On periodic entropy sub- and super-solutions to a degenerate elliptic-hyperbolic equation [1]

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Abstract

We establish a necessary and sufficient condition for existence of nonconstant periodic entropy sub- and super-solutions to a multidimensional degenerate elliptic-hyperbolic equation with merely continuous nonlinearities.

1 Introduction

In the space $\mathbb{R}^n$ we consider a quasilinear degenerated elliptic-hyperbolic equation

$$\sum_{i=1}^{k}(\varphi_i(u))_{x_i} + \sum_{i,j=k+1}^{n}(b_{ij}(u))_{x_ix_j} = 0, \ u = u(x), \tag{1}$$

where $0 \le k \le n$ and the functions $\varphi_i(u)$, $b_{ij}(u)$ are supposed to be merely continuous. It is assumed that the $(n-k)\times(n-k)$ matrix $B(u) = (b_{ij}(u))_{i,j=k+1}^n$ is symmetric and for all $u, v \in \mathbb{R}$ the matrix $(u-v)(B(u)-B(v))$ is positive semidefinite. We can extend the matrix $B(u)$ to $n \times n$ symmetric matrix $(b_{ij}(u))_{i,j=1}^n$, setting $b_{ij}(u) = 0$ for $\min(i,j) \le k$. Then (1) can be written in the form

$$\mathrm{div}\varphi(u) + D^2B(u) = 0, \tag{2}$$

where the vector $\varphi(u) = (\varphi_1(u), \dots, \varphi_k(u), 0, \dots, 0) \in \mathbb{R}^n$ and we use the notation

$$D^2B(u) = \sum_{i,j=1}^{n}(b_{ij}(u))_{x_ix_j}.$$

The notions of entropy sub-solution, super-solution and solution of (1) can be introduced in the same way as for conservation laws

$$\mathrm{div}\varphi(u) = 0 \tag{3}$$

when $k = n$ (see [5, 6]). Let $\mathrm{sign}^+(u) = \max(\mathrm{sign}\, u, 0)$ be the Heaviside function, and $\mathrm{sign}^-(u) = \min(\mathrm{sign}\, u, 0) = -\mathrm{sign}^+(-u)$.

Definition 1.1 (cf. [9, 4])**.** A measurable function $u = u(x)$ is called an entropy sub-solution (e.sub-s. for short) of (1) if $\mathrm{sign}^+(u)\varphi_i(u), \mathrm{sign}^+(u)b_{ij}(u) \in L^1_{loc}(\mathbb{R}^n)$, $i, j = 1, \dots, n$, and for all $p \in \mathbb{R}$

$$\mathrm{div}[\mathrm{sign}^+(u-p)(\varphi(u)-\varphi(p))] + D^2[\mathrm{sign}^+(u-p)(B(u)-B(p))] \le 0 \tag{4}$$

in the sense of distributions on $\mathbb{R}^n$ (in $\mathcal{D}'(\mathbb{R}^n)$);

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A measurable function $u = u(x)$ is called an entropy super-solution (e.super-s. for short) of (1) if $\operatorname{sign}^-(u)\varphi_i(u), \operatorname{sign}^-(u)b_{ij}(u) \in L^1_{loc}(\mathbb{R}^n)$, $i, j = 1, \dots, n$, and for all $p \in \mathbb{R}$

$$\operatorname{div}[\operatorname{sign}^-(u-p)(\varphi(u)-\varphi(p))] + D^2[\operatorname{sign}^-(u-p)(B(u)-B(p))] \le 0 \qquad (5)$$

in $\mathcal{D}'(\mathbb{R}^n)$;

Finally, a measurable function $u = u(x)$ is called an entropy solution (e.s. for short) of (1) if it is an e.s.sub-s. and an e.super-s. of (1) simultaneously.

Condition (4), (5) mean that for all non-negative test functions $f = f(x) \in C_0^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \operatorname{sign}^{\pm}(u-p)\left\{(\varphi(u)-\varphi(p)) \cdot \nabla f(x) - (B(u)-B(p)) \cdot D^2 f(x)\right\} dx \ge 0.$$

We use the notation $D^2 f$ for the matrix $\{\partial^2_{x_i x_j} f\}_{i,j=1}^n$ and "$\cdot$" denotes the scalar product of vectors or matrices. In particular,

$$(\varphi(u)-\varphi(p)) \cdot \nabla f(x) = \sum_{i=1}^n (\varphi_i(u)-\varphi_i(p))\partial_{x_i} f,$$
$$(B(u)-B(p)) \cdot D^2 f = \sum_{i,j=1}^n (b_{ij}(u)-b_{ij}(p))\partial^2_{x_i x_j} f.$$

It is rather well-known that a function $u(x)$ is an e.s. of (1) if and only if $\varphi_i(u), b_{ij}(u) \in L^1_{loc}(\mathbb{R}^n)$, $i, j = 1, \dots, n$, and for all $p \in \mathbb{R}$

$$\operatorname{div}[\operatorname{sign}(u-p)(\varphi(u)-\varphi(p))] + D^2[\operatorname{sign}(u-p)(B(u)-B(p))] \le 0 \qquad (6)$$

in $\mathcal{D}'(\mathbb{R}^n)$, that is, $u(x)$ is an e.s. in the sense of [9, 4]. Remark that in the case of conservation laws (3) relation (6) coincides with the known Kruzhkov entropy condition, see [5].

Observe also that $u(x)$ is an e.super-s. of (1) if and only if $v = -u(x)$ is an e.sub-s. to the equation

$$\operatorname{div}(-\varphi(-v)) + D^2(-B(-v)) = 0. \qquad (7)$$

In the present paper we assume that the requirement of space-periodicity holds: $u(x + e_i) = u(x)$ for almost all $x \in \mathbb{R}^n$ and all $i = 1, \dots, n$, where $\{e_i\}_{i=1}^n$ is a fixed basis of periods in $\mathbb{R}^n$. Without loss of generality, we may suppose that this basis is canonical. We denote by $P = [0, 1)^n$ the corresponding fundamental parallelepiped (cube).

