Initial-boundary Value Problems to the One-dimensional Compressible Navier-Stokes Equations with Degenerate Transport Coefficients

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Abstract

This paper is concerned with the construction of global, non-vacuum solutions with large amplitude to the initial-boundary value problems of the one-dimensional compressible Navier-Stokes equations with degenerate transport coefficients. The main ingredient of the analysis is to derive the positive lower and upper bounds on the specific volume and the absolute temperature.

Keywords & Phrases: One-dimensional compressible Navier-Stokes equations; initial-boundary value problems; global solutions with large data; degenerate transport coefficients.

AMS Subject Classifications: 35D35, 35Q35, 76N10

1 Introduction and our main results

The one-dimensional compressible Navier-Stokes equations in the Lagrangian coordinates can be written as:

$$\begin{cases} v_t - u_x = 0, \\ u_t - \sigma_x = 0, \\ \left(e + \frac{u^2}{2}\right)_t - (\sigma u - q)_x = 0. \end{cases} \tag{1.1}$$

Here v, u, σ, e, and q denote the specific volume (deformation gradient), velocity, stress, (specific) internal energy, and heat flux, respectively. For Newtonian fluid, σ is given by

$$\sigma(v,\theta,u_x) = -p(v,\theta) + \frac{\mu(v,\theta)}{v}u_x$$

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and Fourier's law tells us that heat flux q satisfies

$$q(v,\theta,u_x) = -\frac{\kappa(v,\theta)}{v}\theta_x$$

with p and θ being the pressure and the absolute temperature respectively.

The thermodynamic variables p, v, θ, and e are related by the Gibbs equation $de = \theta ds - pdv$ with s being the specific entropy. $\kappa(v,\theta) > 0$ and $\mu(v,\theta) > 0$ denote the heat conductivity coefficient and viscosity coefficient, respectively.

This manuscript is concerned with the construction of global, non-vacuum, large, smooth solutions to the one-dimensional compressible Navier-Stokes equation (1.1) in the domain $\{(x,t)| x \in I = [0,1], t \geq 0\}$ with prescribed initial condition

$$(v(x,0), u(x,0), \theta(x,0)) = (v_0(x), u_0(x), \theta_0(x)), \quad x \in [0,1] \tag{1.2}$$

and one of the following three boundary conditions

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ q(0,t) = q(1,t) = 0, \end{cases} \tag{1.3}$$

$$\begin{cases} \sigma(0,t) = \sigma(1,t) = 0, \\ q(0,t) = q(1,t) = 0, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \sigma(0,t) = \sigma(1,t) = -Q(t) < 0, \\ q(0,t) = q(1,t) = 0. \end{cases} \tag{1.5}$$

Here the outer pressure $Q(t) \in C^1(\mathbf{R}_+)$ is a given function.

Throughout this manuscript, we will focus on the ideal, polytropic gases which contain the case of gases for which kinetic theory provides constitutive relations, cf. [3, 4, 7, 26]

$$e = C_v\theta = \frac{R\theta}{\gamma-1}, \quad p(v,\theta) = \frac{R\theta}{v} = Av^{-\gamma}\exp\left(-\frac{\gamma-1}{R}s\right) \tag{1.6}$$

with suitable positive constants $\gamma > 1$, R, and A. And our main interest concerns the case when the transport coefficients μ and κ may depend on the specific volume and/or the absolute temperature which are degenerate in the sense that κ and/or μ are not uniformly bounded from below or above by some positive constants for all $v > 0$ and $\theta > 0$.

Compressible Navier-Stokes type equations with density and temperature dependent transport coefficients arise in many applied sciences, such as certain class of solid-like materials [5, 6], gases at very high temperatures [27, 14], etc. Such a dependence of μ and κ on v and θ will obviously influence the solutions of the field equations as well as the mathematical analysis and to establish the corresponding well-posedness theory has been the subject of many recent researchs, cf. [5, 6, 10, 9, 14, 21, 17, 22, 23] and the references cited therein. These studies indicate that temperature dependence of the viscosity μ is especially challenging but one can incorporate various forms of density dependence in μ and also temperature dependence in κ. For results in this direction, Dafermos [5] and Dafermos and Hsiao [6] considered certain classes of solid-like materials in which the viscosity and/or the heat conductivity depend on density and where the heat conductivity may depend on temperature. However, the latter is assumed to be bounded as well as uniformly bounded away from zero. Kawohl [14] and Luo [17] considered a gas model that incorporates real-gas effects that occur in high-temperature regimes. In [14, 17] the viscosity depends only on density (or is constant) and it is uniformly bounded away from zero, while the thermal conductivity may depend on both density and temperature. For example, one of the assumptions in [14] is that there are constants $\kappa_0 > 0,\ \kappa_1 >$ such that $\kappa(v,\theta)$ satisfies $\kappa_0(1+\theta^q) \leq \kappa(v,\theta) \leq \kappa_1(1+\theta^q)$, where $q \geq 2$. This type of temperature dependence is motivated by experimental results for gases at very high temperatures, cf. Zeldovich and Raizer [27]. Jenssen and Karper [9] and Pan [22] studied the case when μ is a positive constant and $\kappa = \bar{k}\theta^b$ for some positive constant $\bar{k} > 0$. Such a study is motivated by the first level of approximation in kinetic theory, in which the viscosity μ and heat conductivity κ are power functions of the temperature alone.

We note, however, that in all the above studies although the viscosity coefficient μ may depend on v and the heat conductivity κ may depend on both v and θ, they ask that at least one of μ and κ is non-degenerate. What we are interested in this paper focuses on the case when μ is a function of v and κ depends on v and/or θ and both μ and κ are degenerate. To simplify the presentation, we will mainly concentrated on the case

$$\mu = v^{-a}, \quad \kappa = \theta^b \tag{1.7}$$

for some positive constants $a > 0,\ b > 0$ or for the case $a = 0$ but κ is a general smooth function of v and θ satisfying $\kappa(v,\theta) > 0$ for $v > 0,\ \theta > 0$. For such a case, it is worth to point out that for ideal polytropic gases, the assumptions imposed on μ in [14, 17] hold only when $a = 0$. That is the viscosity coefficient μ is a positive constant.

Now we turn to state the main results obtained in this paper. The first result is concerned with the initial-boundary value problem (1.1), (1.2), (1.3). In such a case, the transport coefficients μ and κ are assumed to satisfy one of the following two conditions

(i). μ is a positive constant and $\kappa(v,\theta)$ is a smooth function of v and θ satisfying $\kappa(v,\theta) > 0$ for $v > 0,\ \theta > 0$ and there exist positive constants μ_0 and $K\left(\widetilde{v},\widetilde{\theta}\right)$ such that

$$\mu(v,\theta) = \mu_0 > 0, \qquad \min_{v\geq\widetilde{v}>0,\theta\geq\widetilde{\theta}>0} \kappa(v,\theta) = K\left(\widetilde{v},\widetilde{\theta}\right) > 0 \tag{1.8}$$

hold true for each given positive constants $\widetilde{v} > 0$ and $\widetilde{\theta} > 0$;

(ii). μ and κ are given by (1.7) with the two positive constants a and b satisfying one of the following conditions

$$\begin{cases} \frac{1}{3} < a < \frac{1}{2}, & 1 \leq b < \frac{2a}{1-a}; \\ \frac{1}{3} < a < \frac{1}{2}, & \frac{2}{1+5a-6a^2} < b < 1. \end{cases} \tag{1.9}$$

And our result in this direction can be stated as

Theorem 1.1 *Suppose that $(v_0(x), u_0(x), \theta_0(x)) \in H^1(I)$. Let $\inf\limits_{x\in I} v_0(x) > 0,\ \inf\limits_{x\in I}\theta_0(x) > 0$ and assume that the initial data $(v_0(x), u_0(x), \theta_0(x))$ are compatible with the boundary condition (1.3). Then if the transport coefficients μ and κ are assumed to satisfy (1.8) or (1.7), (1.9), there exists a unique global solution $(v(x,t), u(x,t), \theta(x,t))$ to the initial-boundary value problem (1.1), (1.2), (1.3) which satisfies*

$$\begin{aligned} &(v(x,t), u(x,t), \theta(x,t)) \in C^0\left(0,T; H^1(I)\right), \\ &(u_x(x,t), \theta_x(x,t)) \in L^2\left(0,T; H^1(I)\right), \\ &\underline{V} \leq v \leq \overline{V}, \quad \underline{\Theta} \leq \theta \leq \overline{\Theta}, \quad \forall (x,t) \in I\times[0,T]. \end{aligned} \tag{1.10}$$

Here T is any given positive constant and $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ are some positive constants which may depend on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$.

