}. \end{gathered} \tag{1.6}$$

*Here $T > 0$ is any given positive constant and $V_0, \Theta_0$ are some positive constants which may depend on $T$.*

Note that the assumptions imposed on $a$ and $b$ in Theorem 1.1 exclude the case when the viscous coefficient $\mu$ and the heat conductivity coefficient $\kappa$ are positive constants. The next result will recover this in another setting. The main idea is to use the smallness of $\gamma-1$ to deduce uniform lower and upper bounds on the absolute temperature. This can be achieved by showing that $(v_0(x)-1, u_0(x), s_0(x)-\overline{s}) \in H^1(\mathbf{R})$ are bounded in $H^1(\mathbf{R})$ independent of $\gamma-1$ so that $\|\theta_0(x)-1\|_{L^\infty(\mathbf{R})}$ can be chosen to be small when $\gamma$ is close to 1. Here $\overline{s} = \frac{R}{\gamma-1}\ln\frac{R}{A}$ is the far field of the initial entropy $s_0(x)$, that is,

$$\lim_{|x|\to+\infty} s_0(x) = \lim_{|x|\to+\infty} \frac{R}{\gamma-1} \ln \frac{R\theta_0(x)v_0(x)^{\gamma-1}}{A} = \overline{s}.$$

Take $(v,u,s)$ as the unknown function, the second global existence theorem can be stated as follows.

**Theorem 1.2** *Suppose*

- $\|(v_0(x)-1, u_0(x), s_0(x)-\overline{s}, \Phi_{0x}(x))\|_{H^1(\mathbf{R})}$ *is bounded by some positive constant independent of* $\gamma-1$ *and (1.5) holds for some* $\gamma-1-$*independent positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$*;*
- $(\gamma-1)\|s_0(x)\|_{L^\infty(\mathbf{R})}$ *is bounded by some constant independent of* $\gamma-1$*;*
- *The smooth function* $\mu(v)$ *satisfies* $\mu(v)>0$ *for all* $v>0$ *and*

$$\lim_{v\to0_+}\Psi(v) = -\infty, \qquad \lim_{v\to+\infty}\Psi(v) = +\infty. \tag{1.7}$$

  *Here*

$$\Psi(v) = \int_1^v \frac{\sqrt{z-\ln z-1}}{z}\mu(z)dz; \tag{1.8}$$

- *For the heat conductivity coefficient, there are two cases. If* $\kappa(v,\theta)=\kappa(\theta)$ *depends only on* $\theta$*, we only assume* $\kappa(\theta)>0$ *for* $\theta>0$ *with some smoothness condition. If it depends on both* $v$ *and* $\theta$*, then in addition to* $\kappa(v,\theta)>0$ *for all* $v>0, \theta>0$*, we also assume the following. Set* $\kappa_1(v) = \min_{\underline{\Theta}\le\theta\le\overline{\Theta}} \kappa(v,\theta)$*, assume*

$$\kappa_{\theta\theta}(v,\theta) < 0, \quad \forall v>0,\ \theta>0, \tag{1.9}$$

  *and*

$$\overline{\lim_{v\to0_+}} \frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2} = \overline{\lim_{v\to+\infty}} \frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2} = 0; \tag{1.10}$$

- $\gamma-1$ *is sufficiently small.*

*Then the Cauchy problem (1.1), (1.3) admits a unique global solution* $(v(t,x), u(t,x), \theta(t,x))$ *satisfying (1.6) and*

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\Big|(v(t,x)-1, u(t,x), \theta(t,x)-1)\Big| = 0. \tag{1.11}$$

Although in Theorem 1.2, the case $\mu$ and $\kappa$ are positive constants can be covered, it does ask that $\gamma-1$ to be sufficiently small, our final result in this paper shows that for the case when $\mu$ is a positive constant, similar result hold provided that $\kappa(v,\theta)$ satisfies

$$\kappa(v,\theta)>0 \quad \forall v>0,\ \theta>0, \qquad \min_{v\ge\widetilde{V}>0,\ \theta\ge\widetilde{\Theta}>0} \kappa(v,\theta) \ge \underline{\kappa}\left(\widetilde{V},\widetilde{\Theta}\right) > 0. \tag{1.12}$$

**Theorem 1.3** *Suppose*

- $(v_0(x)-1, u_0(x), \theta_0(x)-1, \Phi_{0x}(x)) \in H^1(\mathbf{R})$, *and there exist positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *satisfying (1.5);*
- $\mu$ *is a positive constant and* $\kappa(v,\theta)$ *satisfies (1.12).*

*Then the Cauchy problem (1.1), (1.3) admits a unique global solution* $(v(t,x), u(t,x), \theta(t,x))$ *satisfying (1.6).*

**Remark 1.1** *We give the following remarks on Theorem 1.1-Theorem 1.3.*

- *From the proof of Theorem 1.2, one will notice that the assumption (1.10) can be replaced by the following weaker assumption*

$$\overline{\lim_{v\to 0_+}} \frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2} \leq \varepsilon_0, \quad \overline{\lim_{v\to +\infty}} \frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2} \leq \varepsilon_0. \tag{1.13}$$

  *Here* $\varepsilon_0 > 0$ *is a suitably chosen sufficiently small positive constant.*

- *Under the assumptions in Theorem 1.2, when* $\gamma - 1$ *is sufficiently small, although* $\|\theta_0 - 1\|_{H^1(\mathbf{R})}$ *is small,* $\|(v_0 - 1, u_0, s_0 - \overline{s})\|_{H^1(\mathbf{R})}$ *can be large.*

- *When* $\mu(v)$ *satisfies certain growth conditions when* $v \to 0_+$ *and* $v \to +\infty$, *for example,* $\mu(v) \sim v^a$ *as* $v \to 0_+$ *and* $\mu(v) \sim v^b$ *as* $v \to +\infty$ *with* $a < 0, b > -\frac{1}{2}$, *then similar result to Theorem 1.2 also holds even when* $(v_0 - 1, u_0, s_0 - \overline{s})\|_{H^1(\mathbf{R})}$, $\underline{V}$, *and* $\overline{V}$ *depend on* $\frac{1}{\gamma-1}$ *with certain growth condition as* $\gamma \to 1_+$.

- *The same arguments for Theorem 1.1-Theorem 1.3 can be applied directly the compressible Navier-Stokes equations which generalize the previous results [18] and [23] where the viscosity coefficient is assumed to be a positive constant and/or is bounded from below and above by some positive constants, which means that viscosity coefficient* $\mu(v)$ *is non-degenerate.*

- *It is worth to pointing out that since the fact that*

$$\left(\frac{\mu(v)u_x}{v}\right)_x = \left(\frac{\mu(v)v_t}{v}\right)_x = \left(\frac{\mu(v)v_x}{v}\right)_t, \tag{1.14}$$

  *plays an important role in the following analysis, we can only treat the case when* $\mu(v)$ *is a smooth function of* $v$. *Hence, it is interesting to study the case when* $\mu$ *depends on* $\theta$.

We now review some related results. Firstly, recently there are some results on the construction of non-vacuum, large solutions to the one-dimensional compressible Navier-Stokes equations with constant viscosity coefficient $\mu$ and density and temperature dependent heat conductivity coefficient $\kappa$, cf. [18], [23]. A key ingredient in these works is the pointwise *a priori* estimates on the specific volume which guarantees that no vacuum nor concentration of mass occur. It is worth pointing out that it was in deducing the above mentioned upper and lower bounds on the specific volume that the viscosity coefficient $\mu(v)$ is assumed to be non-degenerate, for example $\mu(v)$ is assumed to satisfy $0 < \mu_0 \leq \mu(v) \leq \mu_1$ in [23] and $\mu(v) \equiv \mu_0 > 0$ in [18], .

The strategy to prove Theoerem 1.1 can be stated as follows. We will firstly apply the maximum principle for second order parabolic equation to obtain a lower bound estimate on

$\theta(t,x)$ in terms of the lower bound on $v(t,x)$ in Lemma 2.4. And then by combining the arguments used in [21] and [25], we can deduce a lower bound and an upper bound on $v(t,x)$ in terms of $\|\theta^{1-b}\|_{L^\infty([0,T]\times\mathbf{R})}$, that is, the estimates (2.35) and (2.36). These two estimates together with the $L^\infty([0,T]\times\mathbf{R})-$estimate on $\theta(t,x)$ given in Lemma 4.9 then yield the desired lower and upper bound on $v(t,x)$ and $\theta(t,x)$ provided that the parameters $a$ and $b$ satisfy certain conditions.

To prove Theorem 1.2, the main idea is to assume the following *a priori assumption* on the absolute temperature $\theta(t,x)$

$$\frac{1}{2}\underline{\Theta} \leq \theta(t,x) \leq 2\overline{\Theta}, \tag{1.15}$$

for $(t,x)\in[0,T]\times\mathbf{R}$. Then by some delicate energy type estimates and by using the argument initiated in [21], we can deduce an uniform (with respect to the time variable $t$) lower and upper bound on $v(t,x)$ and some uniform energy estimates on $\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_{H^1(\mathbf{R})}$ in terms of $\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_{H^1(\mathbf{R})}$, $\inf_{x\in\mathbf{R}} v_0(x)$, and $\sup_{x\in\mathbf{R}} v_0(x)$. At the end, to extend the solution globally in time, we only need to close the *a priori assumption* (1.15) where we need the smallness of $\gamma-1$.

For Theorem 1.3, since the viscosity coefficient $\mu$ is assumed to be a positive constant, even though the heat conductivity coefficient $\kappa(v,\theta)$ may depend on $v$ and $\theta$ and there is a nonlocal term $\frac{\Phi_x}{v}$ in the momentum equation, the argument in [25] can be adopted to deduce an explicit formula for the specific volume $v(t,x)$. Based on this formula, we can deduce a lower bound on $v(t,x)$ and from which and the maximum principle for the absolute temperature $\theta(t,x)$, we can deduce a lower bound on $\theta(t,x)$. Having obtained the lower bound estimates on both $v(t,x)$ and $\theta(t,x)$, we can deduce the desired upper bound on $v(t,x)$ by employing the explicit formula for $v(t,x)$ again provided that $\kappa(v,\theta)$ satisfies (1.12). With these estimates in hand, the upper bound on $\theta(t,x)$ can be obtained by following the argument used in the proof of Theorem 1.1 by considering the case $\kappa(v,\theta)$ is uniformly bounded for $0<V_0^{-1}\leq v\leq V_0,\ \theta\geq\Theta_0^{-1}$ (such a bound can depends on $V_0$ and $\Theta_0$) and the case $\overline{\lim_{\theta\to+\infty}}\kappa(v,\theta)=+\infty$ for $0<V_0^{-1}\leq v\leq V_0,\ \theta\geq\Theta_0^{-1}$ respectively.

Before concluding the introduction, we point out that there are many results on the construction of global solutions to the NSP system (1.1). In particular, recently, the global existence of smooth small perturbative solutions away from vacuum with the optimal time decay estimates was obtained in [26] for the isentropic flow, and in [37], [16] for the non-isentropic flow. There, it is observed that the electric field affects the large time behavior of the solution so that the momentum decays at the rate $(1+t)^{-\frac{1}{4}}$ which is slower than the rate $(1+t)^{-\frac{3}{4}}$ for the compressible Navier-Stokes system, while the density tends to its asymptotic state at the rate $(1+t)^{-\frac{3}{4}}$ just like the compressible Navier-Stokes system. Moreover, the global existence of strong solution in Besov type space was obtained in [15]. On the other hand, it is quite different for the compressible Euler-Poisson (EP) system. In fact, it was shown in [14] that the long time convergence rate of global irrotational solution is enhanced by the dispersion effect due to the coupling of electric field, namely, both density and velocity tend to the equilibrium constant state at the rate $(1+t)^{-p}$ for any $p\in(1,\frac{3}{2})$.

Note that even though most of the results for the small perturbative solutions are considered for the case when $\mu,\ \nu$, and $\kappa$ are constants, it is straightforward to show that they hold when $\mu,\nu$, and $\kappa$ are smooth functions of density and temperature.