We propose the following necessary and sufficient condition for the nonexistence of nonconstant periodic entropy sub- and super-solutions of (1) (by $\mathbb{Z}$ we denote the set of integers)

$$\begin{array}{l}\forall \xi \in \mathbb{Z}^n, \xi \ne 0, \text{ the functions } u \mapsto \varphi(u)\cdot\xi,\ u \mapsto B(u)\xi\cdot\xi \\ \text{are not constant simultaneously on non-empty intervals.}\end{array} \qquad (8)$$

Thus, our main result is the following

Theorem 1.2. *If requirement (8) holds then any periodic e.sub-s (or e.super-s.) $u(x)$ is constant. Conversely, if (8) fails then there exists a nonconstant periodic e.s. of (1).*

Remark that if the basis of periods is not fixed and may depend on a solution, Theorem 1.2 remains valid after replacement of condition (8) by the following stronger one:

$$\begin{aligned}&\forall \xi \in \mathbb{R}^n, \xi \neq 0, \text{ the functions } u \mapsto \varphi(u) \cdot \xi, \ u \mapsto B(u)\xi \cdot \xi \\ &\text{are not constant simultaneously on non-empty intervals.}\end{aligned} \tag{9}$$

2 Preliminaries

We denote by X the space

$$\mathbb{R}^k = \{ \ x = (x_1, \dots, x_n) \mid x_i = 0 \ \forall i > k \ \}, \tag{10}$$

and let $X^\perp$ be its orthogonal complement,

$$X^\perp = \mathbb{R}^{n-k} = \{ \ x = (x_1, \dots, x_n) \mid x_i = 0 \ \forall i \leq k \ \}.$$

We also denote by P_1, P_2 operators of orthogonal projections on X and $X^\perp$, respectively. To prove Theorem 1.2 we derive the strong pre-compactness property for the self-similar scaling sequence $u(k^2 P_1 x + k P_2 x)$, $k \in \mathbb{N}$, which can be satisfied only for constant $u(x)$. This pre-compactness property will be obtained under condition (8) on the base of localization principles for ultra-parabolic H-measures with "continuous indexes", introduced in [9]. The strong pre-compactness property for arbitrary sequences of e.s. of (1) under exact non-degeneracy condition (9) follows from general results of papers [9, 4]. In the present paper we also take into account the periodicity condition, which allow to refine the localization principle.

First, we recall the original concept of H-measure introduced by L. Tartar [12] and P. Gerárd [3]. Let $F(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u(x) dx$, $\xi \in \mathbb{R}^n$, be the Fourier transform extended as a unitary operator in the Hilbert space of functions $u(x) \in L^2(\mathbb{R}^n)$, $S = S^{n-1} = \{ \ \xi \in \mathbb{R}^n \ \mid \ |\xi| = 1 \ \}$ be the unit sphere in $\mathbb{R}^n$. Denote by $u \to \overline{u}$, $u \in \mathbb{C}$ the complex conjugation.

Let $U_k(x) = \left(U_k^1(x), \dots, U_k^l(x)\right) \in L^2_{loc}(\mathbb{R}^n, \mathbb{R}^l)$ be a sequence of vector-functions weakly convergent to the zero vector.

Proposition 1 (see [12], Theorem 1.1)**.** *There exists a family of complex locally finite Borel measures $\mu = \left\{\mu^{ij}\right\}_{i,j=1}^l$ in $\mathbb{R}^n \times S$ and a subsequence $U_r(x) = U_k(x)$, $k = k_r$, such that*

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(U_r^i \Phi_1)(\xi) \overline{F(U_r^j \Phi_2)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi \tag{11}$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\mathbb{R}^n)$ and $\psi(\xi) \in C(S)$.

The family $\mu = \left\{\mu^{ij}\right\}_{i,j=1}^l$ is called the H-measure corresponding to $U_r(x)$.

In [1] the new concept of parabolic H-measures was introduced. Here we need the more general variant of this concept recently developed in [9], see also [10].

Suppose that $X \subset \mathbb{R}^n$ is a linear subspace, $X^\perp$ is its orthogonal complement, P_1, P_2 are orthogonal projections on X, $X^\perp$, respectively. We denote for $\xi \in \mathbb{R}^n$ $\tilde{\xi} = P_1\xi$, $\bar{\xi} = P_2\xi$, so that $\tilde{\xi} \in X$, $\bar{\xi} \in X^\perp$, $\xi = \tilde{\xi} + \bar{\xi}$. Under the above notations we define the set

$$S_X = \{ \ \xi \in \mathbb{R}^n \mid |\tilde{\xi}|^2 + |\bar{\xi}|^4 = 1 \ \}$$

and the projection $\pi_X : \mathbb{R}^n \setminus \{0\} \to S_X$

$$\pi_X(\xi) = \frac{\tilde{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} + \frac{\bar{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}}.$$

Obviously, S_X is a compact smooth manifold of codimension 1, in the case when $X = \{0\}$ or $X = \mathbb{R}^n$, it coincides with the unit sphere $S = \{\xi \in \mathbb{R}^n \mid |\xi| = 1 \ \}$ and then $\pi_X(\xi) = \xi/|\xi|$ is the orthogonal projection on the sphere.

The following analogue of Proposition 1 holds.

Proposition 2. *There exist a family of complex Borel measures* $\mu = \left\{\mu^{ij}\right\}_{i,j=1}^{l}$ *in* $\Omega \times S_X$ *and a subsequence* $U_r(x)$ *of* $U_k(x)$ *such that for all* $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^i)(\xi)\overline{F(\Phi_2 U_r^j)(\xi)}\psi(\pi_X(\xi))d\xi. \tag{12}$$

Besides, the matrix-valued measure μ *is Hermitian and positive definite, that is, for each* $\zeta = (\zeta_1, \ldots, \zeta_l) \in \mathbb{C}^l$ *the measure* $\mu\zeta \cdot \zeta = \sum_{i,j=1}^{l} \mu^{ij}\zeta_i\overline{\zeta_j} \ge 0$.