Remark 1.1 *Several remarks concerning Theorem 1.1 are listed below:*

- *The initial-boundary value problem (1.1), (1.2), (1.3) has been studied in [14]. Since the argument developed by Kazhikhov and Shelukhin in [15] is used in [14] to deduce the desired lower and upper bounds on the specific volume v, the assumption that μ is a positive constant should be imposed. But in our Theorem 1.1, if we focus on the ideal polytropic gas, then, on the one hand, we can deal with the case when μ and κ are given by (1.7) with the two parameters a and b satisfying (1.9) (in such a case, both of them are degenerate) and on the other hand, even for the case when μ is a positive constant, we only need to ask the heat conductivity κ to satisfy (1.8) which can be degenerate.*

- *Note that for the case when the transport coefficients μ and κ are given by (1.7), the assumptions imposed on a and b in Theorem 1.1 exclude the case when $0 < a \leq \frac{1}{3}$. We are convinced that the arguments used here can be modified to cover such a case.*

For the initial-boundary value problem (1.1), (1.2), (1.4), we have the following result

Theorem 1.2 *Suppose that*

(i). $(v_0(x), u_0(x), \theta_0(x)) \in H^1(I)$, $\inf_{x\in I} v_0(x) > 0$, $\inf_{x\in I} \theta_0(x) > 0$, and the initial data $(v_0(x)$, $u_0(x)$, $\theta_0(x))$ are compatible with the boundary condition (1.4);

(ii). The transport coefficients μ and κ are assumed to satisfies one of the following two conditions

- *μ is a positive constant and κ satisfies $\kappa(v,\theta) > 0$ for $v > 0$, $\theta > 0$ and*

$$0 \leq \kappa(v,\theta) \leq C(V)(1+\theta^c), \quad 0 < V^{-1} \leq v \leq V \tag{1.11}$$

  holds for some positive constant $C(V) > 0$ and $\theta > 0$ sufficiently large. Here $0 \leq c < 1$ is a constant and $V > 0$ is any given positive constant;

- *μ and κ are given by (1.7) with a and b satisfying*

$$0 \leq a < \frac{1}{5}, \quad b \geq 2. \tag{1.12}$$

Then the initial-boundary value problem (1.1), (1.2), (1.4) admits a unique global solution $(v(x,t),\ u(x,t), \theta(x,t))$ such that (1.10) holds.

Remark 1.2 *The initial-boundary value problem (1.1), (1.2), (1.4) has also been studied in [14]. To deduce the desired lower and upper bound on the specific volume v, the viscosity coefficient $\mu(v)$ is assumed to satisfy*

$$0 < \mu_0 \leq \mu(v) \leq \mu_1 \tag{1.13}$$

and the entropy $s(v,\theta)$ and the internal energy $e(v,\theta)$ are assumed to satisfy

$$s(v,\theta) \leq \left(\left| \int_1^v \frac{\mu(z)}{z} dz \right|^r + 1 \right) e(v,\theta) \tag{1.14}$$

in [14]. Here μ_0, μ_1, and $r < 2$ are some positive constants. For the ideal polytropic gas, if the transport coefficients μ and κ are assumed to satisfy (1.7), (1.14) holds only if $a = 0$.

Finally, we consider the outer pressure problem (1.1), (1.2), (1.5). Under the assumption that the transport coefficients μ and κ satisfy (1.7) with

$$0 \leq a < \frac{1}{2}, \quad b \geq \frac{1}{2}, \tag{1.15}$$

we have

Theorem 1.3 *Suppose that $(v_0(x), u_0(x), \theta_0(x)) \in H^1(I)$. Let $\inf_{x\in I} v_0(x) > 0$, $\inf_{x\in I} \theta_0(x) > 0$, and assume that the initial data $(v_0(x), u_0(x), \theta_0(x))$ are compatible with the boundary condition (1.5). Then if the transport coefficients μ and κ are given by (1.7) with the two parameters a and b satisfying (1.15), the the initial-boundary value problem (1.1), (1.2) and (1.5) has a unique global solution $(v(x,t), u(x,t), \theta(x,t))$ satisfying (1.10).*

Remark 1.3 *In fact the outer pressure problem (1.1), (1.2), (1.5) was studied in [17] and the main purpose of [17] is to remove the assumption (1.14) needed in [14] in the study of the initial-boundary value problem (1.1), (1.2), (1.4). We note, however, that the assumption (1.13) is still imposed in [17] together with the assumption that the heat conductivity coefficient $\kappa(v,\theta)$ is non-degenerate.*

Before concluding this section, we outline the main ideas used to deduce our main results. Our analysis is based on the continuation argument and the main difficulty lies in how to control the possible growth of the solutions to the one-dimensional compressible Navier-Stokes equation (1.1) caused by the nonlinearities of the equation. If the initial data $(v_0(x), u_0(x), \theta_0(x))$ is a small perturbation of the non-vacuum constant state $(v, u, \theta) = \left(\overline{v}, 0, \overline{\theta}\right)$ with $\overline{v} > 0$ and $\overline{\theta} > 0$ being two given positive constants, even for the case when the transport coefficients μ and κ are general smooth functions of v and θ satisfying $\mu(v,\theta) > 0$ and $\kappa(v,\theta) > 0$ for $v > 0$, $\theta > 0$, the argument developed by Matsumura and Nishida in [18] can be used to deduce a satisfactory well-posedness theory in the class of functions which is a small perturbation of the constant state $(v, u, \theta) = \left(\overline{v}, 0, \overline{\theta}\right)$.

But for the construction of global non-vacuum solutions to the one-dimensional compressible Navier-Stokes equation with large amplitude, the story is quite different and the key point is to deduce the positive lower and upper bounds on the specific volume v and the absolute temperature θ. To give the main ideas used to deduce our main results, we first outline the main ideas used in [5, 6, 9, 14, 17, 22]: A key ingredient in all of these proofs in [5, 6, 9, 14, 17, 22] is to deduce the pointwise a priori estimates on the specific volume first which guarantee that no vacuum nor concentration of mass occur, and then based on some sophisticated energy type estimates, the upper bound on the absolute temperature can be obtained. The arguments used in [9, 14, 17, 22] to deduce the desired positive lower and upper bounds on the specific volume can be outlined as in the following:

- For the initial-boundary value problem (1.1), (1.2), (1.3), the viscosity coefficient μ is assumed to be a positive constant in [14] so that the argument developed in [15] together with the non-degenerate assumption on the heat conductivity coefficient κ can indeed yield the lower and upper bounds on v, cf. [9, 14, 22];

- For the initial-boundary value problem (1.1), (1.2), (1.4), the viscosity coefficient μ and the entropy $s(v,\theta)$, the internal energy $e(v,\theta)$ are assumed to satisfy (1.13) and (1.14) so that a upper bound on the term $|\int_1^v \mu(z)/z dz|$ can be obtained in [14], from which the desired estimates on v follow immediately. It is worth to pointing out that similar argument works for the out pressure problem (1.1), (1.2), (1.4), cf. [17]. In fact, as pointed out before, one of the main purpose of [17] is to remove the assumption (1.14) needed in [14].