Finally, for the results with large initial data, the existence of re-normalized solutions to the NSP system are obtained in [6], [33], [38]. Note that for the compressible NSP system related to the dynamics of selfgravitating gases stars, some existence results on the weak solution (renormalized solution) were given in [8], [9], [38]. Since the analysis in these works is based on the weak convergence argument, only isentropic polytropic gas was studied with a special requirement on the range of adiabatic exponent, i.e. $\gamma > \frac{3}{2}$ with constant viscosity coefficient. For the non-isentropic case, even for the compressible Navier-Stokes system, the only available global existence theory for large data is the construction of the so called "variational solution", cf. [11].

The rest of the paper is organized as follows. The proofs of Theorem 1.1 and Theorem 1.2 will be given in Section 2 and Section 3 respectively.

**Notations:** O(1) or $C_i (i \in \mathbf{N})$ stands for a generic positive constant which is independent of $t$ and $x$, while $C_i(\cdot, \cdots, \cdot)$ $(i \in \mathbf{N})$ is used to denote some positive constant depending on the arguments listed in the parenthesis. Note that all these constants may vary from line to line. $\|\cdot\|_s$ represents the norm in $H^s(\mathbf{R})$ with $\|\cdot\| = \|\cdot\|_0$ and for $1 \le p \le +\infty$, $L^p(\mathbf{R})$ denotes the standard Lebesgue space.

# 2 The proof of Theorem 1.1

To prove Theorem 1.1, we first define the following function space for the solution to the Cauchy problem (1.1), (1.3)

$$X(0,T;M_0,M_1;N_0,N_1) = \left\{ \Big(v,u,\theta,\Phi\Big)(t,x) \left| \begin{array}{c} \Big(v-1,u,\theta-1\Big)(t,x) \in C^0\left(0,T;H^1(\mathbf{R})\right) \\ \Big(u_x,\theta_x\Big)(t,x) \in L^2\left(0,T;H^2(\mathbf{R})\right) \\ \Phi(t,x) \in C^0\left(0,T;H^3(\mathbf{R})\right) \\ M_0 \le v(t,x) \le M_1, \quad N_0 \le \theta(t,x) \le N_1 \end{array} \right. \right\}. \tag{2.1}$$

Here $T > 0, M_1 \ge M_0 > 0, N_1 \ge N_0 > 0$ are some positive constants.

Under the assumptions given in either Theorems 1.1 or 1.2, we can get the following local existence result.

**Lemma 2.1 (Local existence)** *Under the assumptions in either Theorems 1.1 or 1.2, there exists a sufficiently small positive constant $t_1$, which depends only on $\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}$ and $\|(v_0 - 1, u_0, \theta_0 - 1)\|_1$, such that the Cauchy problem (1.1), (1.3) admits a unique smooth solution $(v(t,x), u(t,x), \theta(t,x), \ \Phi(t,x)) \in X\left(0, t_1; \frac{1}{2}\underline{V}, 2\overline{V}; \frac{1}{2}\underline{\Theta}, 2\overline{\Theta}\right)$ and $(v(t,x), u(t,x), \theta(t,x), \Phi(t,x))$ satisfies*

$$\begin{cases} 0 < \frac{\underline{V}}{2} \le v(t,x) \le 2\overline{V}, \\ 0 < \frac{\underline{\Theta}}{2} \le \theta(t,x) \le 2\overline{\Theta}, \end{cases} \tag{2.2}$$

$$\sup_{[0,t_1]} \Big( \|(v-1,u,\theta-1,\Phi_x)(t)\|_1 \Big) \le 2\|(v_0-1,u_0,\theta_0-1,\Phi_0)\|_1, \tag{2.3}$$

*and*

$$\lim_{|x|\to\infty} \Big(v(t,x)-1, u(t,x), \theta(t,x)-1, \Phi_x(t,x)\Big) = (0,0,0,0). \tag{2.4}$$

Lemma 2.1 can be proved by the standard iteration argument as in [32] for the one-dimensional compressible Navier-Stokes system, we thus omit the details for brevity.

Now we give some properties on the local solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ constructed above. Noticing that

$$\left(u+\left(\frac{\Phi_x}{v}\right)_t\right)_x=u_x+\left[\left(\frac{\Phi_x}{v}\right)_x\right]_t=u_x+(1-v)_t=u_x-v_t=0,$$

we have the following lemma from (2.4).

**Lemma 2.2** *Under the conditions in Lemma 2.1, we have*

$$u(t,x)=-\left(\frac{\Phi_x(t,x)}{v(t,x)}\right)_t. \tag{2.5}$$

Now we turn to prove Theorem 1.1. Recall that $\mu(v)=v^{-a},\ \kappa(v,\theta)=\theta^b$, and the constitutive equations (1.2), the Cauchy problem (1.1), (1.3) can be rewritten as

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v,\theta)_x=\left(\frac{u_x}{v^{1+a}}\right)_x+\frac{\Phi_x}{v},\\ C_v\theta_t+p(v,\theta)u_x=\frac{u_x^2}{v^{1+a}}+\left(\frac{\theta^b\theta_x}{v}\right)_x,\\ \left(\frac{\Phi_x}{v}\right)_x=1-v, \qquad \lim\limits_{|x|\to+\infty}\Phi(t,x)=0, \end{cases} \tag{2.6}$$

$$(v(0,x),u(0,x),\theta(0,x))=(v_0(x),u_0(x),\theta_0(x)), \quad \lim_{|x|\to+\infty}(v_0(x),u_0(x),\theta_0(x))=(1,0,1). \tag{2.7}$$

Suppose that the local solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ constructed in Lemma 2.1 has been extended to $t=T\geq t_1$ and satisfies the *a priori* assumption

$$(H_1) \qquad \overline{V}_0\leq v(t,x)\leq \overline{V}_1, \quad \overline{\Theta}_0\leq\theta(t,x)\leq\overline{\Theta}_1$$

for all $x\in\mathbf{R},\ 0\leq t\leq T$, and some positive constants $0<\overline{\Theta}_0\leq\overline{\Theta}_1, 0<\overline{V}_0\leq\overline{V}_1$, we now deduce certain *a priori* estimates on $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ which are independent of $\overline{\Theta}_0,\overline{\Theta}_1,\overline{V}_0,\overline{V}_1$ but may depend on $T$.

The first one is concerned with the basic energy estimate. For this, note that

$$\eta(v,u,\theta)=R\phi(v)+\frac{u^2}{2}+\frac{R\phi(\theta)}{\gamma-1}, \quad \text{with} \quad \phi(x)=x-\ln x-1,$$

is a convex entropy to (2.6) which satisfies

$$\eta(v,u,\theta)_t+\left\{\left(\frac{R\theta}{v}-R\right)u\right\}_x-\left\{\frac{uu_x}{v^{1+a}}+\frac{(\theta-1)\theta_x}{v\theta^{1-b}}\right\}_x+\left\{\frac{u_x^2}{v^{1+a}\theta}+\frac{\theta_x^2}{v\theta^{2-b}}\right\}=\frac{u\Phi_x}{v}. \tag{2.8}$$

With (2.8), since

$$\frac{u\Phi_x}{v}=\left(\frac{u\Phi}{v}+\frac{\Phi}{v}\left(\frac{\Phi_x}{v}\right)_t\right)_x-\frac{1}{2}\left[\left(\frac{\Phi_x}{v}\right)^2\right]_t+\frac{\Phi v_x}{v^2}\left[u+\left(\frac{\Phi_x}{v}\right)_t\right],$$

we can deduce the following lemma by integrating (2.8) with respect to $t$ and $x$ over $[0,T]\times\mathbf{R}$ and from (2.5).

**Lemma 2.3 (Basic energy estimates)** *Let the conditions in Lemma 2.1 hold and suppose that the local solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ constructed in Lemma 2.1 satisfies the a priori assumption ($H_1$), then we have for $0\le t\le T$ that*

$$\begin{aligned}&\int_{\mathbf{R}}\left(\eta(v,u,\theta)+\frac{1}{2}\left(\frac{\Phi_x}{v}\right)^2\right)(t,x)dx+\int_0^t\int_{\mathbf{R}}\left(\frac{u_x^2}{v^{1+a}\theta}+\frac{\theta_x^2}{v\theta^{2-b}}\right)(\tau,x)dxd\tau\\ =&\int_{\mathbf{R}}\left(\eta(v_0,u_0,\theta_0)+\frac{1}{2}\left(\frac{\Phi_{0x}}{v_0}\right)^2\right)(x)dx.\end{aligned}\tag{2.9}$$

The next estimate is concerned with a lower bound estimate on $\theta(t,x)$ in terms of the lower bound on $v(t,x)$.

**Lemma 2.4** *Under the assumptions in Lemma 2.3, we have for $a<1$ that*

$$\frac{1}{\theta(t,x)}\le O(1)+O(1)\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a},\quad x\in\mathbf{R},\ 0\le t\le T.\tag{2.10}$$

**Proof:** First of all, $(2.6)_3$ implies

$$\begin{aligned}C_v\left(\frac{1}{\theta}\right)_t&=-\frac{u_x^2}{\theta^2v^{1+a}}+\frac{Ru_x}{v\theta}-\frac{2\theta^{1+b}}{v}\left[\left(\frac{1}{\theta}\right)_x\right]^2+\left[\left(\frac{\theta^b}{v}\right)\left(\frac{1}{\theta}\right)_x\right]_x\\ &=\left[\left(\frac{\theta^b}{v}\right)\left(\frac{1}{\theta}\right)_x\right]_x-\left\{\frac{2\theta^{1+b}}{v}\left[\left(\frac{1}{\theta}\right)_x\right]^2+\frac{1}{v^{1+a}\theta^2}\left(u_x-\frac{R\theta v^a}{2}\right)^2\right\}\\ &\quad+\frac{R^2}{4v^{1-a}}.\end{aligned}\tag{2.11}$$

Set

$$h(t,x)=\frac{1}{\theta}-\frac{R^2t}{4C_v}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a},$$

we can deduce that $h(t,x)$ satisfies

$$\begin{cases}C_vh_t\le\left(\frac{\theta^b}{v}h_x\right)_x, & x\in\mathbf{R},\ 0\le t\le T,\\ h(0,x)=\frac{1}{\theta_0(x)}\le\frac{1}{\underline{\Theta}},\end{cases}\tag{2.12}$$

and the standard maximum principle for parabolic equation implies that $h(t,x)\le\frac{1}{\underline{\Theta}}$ holds for all $(t,x)\in[0,T]\times\mathbf{R}$. That is, for $x\in\mathbf{R},0\le t\le T$

$$\frac{1}{\theta}-\frac{R^2t}{4C_v}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}\le\frac{1}{\underline{\Theta}}.\tag{2.13}$$

This is (2.10) and the proof of Lemma 2.4 is completed.

To use Y. Kanel's method to deduce a lower bound and an upper bound on $v(t,x)$, we need to deduce an estimate on $\left\|\frac{v_x}{v^{1+a}}\right\|$, which is the main concern of our next lemma. It is worth to pointing out that it is in this step that we ask the viscous coefficient $\mu$ depends only on $v$.