The proof of Proposition 2 can be found in [10].

Definition 2.1. The family μ^{ij}, $i, j = 1, \ldots, l$, is called the ultra-parabolic H-measure corresponding to a subspace $X \subset \mathbb{R}^n$ and a subsequence $U_r(x)$.

Remark 1. In the case when the sequence $U_k(x)$ is bounded in $L^\infty(\Omega)$ it follows from (12) and the Plancherel identity that $\mathrm{pr}_x|\mu^{ij}| \le C\,\mathrm{meas}$, and that (12) remains valid for all $\Phi_1(x), \Phi_2(x) \in L^2(\mathbb{R}^n)$. Here we denote by $|\mu|$ the variation of measure μ (it is a nonnegative measure), and by meas the Lebesgue measure on Ω.

We need also the concept of measure valued functions (Young measures). Let $\Omega \subset \mathbb{R}^n$ be an open domain. Recall (see [2, 11]) that a measure-valued function on Ω is a weakly measurable map $x \mapsto \nu_x$ of Ω into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of ν_x means that for each continuous function $g(\lambda)$ the function $x \to \langle \nu_x, g(\lambda)\rangle = \int g(\lambda)d\nu_x(\lambda)$ is measurable on Ω.

Measure-valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $u(x) \in L^\infty(\Omega)$ and $\delta(\lambda - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are called *regular*. We identify these measure-valued functions and the corresponding functions $u(x)$, so that there is a natural embedding of $L^\infty(\Omega)$ into the set $\mathrm{MV}(\Omega)$ of measure-valued functions on Ω.

Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem by L. Tartar (see [11]).

Theorem 2.2. *Let $u_k(x) \in L^\infty(\Omega)$, $k \in \mathbb{N}$, be a bounded sequence. Then there exist a subsequence (we keep the notation $u_k(x)$ for this subsequence) and a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that*

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_k) \underset{k\to\infty}{\to} \langle \nu_x, g(\lambda)\rangle \quad \textit{weakly-}* \textit{ in } L^\infty(\Omega). \tag{13}$$

Besides, ν_x is regular, i.e., $\nu_x(\lambda) = \delta(\lambda - u(x))$ if and only if $u_k(x) \underset{k\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ (strongly).

In [7] the new concept of H-measures with "continuous indexes" was introduced, corresponding to sequences of measure valued functions. Later in [9] ultra-parabolic variant of such H-measures was developed. We describe this concept in the particular case of "usual" sequences in $L^\infty(\mathbb{R}^n)$. Let $u_k(x)$ be a bounded sequence in $L^\infty(\mathbb{R}^n)$. Passing to a subsequence if necessary, we can suppose that this sequence converges to a measure valued function $\nu_x \in \mathrm{MV}(\mathbb{R}^n)$ in the sense of relation (13). We introduce the measures $\gamma_x^k(\lambda) = \delta(\lambda - u_k(x)) - \nu_x(\lambda)$ and the corresponding distribution functions $U_k(x,p) = \gamma_x^k((p,+\infty))$, $u_0(x,p) = \nu_x((p,+\infty))$ on $\mathbb{R}^n \times \mathbb{R}$. Observe that $U_k(x,p), u_0(x,p) \in L^\infty(\mathbb{R}^n)$ for all $p \in \mathbb{R}$, see [7, Lemma 2]. We define the set

$$E = E(\nu_x) = \left\{ \, p_0 \in \mathbb{R} \;\mid\; u_0(x,p) \underset{p\to p_0}{\to} u_0(x,p_0) \ \text{ in } L^1_{loc}(\mathbb{R}^n) \, \right\}.$$

As was shown in [7, Lemma 4], the complement $\bar{E} = \mathbb{R} \setminus E$ is at most countable and if $p \in E$ then $U_k(x,p) \underset{k\to\infty}{\rightharpoonup} 0$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$.

The next result, similar to Proposition 2, has been established in [9, Proposition 2, Lemma 4].

Proposition 3. *1) There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\mathbb{R}^n \times S_X$ and a subsequence $U_r(x,p) = U_{k_r}(x,p)$ such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\mathbb{R}^n)$ and $\psi(\xi) \in C(S_X)$*

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r(\cdot,p))(\xi)\overline{F(\Phi_2 U_r(\cdot,q))(\xi)}\psi(\pi_X(\xi))d\xi;$$

2) The correspondence $(p,q) \to \mu^{pq}$ is a continuous map from $E \times E$ into the space $\mathrm{M}_{loc}(\mathbb{R}^n \times S_X)$ of locally finite Borel measures on $\Omega \times S_X$ (with the standard locally convex topology);

3) For any $p_1, \ldots, p_l \in E$ the matrix $\{\mu^{p_i p_j}\}_{i,j=1}^l$ is Hermitian and positive semidefinite, that is, for all $\zeta_1, \ldots, \zeta_l \in \mathbb{C}$ the measure

$$\sum_{i,j=1}^{l} \mu^{p_i p_j} \zeta_i \overline{\zeta_j} \geq 0.$$

We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the H-measure corresponding to the subsequence $u_r(x) = u_{m_r}(x)$ (and the subspace X).

As was demonstrated in [9], the H-measure $\mu^{pq} = 0$ for all $p, q \in E$ if and only if the subsequence $u_r(x)$ converges as $r \to \infty$ strongly (in $L^1_{loc}(\mathbb{R}^n)$). Observe also that assertion 3) in Proposition 3 implies that measures $\mu^{pp} \geq 0$ for all $p \in E$, and that

$$|\mu^{pq}(A)| \leq \sqrt{\mu^{pp}(A)\mu^{qq}(A)} \tag{14}$$

for any Borel set $A \subset \mathbb{R}^n \times S_X$ and all $p, q \in E$. Indeed, this directly follows from the fact that the matrix $\begin{pmatrix} \mu^{pp}(A) & \mu^{pq}(A) \\ \mu^{qp}(A) & \mu^{qq}(A) \end{pmatrix}$ is Hermitian and positive semidefinite.