But for the cases considered in this manuscript, the gas is assumed to be ideal polytropic and the transport coefficients μ and κ are degenerate, the above argument can not used to deduce the desired estimates on v first any longer. To overcome such a difficulty, our main tricks are the following:

(i). The first is to control the lower bound of the absolute temperature in terms of the lower bound of the specific volume;

(ii). Even for the case when the viscosity coefficient μ is a positive constant as in one of the two cases considered in Theorem 1.1, since the heat conductivity κ may be degenerate, we can not hope to deduce the desired bounds on v and θ as in [9, 14, 22]. That is to deduce the lower and upper bounds on v first and then to bound θ. Our trick is motivated by [23] and we first deduce the lower bound on v based on the explicit formula for v which is given in [15] for the case when both μ and κ are positive constants, from which and the first trick mentioned above we can deduce the lower bound on θ. With the lower bounds on v and θ in hand, we can then deduce an upper bound on v if the heat conductivity coefficient $\kappa(v,\theta)$ satisfies the assumption $\min\limits_{v\geq\widetilde{v}>0,\theta\geq\widetilde{\theta}>0}\kappa(v,\theta)=K\left(\widetilde{v},\widetilde{\theta}\right)>0$ for any given positive constants $\widetilde{v}>0$ and $\widetilde{\theta}>0$. Having obtained these bounds, the only thing left is to get the desired upper bound on θ and the argument used here to deduce such a bound is similar to those used in [5, 6, 9, 14, 17, 22];

(iii). When the transport coefficients μ and κ are given by (1.7) with $a>0, b>0$ as in the other case considered in Theorem 1.1, we had to estimate the lower and upper bounds on v and θ simultaneously. Our main idea is first to estimate the lower bound of θ in terms of the lower bound of v, cf. Lemma 2.2, then by employing Kanel's argument, cf. [12], to control the lower and upper bounds of v in terms of $\left\|\theta^{1-b}\right\|_\infty$ as in (2.68) and (2.69). These estimates together with the estimate on $\|\theta(t)\|_{L^\infty(I)}$, cf. (2.72), can yield the desired lower and upper bounds on v and θ provided that the two parameters a and b satisfy certain relations stated in Theorem 1.1;

(iv). The discussion on the initial-boundary value problem (1.1), (1.2), (1.4) is more subtle due to the boundary condition (1.4). Our main trick here is to recover the $L^1([0,1])-$estimate on v which is not obvious under the boundary condition (1.4).

The rest of the paper is organized as follows. The proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3 will be given in Section 2, Section 3, and Section 4, respectively.

Notations: Throughout this manuscript, $C>1$ is used to denote a generic constant, which may depend on $\inf\limits_{x\in I}v_0(x)$, $\inf\limits_{x\in I}\theta_0(x)$, T, and $\|(v_0,u_0,\theta_0)\|_{H^1(I)}$. Here $T>0$ is some given constant. $C(\cdot,\cdots,\cdot)$ is used to denote some positive constant depending only on the arguments listed in the parenthesis. Note that all these constants may vary in different places. $H^s(I)$ represents the usual Sobolev spaces on I with the standard norm $\|\cdot\|_{H^s(I)}$ and for $1\leq p\leq+\infty$, $L^p(I)$ denotes the usual L^p spaces equipped with the usual norm $\|\cdot\|_{L^p(I)}$. For simplicity, we use $\|\cdot\|_\infty$ to denote the norm in $L^\infty(I\times[0,T])$.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1 based on the continuation argument. Such an argument is a combination of the local existence result with certain a priori estimates on the local solutions constructed. Firstly we state the local solvability result as

Theorem 2.1 (Local existence result). *Under the assumptions in Theorem 1.1, there exists a sufficiently small positive constant t_1, which depends on $\inf_{x\in I} v_0(x)$, $\inf_{x\in I}\theta_0(x)$, and $\|(v_0,u_0,\theta_0)\|_{H^1(I)}$, such that the initial-boundary value problem (1.1), (1.2), (1.3) admits a unique smooth solution $(v(x,t),u(x,t),\theta(x,t))$ defined on $I\times[0,t_1]$.*

Moreover, $(v(x,t),u(x,t),\theta(x,t))$ satisfies

$$\begin{aligned}&(v(x,t),u(x,t),\theta(x,t))\in C^0\left(0,t_1;H^1(I)\right),\\ &(u_x(x,t),\theta_x(x,t))\in L^2\left(0,t_1;H^1(I)\right),\\ &\tfrac{1}{2}\inf_{x\in I} v_0(x)\le v(x,t)\le 2\sup_{x\in I} v_0(x),\quad \forall (x,t)\in I\times[0,t_1],\\ &\tfrac{1}{2}\inf_{x\in I} \theta_0(x)\le \theta(x,t)\le 2\inf_{x\in I} \theta_0(x),\quad \forall (x,t)\in I\times[0,t_1],\end{aligned}\tag{2.1}$$

and

$$\sup_{[0,t_1]}\left(\|(v,u,\theta)(t)\|_{H^1(I)}\right)\le 2\|(v_0,u_0,\theta_0)\|_{H^1(I)}.\tag{2.2}$$

Theorem 2.1 can be obtained by using a similar approach as in [15] or [24] in the three-dimensional case. We thus omit the details for brevity.

Suppose that the local solution $(v(x,t),u(x,t),\theta(x,t))$ constructed in Theorem 2.1 has been extended to the time step $t=T\ge t_1$ and satisfies the a priori assumption

$$(H)\qquad \underline{V}'\le v(x,t)\le \overline{V}',\quad \underline{\Theta}'\le\theta(x,t)\le\overline{\Theta}',\quad \forall (x,t)\in I\times[0,T].$$

Here $\underline{V}'$, $\overline{V}'$, $\underline{\Theta}'$, and $\overline{\Theta}'$ are some positive constants. To extend such a solution step by step to a global one, we only need to deduce certain a priori estimates on $(v(x,t),u(x,t),\theta(x,t))$ which are independent of $\underline{V}'$, $\overline{V}'$, $\underline{\Theta}'$, and $\overline{\Theta}'$ but may depend on T and the initial data $(v_0(x),u_0(x),\theta_0(x))$.

Using (1.6), we can rewrite (1.1) as

$$\begin{cases} v_t-u_x=0,\\ u_t+p_x=\left(\frac{\mu(v)u_x}{v}\right)_x,\\ C_v\theta_t+pu_x=\frac{\mu(v)u_x^2}{v}+\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x.\end{cases}\tag{2.3}$$

Set

$$\phi(x)=x-\ln x-1.\tag{2.4}$$

Note that

$$\eta(v,u,\theta)=R\phi(v)+\frac{u^2}{2}+C_v\phi(\theta)\tag{2.5}$$

is a convex entropy to (2.3) and satisfies

$$\eta(v,u,\theta)_t+(pu)_x+\frac{\mu(v)u_x^2}{v\theta}+\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}=\left(\frac{\mu(v)uu_x}{v}+p(1,1)u+\frac{\kappa(v,\theta)\theta_x(\theta-1)}{v\theta}\right)_x.\tag{2.6}$$

Then by integrating (2.6) with respect to x and t over $I\times[0,T]$ and with the help of integrations by parts and the boundary condition (1.3), we can deduce the following lemma:

Lemma 2.1 (Basic energy estimates). *Let the conditions in Theorem 2.1 hold and suppose that the local solution $(v(x,t),u(x,t),\theta(x,t))$ constructed in Theorem 2.1 satisfies the a priori assumption (H), then we have for $0\leq t\leq T$ that*

$$\int_0^1\eta(v,u,\theta)dx+\int_0^t\int_0^1\left(\frac{\mu(v)u_x^2}{v\theta}+\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\right)dxds=\int_0^1\eta(v_0,u_0,\theta_0)dx. \tag{2.7}$$

The next lemma is concerned with estimating the lower bound of $\theta(x,t)$ in terms of the lower bound of $v(x,t)$.