**Lemma 2.5** *Under the assumptions in Lemma 2.3, we have*

$$
\begin{aligned}
&\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\theta v_x^2}{v^{3+a}}+g(v)(v-1)\right)dxds\\
&\le\left(\|v_{0x}\|^2+\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\|^2\right)+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds+O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1+a}\theta}dxds,
\end{aligned}
\tag{2.14}
$$

*and*

$$
g(v)=\int_1^v\frac{dz}{z^{1+a}}=\frac{1-v^{-a}}{a}.
$$

**Proof:** Notice that

$$
\left(\frac{v_x}{v^{1+a}}\right)_t=\left(\frac{v_t}{v^{1+a}}\right)_x=\left(\frac{u_x}{v^{1+a}}\right)_x=u_t+p(v,\theta)_x-\frac{\Phi_x}{v},
$$

we have by multiplying the above identity by $\frac{v_x}{v^{1+a}}$ and integrating the resulting equation with respect to $t$ and $x$ over $[0,T]\times\mathbf{R}$ that

$$
\begin{aligned}
&\frac12\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{R\theta v_x^2}{v^{3+a}}dxds\\
&\le O(1)\|v_{0x}\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{R\theta_x v_x}{v^{2+a}}dxds}_{I_1}+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{u_t v_x}{v^{1+a}}dxds}_{I_2}-\underbrace{\int_0^t\int_{\mathbf{R}}\frac{v_x}{v^{1+a}}\frac{\Phi_x}{v}dxds}_{I_3}.
\end{aligned}
\tag{2.15}
$$

Now we estimate $I_1, I_2$ and $I_3$ term by term. First, we have from $(2.6)_4$ and the Cauchy-Schwarz inequality that

$$
I_3=\int_0^t\int_{\mathbf{R}}g(v)_x\left(\frac{\Phi_x}{v}\right)dxds=-\int_0^t\int_{\mathbf{R}}g(v)\left(\frac{\Phi_x}{v}\right)_x dxds=-\int_0^t\int_{\mathbf{R}}g(v)(1-v)dxds\ge 0,
\tag{2.16}
$$

and

$$
I_1\le\frac12\int_0^t\int_{\mathbf{R}}\frac{R\theta v_x^2}{v^{3+a}}dxds+O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1+a}\theta}dxds.
\tag{2.17}
$$

As to $I_2$, we have from (2.9) that

$$
\begin{aligned}
I_2&=\int_{\mathbf{R}}\frac{uv_x}{v^{1+a}}dx\Big|_0^t-\int_0^t\int_{\mathbf{R}}u\left(\frac{v_x}{v^{1+a}}\right)_t dxds\\
&\le\int_{\mathbf{R}}\frac{uv_x}{v^{1+a}}dx+O(1)\left\|(u_0,v_{0x})\right\|^2-\int_0^t\int_{\mathbf{R}}u\left(\frac{u_x}{v^{1+a}}\right)_x dxds\\
&\le\frac12\left\|\frac{v_x}{v^{1+a}}\right\|^2+O(1)\left\|(v_0-1,v_{0x},u_0,\theta_0-1,\Phi_{0x})\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds.
\end{aligned}
\tag{2.18}
$$

Inserting (2.16)-(2.18) into (2.15), we can deduce (2.14) immediately. This completes the proof of Lemma 2.5.

To bound the two terms on the right hand side of (2.14), we now estimate $\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds$ in the following lemma.

**Lemma 2.6** *Under the assumptions in Lemma 2.3, we have*

$$\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds\le O(1)\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\|^2+O(1)\int_0^t\int_{\mathbf{R}}\frac{(\theta-1)^2}{v^{1-a}}dxds. \tag{2.19}$$

**Proof:** Multiplying $(2.6)_2$ by $u$, we have by integrating the resulting equation with respect to $t$ and $x$ over $[0,T]\times\mathbf{R}$ that

$$\begin{aligned}&\frac12\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds\\ &\le O(1)\|u_0\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{R(\theta-1)u_x}{v}dxds}_{I_4}+\underbrace{\int_0^t\int_{\mathbf{R}}R\left(1-\frac1v\right)u_xdxds}_{I_5}+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{u\Phi_x}{v}dxds}_{I_6}\end{aligned} \tag{2.20}$$

From the basic energy estimate (2.9) and the Cauchy-Schwarz inequality, we can bound $I_j$ $(j=4,5,6)$ as follows:

$$I_6\le\int_0^t\|u(s)\|\left\|\left(\frac{\phi_x}{v}\right)(s)\right\|ds\le C(T)\|(u_0,v_0-1,\theta_0-1,\Phi_{0x})\|^2,$$

$$\begin{aligned}I_5&=\int_0^t\int_{\mathbf{R}}R\left(1-\frac1v\right)v_tdxds=R\int_{\mathbf{R}}\phi(v)dx\Big|_0^t\\ &=R\left(\int_{\mathbf{R}}\phi(v)dx-\int_{\mathbf{R}}\phi(v_0)dx\right)\\ &\le O(1)\|(u_0,v_0-1,\theta_0-1,\Phi_{0x})\|^2,\end{aligned}$$

$$I_4\le\frac12\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds+O(1)\int_0^t\int_{\mathbf{R}}\frac{(\theta-1)^2}{v^{1-a}}dxds.$$

Substituting the above estimates into (2.20), we can deduce (2.19) and complete the proof of the lemma.

To bound the terms appearing on the right hand side of (2.19) and (2.14), we need the following

**Lemma 2.7** *Under the assumptions in Lemma 2.3, we have for $b\neq 0,-1$ that*

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^bds\le C(T), \tag{2.21}$$

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^{b+1}ds\le C(T)\left(1+\|\theta\|_{L^\infty_{T,x}}\right), \tag{2.22}$$

*and*

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^{b+1}ds\le C(T)\left(1+\|v\|_{L^\infty_{T,x}}\right). \tag{2.23}$$

**Proof:** We only prove (2.22) because (2.21)and (2.23) can be proved similarly.

From the argument used in [25], we have from the basic energy estimate (2.9), the Jenssen inequality that from each $i\in\mathbf{Z}$, there are positive constants $A_0>0, A_1>0$ such that

$$A_0\le\int_i^{i+1}v(t,x)dx,\ \int_i^{i+1}\theta(t,x)dx\le A_1,\quad\forall t\in[0,T]. \tag{2.24}$$

Hence, there exist $a_i(t) \in [i, i+1]$, $b_i(t) \in [i, i+1]$ such that

$$A_0 \leq v(t, a_i(t)),\ \theta(t, b_i(t)) \leq A_1. \tag{2.25}$$

Define

$$g_1(\theta) = \int_1^\theta s^{\frac{b-1}{2}} ds = \frac{2}{b+1}\left(\theta^{\frac{b+1}{2}} - 1\right),$$

for each $x \in \mathbf{R}$, there exists an integer $i \in \mathbf{Z}$ such that $x \in [i, i+1]$ and we can assume without loss of generality that $x \geq b_i(t)$. Thus

$$\begin{aligned}
g_1(\theta(t,x)) &= g_1(\theta(t, b_i(t))) + \int_{b_i(t)}^x g_1(\theta(t,y))_y dy \\
&\leq O(1) + \int_i^{i+1} \left|\theta^{\frac{b-1}{2}} \theta_x\right| dx \\
&\leq O(1) + \left(\int_{\mathbf{R}} \frac{\theta_x^2}{v\theta^{2-b}} dx\right)^{\frac{1}{2}} \left(\int_i^{i+1} v\theta dx\right)^{\frac{1}{2}} \\
&\leq O(1) + \|\theta\|_{L^\infty_{T,x}}^{\frac{1}{2}} \left(\int_{\mathbf{R}} \frac{\theta_x^2}{v\theta^{2-b}} dx\right)^{\frac{1}{2}}.
\end{aligned}$$

The above estimate and (2.9) give (2.22) and then complete the proof of the lemma.

As a direct corollary of (2.21)-(2.23), we have

**Corollary 2.1** *Under the conditions in Lemma 2.3, we have*

$$\int_0^t \int_{\mathbf{R}} (\theta - 1)^2 dx ds \leq O(1) \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}. \tag{2.26}$$

**Proof:** In fact (2.9) together with (2.21) imply that

$$\begin{aligned}
\int_0^t \int_{\mathbf{R}} (\theta-1)^2 dx d\tau &\leq O(1) \int_0^t \int_{\mathbf{R}} (\theta+1)\phi(\theta) dx d\tau \\
&\leq O(1) \int_0^t \max_{x\in\mathbf{R}} \theta(\tau, x) d\tau + O(1) \\
&= O(1) \int_0^t \max_{x\in\mathbf{R}} \left(\theta^{1-b}\theta^b\right) d\tau + O(1) \\
&\leq O(1) \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}} \int_0^t \max_{x\in\mathbf{R}} \theta^b(\tau, x) d\tau + O(1) \\
&\leq O(1) \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}} + O(1),
\end{aligned}$$

and this completes the proof of corollary.

Having obtained (2.26), we can deduce that

$$\int_0^t \int_{\mathbf{R}} \frac{(\theta-1)^2}{v^{1-a}} dx d\tau \leq \left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a} \int_0^t \int_{\mathbf{R}} (\theta-1)^2 dx d\tau \leq O(1) \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}} \left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}. \tag{2.27}$$

On the other hand, from (2.9), we have

$$
\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{\theta v^{1+a}}dxd\tau &= \int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}\frac{1}{v^a\theta^{b-1}}dxd\tau\\
&\leq \left\|\frac{1}{v}\right\|^a_{L^\infty_{T,x}}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}dxd\tau\\
&\leq O(1)\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\|^2\left\|\frac{1}{v}\right\|^a_{L^\infty_{T,x}}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}. \qquad (2.28)
\end{aligned}
$$

Substituting (2.27) and (2.28) into (2.19) and (2.14), we have

**Corollary 2.2** *Under the assumptions in Lemma 2.3, we have*

$$
\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxd\tau \leq O(1)\left\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\right\|^2+O(1)\left\|\frac{1}{v}\right\|^a_{L^\infty_{T,x}}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}, \qquad (2.29)
$$

$$
\begin{aligned}
&\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\theta v_x^2}{v^{3+a}}+g(v)(v-1)\right)dxd\tau\\
&\leq O(1)\left\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\right\|^2+O(1)\left(\left\|\frac{1}{v}\right\|^a_{L^\infty_{T,x}}+\left\|\frac{1}{v}\right\|^{1-a}_{L^\infty_{T,x}}\right)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}. \qquad (2.30)
\end{aligned}
$$

Now we apply Y. Kanel's approach to deduce a lower bound and an upper bound on $v(t,x)$ in terms of $\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}$. To this end, set

$$
\Psi(v)=\int_1^v\frac{\sqrt{\phi(z)}}{z^{1+a}}dz. \qquad (2.31)
$$

Note that there exist positive constants $A_2, A_3$ such that

$$
|\Psi(v)|\geq A_2\left(v^{-a}+v^{\frac{1}{2}-a}\right)-A_3. \qquad (2.32)
$$

Since

$$
\begin{aligned}
|\Psi(v)| &= \left|\int_{-\infty}^x\Psi(v(t,y))_y dy\right|\\
&\leq \int_{\mathbf{R}}\left|\frac{\sqrt{\phi(v)}}{v^{1+a}}v_x\right|dx\\
&\leq \left\|\sqrt{\phi(v)}\right\|\left\|\frac{v_x}{v^{1+a}}\right\| \qquad (2.33)\\
&\leq O(1)\left(1+\left(\left\|\frac{1}{v}\right\|^{\frac{a}{2}}_{L^\infty_{T,x}}+\left\|\frac{1}{v}\right\|^{\frac{1-a}{2}}_{L^\infty_{T,x}}\right)\left\|\theta^{1-b}\right\|^{\frac{1}{2}}_{L^\infty_{T,x}}\right),
\end{aligned}
$$

we have from (2.32) and (2.33) that

$$
\left\|\frac{1}{v}\right\|^a_{L^\infty_{T,x}}+\|v\|^{\frac{1}{2}-a}_{L^\infty_{T,x}}\leq O(1)\left(1+\left(\left\|\frac{1}{v}\right\|^{\frac{a}{2}}_{L^\infty_{T,x}}+\left\|\frac{1}{v}\right\|^{\frac{1-a}{2}}_{L^\infty_{T,x}}\right)\left\|\theta^{1-b}\right\|^{\frac{1}{2}}_{L^\infty_{T,x}}\right). \qquad (2.34)
$$

Thus if $\frac{1}{3}<a<\frac{1}{2}$, we can deduce from (2.34)

**Corollary 2.3** *Under the conditions in Lemma 2.3, if we assume further that* $\frac{1}{3} < a < \frac{1}{2}$, *then we have*

$$\frac{1}{v(t,x)} \leq O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{1}{3a-1}}\right), \tag{2.35}$$

*and*

$$v(t,x) \leq O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{(3a-1)(1-2a)}}\right) \tag{2.36}$$

*hold for any* $(t,x) \in [0,T] \times \mathbf{R}$.