3 Main results

Let $v = v(x)$ be a periodic function on $\mathbb{R}^n$ such that $v(x) \in L^2(P)$, and let

$$v(x) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa e^{2\pi i \kappa \cdot x} \tag{15}$$

be the Fourier series of $v(x)$ in $L^2(P)$, so that $a_\kappa = \int_P e^{-2\pi i \kappa \cdot x} v(x) dx$. Then this series converges to $v(x)$ in $L^2(P)$. We consider the subspace $X = \mathbb{R}^k$ defined in (10) and the sequence $v_k = v(k^2 \tilde{x} + k\bar{x})$, $k \in \mathbb{N}$, where $\tilde{x} = P_1 x$, $\bar{x} = P_2 x$ are the orthogonal projection of x on the spaces X, $X^\perp$, respectively. We define the ultra-parabolic H-measure $\bar{\mu}$ corresponding to the subspace X and the scalar sequence $v_r - v^*$, where $v_r = v_{k_r}(x)$ is a subsequence of v_k, and $v^* = v^*(x)$ is a weak limit of v_r as $r \to \infty$ in $L^2(P)$ (remark that, in view of periodicity of v_k, $\|v_k\|_{L^2(P)} = \|v\|_{L^2(P)} = \text{const}$).

Lemma 3.1. *(i) The function $v^*(x) \equiv a_0$ is constant;*

(ii) $v_r \underset{r\to\infty}{\to} v^*$ in $L^2(P)$ (strongly) if and only if $v \equiv \text{const} = a_0$;

(iii) $\operatorname{supp} \bar{\mu} \subset \mathbb{R}^n \times S_0$, where

$$S_0 = \{ \pi_X(\xi) \mid \xi \in \mathbb{Z}^n, \xi \neq 0 \} \subset S_X.$$

Proof. By (15)

$$v_k(x) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa e^{2\pi i \kappa \cdot (k^2 \tilde{x} + k\bar{x})} = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa e^{2\pi i (k^2 \tilde{\kappa} + k\bar{\kappa}) \cdot x}, \tag{16}$$

where $\tilde{\kappa} = P_1 \kappa$, $\bar{\kappa} = P_2 \kappa$. It readily follows from (16) that $v_r \underset{r\to\infty}{\rightharpoonup} a_0$ and this convergence is strong in $L^2(P)$ if and only if $v \equiv a_0$. Thus, statements (i), (ii) are proved.

Let $\Phi(x) \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ be such that its Fourier transform is a continuous compactly supported function:

$$\hat{\Phi}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \Phi(x) dx \in C_0(\mathbb{R}^n). \tag{17}$$

We take $R = \max\limits_{\xi \in \operatorname{supp} \hat{\Phi}} |\xi|$. By (16) we find that

$$(v_r(x) - v^*)\Phi(x) = \sum_{\kappa \in \mathbb{Z}^n, \kappa \neq 0} a_\kappa e^{2\pi i (k^2 \tilde{\kappa} + k\bar{\kappa}) \cdot x} \Phi(x). \tag{18}$$

Observe that the Fourier transform of $e^{2\pi i (k^2 \tilde{\kappa} + k\bar{\kappa}) \cdot x} \Phi(x)$ in $\mathbb{R}^n$ coincides with $\hat{\Phi}(\xi - (k_r^2 \tilde{\kappa} + k_r \bar{\kappa}))$. Since for $k_r > 2R$ supports of these functions do not intersect, then for such r the series

$$\sum_{\kappa \in \mathbb{Z}^n, \kappa \neq 0} a_\kappa \hat{\Phi}(\xi - (k_r^2 \tilde{\kappa} + k_r \bar{\kappa})) \tag{19}$$

is orthogonal in $L^2(\mathbb{R}^n)$. Besides, by the Plancherel equality

$$\|\hat{\Phi}(\xi - (k_r^2 \tilde{\kappa} + k_r \bar{\kappa}))\|_{L^2(\mathbb{R}^n)} = \|\hat{\Phi}\|_2 = \|\Phi\|_2,$$

and

$$\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2\|\hat{\Phi}(\xi-(k_r^2\tilde{\kappa}+k_r\bar{\kappa}))\|^2_{L^2(\mathbb{R}^n)}=$$

$$\|\Phi\|_2^2\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2=\|\Phi\|_2^2\cdot\|v-v^*\|^2_{L^2(P)}<+\infty.$$

Therefore, orthogonal series (19) converges in $L^2(\mathbb{R}^n)$. Since the Fourier transformation is an isomorphism on $L^2(\mathbb{R}^n)$, we conclude that series (18) also converges in $L^2(\mathbb{R}^n)$ (not only in $L^2(P)$. This implies that

$$F((v_r-v^*)\Phi)(\xi)=\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}a_\kappa\hat{\Phi}(\xi-(k_r^2\tilde{\kappa}+k_r\bar{\kappa})). \tag{20}$$

Since supports of functions $\Phi(\xi-(k_r^2\tilde{\kappa}+k_r\bar{\kappa}))$ do not intersect for different κ if $k_r>2R$, we derive from (20) that

$$|F((v_r-v^*)\Phi)(\xi)|^2=\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2|\hat{\Phi}(\xi-(k_r^2\tilde{\kappa}+k_r\bar{\kappa}))|^2. \tag{21}$$