Lemma 2.2 *Under the condition listed in Lemma 2.1, we have*

$$\frac{1}{\theta(x,t)}\leq C+C\left\|\frac{1}{\mu(v)v}\right\|_\infty,\quad \forall(x,t)\in I\times[0,T]. \tag{2.8}$$

Proof: First of all, $(2.3)_3$ implies

$$C_v\left(\frac{1}{\theta}\right)_t=-\frac{\mu(v)u_x^2}{v\theta^2}+\frac{Ru_x}{v\theta}-\frac{1}{\theta^2}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x. \tag{2.9}$$

From (2.9), we can get for each $p>1$ that

$$\begin{aligned}&C_v\left[\left(\tfrac{1}{\theta}\right)^{2p}\right]_t+\tfrac{2p(2p+1)\kappa(v,\theta)\theta_x^2}{v\theta^{2p+2}}\\&=-2p\left(\tfrac{1}{\theta}\right)^{2p-1}\left[\tfrac{\mu(v)}{v}\left(\tfrac{u_x}{\theta}-\tfrac{R}{2\mu(v)}\right)^2-\tfrac{R^2}{4\mu(v)v}\right]-\left(\tfrac{2p\kappa(v,\theta)\theta_x}{v\theta^{2p+1}}\right)_x.\end{aligned} \tag{2.10}$$

Integrating (2.10) with respect to x over I, we have

$$C_v\left(\left\|\frac{1}{\theta}\right\|_{L^{2p}}^{2p}\right)_t\leq 2p\int_0^1\frac{R^2}{4\mu(v)v}\left(\frac{1}{\theta}\right)^{2p-1}dx\leq 2pC\left\|\frac{1}{\mu(v)v}\right\|_{L^{2p}}\left\|\frac{1}{\theta}\right\|_{L^{2p}}^{2p-1}, \tag{2.11}$$

which implies

$$\left\|\frac{1}{\theta}\right\|_{L^{2p}}\leq C\left(\inf_{x\in I}\theta_0(x)\right)^{-1}+C\int_0^t\left\|\frac{1}{\mu(v)v}\right\|_{L^{2p}}ds. \tag{2.12}$$

Letting $p\to+\infty$ in (2.12), we can deduce (2.8) immediately. This completes the proof of Lemma 2.2.

To derive bounds on the specific volume v, we first define

$$g(v):=\int_1^v\frac{\mu(\xi)}{\xi}d\xi. \tag{2.13}$$

Then we get

$$\left(\frac{\mu(v)u_x}{v}\right)_x=\left(\frac{\mu(v)v_t}{v}\right)_x=[g(v)]_{xt} \tag{2.14}$$

and $(2.3)_2$ can be rewritten as

$$u_t+p_x=[g(v)]_{xt}. \tag{2.15}$$

Integrating (2.15) over $[y,x]\times[0,t]$ yields

$$\begin{aligned}&-g(v(x,t))+\int_0^t p(x,s)ds\\&=\int_y^x(u_0(z)-u(z,t))dz-g(v(y,t))-g(v(x,0))+g(v(y,0))+\int_0^t p(y,s)ds.\end{aligned} \tag{2.16}$$

For the case when the transport coefficients $\mu(v)$, $\kappa(v,\theta)$ satisfy (1.8), we have the following result

Lemma 2.3 *Under the conditions listed in Lemma 2.1 and assume that the transport coefficients $\mu(v)$, $\kappa(v,\theta)$ satisfy (1.8), there exist positive constants $\underline{V}_1$, $\overline{V}_1$, and $\underline{\Theta}_1$ depending only on T and $(v_0(x),\ u_0(x), \theta_0(x))$ such that*

$$\underline{V}_1 \le v(x,t) \le \overline{V}_1, \quad \theta(x,t) \ge \underline{\Theta}_1, \quad \forall (x,t) \in I \times [0,T]. \tag{2.17}$$

Proof: Note that when the transport coefficients $\mu(v) \equiv \mu_0$ is a positive constant, we have

$$g(v) = \mu_0 \log v. \tag{2.18}$$

Without loss of generality, we assume $\int_0^1 v_0(x)dx = 1$. Thus integrating $(2.3)_1$ over $I \times [0,t]$ and using the boundary condition (1.3), we have

$$\int_0^1 v(x,t)dx = 1. \tag{2.19}$$

Hence for each $t \in [0,T]$, there exists at least one number $a(t) \in [0,1]$ such that

$$v(a(t),t) = 1. \tag{2.20}$$

Set $y = a(t)$ in (2.16), then by (2.18) and (2.20) we can obtain

$$\begin{aligned} &-\mu_0 \log v(x,t) + \int_0^t p(x,s)ds \\ &= \int_{a(t)}^x (u_0(z) - u(z,t))dz - \mu_0 \log v(x,0) + \mu_0 \log v(a(t),0) + \int_0^t p(a(t),s)ds. \end{aligned} \tag{2.21}$$

Multiplying (2.21) by μ_0^{-1} and then taking the exponentials on the resulting identity, we arrive at

$$\frac{1}{v(x,t)} \exp\left\{\frac{1}{\mu_0}\int_0^t p(x,s)ds\right\} = Y(t)B(x,t), \tag{2.22}$$

where

$$Y(t) = v_0(a(t)) \exp\left\{\frac{1}{\mu_0}\int_0^t p(a(t),s)ds\right\}, \quad B(x,t) = \frac{1}{v_0(x)} \exp\left\{\frac{1}{\mu_0}\int_{a(t)}^x (u_0(z) - u(z,t))dz\right\}. \tag{2.23}$$

For $Y(t)$, we can deduce immediately that

$$Y(t) \ge v_0(a(t)) \ge C^{-1} > 0, \quad \forall t \in [0,T], \tag{2.24}$$

and by (2.7) we have

$$C^{-1} \le B(x,t) \le C, \quad \forall (x,t) \in I \times [0,T]. \tag{2.25}$$

Now we turn to estimate the upper bound on $Y(t)$. Using the argument in [15] and by (2.22) we have

$$v(x,t)Y(t) = B^{-1}(x,t)\left(1 + \frac{1}{\mu_0}\int_0^t p(x,s)v(x,s)Y(s)B(x,s)ds\right). \tag{2.26}$$

Integrating (2.26) with respect to x over I and using (2.7), (2.19), and (2.25), we have

$$\begin{aligned} Y(t) &\leq C + C\int_0^t Y(s)\int_0^1 p(x,s)v(x,s)dxds \\ &\leq C + C\int_0^t Y(s)\int_0^1 \theta dxds \\ &\leq C + C\int_0^t Y(s)ds, \end{aligned} \tag{2.27}$$

then by Gronwall inequality, we get

$$Y(t) \leq C, \quad \forall t \in [0,T]. \tag{2.28}$$

This together with (2.26), we arrive at the lower bound on v, i.e.

$$v(x,t) \geq \underline{V}_1, \quad \forall (x,t) \in I \times [0,T] \tag{2.29}$$

holds for some positive constant $\underline{V}_1$.

(2.29) together with (2.8), we can easily get the lower bound on $\theta(x,t)$. That is, there exists a positive constant $\overline{\Theta}_1$ depending on T and $(v_0(x), u_0(x), \theta_0(x))$ such that

$$\theta(x,t) \geq \underline{\Theta}_1, \quad \forall (x,t) \in I \times [0,T]. \tag{2.30}$$

Next we have to estimate the upper bound on $v(x,t)$ to finish the proof of Lemma 2.3. First the assumption (1.8) together with the estimates (2.29)–(2.30) imply that

$$\kappa(v,\theta) \geq K \tag{2.31}$$

holds for some positive number K depending on $\underline{\Theta}_1$ and $\underline{V}_1$ for all v and θ under our consideration.