*Consequently, (2.29) and (2.30) can be rewritten as*

$$\|u(t)\|^2 + \int_0^t \int_{\mathbf{R}} \frac{u_x^2}{v^{1+a}} dx d\tau \leq O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right), \tag{2.37}$$

$$\left\|\frac{v_x}{v^{1+a}}\right\|^2 + \int_0^t \int_{\mathbf{R}} \left(\frac{\theta v_x^2}{v^{3+a}} + g(v)(v-1)\right) dx d\tau \leq O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right). \tag{2.38}$$

To get an upper bound on $\theta(t,x)$, we need also the estimate on $\|u_x(t)\|$ which is given in the following lemma.

**Lemma 2.8** *Under the conditions listed in Lemma 2.3, we have for* $0 \leq t \leq T$, *that*

$$\begin{aligned}
&\|u_x(t)\|^2 + \|v(t)\|^2 + \int_0^t \int_{\mathbf{R}} \frac{u_{xx}^2}{v^{1+a}} dx d\tau \\
\leq\ & O(1)\left\|(v_0 - 1, u_0, \theta_0 - 1, \Phi_{0x})\right\|^2 \\
&+ O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a^2}{(3a-1)(1-2a)}}\right) \\
&+ O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2(2a-2a^2+1)}{(3a-1)(1-2a)}}\right).
\end{aligned} \tag{2.39}$$

**Proof:** By differentiating $(2.6)_2$ with respect to $x$, multiplying the resulting identity by $u_x$, and integrating the result with respect to $t$ and $x$ over $[0,T] \times \mathbf{R}$, we have

$$\begin{aligned}
&\|u_x(t)\|^2 + \int_0^t \int_{\mathbf{R}} \frac{u_{xx}^2}{v^{1+a}} dx d\tau + \|v-1\|^2 \\
\leq\ & O(1)\|u_{0x}\|^2 + \underbrace{2\int_0^t \int_{\mathbf{R}} u_{xx} p(v,\theta)_x dx d\tau}_{I_7} + \underbrace{2(1+a)\int_0^t \int_{\mathbf{R}} \frac{u_x v_x u_{xx}}{v^{2+a}} dx d\tau}_{I_8}.
\end{aligned} \tag{2.40}$$

For $I_7$, we have from (2.9) that

$$\begin{aligned}
I_7 &= 2R\int_0^t \int_{\mathbf{R}} u_{xx}\left(\frac{\theta_x}{v} - \frac{\theta v_x}{v^2}\right) dx d\tau \\
&\leq \frac{1}{4}\int_0^t \int_{\mathbf{R}} \frac{u_{xx}^2}{v^{1+a}} dx d\tau + O(1)\int_0^t \int_{\mathbf{R}} \frac{\theta_x^2}{v^{1-a}} dx d\tau + O(1)\int_0^t \int_{\mathbf{R}} \frac{\theta^2 v_x^2}{v^{3-a}} dx d\tau \\
&\leq \frac{1}{4}\int_0^t \int_{\mathbf{R}} \frac{u_{xx}^2}{v^{1+a}} dx d\tau + O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a^2}{(3a-1)(1-2a)}}\right) \\
&\quad + O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{7a-4a^2-1}{(3a-1)(1-2a)}}\right).
\end{aligned} \tag{2.41}$$

Here we have used the fact that

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1-a}}dxd\tau &= \int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}v^a\theta^{2-b}dxd\tau\\
&\leq O(1)\|v\|^a_{L^\infty_{T,x}}\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}dxd\tau\\
&\leq O(1)\|v\|^a_{L^\infty_{T,x}}\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\\
&\leq O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{\theta^2v_x^2}{v^{3-a}}dxd\tau &= \int_0^t\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}\frac{\theta^2}{v^{1-3a}}dxd\tau\\
&\leq \int_0^t\int_{\mathbf{R}}\left(\max_{x\in\mathbf{R}}\theta^2(s,x)ds\right)\|v\|^{3a-1}_{L^\infty_{T,x}}\left(\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}dx\right)d\tau\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{2a}{3a-1}}_{L^\infty_{T,x}}\right)\|v\|^{3a-1}_{L^\infty_{T,x}}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^2(s,x)ds\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{2a}{3a-1}}_{L^\infty_{T,x}}\right)\|v\|^{3a-1}_{L^\infty_{T,x}}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^{1-b}\theta^{1+b}(s,x)ds\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{5a-1}{3a-1}}_{L^\infty_{T,x}}\right)\|v\|^{3a-1}_{L^\infty_{T,x}}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^{1+b}(s,x)ds\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{5a-1}{3a-1}}_{L^\infty_{T,x}}\right)\|v\|^{3a-1}_{L^\infty_{T,x}}\left(1+\|v\|_{L^\infty_{T,x}}\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{7a-4a^2-1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right),
\end{aligned}$$

where (2.9), (2.21)-(2.23), and (2.38) are used.

As for $I_8$, since (2.36), (2.37) together with the Sobolev inequality imply

$$\begin{aligned}
\int_0^t\|u_x(\tau)\|^2_{L^\infty_x}d\tau &\leq \int_0^t\|u_x(\tau)\|\|u_{xx}(\tau)\|d\tau\\
&\leq\left(\int_0^t\|u_x(\tau)\|^2d\tau\right)^{\frac12}\left(\int_0^t\|u_{xx}(\tau)\|^2d\tau\right)^{\frac12}\\
&\leq\|v\|^{1+a}_{L^\infty_{T,x}}\left(\int_0^t\left\|\frac{u_x}{v^{\frac{1+a}{2}}}(\tau)\right\|^2d\tau\right)^{\frac12}\left(\int_0^t\left\|\frac{u_{xx}}{v^{\frac{1+a}{2}}}(\tau)\right\|^2d\tau\right)^{\frac12}\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{3a}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\left(\int_0^t\left\|\frac{u_{xx}}{v^{\frac{1+a}{2}}}(\tau)\right\|^2d\tau\right)^{\frac12},
\end{aligned}\tag{2.42}$$

we can deduce from (2.35)-(2.38) that

$$I_8\leq\frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\frac{u_x^2v_x^2}{v^{3+a}}dxd\tau$$

$$
\begin{aligned}
&\leq \frac{1}{4}\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau+O(1)\int_0^t\left\|\frac{u_x^2}{v^{1-a}}\right\|_{L_x^\infty}\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}dxd\tau\\
&\leq \frac{1}{4}\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau+O(1)\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right)\int_0^t\|u_x(\tau)\|_{L_x^\infty}^2d\tau\\
&\leq \frac{1}{4}\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau+O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a-2a^2+1}{(3a-1)(1-2a)}}\right)\left(\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau\right)^{\frac{1}{2}}\\
&\leq \frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau+O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2(2a-2a^2+1)}{(3a-1)(1-2a)}}\right).
\end{aligned}\tag{2.43}
$$

Putting (2.40), (2.41), and (2.43) together and noticing that $2(2a-2a^2+1)>7a-4a^2-1$ imply (2.39), and this completes the proof of Lemma 2.8.

Now we turn to deduce the upper bound on $\theta(t,x)$.

**Lemma 2.9** *Under the conditions in Lemma 2.3, we have*

$$
\|\theta\|_{L^\infty_{T,x}}\leq O(1)\left\{1+\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L_x^\infty}+\left\|\frac{u_x^2}{v^2}\right\|_{L_x^\infty}+\|\theta\|_{L_x^\infty}^2\right)d\tau\right\}.\tag{2.44}
$$

**Proof:** From $(2.6)_3$, it is easy to see that for each $p>1$,

$$
\begin{aligned}
&C_v\left[(\theta-1)^{2p}\right]_t+2p(2p-1)(\theta-1)^{2(p-1)}\frac{\theta^b\theta_x^2}{v}\\
=&\left\{\frac{2p(\theta-1)^{2p-1}\theta^b\theta_x}{v}\right\}_x+\frac{2p(\theta-1)^{2p-1}}{v^{1+a}}u_x^2-\frac{2pR\theta}{v}u_x(\theta-1)^{2p-1}.
\end{aligned}\tag{2.45}
$$

Integrating (2.45) with respect to $x$ over $\mathbf{R}$, we have

$$
C_v\left(\|\theta-1\|_{L^{2p}}^{2p}\right)_t\leq \underbrace{2p\int_{\mathbf{R}}\frac{u_x^2(\theta-1)^{2p-1}}{v^{1+a}}dx}_{I_9}-\underbrace{2pR\int_{\mathbf{R}}\frac{\theta u_x(\theta-1)^{2p-1}}{v}dx}_{I_{10}}.\tag{2.46}
$$

Since

$$
\begin{aligned}
I_9&\leq 2pO(1)\|\theta-1\|_{L^{2p}}^{2p-1}\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^{2p}},\\
I_{10}&\leq 2pO(1)\|\theta-1\|_{L^{2p}}^{2p-1}\left\|\frac{\theta u_x}{v}\right\|_{L^{2p}}
\end{aligned}
$$

hold for some positive constant $O(1)$ independent of $p$, we have

$$
\|\theta-1\|_{L^{2p}}\leq O(1)+O(1)\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^{2p}}+\left\|\frac{\theta u_x}{v}\right\|_{L^{2p}}\right)d\tau.\tag{2.47}
$$

Letting $p\to\infty$ in (2.47) and by exploiting the Cauchy inequality, we can deduce (2.44) immediately and the proof of Lemma 2.9 is complete.

We are now ready to use (2.35), (2.36), and (2.44) to deduce a lower bound and an upper bound on $\theta(t,x)$. Firstly, we have from (2.42) and (2.39) that

$$\begin{aligned}
&\int_0^t \|u_x(s)\|^2_{L^\infty_x} ds\\
\leq\ & O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{3a}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\
&\times\left[\left\|\theta^{2-b}\right\|^{\frac12}_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)+1+\left\|\theta^{1-b}\right\|^{\frac{2a-2a^2+1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right]\\
\leq\ & O(1)\left\|\theta^{2-b}\right\|^{\frac12}_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{3a+a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)+O(1)\left\|\theta^{1-b}\right\|^{\frac{5a-2a^2+1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}+O(1).
\end{aligned}\tag{2.48}$$

Thus, we have from (2.35)-(2.36), (2.48) that

$$\begin{aligned}
&\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty_x}+\left\|\frac{u_x^2}{v^2}\right\|_{L^\infty_x}\right)d\tau\\
\leq\ & O(1)\left(\left\|\frac1v\right\|^{1+a}_{L^\infty_{T,x}}+\left\|\frac1v\right\|^{2}_{L^\infty_{T,x}}\right)\int_0^t\|u_x(\tau)\|^2_{L^\infty_x}d\tau\\
\leq\ & O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{2}{3a-1}}_{L^\infty_{T,x}}\right)\int_0^t\|u_x(\tau)\|^2_{L^\infty_x}d\tau\\
\leq\ & O(1)\left\|\theta^{2-b}\right\|^{\frac12}_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{a^2-a+2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)+O(1)\left\|\theta^{1-b}\right\|^{\frac{3+a-2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}+O(1),
\end{aligned}\tag{2.49}$$

and

$$\begin{aligned}
\int_0^t\max_{x\in\mathbf{R}}\theta^2(s,x)ds &\leq \int_0^t\max_{x\in\mathbf{R}}\left(\theta^{1-b}(s,x)\theta^{b+1}(s,x)\right)ds\\
&\leq\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\int_0^t\max_{x\in\mathbf{R}}\theta^{1+b}(s,x)ds\\
&\leq O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\left(1+\|v\|_{L^\infty_{T,x}}\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{7a-6a^2-1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right).
\end{aligned}\tag{2.50}$$