It now follows from (21) that for $k_r>2R$

$$\int_{\mathbb{R}^n}|F(\Phi(v_r-v^*))(\xi)|^2\psi(\pi_X(\xi))d\xi=$$

$$\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2\int_{\mathbb{R}^n}|\hat{\Phi}(\xi-(k_r^2\tilde{\kappa}+k_r\bar{\kappa}))|^2\psi(\pi_X(\xi))d\xi=$$

$$\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2\int_{\mathbb{R}^n}|\hat{\Phi}(\xi)|^2\psi(\pi_X(\xi+(k_r^2\tilde{\kappa}+k_r\bar{\kappa})))d\xi. \tag{22}$$

Since $\psi(\xi)\in C(S_X)$ it is uniformly continuous on S_X and, therefore,

$$\psi(\pi_X(\xi+(k_r^2\tilde{\kappa}+k_r\bar{\kappa})))=\psi(\pi_X(\kappa+(k_r^{-2}\tilde{\xi}+k_r^{-1}\bar{\xi})))\underset{r\to\infty}{\to}\psi(\pi_X(\kappa))$$

uniformly in the ball $|\xi|\le R$. Hence,

$$\int_{\mathbb{R}^n}|\hat{\Phi}(\xi)|^2\psi(\pi_X(\xi+(k_r^2\tilde{\kappa}+k_r\bar{\kappa})))d\xi\underset{r\to\infty}{\to}\psi(\pi_X(\kappa))\int_{\mathbb{R}^n}|\hat{\Phi}(\xi)|^2d\xi=\psi(\pi_X(\kappa))\|\Phi\|_2^2. \tag{23}$$

Taking into account that

$$\int_{\mathbb{R}^n}|\hat{\Phi}(\xi)|^2\psi(\pi_X(\xi+(k_r^2\tilde{\kappa}+k_r\bar{\kappa})))d\xi\le\|\psi\|_\infty\|\Phi\|_2^2=\text{const}$$

and that the series $\sum\limits_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2$ converges, we derive from (22), (23) that

$$\langle\bar{\mu},|\Phi(x)|^2\psi(\xi)\rangle=\lim_{r\to\infty}\int_{\mathbb{R}^n}|F(\Phi(v_r-v^*))(\xi)|^2\psi(\pi_X(\xi))d\xi=$$

$$\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2\psi(\pi_X(\kappa))\|\Phi\|_2^2=\int_{\mathbb{R}^n}|\Phi(x)|^2dx\sum_{\kappa\in\mathbb{Z}^n,\kappa\neq 0}|a_\kappa|^2\psi(\pi_X(\kappa)). \tag{24}$$

Observe that by Remark 1 we may use test functions $\Phi(x) \in L^2(\mathbb{R}^n)$ in the definition of H-measure $\bar{\mu}$. Since the set of functions $\Phi(x)$ with the prescribed above properties is dense in $L^2(\mathbb{R}^n)$ we derive from (24) that $\bar{\mu}$ is a the product of the Lebesgue measure dx and the singular measure $\sigma = \sum\limits_{\kappa\in\mathbb{Z}^n,\kappa\neq 0} |a_\kappa|^2\delta(\xi - \pi_X(\kappa))$: $\bar{\mu} = dx \times \sigma$. In particular, $\operatorname{supp}\bar{\mu} \subset \mathbb{R}^n \times S_0$, as was to be proved. □

Now let $u(x) \in L^\infty(\mathbb{R}^n)$. We consider the H-measure $\{\mu^{pq}\}_{p,q\in E}$ corresponding to a subsequence $u_r = u_{k_r}(x)$ of the sequence $u_k(x) = u(k^2\tilde{x} + k\bar{x})$, $k \in \mathbb{N}$, defined in accordance with Proposition 3.

Theorem 3.2. *For every* $p, q \in E$ $\operatorname{supp}\mu^{pq} \subset \mathbb{R}^n \times S_0$.

Proof. Let ν_x be a weak measure valued limit of the sequence u_r (in the sense of Theorem 2.2). We introduce measures

$$\gamma_x^r(\lambda) = \delta(\lambda - u_r(x)) - \nu_x(\lambda),$$

and set $U_r(x, p) = \gamma_x^r((p, +\infty))$. Let $s(u) \in C^1(\mathbb{R})$ be such that its derivative $s'(u)$ is compactly supported, and $v_r(x) = s(u_r(x))$, $r \in \mathbb{N}$. Then $v_r \rightharpoonup v^* = \int s(\lambda)d\nu_x(\lambda)$ as $r \to \infty$ weakly-$*$ in $L^\infty(\Pi)$ (by Lemma 3.1(i), the limit function v^* is constant). Integrating by parts, we find that

$$v_r(x) - v^* = \int s(\lambda)d\gamma_x^r(\lambda) = \int s'(\lambda)U_r(x,\lambda)d\lambda. \tag{25}$$

Let $\Phi(x) \in C_0(\mathbb{R}^n)$, $\psi(\xi) \in C(S_X)$. Then, in view of (25), we find

$$\int_{\mathbb{R}^n} |F(\Phi(v_r - v^*))(\xi)|^2\psi(\pi_X(\xi))d\xi =$$
$$\int\int s'(p)s'(q)\left(\int_{\mathbb{R}^n} F(\Phi U_r(\cdot,p))(\bar{\xi})\overline{F(\Phi U_r(\cdot,q))(\xi)}\psi(\pi_X(\xi))d\xi\right)dpdq. \tag{26}$$

By the definition of H-measure, for each $p, q \in E$

$$\lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi U_r(\cdot,p))(\bar{\xi})\overline{F(\Phi U_r(\cdot,q))(\xi)}\psi(\pi_X(\xi))d\xi = \langle\mu^{pq}, |\Phi(x)|^2\psi(\xi)\rangle.$$