From (2.7) we have that for each t, there exists at least one number $b(t) \in I$ such that $\theta(b(t),t) \leq C$. Then we have

$$\begin{aligned} \theta(x,t) &\leq 4\theta(b(t),t) + 2\left(\sqrt{\theta(x,t)} - \sqrt{\theta(b(t),t)}\right)^2 \\ &\leq 4\theta(b(t),t) + \frac{1}{2}\left(\int_{b(t)}^x \frac{\theta_y(y,s)}{\sqrt{\theta(y,s)}}dy\right)^2 \\ &\leq C + C\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dx \int_0^1 \frac{v\theta}{\kappa(v,\theta)}dx \\ &\leq C + C\|v(t)\|_{L^\infty(I)}\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dx. \end{aligned} \tag{2.32}$$

This together with (2.24)–(2.25) and (2.28), we can deduce from (2.26) that

$$\begin{aligned} v(x,t) &\leq Y^{-1}(t)B^{-1}(x,t)\left(1 + \frac{R}{\mu_0}\int_0^t \theta(x,s)Y(s)B(x,s)ds\right) \\ &\leq C + C\int_0^t \|\theta(s)\|_{L^\infty(I)}ds \\ &\leq C + C\int_0^t \|v(s)\|_{L^\infty(I)}\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dxds. \end{aligned} \tag{2.33}$$

Thus with the aid of the Gronwall inequality and (2.7), we can get the upper bound on $v(x,t)$, which completes the proof of Lemma 2.3.

Now we turn to deduce the upper bound on $\theta(x,t)$. For this purpose, an immediate consequence of (2.32) and (2.17) is

Corollary 2.1 *Under the conditions listed in Lemma 2.3, we have for $0 \le t \le T$ that*

$$\int_0^t \|\theta(s)\|_{L^\infty(I)} ds \le C \tag{2.34}$$

and

$$\int_0^t \int_0^1 \theta^2(x,s) dx ds \le C. \tag{2.35}$$

By (2.35), we can obtain

Lemma 2.4 *Under the conditions listed in Lemma 2.3, we have for $0 \le t \le T$ that*

$$\int_0^1 u^2 dx + \int_0^t \int_0^1 u_x^2 dx ds \le C. \tag{2.36}$$

Proof: Multiplying $(2.3)_2$ by u and integrating the resulting equation with respect to x and t over $I \times [0,t]$, one has

$$\int_0^1 \frac{u^2}{2} dx + \int_0^t \int_0^1 \frac{\mu u_x^2}{v} dx ds \le C\|u_0\|_{L^2}^2 + C \int_0^t \int_0^1 \frac{\theta^2}{\mu v} dx ds. \tag{2.37}$$

Thus applying (2.17) and (2.35), we get (2.36). This proves Lemma 2.4.

To estimate the upper bound on θ, we get by employing the argument used in [23] that

Lemma 2.5 *Under the conditions listed in Lemma 2.3, we have for $0 \le t \le T$ that*

$$\|\theta(t)\|_{L^\infty(I)} \le C + C \int_0^t \left(\|u_x(s)\|_{L^\infty(I)}^2 + \|\theta(s)\|_{L^\infty(I)}^2 \right) ds. \tag{2.38}$$

Proof: From $(2.3)_3$, we can easily deduce for each $p > 1$ that

$$C_v \left(\theta^{2p}\right)_t + 2p(2p-1)\theta^{2p-2} \frac{\kappa \theta_x^2}{v} = \left(2p\theta^{2p-1} \frac{\kappa \theta_x}{v}\right)_x + 2p\theta^{2p-1} \frac{\mu u_x^2}{v} - 2p\theta^{2p-1} \frac{R\theta u_x}{v}. \tag{2.39}$$

Integrating (2.39) with respect to x over I, one has

$$C_v \left(\|\theta(t)\|_{L^{2p}}^{2p}\right)_t \le 2p \int_0^1 \theta^{2p-1} \frac{\mu u_x^2}{v} dx - 2pR \int_0^1 \theta^{2p-1} \frac{R\theta u_x}{v} dx. \tag{2.40}$$

By exploiting the Holder inequality and letting $p \to +\infty$, we get from (2.40) that

$$\|\theta(t)\|_{L^\infty(I)} \le C\|\theta_0\|_{L^\infty(I)} + C \int_0^t \left(\left\| \frac{\mu u_x^2}{v} \right\|_{L^\infty(I)} + \left\| \frac{\theta u_x}{v} \right\|_{L^\infty(I)} \right) ds. \tag{2.41}$$

Then with the help of (2.29) and Cauchy's inequality, we can deduce (2.38) and the proof of Lemma 2.5 is complete.

To estimate $\int_0^t \|\theta(s)\|_{L^\infty(I)}^2 ds$, we need the following result

Lemma 2.6 *Under the conditions listed in Lemma 2.3, we have for $0 \le t \le T$ that*

$$\int_0^t \int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{\theta^{1-r}} dx ds \le C + C\|\theta\|_\infty^r, \quad \forall r \in (0,1). \tag{2.42}$$

Proof: Multiplying $(2.3)_3$ by θ^r and integrating the resulting equation with respect to x over I yield

$$\begin{aligned}
&C_v\int_0^1\theta^{1+r}dx+\int_0^t\int_0^1\frac{r\kappa(v,\theta)\theta_x^2}{v\theta^{1-r}}dxds\\
&=C_v\int_0^1\theta_0^{1+r}dx+\int_0^t\int_0^1\frac{\mu_0\theta^r u_x^2}{v}dxds-R\int_0^t\int_0^1\frac{\theta^{r+1}u_x}{v}dxds\\
&\le C\|\theta_0\|_{L^\infty}^{1+r}+C\|\theta\|_\infty^r\left(\int_0^t\int_0^1\theta^2dxds+\int_0^t\int_0^1u_x^2dxds\right)\\
&\le C+C\|\theta\|_\infty^r,
\end{aligned}\tag{2.43}$$

where (2.35) and (2.36) are used. This is (2.42) and completes the proof of Lemma 2.6.

A direct consequence of (2.42) is

Lemma 2.7 *Under the conditions listed in Lemma 2.3, we have for $0\le t\le T$ that*

$$\int_0^t\|\theta(s)\|_{L^\infty(I)}^2ds\le C+C\|\theta\|_\infty^{\frac12}.\tag{2.44}$$

Proof: Observe that (2.31) and (2.7) imply

$$\begin{aligned}
\theta^2(x,t)&=\theta^2(b(t),t)+\int_{b(t)}^x2\theta(y,t)\theta_y(y,t)dy\\
&\le C+C\|\theta(t)\|_{L^\infty(I)}^{1-\frac r2}\left(\int_0^1\theta(x,t)dx\right)^{\frac12}\left(\int_0^1\left(\frac{\theta_x^2}{\theta^{1-r}}\right)(x,t)dx\right)^{\frac12}\\
&\le C+C\|\theta(t)\|_{L^\infty(I)}^{1-\frac r2}\left(\int_0^1\left(\frac{\theta_x^2}{\theta^{1-r}}\right)(x,t)dx\right)^{\frac12}.
\end{aligned}$$

From the above inequality together with the estimates (2.42) and (2.34), we can get that

$$\begin{aligned}
\int_0^t\|\theta(s)\|_{L^\infty(I)}^2ds&\le C+C\int_0^t\left(\|\theta(s)\|_{L^\infty(I)}^{1-\frac r2}\left(\int_0^1\left(\frac{\theta_x^2}{\theta^{1-r}}\right)(x,s)dx\right)^{\frac12}\right)ds\\
&\le C+C\left(\int_0^t\|\theta(s)\|_{L^\infty(I)}^{2-r}ds\right)^{\frac12}\left(\int_0^t\int_0^1\left(\frac{\theta_x^2}{\theta^{1-r}}\right)(x,s)dxds\right)^{\frac12}\\
&\le C+C\|\theta\|_\infty^{\frac{1-r}{2}}\left(\int_0^t\|\theta(s)\|_{L^\infty(I)}ds\right)^{\frac12}\left(\int_0^t\int_0^1\left(\frac{\theta_x^2}{\theta^{1-r}}\right)(x,s)dxds\right)^{\frac12}\\
&\le C+C\|\theta\|_\infty^{\frac12}.
\end{aligned}\tag{2.45}$$

This is exactly (2.44) and the proof of Lemma 2.7 is complete.