Inserting (2.49) and (2.50) into (2.44) yields

$$\begin{aligned}
\|\theta\|_{L^\infty_{T,x}} \leq\ & O(1)+O(1)\left\|\theta^{2-b}\right\|^{\frac12}_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{a^2-a+2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\
&+O(1)\left\|\theta^{1-b}\right\|^{\frac{3+a-2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}+O(1)\left\|\theta^{1-b}\right\|^{\frac{7a-6a^2-1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\\
\leq\ & O(1)+O(1)\left\|\theta^{2-b}\right\|^{\frac12}_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{a^2-a+2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\
&+O(1)\left\|\theta^{1-b}\right\|^{\frac{3+a-2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}.
\end{aligned}\tag{2.51}$$

Based on the estimate (2.10), (2.35), (2.36) and (2.51), we have

**Corollary 2.4** *Under the assumptions in Lemma 2.3, we further assume that* $\frac{1}{3} < a < \frac{1}{2}$ *and one of the following conditions holds*

*(i).* $1 \leq b < \frac{2a}{1-a} < 2$;

*(ii).* $0 < b < 1, \frac{2-b}{2} + \frac{(a^2-a+2)(1-b)}{(3a-1)(1-2a)} < 1, \frac{(1-b)(3+a-2a^2)}{(3a-1)(1-2a)} < 1.$

*Then there exists positive constants* $V_1 > 0, \Theta_1 > 0$*, such that*

$$\begin{cases} V_1^{-1} \leq v(t,x) \leq V_1, \\ \Theta_1^{-1} \leq \theta(t,x) \leq \Theta_1. \end{cases} \tag{2.52}$$

**Proof:** We first consider the case $b \geq 1$. In this case, we have from (2.10), (2.35), and (2.36) that

$$\left\| \frac{1}{\theta} \right\|_{L^\infty_{T,x}} \leq O(1) + O(1) \left\| \theta^{1-b} \right\|_{L^\infty_{T,x}}^{\frac{1-a}{3a-1}} \leq O(1) + O(1) \left\| \frac{1}{\theta} \right\|_{L^\infty_{T,x}}^{\frac{(1-a)(b-1)}{3a-1}},$$

which, together with the assumption $b < \frac{2a}{1-a}$, implies that there exists a positive constant $\Theta_1 > 0$ such that

$$\theta(t,x) \geq \Theta_1^{-1} > 0, \ \forall (t,x) \in [0,T] \times \mathbf{R}. \tag{2.53}$$

Moreover, (2.35), (2.36), (2.53) together with the fact that $b \geq 1$ imply that there exists a positive constant $V_1 > 0$, which may depends on $T$, such that

$$V_1^{-1} \leq v(t,x) \leq V_1, \quad \forall (t,x) \in [0,T] \times \mathbf{R}. \tag{2.54}$$

Thus to prove (2.52), we only need to deduce the upper bound on $\theta(t,x)$. For this purpose, we have from the fact $1 \leq b < \frac{2a}{1-a} < 2$, (2.53), and (2.51) that

$$\begin{aligned} \|\theta\|_{L^\infty_{T,x}} &\leq O(1) + O(1) \|\theta\|_{L^\infty_{T,x}}^{\frac{2-b}{2}} \left( 1 + \left\| \frac{1}{\theta} \right\|_{L^\infty_{T,x}}^{\frac{(a^2-a+2)(b-1)}{(3a-1)(1-2a)}} \right) + O(1) \left\| \frac{1}{\theta} \right\|_{L^\infty_{T,x}}^{\frac{(3+a-2a^2)(b-1)}{(3a-1)(1-2a)}} \\ &\leq O(1) \left( 1 + \|\theta\|_{L^\infty_{T,x}}^{\frac{2-b}{2}} \right). \end{aligned} \tag{2.55}$$

From (2.55) and the fact that $0 < \frac{2-b}{2} < 1$, one can easily deduce an upper bound on $\theta(t,x)$. This completes the proof of (2.52) for the case $1 \leq b < \frac{2a}{1-a}$.

When $b < 1$, we have from (2.51) that

$$\|\theta\|_{L^\infty_{T,x}} \leq O(1) + O(1) \|\theta\|_{L^\infty_{T,x}}^{\frac{2-b}{2} + \frac{(a^2-a+2)(1-b)}{(3a-1)(1-2a)}} + O(1) \|\theta\|_{L^\infty_{T,x}}^{\frac{(3+a-2a^2)(1-b)}{(3a-1)(1-2a)}}. \tag{2.56}$$

From (2.56) and the assumption (ii) of Corollary 2.4, we can deduce an upper bound on $\theta(t,x)$. With this, the lower and upper bound on $v(t,x)$ can be deduced from (2.35) and (2.36). And then (2.10) implies the lower bound on $\theta(t.x)$. This completes the proof of the corollary.

With Corollary 2.4, Theorem 1.1 follows from the standard continuation argument.

# 3 The proof of Theorem 1.2

First of all, the local solvability of the Cauchy problem (1.1),(1.3) in the function space $X\left(0,t_1;\frac{1}{2}\underline{V},2\overline{V};\frac{1}{2}\underline{\Theta},2\overline{\Theta}\right)$ with $t_1$ depending on $\underline{V},\overline{V},\underline{\Theta},\overline{\Theta},\|(v_0-1,v_0,\theta_0-1,\Phi_{0x})\|_1$ can be proved as in Lemma 3.1. Suppose this solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ is extended to $t=T\geq t_1$. To apply the continuation argument for global existence, we first set the following *a priori estimate*:

$$(H_2)\qquad \frac{1}{2}\underline{\Theta}\leq\theta(t,x)\leq 2\overline{\Theta},\quad (t,x)\in[0,T]\times\mathbf{R}.$$

Here without loss of generality, we can assume that $0<\underline{\Theta}<1<\overline{\Theta}$.

Note that the smallness of $\gamma-1$ is needed to close the *a priori estimate*, the generic constants used later are independent of $\gamma-1$ and whenever the dependence on this factor will be clearly stated in the estimates.

Similar to Lemma 2.3 we have the following basic energy estimate.

**Lemma 3.1** *Under the conditions in Theorem 1.2, we have for $0\leq t\leq T$ that*

$$\begin{aligned}&\int_{\mathbf{R}}\left\{R\phi(v)+\frac{u^2}{2}+\frac{R}{\gamma-1}\phi(\theta)+\frac{\Phi_x^2}{2v^2}\right\}(t,x)dx+\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(v)u_x^2}{v\theta}+\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\right)dxd\tau\\&\quad=\int_{\mathbf{R}}\left(R\phi(v_0)+\frac{u_0^2}{2}+\frac{R}{\gamma-1}\phi(\theta_0)+\frac{\Phi_{0x}^2}{2v_0^2}\right)(x)dx.\end{aligned}\tag{3.1}$$

*Here, as in Section 2, $\phi(x)=x-\ln x-1$.*

Now, we turn to deduce an estimate on $\left\|\frac{\mu(v)v_x}{v}\right\|$. For this, similar to Lemma 2.5, we can deduce

$$\begin{aligned}&\left\|\frac{\mu(v)v_x}{v}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)\theta v_x^2}{v^3}dxd\tau+\int_0^t\int_{\mathbf{R}}g(v)(1-v)dxd\tau\\&\leq O(1)\|v_{0x}\|^2+O(1)\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\mu(v)u_x^2}{v}dxd\tau}_{J_1}+O(1)\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\mu(v)\theta_x^2}{v\theta}dxd\tau}_{J_2}.\end{aligned}\tag{3.2}$$

If the *a priori estimate* ($\mathrm{H}_2$) holds, we have from (3.1) and the assumptions imposed on $\kappa(v,\theta)$ in Theorem 1.2 that

$$J_1\leq O(1)\int_0^t\int_{\mathbf{R}}\frac{\mu(v)u_x^2}{v\theta}dxd\tau\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|^2,\tag{3.3}$$

and

$$\begin{aligned}J_2&\leq\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\frac{\theta\mu(v)}{\kappa(v,\theta)}dxd\tau\\&\leq O(1)\left\|\frac{\mu(v)}{\kappa_1(v)}\right\|_{L^\infty}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dxd\tau\\&\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|^2\left\|\frac{\mu(v)}{\kappa_1(v)}\right\|_{L^\infty}.\end{aligned}\tag{3.4}$$

Putting (3.2), (3.3) and (3.4) together, we obtain

**Lemma 3.2** *Under the assumptions in Lemma 3.1 and the a priori assumption ($H_2$), we have*

$$\left\|\frac{\mu(v)v_x}{v}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)v_x^2}{v^3}dxd\tau\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|^2\left(1+\left\|\frac{\mu(v)}{\kappa_1(v)}\right\|_{L^\infty_{T,x}}\right). \tag{3.5}$$

Having obtained (3.1) and (3.5), we can use Y. Kanel's argument, cf. [21], to deduce the lower and upper bounds on $v(t,x)$ as follows.

**Lemma 3.3** *Under the assumptions in Theorem 1.2 and Lemma 3.2, there exists a positive constant $V_2\geq 1$, which depends only on $\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|$, $\underline{V},\overline{V},\underline{\Theta}$, and $\overline{\Theta}$, but is independent of $T$, such that*

$$V_2^{-1}\leq v(t,x)\leq V_2,\quad (t,x)\in[0,T]\times\mathbf{R}. \tag{3.6}$$

**Proof:** Define

$$\Psi(v)=\int_1^v\frac{\sqrt{\phi(z)}}{z}\mu(z)dz,\quad \phi(z)=z-\ln z-1,$$

and notice that

$$\begin{aligned}|\Psi(v)|&=\left|\int_{-\infty}^x\Psi(v(t,y))_ydy\right|\\ &\leq\int_{\mathbf{R}}\left|\sqrt{\phi(v)}\frac{\mu(v)v_x}{v}\right|dx\\ &\leq\|\phi(v)\|_{L^1}^{\frac12}\left\|\frac{\mu(v)v_x}{v}\right\|\\ &\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|^2\left(1+\left\|\frac{\mu(v)}{\kappa_1(v)}\right\|_{L^\infty_{T,x}}\right)^{\frac12}.\end{aligned}$$

It is straightforward to deduce (3.6) from the assumptions in Theorem 1.2. This completes the proof of the lemma.

The next lemma is about the estimate on $\|u_x(t)\|$.

**Lemma 3.4** *Under the assumptions in Lemma 3.3, we have for each $0\leq t\leq T$ that*

$$\|u_x(t)\|^2+\|v(t)-1\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau\leq O(1)\left\|\left(v_0-1,v_{0x},u_x,u_{0x},\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|^6. \tag{3.7}$$

Since $v(t,x)$ satisfies (3.6) and $\theta(t,x)$ is assumed to satisfy the *a priori estimate* ($H_2$), (3.7) can be proved by applying the argument used in the proof of Lemma 2.8. Thus, we omit the detail for brevity.