Using Lebesgue dominated convergence theorem, we can pass to the limit as $r \to \infty$ in equality (26) and arrive at

$$\langle\bar{\mu}, |\Phi(x)|^2\psi(\xi)\rangle = \lim_{r\to\infty}\int_{\mathbb{R}^n} |F(\Phi(v_r - v))(\xi)|^2\psi(\pi_X(\xi))d\xi =$$
$$\int\int s'(p)s'(q)\langle\mu^{pq}, |\Phi(x)|^2\psi(\xi)\rangle dpdq, \tag{27}$$

where $\bar{\mu} = \bar{\mu}(x, \xi)$ is the ultra-parabolic H-measure, corresponding to the scalar sequence $U_r = v_r - v^*$ in accordance with Definition 2.1. Clearly, the equality

$$\langle\bar{\mu}, |\Phi(x)|^2\psi(\xi)\rangle = \int\int s'(p)s'(q)\langle\mu^{pq}, |\Phi(x)|^2\psi(\xi)\rangle dpdq$$

remains valid for every Borel function $\psi(\xi)$. Taking $\psi(\xi)$ being the indicator function of the set $S_X \setminus S_0$ and using Lemma 3.1 (iii), we obtain the relation

$$\int\int s'(p)s'(q)\langle\mu^{pq}, |\Phi(x)|^2\psi(\xi)\rangle dpdq = 0. \tag{28}$$

Now we take in (28) $s'(p) = l\omega(l(p-p_0))$, where $p_0 \in E$, $l \in \mathbb{N}$, and $\omega(y) \in C_0((0,1))$ is a non-negative function such that $\int \omega(y)dy = 1$. Since the H-measure μ^{pq} is strongly continuous with respect to (p,q) at the point (p_0,p_0), we derive from (28) in the limit as $l \to \infty$ that

$$\langle \mu^{p_0p_0}, |\Phi(x)|^2\psi(\xi)\rangle = \lim_{l\to\infty} l^2 \int\int \omega(l(p-p_0))\omega(l(q-p_0))\langle \mu^{pq}, |\Phi(x)|^2\psi(\xi)\rangle dpdq = 0.$$

Since $\Phi(x) \in C_0(\mathbb{R}^n)$ is arbitrary, we conclude that $\mu^{p_0p_0}(\mathbb{R}^n \times (S_X \setminus S_0)) = 0$ (remark that $\mu^{p_0p_0} \geq 0$). Hence, for every $p = p_0 \in E$ $\operatorname{supp}\mu^{pp} \subset \mathbb{R}^n \times S_0$. Finally, as directly follows from (14), for $p,q \in E$ $\operatorname{supp}\mu^{pq} \subset \operatorname{supp}\mu^{pp} \subset \mathbb{R}^n \times S_0$. The proof is complete. □

Now, suppose that $u(x)$ is an e.sub-s. of (1). Then, as is easy to verify, the sequence $u_k(x) = u(k^2\tilde{x} + k\bar{x})$ consists of e.sub-s. of (1). Indeed, for all $p \in \mathbb{R}$

$$\begin{aligned}\operatorname{div}[\operatorname{sign}^+(u_k-p)(\varphi(u_k)-\varphi(p))] - D^2[\operatorname{sign}^+(u_k-p)(B(u_k)-B(p))] = \\ k^2\{\operatorname{div}[\operatorname{sign}^+(u-p)(\varphi(u)-\varphi(p))] - \\ D^2[\operatorname{sign}^+(u-p)(B(u)-B(p))]\}(k^2\tilde{x}+k\bar{x}) \leq 0\end{aligned}$$

in $\mathcal{D}'(\mathbb{R}^n)$. We need the following simple

Lemma 3.3. *Let $u = u(x)$ be an e.sub-s. of (1), $M = \|u\|_\infty$. Then for each $p \in \mathbb{R}$*

$$\mathcal{L}_p \doteq \operatorname{div}[\operatorname{sign}^+(u-p)(\varphi(u)-\varphi(p))] - D^2[\operatorname{sign}^+(u-p)(B(u)-B(p))] = -\gamma_p \quad (29)$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\gamma_p \in \mathrm{M}_{loc}(\mathbb{R}^n)$ is a nonnegative locally finite Borel measure on $\mathbb{R}^n$. Besides, for each compact $K \subset \mathbb{R}^n$

$$\begin{aligned}\gamma_p(K) \leq C(K)\big(\|\operatorname{sign}^+(u-p)(\varphi(u)-\varphi(p))\|_{L^1(K)} + \\ \|\operatorname{sign}^+(u-p)(B(u)-B(p))\|_{L^1(K)}\big),\ u = u(x).\end{aligned} \quad (30)$$

where the constant $C(K)$ depends only on K.

Proof. By the known representation of nonnegative distributions,

$$-\mathcal{L}_p = -\{\operatorname{div}[\operatorname{sign}^+(u-p)(\varphi(u)-\varphi(p))] - D^2[\operatorname{sign}^+(u-p)(B(u)-B(p))]\} = \gamma_p,$$

where $\gamma_p \in \mathrm{M}_{loc}(\mathbb{R}^n)$, $\gamma_p \geq 0$, and (29) follows.

Further, for a compact set $K \subset \mathbb{R}^n$ we choose a nonnegative function $f_K(x) \in C_0^\infty(\mathbb{R}^n)$, which equals 1 on K. Then

$$\begin{aligned}\gamma_p(K) \leq \int f_k(x)d\gamma_p(x) = -\langle \mathcal{L}_p, f_K\rangle = \\ \int \operatorname{sign}^+(u-p)[(\varphi(u)-\varphi(p))\cdot\nabla f_K(x) + (B(u)-B(p))\cdot D^2 f_K(x)]dx \leq \\ \max\big(\|\nabla f_K\|_\infty, \|D^2 f_K\|_\infty\big)\big(\|\operatorname{sign}^+(u-p)(\varphi(u)-\varphi(p))\|_{L^1(K)} + \\ \|\operatorname{sign}^+(u-p)(B(u)-B(p))\|_{L^1(K)}\big),\end{aligned}$$

and estimate (30) follows with $C(K) = \max\big(\|\nabla f_K\|_\infty, \|D^2 f_K\|_\infty\big)$. The proof is complete. □

For $a, b \in \mathbb{R}$, $a \le b$ we define the cut-off function $s_{a,b}(u) = \max(a, \min(b, u))$.