Now we turn to estimate the term $\int_0^t\|u_x(s)\|_{L^\infty(I)}^2ds$ on the right-hand side of (2.38). To do so, we shall estimate $\int_0^1v_x^2dx$ first which is main content of the following lemma

Lemma 2.8 *Under the conditions listed in Lemma 2.3, we have for $0\le t\le T$ that*

$$\int_0^1v_x^2dx+\int_0^t\int_0^1\theta v_x^2dxds\le C+C\|\theta\|_\infty^r,\quad\forall r\in(0,1).\tag{2.46}$$

Proof: As in (2.14), we can rewrite $(2.3)_2$ as

$$u_t + \left(\frac{R\theta}{v}\right)_x = \left(\frac{\mu u_x}{v}\right)_x = \left(\frac{\mu v_t}{v}\right)_x = \left(\frac{\mu v_x}{v}\right)_t. \tag{2.47}$$

Multiplying the identity (2.47) by $\frac{\mu v_x}{v}$, we get that

$$\left(\frac{\mu^2 v_x^2}{2v^2}\right)_t = \left(\frac{\mu u v_x}{v}\right)_t - \left(\frac{\mu u u_x}{v}\right)_x + \frac{\mu u_x^2}{v} + \frac{R\mu v_x \theta_x}{v^2} - \frac{R\mu\theta v_x^2}{v^3}. \tag{2.48}$$

Integrating (2.48) with respect to x and t over $I\times[0,t]$ and with the aid of (2.7) and Cauchy's inequality, we get

$$\begin{aligned}\int_0^1 v_x^2 dx + \int_0^t\int_0^1 \theta v_x^2 dxds &\le C\left(\underline{V}_1, \overline{V}_1, \|v_{0x}\|_{L^2}\right) + C\int_0^t\int_0^1 \frac{u_x^2}{v} dxds + C\int_0^t\int_0^1 \frac{\theta_x^2}{\theta} dxds\\ &\le C + C\int_0^t\int_0^1 \frac{\theta_x^2}{\theta^{1-r}} dxds\\ &\le C + C\|\theta\|_\infty^r, \quad \forall r\in(0,1).\end{aligned} \tag{2.49}$$

This is (2.46) and the proof of Lemma 2.8 is completed.

On the other hand, noticing that

$$u_x^2(y,t) \le \int_0^1 u_x^2(x,t)dx + 2\int_0^1 |u_x(x,t)||u_{xx}(x,t)|dx, \tag{2.50}$$

we have from (2.36) and Holder's inequality that

Lemma 2.9 *Under the conditions listed in Lemma 2.3, we have for $0\le t\le T$ that*

$$\int_0^t \|u_x(s)\|^2_{L^\infty(I)} ds \le C + C\left(\int_0^t\int_0^1 u_{xx}(x,s)^2 dxds\right)^{\frac{1}{2}}. \tag{2.51}$$

Next we need to estimate $\int_0^t\int_0^1 u_{xx}^2 dxds$. To this end, differentiating $(2.3)_2$ with respect to x and multiplying the resulting equation by u_x, one has

$$\left(\frac{u_x^2}{2}\right)_t = \left[\left(\frac{\mu u_x}{v} - \frac{R\theta}{v}\right)_x u_x\right]_x - \left(\frac{\mu u_{xx}}{v} - \frac{\mu u_x v_x}{v^2} - \frac{R\theta_x}{v} + \frac{R\theta v_x}{v^2}\right)u_{xx}. \tag{2.52}$$

Note that the term $\left(\frac{\mu u_x}{v} - \frac{R\theta}{v}\right)_x\Big|_{x=0,1} = u_t|_{x=0,1} = 0$, one has by integrating (2.52) with respect to x and t over $I\times[0,t]$ that

$$\begin{aligned}&\int_0^1 u_x^2 dx + \int_0^t\int_0^1 u_{xx}^2 dxds\\ &\le C\left(\underline{V}_1, \overline{V}_1, \|u_{0x}\|_{L^2}\right) + C\int_0^t\int_0^1 \left(u_x^2 v_x^2 + \theta_x^2 + \theta^2 v_x^2\right) dxds\\ &\le C + C\int_0^t \left(\|u_x(s)\|^2_{L^\infty(I)} + \|\theta(s)\|^2_{L^\infty(I)}\right)\int_0^1 v_x^2 dxds + C\int_0^t\int_0^1 \theta_x^2 dxds\\ &\le C\left(1 + \|\theta\|_\infty^{\max\{\frac{1}{2}+r, 2r, 1\}}\right) + \frac{1}{2}\int_0^t\int_0^1 u_{xx}^2 dxds.\end{aligned} \tag{2.53}$$

Here we use the fact that

$$\begin{aligned}
&\int_0^t \left(\|u_x(s)\|^2_{L^\infty(I)} + \|\theta(s)\|^2_{L^\infty(I)}\right) \int_0^1 v_x^2 dxds\\
&\leq C\left(1+\|\theta\|_\infty^r\right)\left(1+\|\theta\|_\infty^{\frac{1}{2}}+\int_0^t \|u_{xx}(s)\|_{L^2(I)}\, ds\right)\\
&\leq C\left(1+\|\theta\|_\infty^{\max\{\frac{1}{2}+r,2r\}}\right)+\frac{1}{2}\int_0^t\int_0^1 u_{xx}^2 dxds,
\end{aligned} \tag{2.54}$$

and

$$\int_0^t\int_0^1 \theta_x^2 dxds \leq \|\theta\|_\infty^{1-r}\int_0^t\int_0^1 \frac{\theta_x^2}{\theta^{1-r}}dxds \leq C\left(1+\|\theta\|_\infty\right). \tag{2.55}$$

Thus we can immediately derive from (2.53) that

$$\int_0^t\int_0^1 u_{xx}^2 dxds \leq C\left(1+\|\theta\|_\infty^{\max\{\frac{1}{2}+r,2r,1\}}\right), \tag{2.56}$$

which combining with (2.51) implies

$$\int_0^t \|u_x(s)\|^2_{L^\infty(I)} ds \leq C\left(1+\|\theta\|_\infty^{\max\{\frac{1+2r}{4},r,\frac{1}{2}\}}\right). \tag{2.57}$$

Hence together with (2.38), (2.44), and (2.57), we can obtain the upper bound on $\theta(x,t)$ immediately since the parameter $r>0$ can be chosen sufficiently small.