To close the *a priori estimate* ($H_2$), we need to deduce an estimate on $\|\theta_x(t)\|$. For the case when $\kappa(v,\theta)\equiv\kappa(\theta)$, we have

**Lemma 3.5** *Under the assumptions in Theorem 1.2 and Lemma 3.3, we have*

$$\int_{\mathbf{R}}\frac{|K(\theta)_x|^2}{\gamma-1}dx+\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10}. \tag{3.8}$$

*Here*

$$K(\theta)=\int_1^\theta\kappa(z)dz. \tag{3.9}$$

**Proof:** Multiplying $(1.1)_3$ by $\kappa(\theta)$ and differentiating the resulting equation with respect to $x$, we get

$$C_v K(\theta)_{tx} + (\kappa(\theta)p(v,\theta)u_x)_x = \left(\frac{\kappa(\theta)\mu(v)u_x^2}{v}\right)_x + \left[\kappa(\theta)\left(\frac{K(\theta)_x}{v}\right)_x\right]_x. \tag{3.10}$$

Multiplying (3.10) by $K(\theta)_x$ and integrating with respect to $t$ and $x$ over $[0,t]\times\mathbf{R}$ give

$$\begin{aligned}
&\int_{\mathbf{R}} \frac{C_v}{2}|K(\theta)_x|^2\,dx + \int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau \\
&\le O(1)\left\|\frac{\theta_{0x}}{\sqrt{\gamma-1}}\right\|^2 + O(1)\underbrace{\int_0^t\int_{\mathbf{R}}\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|\left|\frac{K(\theta)_x}{v}v_x\right| dxd\tau}_{J_3} \\
&\quad + \underbrace{\int_0^t\int_{\mathbf{R}} K(\theta)_x\left(\frac{\kappa(\theta)\mu(v)u_x^2}{v}\right)_x dxd\tau}_{J_4} - \underbrace{\int_0^t\int_{\mathbf{R}} K(\theta)_x\left(\kappa(\theta)p(v,\theta)u_x\right)_x dxd\tau}_{J_5}.
\end{aligned} \tag{3.11}$$

Notice that

$$\|\theta_x\|_{L_x^\infty} \le O(1)\left\|\frac{K(\theta)_x}{v}\right\|_{L_x^\infty} \le O(1)\left\|\frac{K(\theta)_x}{v}\right\|^{\frac12}\left\|\left(\frac{K(\theta)_x}{v}\right)_x\right\|^{\frac12} \le O(1)\|\theta_x\|^{\frac12}\left\|\left(\frac{K(\theta)_x}{v}\right)_x\right\|^{\frac12}, \tag{3.12}$$

we have from (3.1), (3.5), (3.6), and the *a priori estimate* ($\mathrm{H}_2$) that

$$\begin{aligned}
J_3 &\le \frac{1}{12}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\int_0^t\int_{\mathbf{R}} v_x^2\theta_x^2 dxd\tau \\
&\le \frac{1}{12}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\int_0^t \|v_x(\tau)\|^2\|\theta_x(\tau)\|\left\|\sqrt{v\kappa(\theta)}\left(\frac{K(\theta)_x}{v}\right)_x\right\| d\tau \\
&\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\int_0^t \|v_x(\tau)\|^4\|\theta_x(\tau)\|^2 d\tau \\
&\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^6,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
J_4 &= -\int_0^t\int_{\mathbf{R}} \kappa(\theta)\mu(v)u_x^2\left(\frac{K(\theta)_x}{v}\right)_x dxd\tau - \int_0^t\int_{\mathbf{R}} \frac{K(\theta)_x v_x\kappa(\theta)\mu(v)u_x^2}{v^2} dxd\tau \\
&\le \frac{1}{12}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\int_0^t\int_{\mathbf{R}} u_x^4 dxd\tau + O(1)\int_0^t\int_{\mathbf{R}} \theta_x^2 v_x^2 dxd\tau \\
&\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^{10},
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
J_5 &= \int_0^t\int_{\mathbf{R}} \left(\frac{K(\theta)_x}{v}\right)_x v\kappa(\theta)p(v,\theta)u_x dxd\tau + \int_0^t\int_{\mathbf{R}} \frac{K(\theta)_x}{v} v_x\kappa(\theta)p(v,\theta)u_x dxd\tau \\
&\le \frac{1}{12}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\int_0^t\int_{\mathbf{R}} \left(u_x^2 + \theta_x^2 v_x^2\right) dxd\tau \\
&\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\Phi_{0x}\right)\right\|_1^6.
\end{aligned} \tag{3.15}$$

Here we have used the fact that

$$\begin{aligned}\int_0^t\int_{\mathbf{R}} u_x^4 dxd\tau &\leq O(1)\int_0^t \|u_x(\tau)\|^2\|u_x(\tau)\|^2_{L^\infty_x}d\tau\\ &\leq O(1)\int_0^t \|u_x(\tau)\|^3\|u_{xx}(\tau)\|d\tau\\ &\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\Phi_{0x}\right)\right\|_1^{10}. \end{aligned}\tag{3.16}$$

Inserting (3.13)-(3.15) into (3.11), we deduce (3.8) and complete the proof of the lemma.

Now we turn to the case when $\kappa(v,\theta)$ depends on both $v$ and $\theta$. For this , we have

**Lemma 3.6** *Under the assumptions in Lemma 3.5, if $\kappa_{\theta\theta}(v,\theta)<0$ holds for $v>0,\theta>0$, then we have*

$$\begin{aligned}&\left\|\frac{\theta_x(t)}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau-\int_0^t\int_{\mathbf{R}}\frac{\kappa_{\theta\theta}(v,\theta)}{v}\theta_x^4dxd\tau\\ &\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^6.\end{aligned}\tag{3.17}$$

**Proof:** Differentiating $(1.1)_3$ with respect to $x$ and multiplying the resulting equation by $\theta_x$, we have by integrating it over $[0,t]\times\mathbf{R}$ that

$$\begin{aligned}&\frac{C_v}{2}\|\theta_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau\\ =&\frac{C_v}{2}\|\theta_{0x}\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\theta_x\left(\frac{\mu(v)}{v}u_x^2\right)_x dxd\tau}_{J_6}-\underbrace{\int_0^t\int_{\mathbf{R}}\theta_x\left(\frac{\kappa(v,\theta)}{v}\right)_x\theta_{xx}dxd\tau}_{J_7}\\ &+\underbrace{\int_0^t\int_{\mathbf{R}}\theta_{xx}p(v,\theta)u_xdxd\tau}_{J_8}.\end{aligned}\tag{3.18}$$

For $J_6, J_7$ and $J_8$, we have from Lemma 3.1-Lemma 3.4 and the *a priori estimate* $(\mathrm{H}_2)$ that

$$\begin{aligned}J_6&=-\int_0^t\int_{\mathbf{R}}\frac{\mu(v)}{v}u_x^2\theta_{xx}dxd\tau\\ &\leq\frac16\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}u_x^4dxd\tau\\ &\leq\frac16\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\|u_x(\tau)\|^3\|u_{xx}(\tau)\|dxd\tau\\ &\leq\frac16\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau+O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10},\end{aligned}\tag{3.19}$$

$$\begin{aligned}J_8&\leq\frac16\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}u_x^2dxd\tau\\ &\leq\frac16\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau+O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^2,\end{aligned}\tag{3.20}$$

and

$$
\begin{aligned}
J_7 =& -\int_0^t\int_{\mathbf{R}}\theta_x^2\left(\frac{\kappa(v,\theta)}{v}\right)_\theta\theta_{xx}dxd\tau-\int_0^t\int_{\mathbf{R}}\theta_xv_x\left(\frac{\kappa(v,\theta)}{v}\right)_v\theta_{xx}dxd\tau\\
=& \frac13\int_0^t\int_{\mathbf{R}}\theta_x^4\frac{\kappa_{\theta\theta}(v,\theta)}{v}dxd\tau+\frac13\int_0^t\int_{\mathbf{R}}\theta_x^3v_x\left(\frac{\kappa_\theta(v,\theta)}{v}\right)_vdxd\tau-\int_0^t\int_{\mathbf{R}}\theta_xv_x\left(\frac{\kappa(v,\theta)}{v}\right)_v\theta_{xx}dxd\tau\\
\leq& \frac16\int_0^t\int_{\mathbf{R}}\theta_x^4\frac{\kappa_{\theta\theta}(v,\theta)}{v}dxd\tau+\frac1{12}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\theta_x^2v_x^2dxd\tau\\
\leq& \frac16\int_0^t\int_{\mathbf{R}}\theta_x^4\frac{\kappa_{\theta\theta}(v,\theta)}{v}dxd\tau+\frac16\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau\\
&+O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^6. \qquad (3.21)
\end{aligned}
$$

Here we have used the fact that

$$
\begin{aligned}
&\int_0^t\int_{\mathbf{R}}\theta_x^2v_x^2dxd\tau\\
\leq& \int_0^t\|\theta_x(\tau)\|_{L_x^\infty}^2\|v_x(\tau)\|^2d\tau\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^2\int_0^t\|\theta_x(\tau)\|_{L_x^\infty}^2d\tau\\
\leq& O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^2\int_0^t\|\theta_x(\tau)\|\|\theta_{xx}(\tau)\|d\tau\\
\leq& \frac1{12}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau+O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^4\int_0^t\int_{\mathbf{R}}\theta_x^2dxd\tau.
\end{aligned}
$$

Inserting (3.19)-(3.21) into (3.18), we obtain

$$
\begin{aligned}
&\frac{C_v}{2}\|\theta_x(t)\|^2+\frac12\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau-\frac16\int_0^t\int_{\mathbf{R}}\frac{\kappa_{\theta\theta}(v,\theta)}{v}\theta_x^4dxd\tau \qquad (3.22)\\
&\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10}.
\end{aligned}
$$

This is (3.17) and the proof of Lemma 3.6 is completed.

Lemma 3.1-Lemma 3.6 imply that under the *a priori estimate* ($\mathrm{H}_2$), there exist two positive constants $V_2\geq1$ and $C_1\geq1$ with $V_2$ depending only on $\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|$, $\underline{V},\overline{V},\underline{\Theta}$, and $\overline{\Theta}$ but independent of $T$ and $\gamma-1$, and $C_1$ depending only on $V_2$ but independent of $T>0$ and $\gamma-1$, such that

$$
\begin{aligned}
&V_2^{-1}\leq v(t,x)\leq V_2,\quad (t,x)\in[0,T]\times\mathbf{R},\\
&\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}},\Phi_x\right)(t)\right\|^2+\int_0^t\int_{\mathbf{R}}\left(u_x^2+\theta_x^2\right)(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|^2,\\
&\|v_x(t)\|^2+\int_0^t\int_{\mathbf{R}}v_x^2(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^2,\\
&\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}u_{xx}^2(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^6,\\
&\left\|\frac{\theta_x(\tau)}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\theta_{xx}^2(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10} \qquad (3.23)
\end{aligned}
$$

hold for $0 \le t \le T$.

To obtain the global existence of solutions, we only need to close the *a priori estimate* ($\mathrm{H}_2$). For this, we need the smallness of $\gamma - 1 > 0$. In fact, we have from $(3.23)_2$, $(3.23)_5$ and Sobolev's inequality that

$$\|\theta(t) - 1\|_{L^\infty_{T,x}} \le \|\theta(t) - 1\|^{\frac{1}{2}}\|\theta_x(t)\|^{\frac{1}{2}} \le C_1(\gamma - 1)^{\frac{1}{2}} \left\|\left(v_0 - 1, u_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}}, \Phi_{0x}\right)\right\|_1^4. \tag{3.24}$$

On the other hand, since $\theta = \frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right)$, if we set $\overline{s} = \frac{R}{\gamma-1}\ln\frac{R}{A}$, we have

$$\begin{aligned}
\theta - 1 &= \frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right) - 1\\
&= \frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right) - \frac{A}{R}\exp\left(\frac{\gamma-1}{R}\overline{s}\right)\\
&= \frac{A}{R}\left(v^{1-\gamma} - 1\right)\exp\left(\frac{\gamma-1}{R}s\right) + \frac{A}{R}\left(\exp\left(\frac{\gamma-1}{R}s\right) - \exp\left(\frac{\gamma-1}{R}\overline{s}\right)\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\theta_0 - 1\| &\le O(1)\frac{A(\gamma-1)}{R}\exp\left(\frac{\gamma-1}{R}\|s_0\|_{L^\infty_x}\right)\left[\left\|v_0^{-\gamma}\right\|_{L^\infty_x}\|v_0 - 1\| + \frac{1}{R}\|s_0(x) - \overline{s}\|\right], \\
\|\theta_{0x}\| &\le O(1)\frac{A(\gamma-1)}{R}\exp\left(\frac{\gamma-1}{R}\|s_0\|_{L^\infty_x}\right)\left[(\inf v_0(x))^{-\gamma}\|v_{0x}\| + \frac{1}{R}\left(\inf_x v_0(x)\right)^{1-\gamma}\|s_{0x}\|\right].
\end{aligned} \tag{3.25}$$

Since $\|v_0(x)\|_{L^\infty_x}, \inf v_0(x), \frac{\gamma-1}{A}\|s_0(x)\|_{L^\infty_x}$ are assumed to be independent of $\gamma - 1$, we have from (3.24) and (3.25) that

$$\|\theta(t) - 1\|_{L^\infty_x} \le C_2(\gamma-1)^{\frac{1}{2}}\|(v_0 - 1, u_0, \Phi_{0x})\|_1^3 + C_3(\gamma-1)^2\|(v_0 - 1, s_0 - \overline{s})\|_1^3 \tag{3.26}$$

holds for $0 \le t \le T$.