Lemma 3.4. *Let $u = u(x)$ be a e.sub-s. of (1), $M = \|u\|_\infty$. Then for each $a, b \in \mathbb{R}$, $a \le b$*

$$\mathrm{div}\varphi(s_{a,b}(u)) - D^2 B(s_{a,b}(u)) = \gamma_b - \gamma_a \quad in\ \mathcal{D}'(\mathbb{R}^n). \tag{31}$$

Proof. One can easily verify that

$$\varphi(s_{a,b}(u)) = \mathrm{sign}^+(u-a)(\varphi(u) - \varphi(a)) - \mathrm{sign}^+(u-b)(\varphi(u) - \varphi(b)) + \varphi(a),$$
$$B(s_{a,b}(u)) = \mathrm{sign}^+(u-a)(B(u) - B(a)) - \mathrm{sign}^+(u-b)(B(u) - B(b)) + B(b).$$

Therefore,

$$\mathrm{div}\varphi(s_{a,b}(u)) - D^2 B(s_{a,b}(u)) = \mathcal{L}_a - \mathcal{L}_b = \gamma_b - \gamma_a,$$

by Lemma 3.3, as was to be proved. □

Let us fix $m \in \mathbb{R}$, $m > 0$. We consider the H-measure $\{\mu^{pq}\}_{p,q\in E}$ corresponding to a subsequence $v_r = s_{-m,m}(u_r)$, where $u_r = u(k_r^2\tilde{x} + k_r\bar{x})$. Let $p_0 \in E$. We define the minimal linear subspace $L = L(p_0) \subset \mathbb{R}^n$ such that $\mathrm{supp}\,\mu^{p_0p_0} \subset \mathbb{R}^n \times L$.

Theorem 3.5. *There exists $\delta > 0$ such that the functions $u \mapsto \tilde{\xi}\cdot\varphi(u)$, $u \mapsto B(u)\bar{\xi}\cdot\bar{\xi}$ are constant on the interval $(p_0 - \delta, p_0 + \delta)$ for all $\xi \in L$.*

Proof. Let $D \subset E$ be a countable dense subset such that $p_0 \in D$. By [9, Proposition 3] there exists a family of complex finite Borel measures $\mu_x^{pq} \in \mathrm{M}(S_X)$ on S_X, where $p, q \in D$, $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ being a subset of full Lebesgue measure such that $\mu^{pq} = \mu_x^{pq}dx$, i.e., for all $\Phi(x,\xi) \in C_0(\mathbb{R}^n \times S_X)$ the function

$$x \mapsto \langle \mu_x^{pq}(\xi), \Phi(x,\xi)\rangle = \int_{S_X} \Phi(x,\xi) d\mu_x^{pq}(\xi)$$

is Lebesgue-measurable, bounded, and

$$\langle \mu^{pq}, \Phi(x,\xi)\rangle = \int\limits_{\Omega} \langle \mu_x^{pq}(\xi), \Phi(x,\xi)\rangle dx.$$

Since $u_r(x)$ is a sequence of e.sub-s. of equation (1) then by Lemma 3 for every $a, b \in \mathbb{R}$, $a \le b$ the sequence of distributions

$$\mathcal{L}_{a,b}^r = \mathrm{div}\varphi(s_{a,b}(u)) - D^2 B(s_{a,b}(u)) = \gamma_b^r - \gamma_a^r.$$

Since $u(x)$ is a periodic function then the sequences

$$\|\mathrm{sign}^+(u_r - p)(\varphi(u_r) - \varphi(p))\|_{L^1(K)} + \|\mathrm{sign}^+(u_r - p)(B(u_r) - B(p))\|_{L^1(K)}$$

are bounded for every compact $K \subset \mathbb{R}^n$. In view of (30) the sequences $\mathcal{L}_{a,b}^r$, $r \in \mathbb{N}$ are bounded in $\mathrm{M}_{loc}(\mathbb{R}^n)$. Therefore, these sequences are precompact in the Sobolev space $W_{d,loc}^{-1}(\mathbb{R}^n)$ for each $1 < d < n/(n-1)$. By [9, Theorem 4] the H-measure satisfies the following localization property: for all $p \in D$ and a.e. $x \in \mathbb{R}^n$ $\mathrm{supp}\,\mu_x^{pp} \subset L_1(p)$, where

$$L_1(p) = \{\ \xi \in \mathbb{R}^n \mid \exists\delta > 0\ \forall u \in (p-\delta, p+\delta)$$
$$(\varphi(u) - \varphi(p)) \cdot \tilde{\xi} = (B(u) - B(p))\bar{\xi} \cdot \bar{\xi} = 0\ \}.$$

In view of the representation $\mu^{pp} = \mu_x^{pp} dx$ we derive that $\operatorname{supp} \mu^{pp} \subset \mathbb{R}^n \times L_1$. In particular, $L \subset L_1 = L_1(p_0)$. Let ξ_k, $k = 1, \dots, l = \dim L$, be a basis in L. Since $\xi_k \in L_1$ then there exist $\delta_k > 0$ such that the functions

$$(\varphi(u) - \varphi(p_0)) \cdot \tilde{\xi}_k = (B(u) - B(p_0))\bar{\xi}_k \cdot \bar{\xi}_k = 0 \tag{32}$$

for all $u \in (p_0 - \delta_k, p_0 + \delta_k)$, $k = 1, \dots, l$. Setting $\delta = \min\limits_{k=1,\dots,l} \delta_k$, we find that (32) holds on the interval $u \in (p_0 - \delta, p_0 + \delta)$ for all vectors ξ_k, $k = 1, \dots, l$. Since the linear span of these vectors coincides with L, relation

$$(\varphi(u) - \varphi(p_0)) \cdot \tilde{\xi} = (B(u) - B(p_0))\bar{\xi} \cdot \bar{\xi} = 0$$

remains true for $u \in (p_0 - \delta, p_0 + \delta)$ and every $\xi \in L$. This concludes the proof. $\square$

Now we are ready to prove our main Theorem 1.2.