Now we turn to deal with the case when the transport coefficients μ and κ satisfy (1.7) and (1.9). By (2.8) we have

$$\frac{1}{\theta(x,t)} \leq C + C\left\|\frac{1}{v}\right\|_\infty^{1-a}, \quad \forall (x,t)\in I\times[0,t] \tag{2.58}$$

For $\epsilon > -b$, since

$$\begin{aligned}
\int_0^t \|\theta(s)\|^{b+\epsilon}_{L^\infty(I)} ds &\leq C + C\int_0^t\left(\int_0^1 \theta^{\frac{b+\epsilon}{2}-1}|\theta_x| dx\right)^2 ds\\
&\leq C + C\int_0^t\left(\int_0^1 \frac{\theta^{b-2}\theta_x^2}{v}dx\right)\left(\int_0^1 v\theta^\epsilon dx\right) ds,
\end{aligned} \tag{2.59}$$

we can deduce that

$$\int_0^t \|\theta(s)\|^{b+\epsilon}_{L^\infty(I)} ds \leq C + C\left\|\theta^\epsilon\right\|_\infty, \tag{2.60}$$

or

$$\int_0^t \|\theta(s)\|^{b+\epsilon}_{L^\infty(I)} ds \leq C + C\|v\|_\infty\left\|\theta^{\epsilon-1}\right\|_\infty. \tag{2.61}$$

And

$$\int_0^t\int_0^1 \theta^2 dxds \leq C\int_0^t \|\theta(s)\|_{L^\infty(I)} ds \leq C + C\left\|\theta^{1-b}\right\|_\infty, \tag{2.62}$$

where we have taken $\epsilon = 1-b$ in (2.60).

From (2.37), we have

$$\int_0^1 u^2 dx + \int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}dxds \leq C + C\int_0^t\int_0^1 \frac{\theta^2}{v^{1-a}}dxds \leq C + C\left\|\frac{1}{v}\right\|_\infty^{1-a}\left\|\theta^{1-b}\right\|_\infty. \tag{2.63}$$

On the other hand, integrating (2.48) over $I \times [0,T]$, we get

$$
\begin{aligned}
&\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx + \int_0^t\int_0^1 \frac{\theta v_x^2}{v^{3+a}}dxds \\
&\leq C + C\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}dxds + C\int_0^t\int_0^1 \frac{\theta_x^2}{v^{1+a}\theta}dxds \\
&\leq C + C\left\|\frac{1}{v}\right\|_\infty^{1-a}\left\|\theta^{1-b}\right\|_\infty + C\left\|\frac{1}{v}\right\|_\infty^{a}\left\|\theta^{1-b}\right\|_\infty\int_0^t\int_0^1 \frac{\theta^{b-2}\theta_x^2}{v}dxds \\
&\leq C + C\left(\left\|\frac{1}{v}\right\|_\infty^{1-a} + \left\|\frac{1}{v}\right\|_\infty^{a}\right)\left\|\theta^{1-b}\right\|_\infty.
\end{aligned}
\tag{2.64}
$$

Set

$$
\Phi(v) = \int_1^v \frac{\sqrt{\phi(z)}}{z^{1+a}}dz, \tag{2.65}
$$

it is easy to see that there exist two positive constant C_1 and C_2 such that

$$
|\Phi(v)| \geq C_1\left(v^{-a} + v^{\frac{1}{2}-a}\right) - C_2. \tag{2.66}
$$

Since

$$
\begin{aligned}
|\Phi(v)| &= \left|\int_0^x \Phi(v(y,t))_y dy\right| \\
&\leq \int_0^1 \left|\frac{\sqrt{\phi(v)}}{v^{1+a}}v_x\right| dx \\
&\leq \left(\int_0^1 \phi(v)dx\right)^{\frac{1}{2}}\left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx\right)^{\frac{1}{2}} \\
&\leq C + C\left(\left\|\frac{1}{v}\right\|_\infty^{\frac{1-a}{2}} + \left\|\frac{1}{v}\right\|_\infty^{\frac{a}{2}}\right)\left\|\theta^{1-b}\right\|_\infty^{\frac{1}{2}}.
\end{aligned}
\tag{2.67}
$$

Combining (2.67) with (2.66), and making use of the Young inequality, we have

$$
\frac{1}{v(x,t)} \leq C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{1}{3a-1}} \tag{2.68}
$$

and

$$
v(x,t) \leq C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{2a}{(3a-1)(1-2a)}}. \tag{2.69}
$$

With (2.68) and (2.69) in hand, (2.63)–(2.64) can be rewritten as

$$
\int_0^1 u^2dx + \int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}dxds \leq C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{2a}{3a-1}} \tag{2.70}
$$

and

$$
\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx + \int_0^t\int_0^1 \frac{\theta v_x^2}{v^{3+a}}dxds \leq C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{2a}{3a-1}}. \tag{2.71}
$$

From (2.41), we get

$$
\|\theta(t)\|_{L^\infty(I)} \leq C + C\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty(I)} + \left\|\frac{\theta^2}{v^{1-a}}\right\|_{L^\infty(I)}\right)ds. \tag{2.72}
$$

Thus to deduce a nice bound on $\|\theta(t)\|_{L^\infty(I)}$, we need to estimate $\int_0^t \left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty(I)} ds$ and $\int_0^t \left\|\frac{\theta^2}{v^{1-a}}\right\|_{L^\infty(I)} ds$. The next lemma is concerned with the first term

Lemma 2.10 *Under the conditions listed in Lemma 2.1 and assume that the transport coefficients μ and κ satisfy (1.7) and (1.9), we have for $0 \le t \le T$ that*

$$\int_0^1 u_x^2 dx + \int_0^t\int_0^1 \frac{u_{xx}^2}{v^{1+a}} dxds \le C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{6a-8a^2}{(3a-1)(1-2a)}} + C\left\|\theta^{1-b}\right\|_\infty^{\frac{2a^2}{(3a-1)(1-2a)}}\left\|\theta^{1-b-\delta}\right\|_\infty \|\theta\|_\infty^\delta. \tag{2.73}$$

Here $\delta > 0$ is a positive constant which can be chosen as small as wanted.

Proof: Integrating (2.52) with respect to x and t over $I \times [0,t]$, we have

$$\int_0^1 u_x^2 dx + \int_0^t\int_0^1 \frac{u_{xx}^2}{v^{1+a}} dxds \le C + C\int_0^t\int_0^1 \left(\frac{u_x^2 v_x^2}{v^{3+a}} + \frac{\theta^2 v_x^2}{v^{3+a}} + \frac{\theta_x^2}{v^{1-a}}\right) dxds, \tag{2.74}$$

and the terms on the right-hand side of (2.74) can be estimated term by term as in the following.

First, (2.68)–(2.70) together with (2.50) imply that

$$\begin{aligned}\int_0^t \|u_x(s)\|_{L^\infty(I)}^2 ds &\le C\int_0^t\int_0^1 u_x^2 dxds + C\left(\int_0^t\int_0^1 u_x^2 dxds\right)^{\frac12}\left(\int_0^t\int_0^1 u_{xx}^2 dxds\right)^{\frac12}\\ &\le C\|v\|_\infty^{1+a}\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}} dxds \\ &\quad + C\|v\|_\infty^{1+a}\left(\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}} dxds\right)^{\frac12}\left(\int_0^t\int_0^1 \frac{u_{xx}^2}{v^{1+a}} dxds\right)^{\frac12}\\ &\le C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{4a-2a^2}{(3a-1)(1-2a)}} + C\left\|\theta^{1-b}\right\|_\infty^{\frac{3a}{(3a-1)(1-2a)}}\left(\int_0^t\int_0^1 \frac{u_{xx}^2}{v^{1+a}} dxds\right)^{\frac12},\end{aligned} \tag{2.75}$$

then by (2.71), the first term on right-hand side of (2.74) can be controlled by

$$\begin{aligned}\int_0^t\int_0^1 \frac{u_x^2 v_x^2}{v^{3+a}} dxds &\le \left\|\frac1v\right\|_\infty^{1-a}\int_0^t \|u_x(s)\|_{L^\infty(I)}^2\left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}} dx\right) ds\\ &\le C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{2+4a-4a^2}{(3a-1)(1-2a)}} + \frac12\int_0^t\int_0^1 \frac{u_{xx}^2}{v^{1+a}} dxds.\end{aligned} \tag{2.76}$$