Thus if $\gamma - 1 > 0$ is chosen to be sufficiently small such that

$$C_2(\gamma-1)^{\frac{1}{2}}\|(v_0 - 1, u_0, \Phi_{0x})\|_1^3 + C_3(\gamma-1)^2\|(v_0 - 1, u_0, s_0 - \overline{s})\|_1^3 \le \min\left\{\overline{\Theta} - 1, 1 - \underline{\Theta}\right\}, \tag{3.27}$$

we have from (3.26) and (3.27) that for any $0 \le t \le T,\ x \in \mathbf{R}$,

$$\theta(t,x) \le \|\theta(t,x) - 1\|_{L^\infty_{T,x}} + 1 \le 1 + \min\left\{\overline{\Theta} - 1, 1 - \underline{\Theta}\right\} \le \overline{\Theta}, \tag{3.28}$$

and

$$\theta(t,x) \ge 1 - \|\theta(t,x) - 1\|_{L^\infty_{T,x}} \ge 1 - \min\left\{\overline{\Theta} - 1, 1 - \underline{\Theta}\right\} \ge 1 - (1 - \underline{\Theta}) = \underline{\Theta}. \tag{3.29}$$

That is

$$\underline{\Theta} \le \theta(t,x) \le \overline{\Theta}, \quad x \in \mathbf{R}, \quad 0 \le t \le T. \tag{3.30}$$

This closes the *a priori estimate* ($\mathrm{H}_2$) and then Theorem 1.2 follows from the standard continuation argument.

# 4 The proof of Theorem 1.3

When $\mu$ is a positive constant, the Cauchy Problem (1.1), (1.3) can be rewritten as

$$\begin{cases} v_t - u_x = 0, \\ u_t + \left(\frac{R\theta}{v}\right)_x = \mu\left(\frac{u_x}{v}\right)_x + \frac{\Phi_x}{v}, \\ C_v\theta_t + \frac{R\theta}{v}u_x = \frac{\mu u_x^2}{v} + \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x, \\ \left(\frac{\Phi_x}{v}\right)_x = 1 - v, \quad \lim_{|x|\to+\infty}\Phi(t,x) = 0 \end{cases} \tag{4.1}$$

with prescribed initial data

$$(v(0,x),u(0,x),\theta(0,x)) = (v_0(x),u_0(x),\theta_0(x)), \quad \lim_{x\to\pm\infty}(v_0(x),u_0(x),\theta_0(x)) = (1,0,1). \tag{4.2}$$

Let $(v(t,x),u(t,x),\theta(t,x)) \in X(0,T;M_0,M_1;N_0,N_1)$ be a solution of the Cauchy Problem (1.1), (1.3) which is defined in the time strip $[0,T]$ for some $T>0$, to extend such a solution globally, as pointed out in the proofs of Theorems 1.1 and 1.2, we only need to deduce positive lower and upper bounds on $v(t,x)$ and $\theta(t,x)$ which are independent of $M_0, M_1, N_0$ and $N_1$ but may depend on $T$.

First, similar to the proof of Theorem 1.2, we have the following basic energy estimate

**Lemma 4.1** *Under the conditions in Theorem 1.3, we have for $0\le t\le T$ that*

$$\begin{aligned} &\int_{\mathbf{R}}\left\{R\phi(v) + \frac{u^2}{2} + \frac{R}{\gamma-1}\phi(\theta) + \frac{\Phi_x^2}{2v^2}\right\}(t,x)dx + \int_0^t\int_{\mathbf{R}}\left(\frac{\mu u_x^2}{v\theta} + \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\right)dxd\tau \\ &= \int_{\mathbf{R}}\left(R\phi(v_0) + \frac{u_0^2}{2} + \frac{R}{\gamma-1}\phi(\theta_0) + \frac{\Phi_{0x}^2}{2v_0^2}\right)(x)dx. \end{aligned} \tag{4.3}$$

*Here, as in Section 2, $\phi(x) = x - \ln x - 1$.*

Based on the estimate (4.3), we now turn to deduce the desired lower and upper bounds on $v(t,x)$ and $\theta(t,x)$. To this end, for each $x\in\mathbf{R}$, we can find some $i\in\mathbf{Z}$ such that $x\in[i,i+1]$. Recall $a_i(t), b_i(t)$ defined in (2.24) and (2.25) and notice that in Theorem 1.3, since $\mu$ is a positive constant, we have by employing the argument developed in [25] that

$$v(t,x) = \frac{1 + \frac{R}{\mu}\int_0^t \theta(\tau,x)B_i(\tau,x)Y_i(\tau)A_i(\tau,x)d\tau}{B_i(t,x)Y_i(t)A_i(t,x)}. \tag{4.4}$$

Here

$$A_i(t,x) = \exp\left(-\int_0^t\int_x^{a_i(t)}\left(\frac{\Phi_x}{v}\right)(\tau,y)dyd\tau\right), \tag{4.5}$$

$$B_i(t,x) = \frac{v_0(a_i(t))}{v_0(x)v(t,a_i(t))}\exp\left(\frac{1}{\mu}\int_x^{a_i(t)}(u(t,y)-u_0(y))dy\right), \tag{4.6}$$

$$Y_i(t) = \exp\left(\frac{R}{\mu}\int_0^t\left(\frac{\theta}{v}\right)(\tau,a_i(t))d\tau\right). \tag{4.7}$$

(2.5) implies that

$$\left(\frac{\Phi_x}{v}\right)(t,x) = -\int_0^t u(\tau,x)dx + \left(\frac{\Phi_x}{v}\right)(0,x), \tag{4.8}$$

we have from (4.8) and (4.3) that there exist positive constants $\underline{A}$, $\overline{A}$, $\underline{B}$, $\overline{B}$ such that

$$0 < \underline{A} \leq A_i(t,x) \leq \overline{A}, \quad 0 < \underline{B} \leq B_i(t,x) \leq \overline{B}, \quad Y_i(t) \geq 1. \tag{4.9}$$

On the other hand, notice that (4.4) can be rewritten as

$$v(t,x)Y_i(t) = \frac{1 + \frac{R}{\mu}\displaystyle\int_0^t \theta(\tau,x)B_i(\tau,x)Y_i(\tau)A_i(\tau,x)d\tau}{B_i(t,x)A_i(t,x)}. \tag{4.10}$$

Integrating (4.10) with respect to $x$ over $[i,i+1]$, we can get from (2.24) and (4.9) that

$$\begin{aligned} Y_i(t) &\leq O(1)\left(1 + \int_0^t \left(\int_i^{i+1} \theta(\tau,y)dy\right) Y_i(\tau)d\tau\right) \\ &\leq O(1)\left(1 + \int_0^t Y_i(\tau)d\tau\right). \end{aligned} \tag{4.11}$$

From which and the Gronwall inequality, we can deduce that there exists a positive constant $\overline{Y}$ which is independent of $M_0, M_1, N_0, N_1$ and $i$ but may depend on $T$ such that

$$1 \leq Y_i(t) \leq \overline{Y}, \quad \forall i \in \mathbf{Z}, \quad \forall t \in [0,T]. \tag{4.12}$$

From (4.4), (4.9), and (4.12), one easily deduce that there exists a positive constant $V_3 > 0$ which is independent of $M_0, M_1, N_0, N_1$ but may depend on $T$ such that

$$v(t,x) \geq V_3^{-1} > 0, \quad \forall (t,x) \in [0,T] \times \mathbf{R}. \tag{4.13}$$

(4.13) together with Lemma 2.4 imply that there exists a positive constant $\Theta_3^{-1} > 0$ which is independent of $M_0, M_1, N_0, N_1$ but may depend on $T$ such that

$$\theta(t,x) \geq \Theta_3^{-1} > 0, \quad \forall (t,x) \in [0,T] \times \mathbf{R}. \tag{4.14}$$

Now we turn to deduce an upper bound for $v(t,x)$. For this purpose, notice from (2.24), (2.25), (4.9)-(4.14), (4.12) and the fact

$$\min_{v \geq V_3^{-1},\ \theta \geq \Theta_3^{-1}} \kappa(v,\theta) \geq \underline{\kappa}\left(V_3^{-1}, \Theta_3^{-1}\right) > 0$$

that

$$\begin{aligned} \theta(t,x) &\leq 2\left|\sqrt{\theta(t,x)} - \sqrt{\theta(t,b_i(t))}\right|^2 + 4\theta(t,b_i(t)) \\ &= \left|\int_x^{b_i(t)} \left(\frac{|\theta_x|}{\sqrt{\theta}}\right)(t,y)dy\right|^2 + 4\theta(t,b_i(t)) \\ &\leq O(1)\left(1 + \int_i^{i+1} \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dx \cdot \int_i^{i+1} \frac{v\theta}{\kappa(v,\theta)}dx\right) \\ &\leq O(1)\left(1 + \|v(t)\|_{L_x^\infty} \int_i^{i+1} \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dx\right). \end{aligned} \tag{4.15}$$

(4.4) together with (4.3), (2.24), (2.25), (4.9)-(4.14), (4.12), and (4.15) imply

$$\begin{aligned}\|v(t)\|_{L^\infty_x} &\leq O(1)\left(1+\int_0^t \max_{x\in\mathbf{R}}\theta(\tau,x)d\tau\right)\\ &\leq O(1)\left(1+\int_0^t \|v(\tau)\|_{L^\infty_x}\int_{\mathbf{R}}\left(\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\right)(\tau,x)dxd\tau\right).\end{aligned}$$

Based on the above inequality, we can conclude from (4.3), (4.13), (4.14) and the Gronwall inequality that

**Lemma 4.2** *Under the conditions listed in Lemma 4.1, there exist positive constants $V_3$ and $\Theta_3$ such that*

$$0 < V_3^{-1} \leq v(t,x) \leq V_3, \quad \theta(t,x) \geq \Theta_3^{-1} > 0 \tag{4.16}$$

*hold for $(t,x)\in[0,T]\times\mathbf{R}$. Here the constants $V_3$ and $\Theta_3$ are independent of $M_0, M_1, N_0, N_1$ but may depend on $T$.*

To complete the proof of Theorem 1.3, we only need to deduce an upper bound on $\theta(t,x)$. To do so, as a direct consequence of (4.15), (4.16), we can deduce from (4.3) and the fact

$$(\theta-1)^2 \leq O(1)(1+|\theta-1|)\phi(\theta)$$

that

**Corollary 4.1** *Under the conditions listed in Lemma 4.1, we have*

$$\int_0^t \max_{x\in\mathbf{R}}\theta(\tau,x)d\tau \leq O(1) \tag{4.17}$$

*and*

$$\int_0^t\int_{\mathbf{R}}(\theta(\tau,x)-1)^2 dxd\tau \leq O(1). \tag{4.18}$$

*Here and throughout this section, $O(1)$ is used to denote some positive constant independent of $M_0, M_1, N_0, N_1$ but may depend on $T$.*

With (4.17) and (4.18) in hand, we have from (4.16) and the proof of Lemma 2.6 that

**Lemma 4.3** *Under the conditions listed in Lemma 4.1, we have*

$$\|u(t)\|^2 + \int_0^t \|u_x(s)\|^2 ds \leq O(1). \tag{4.19}$$

As to the estimate on $\|v_x(t)\|$, if we set

$$\kappa_1(\theta) = \min_{V_3^{-1}\leq v\leq V_3}\kappa(v,\theta), \tag{4.20}$$

we have from the proof of Lemma 2.5 and the fact

$$\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{\theta}dxd\tau \leq \left\|\frac{\theta}{\kappa_1(\theta)}\right\|_{L^\infty_{t,x}}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)\theta_x^2}{\theta^2}dxd\tau \leq O(1)\left\|\frac{\theta}{\kappa_1(\theta)}\right\|_{L^\infty_{t,x}}$$

that

**Lemma 4.4** *Under the conditions listed in Lemma 4.1, we have*

$$\|v_x(t)\|^2+\int_0^t\left\|\sqrt{\theta(s)}v_x(s)\right\|^2ds\le O(1)\left(1+\left\|\frac{\theta}{\kappa_1(\theta)}\right\|_{L^\infty_{t,x}}\right). \tag{4.21}$$