Proof of Theorem 1.2. We fix $m > 0$, $p \in E$ and assume that $\mu^{pp} \neq 0$ (recall that $\{\mu^{pq}\}_{p,q \in E}$ is the H-measure corresponding to the subsequence $v_r = s_{-m,m}(u_r)$). Then the space $L = L(p)$ is not trivial: $\dim L > 0$. By Theorem 3.2 there exists a nonzero vector $\xi \in \mathbb{Z}^n \cap L$. Then, by Theorem 3.5 the functions

$$u \mapsto \xi \cdot \varphi(u) = \tilde{\xi} \cdot \varphi(u), \ u \mapsto B(u)\xi \cdot \xi = B(u)\bar{\xi} \cdot \bar{\xi}$$

are constant on some interval $(p-\delta, p+\delta)$, which contradicts to condition (8). Hence $\mu^{pp} = 0$ for all $p \in E$. In view of (14) this implies that the H-measure $\mu^{pq} \equiv 0$. Therefore, the sequence $v_r(x) = s_{-m,m}(u(k_r^2 \tilde{x} + k_r \bar{x}))$ converges strongly as $r \to \infty$ to a function $u^*(x)$. By Lemma 3.1(ii) and arbitrariness of m, this is possible only if $u(x) \equiv c = \text{const}$.

If $u(x)$ is a periodic e.super-s. of (1) when $v = -u$ is a periodic e.sub-s. of equation (7), which obviously satisfies requirement (8), and, as was already proved above, u together with v must be constant.

Conversely, if condition (8) fails then we can find the segment $[a, b]$, $a < b$, and a nonzero vector $\xi \in \mathbb{Z}^n$ such that the functions $u \mapsto \xi \cdot \varphi(u)$, $u \mapsto B(u)\xi \cdot \xi$ are constant on the segment $[a, b]$. Then, as is easy to verify, the function

$$u(x) = \frac{a+b}{2} + \frac{b-a}{2} \sin(2\pi \xi \cdot x)$$

is a nonconstant periodic e.s. of (1). The proof is complete. $\square$

Corollary 1. *Let $k = n$, $u_1(x), u_2(x)$ be bounded e.sub-s. and e.super-s. of (3), respectively, and $h \neq 0$. Introduce the functions*

$$v_h(x) = \sup_{\kappa \in \mathbb{Z}^n} u_1(x + h\kappa), \quad w_h(x) = \inf_{\kappa \in \mathbb{Z}^n} u_2(x + h\kappa).$$

Assume that condition (8) is satisfied. Then the functions $v_h(x)$, $w_h(x)$ are constant: $v_h(x) = \operatorname{ess\,sup} u_1(x)$, $w_h(x) = \operatorname{ess\,inf} u_2(x)$ a.e. on $\mathbb{R}^n$.

Proof. Since the functions $u_1(x + h\kappa)$, $u_2(x + h\kappa)$ are, respectively, e.sub-s. and e.super-s. of conservation law (3) for every $\kappa \in \mathbb{Z}^n$ then by the results of [8, Theorem

1], the functions $v_h(x)$ and $w_h(x)$ are e.sub-s. and e.super-s. of (3) as well. Indeed, introduce the sequences

$$v_r(x) = \max_{\kappa\in\mathbb{Z}^n, |\kappa|<r} u_1(x + h\kappa), \quad w_r(x) = \min_{\kappa\in\mathbb{Z}^n, |\kappa|<r} u_2(x + h\kappa), \quad r \in \mathbb{N}.$$

Then $v_r(x)$, $w_r(x)$ are e.sub-s. and e.super-s. of (3) as a maximum of finite family of e.sub-s. (respectively, a minimum of finite family of e.super-s.). It is clear that $v_r(x) \underset{r\to\infty}{\to} v_h(x)$, $w_r(x) \underset{r\to\infty}{\to} w_h(x)$ pointwise and in $L^1_{loc}(\mathbb{R}^n)$. Obviously, the limit functions $v_h(x)$, $w_h(x)$ are, respectively, an e.sub-s. and an e.super-s. of (3). It is clear that $v_h(hx)$, $w_h(hx)$ are periodic e.sub-s. and e.super-s. of (3). By Theorem 1.2 we claim that these functions are constant. Therefore, $v_h(x) = \alpha(h)$, $w_h(x) = \beta(h)$ for almost every $x \in \mathbb{R}^n$, where $\alpha(h), \beta(h)$ are some constants. Obviously,

$$u_1(x) \le \alpha(h) = v_h(x) \le \operatorname{ess\,sup} u_1(x), \quad \operatorname{ess\,inf} u_2(x) \le \beta(h) = w_h(x) \le u_2(x)$$

a.e. in $\mathbb{R}^n$. This implies that $\alpha(h) = \operatorname{ess\,sup} u_1(x)$ and $\beta(h) = \operatorname{ess\,inf} u_2(x)$. In particular, the constants $\alpha(h), \beta(h)$ do not depend on h. Thus, $v_h(x) = \operatorname{ess\,sup} u_1(x)$, $w_h(x) = \operatorname{ess\,inf} u_2(x)$ a.e. on $\mathbb{R}^n$. This completes the proof. $\square$

Remark 2. For the general basis of periods e_i, $i = 1, \dots, n$, one should replace $\mathbb{Z}^n$ in condition (8) by the set

$$\{ \ \xi \in \mathbb{R}^n \mid \xi \cdot e_i \in \mathbb{Z} \ \forall i = 1, \dots, n \ \}.$$

If the vector $\varphi(u)$ and the matrix $B(u)$ are not simultaneously constant on non-degenerate intervals then one always can choose such a basis e_i, $i = 1, \dots, n$ that condition (8) is satisfied and, therefore, the statement of Theorem 1.2 holds.

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