To deduce an estimate on $\|u_x(t)\|$, noticing that

$$\int_0^t\int_{\mathbf{R}}u_x^2v_x^2dxds\le\varepsilon\int_0^t\|u_{xx}(s)\|^2ds+O(1)\int_0^t\|u_x(s)\|^2\|v_x(s)\|^4ds,$$

we have from the proof of Lemma 2.8, (4.3), (4.19), and (4.21) that

**Lemma 4.5** *Under the conditions listed in Lemma 4.1, we have*

$$\begin{aligned}&\|u_x(t)\|^2+\int_0^t\|u_{xx}(s)\|^2ds\\&\le O(1)+O(1)\int_0^t\int_{\mathbf{R}}\left(\theta_x^2+\theta^2v_x^2+u_x^2v_x^2\right)dxds\\&\le O(1)\left(1+\left\|\frac{\theta}{\kappa_1(\theta)}\right\|^2_{L^\infty_{t,x}}+\|\theta\|_{L^\infty_{t,x}}\left\|\frac{\theta}{\kappa_1(\theta)}\right\|_{L^\infty_{t,x}}+\left\|\frac{\theta}{\sqrt{\kappa_1(\theta)}}\right\|^2_{L^\infty_{t,x}}\right).\end{aligned} \tag{4.22}$$

Now we turn to deduce an upper bound on $\theta(t,x)$ for the case $\overline{\lim_{\theta\to+\infty}}\kappa_1(\theta)=+\infty$ based on the above estimates. In fact (2.44) together with (4.17) and the Gronwall inequality imply

$$\begin{aligned}\|\theta(t)\|_{L^\infty_x}&\le O(1)\left(1+\int_0^t\|u_x(s)\|^2_{L^\infty_x}+\|\theta-1\|^2_{L^\infty_x}ds\right)\\&\le O(1)\left(1+\int_0^t\|u_x(s)\|\|u_{xx}(s)\|+\|\theta-1\|^2_{L^\infty_x}ds\right)\\&\le O(1)\left(1+\left\|\frac{\theta}{\kappa_1(\theta)}\right\|^2_{L^\infty_{t,x}}+\|\theta\|^{\frac12}_{L^\infty_{t,x}}\left\|\frac{\theta}{\kappa_1(\theta)}\right\|^{\frac12}_{L^\infty_{t,x}}+\left\|\frac{\theta}{\sqrt{\kappa_1(\theta)}}\right\|_{L^\infty_{t,x}}+\int_0^t\|\theta-1\|^2_{L^\infty_x}ds\right).\end{aligned} \tag{4.23}$$

Here we have used (4.19) and (4.22).

From (4.23) and the assumption that $\overline{\lim_{\theta\to+\infty}}\kappa_1(\theta)=+\infty$, one easily deduce an upper bound on $\theta(t,x)$.

For the case $\kappa(v,\theta)$ is bounded from above, the above argument does not apply and we have to use another method to deduce an upper bound on $\theta(t,x)$. For this purpose, as in [1], set

$$w(t,x)=\frac12u^2(t,x)+C_v(\theta(t,x)-1), \tag{4.24}$$

we can easily deduce that

$$w_t=\mu\left(\frac{w_x}{v}\right)_x+\left((\kappa(v,\theta)-C_v\mu)\frac{\theta_x}{v}\right)_x-\left(\frac{R\theta u}{v}\right)_x+\frac{u\Phi_x}{v}. \tag{4.25}$$

Multiplying (4.25) by $w$ and integrating the result with respect to $t$ and $x$ over $[0,t]\times\mathbf{R}$, we have by some integrations by parts that

$$\begin{aligned}\frac12\|w(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu w_x^2}{v}dxds&\le O(1)-\int_0^t\int_{\mathbf{R}}(\kappa(v,\theta)-C_v\mu)\frac{w_x\theta_x}{v}dxds\\&\quad+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{R\theta uw_x}{v}+\frac{wu\Phi_x}{v}\right)dxds}_{K_1}.\end{aligned} \tag{4.26}$$

Since

$$
\begin{aligned}
\frac{\mu w_x^2}{v} &= \frac{\mu u^2 u_x^2}{v} + \frac{2\mu C_v u u_x \theta_x}{v} + \frac{C_v^2 \mu \theta_x^2}{v},\\
-\frac{(\kappa(v,\theta) - C_v\mu)\theta_x w_x}{v} &= \frac{(C_v\mu - \kappa(v,\theta))C_v\theta_x^2}{v} + \frac{(c_v\mu - \kappa(v,\theta))\theta_x u u_x}{v},\\
K_1 &\leq O(1) + \varepsilon\int_0^t \|w_x(s)\|^2 ds\\
&\quad + O(1)\int_0^t\int_{\mathbf{R}}\left((\theta-1)^2u^2 + w^2u^2 + \left(\frac{\Phi_x}{v}\right)^2\right)dxds\\
&\leq O(1) + \varepsilon\int_0^t \|w_x(s)\|^2 ds + O(1)\int_0^t\int_{\mathbf{R}}\left((\theta-1)^2u^2 + w^2u^2\right)dxds,\\
\int_0^t\int_{\mathbf{R}}|u u_x\theta_x|dxds &\leq \varepsilon\int_0^t\|\theta_x(s)\|^2ds + O(1)\int_0^t\int_{\mathbf{R}}u^2u_x^2dxds,
\end{aligned}
$$

we have by substituting the above estimates into (4.26) that

$$
\begin{aligned}
&\|w(t)\|^2 + \int_0^t\int_{\mathbf{R}}(C_v\kappa(v,\theta) - \varepsilon)\theta_x^2 dxds \\
&\leq O(1) + O(1)\int_0^t\int_{\mathbf{R}}\left((\theta-1)^2u^2 + w^2u^2 + u^2u_x^2\right)dxds.
\end{aligned}\tag{4.27}
$$

Here and in the above analysis, we have used the fact that $\kappa(v,\theta)$ is uniformly bounded for $0 < V_3^{-1} \leq v \leq V_3,\ \theta \geq \Theta_3^{-1}$.

To estimate $\int_0^t\int_{\mathbf{R}} u^2u_x^2dxds$, we multiply (4.1) by $u^3$ and integrate the result with respect to $t$ and $x$ over $[0,t]\times\mathbf{R}$ to yield

$$
\begin{aligned}
\|u(t)\|_{L^4}^4 + \int_0^t\int_{\mathbf{R}}\frac{\mu u^2u_x^2}{v}dxds &\leq O(1) + \underbrace{O(1)\int_0^t\int_{\mathbf{R}}\left|\frac{R\theta u^2 u_x}{v}\right|dxds}_{K_2}\\
&\quad + \underbrace{O(1)\int_0^t\int_{\mathbf{R}}\left|\frac{\Phi_x u^3}{v}\right|dxds}_{K_3}.
\end{aligned}\tag{4.28}
$$

Due to

$$
\begin{aligned}
K_3 &\leq O(1)\int_0^t\int_{\mathbf{R}}\left|\frac{\Phi_x}{v}\right|^2dxds + O(1)\int_0^t\int_{\mathbf{R}}u^6dxds\\
&\leq O(1) + O(1)\int_0^t\|u(s)\|_{L_x^\infty}^4ds\\
&\leq O(1) + \varepsilon\int_0^t\|u(s)u_x(s)\|^2ds + O(1)\int_0^t\|u(s)\|_{L^4}^4ds,\\
K_2 &\leq \varepsilon\int_0^t\int_{\mathbf{R}}\frac{\mu u^2u_x^2}{v}dxds + O(1)\int_0^t\int_{\mathbf{R}}(\theta-1)^2u^2dxds + O(1),
\end{aligned}
$$

we have by inserting the above estimates into (4.28) and by employing the Gronwall inequality that

$$
\|u(t)\|_{L^4}^4 + \int_0^t\|u(s)u_x(s)\|^2ds \leq O(1) + O(1)\int_0^t\int_{\mathbf{R}}(\theta-1)^2u^2dxds. \tag{4.29}
$$

A suitable linear combination of (4.29) and (4.27) yields

$$\begin{aligned}&\|\theta(t)-1\|^2+\|u(t)\|_{L^4}^4+\int_0^t\left(\|\theta_x(s)\|^2+\|u(s)u_x(s)\|^2\right)ds\\ &\leq O(1)+O(1)\int_0^t\int_{\mathbf{R}}\left(\theta-1)^2u^2+u^2w^2\right)dxds.\end{aligned}\tag{4.30}$$

Here again we have used the fact that $\kappa(v,\theta)$ is bounded both from below and above for $0<V_3^{-1}\leq v\leq V_3,\ \theta\geq\Theta_3^{-1}>0$.

From (4.19), we have

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}(\theta-1)^2u^2dxds&\leq O(1)\int_0^t\|\theta(s)-1\|_{L_x^\infty}^2ds,\\ \int_0^t\int_{\mathbf{R}}u^2w^2dxds&\leq O(1)\int_0^t\|w(s)\|_{L_x^\infty}^2ds\\ &\leq\int_0^t\left(\|\theta(s)\|_{L_x^\infty}^2+\|u(s)\|_{L_x^\infty}^4\right)ds,\\ \int_0^t\|\theta(s)-1\|_{L_x^\infty}^2ds&\leq\varepsilon\int_0^t\|\theta_x(s)\|^2ds+O(1)\int_0^t\|\theta(s)-1\|^2ds,\\ \int_0^t\|u(s)\|_{L_x^\infty}^4ds&\leq\varepsilon\int_0^t\|u(s)u_x(s)\|^2ds+O(1)\int_0^t\|u(s)\|_{L^4}^4ds.\end{aligned}$$

Putting the above estimates into (4.30), we have by the Gronwall inequality that

$$\|\theta(t)-1\|^2+\|u(t)\|_{L^4}^4+\int_0^t\left(\|\theta_x(s)\|^2+\|u(s)u_x(s)\|^2\right)ds\leq O(1).\tag{4.31}$$

A direct consequence of (4.31) is

$$\int_0^t\|\theta(s)\|_{L_x^\infty}^2ds\leq O(1)\tag{4.32}$$

and the estimate (4.21) obtained in Lemma 4.4 can be improved as

$$\|v_x(t)\|^2+\int_0^t\left\|\sqrt{\theta(s)}v_x(s)\right\|^2ds\leq O(1).\tag{4.33}$$

(4.19), (4.32) together with (4.33) imply

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}\theta^2v_x^2dxds&\leq O(1)\int_0^t\|\theta(s)\|_{L_x^\infty}^2\|v_x(s)\|^2ds\\ &\leq O(1)\int_0^t\|\theta(s)\|_{L_x^\infty}^2ds\leq O(1),\\ \int_0^t\int_{\mathbf{R}}u_x^2v_x^2dxds&\leq O(1)\int_0^t\|u_x(s)\|_{L_x^\infty}^2\|v_x(s)\|^2ds\\ &\leq O(1)\int_0^t\|u_x(s)\|_{L_x^\infty}^2ds\\ &\leq\varepsilon\int_0^t\|u_{xx}(s)\|^2ds+O(1)\int_0^t\|u_x(s)\|^2ds\\ &\leq\varepsilon\int_0^t\|u_{xx}(s)\|^2ds+O(1).\end{aligned}$$

Combining the above estimates with the first inequality of (4.22) yields

$$\|u_x(t)\|^2 + \int_0^t \|u_{xx}(s)\|^2 ds \le O(1). \tag{4.34}$$

(4.34) together with (4.19) imply

$$\int_0^t \|u_x(s)\|^2_{L^\infty_x} ds \le O(1). \tag{4.35}$$

Having obtained (4.35), the upper bound on $\theta(t,x)$ can be obtained immediately from (4.23) for the case when $\kappa(v,\theta)$ is uniformly bounded for $0 < V_3^{-1} \le v \le V_3,\ \theta \ge \Theta_3^{-1} > 0$. This completes the proof of Theorem 1.3.

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