Secondly, taking $\epsilon = 2 - b$ in (2.61), we have

$$\int_0^t \|\theta(s)\|_{L^\infty(I)}^2 ds \le C + C\|v\|_\infty\left\|\theta^{1-b}\right\|_\infty \le C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{7a-6a^2-1}{(3a-1)(1-2a)}}, \tag{2.77}$$

and the second term on right-hand side of (2.74) can be estimated as

$$\begin{aligned}\int_0^t\int_0^1 \frac{\theta^2 v_x^2}{v^{3+a}} dxds &\le \left\|\frac1v\right\|_\infty^{1-a}\int_0^t \|\theta(s)\|_{L^\infty(I)}^2\left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}} dx\right) ds\\ &\le C + C\left\|\theta^{1-b}\right\|_\infty^{\frac{6a-8a^2}{(3a-1)(1-2a)}}.\end{aligned} \tag{2.78}$$

To bound the third term on right-hand side of (2.74), we have by multiplying $(2.3)_3$ by θ^δ with δ being an arbitrary positive number, and integrating the result equation with respect to x and t over $I\times[0,t]$ that

$$\begin{aligned}\int_0^1\theta^{1+\delta}dx+\int_0^t\int_0^1\frac{\theta^{b-1+\delta}\theta_x^2}{v}dxds&\le C+C\int_0^t\int_0^1\frac{u_x^2\theta^\delta}{v^{1+a}}+C\int_0^t\int_0^1\frac{\theta^{2+\delta}}{v^{1-a}}dxds\\&\le C+C\left\|\theta^{1-b}\right\|_\infty^{\frac{2a}{3a-1}}\|\theta\|_\infty^\delta.\end{aligned}\tag{2.79}$$

From which we can deduce that

$$\begin{aligned}\int_0^t\int_0^1\frac{\theta_x^2}{v^{1-a}}dxds&\le\|v\|_\infty^a\left\|\theta^{1-b-\delta}\right\|_\infty\int_0^t\int_0^1\frac{\theta^{b-1+\delta}\theta_x^2}{v}dxds\\&\le C+C\left\|\theta^{1-b}\right\|_\infty^{\frac{2a^2}{(3a-1)(1-2a)}}\left\|\theta^{1-b-\delta}\right\|_\infty\|\theta\|_\infty^\delta.\end{aligned}\tag{2.80}$$

Thus (2.74) together with (2.76), (2.78), and (2.80) imply (2.73) and proof of the lemma is complete.

Plunging (2.73) into (2.75), and by using (2.68), we have

Lemma 2.11 *Under the conditions listed in Lemma 2.10, we have for $0\le t\le T$ that*

$$\int_0^t\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty(I)}ds\le C+C\left\|\theta^{1-b}\right\|_\infty^{\frac{1+5a-6a^2}{(3a-1)(1-2a)}}+C\left\|\theta^{1-b}\right\|_\infty^{\frac{1+2a-a^2}{(3a-1)(1-2a)}}\left\|\theta^{1-b-\delta}\right\|_\infty^{\frac12}\|\theta\|_\infty^{\frac{\delta}{2}}.\tag{2.81}$$

Here $\delta>0$ is a sufficiently small positive constant.

Now we turn to estimate $\int_0^t\left\|\frac{\theta^2}{v^{1-a}}\right\|_{L^\infty(I)}ds$. For result in this direction, we have

Lemma 2.12 *Under the conditions listed in Lemma 2.10, we have for $0\le t\le T$ that*

$$\int_0^t\left\|\frac{\theta^2}{v^{1-a}}\right\|_{L^\infty(I)}ds\le C+C\left\|\theta^{1-b}\right\|_\infty^{\frac{4a-4a^2}{(3a-1)(1-2a)}}.\tag{2.82}$$

Proof: Taking $\epsilon=1$ in (2.61), one has

$$\begin{aligned}\int_0^t\left\|\frac{\theta^2}{v^{1-a}}\right\|_{L^\infty(I)}ds&\le\left\|\frac1v\right\|_\infty^{1-a}\int_0^t\left\|\theta^2(s)\right\|_{L^\infty(I)}ds\\&\le C\left(1+\left\|\theta^{1-b}\right\|_\infty^{\frac{1-a}{3a-1}}\right)\left\|\theta^{1-b}\right\|_\infty\int_0^t\left\|\theta^{1+b}(s)\right\|_{L^\infty(I)}ds\\&\le C\left(1+\left\|\theta^{1-b}\right\|_\infty^{\frac{2a}{3a-1}}\right)(1+\|v\|_\infty).\end{aligned}\tag{2.83}$$

Thus with the aid of (2.69), we get (2.82) and completes the proof of Lemma 2.12.

Putting (2.81)–(2.82) together, we derive from (2.72) that for $\forall\delta>0$,

$$\|\theta(t)\|_{L^\infty(I)}\le C+C\left\|\theta^{1-b}\right\|_\infty^{\frac{1+5a-6a^2}{(3a-1)(1-2a)}}+C\left\|\theta^{1-b}\right\|_\infty^{\frac{1+2a-a^2}{(3a-1)(1-2a)}}\left\|\theta^{1-b-\delta}\right\|_\infty^{\frac12}\|\theta\|_\infty^{\frac{\delta}{2}}.\tag{2.84}$$

With the above preparations in hand, we now turn to deduce the desired lower and upper bounds on v and θ for the case when the transport coefficients μ and θ are given by (1.7). In fact we have

Corollary 2.2 *Under the conditions listed in Lemma 2.10, if we further assume that* $\frac{1}{3} < a < \frac{1}{2}$ *and* b *satisfies one of the following two conditions*

(i). $1 \leq b < \dfrac{2a}{1-a}$;

(ii). $\dfrac{2}{1+5a-6a^2} < b < 1.$

Then there exist positive constants $\underline{V}_2$, $\overline{V}_2$, $\underline{\Theta}_2$, *and* $\overline{\Theta}_2$, *such that*

$$\underline{V}_2 \leq v(x,t) \leq \overline{V}_2, \quad \underline{\Theta}_2 \leq \theta(x,t) \leq \overline{\Theta}_2, \quad \forall (x,t) \in I \times [0,t]. \tag{2.85}$$

Proof: We first consider the case $b \geq 1$. In such a case, as a direct consequence of (2.58) and (2.68), we have

$$\frac{1}{\theta(x,t)} \leq C + C \left\| \theta^{1-b} \right\|_\infty^{\frac{1-a}{3a-1}} \leq C + C \left\| \frac{1}{\theta} \right\|_\infty^{\frac{(1-a)(b-1)}{3a-1}}, \tag{2.86}$$

which implies, under the assumption $1 < b < \dfrac{2a}{1-a}$, that there exists one positive constant $\underline{\Theta}_2$ such that

$$\theta(x,t) \geq \underline{\Theta}_2, \quad \forall (x,t) \in I \times [0,t]. \tag{2.87}$$

And (2.68)–(2.69) together with the fact that $b \geq 1$ and (2.87) imply that there exist two positive constants $\underline{V}_2$ and $\overline{V}_2$, such that

$$\underline{V}_2 \leq v(x,t) \leq \overline{V}_2, \quad \forall (x,t) \in I \times [0,t]. \tag{2.88}$$

On the other hand, note that we can choose δ small enough in (2.84), then the upper bound on $\theta(x,t)$ can be obtained by the Young inequality.

When $b < 1$, by choosing some δ belonging to $(0, \frac{1-b}{2}]$, we have from (2.84) that

$$\begin{aligned} \|\theta(t)\|_{L^\infty(I)} &\leq C + C\|\theta\|_\infty^{\frac{(1+5a-6a^2)(1-b)}{(3a-1)(1-2a)}} + C\|\theta\|_\infty^{\frac{(1+2a-a^2)(1-b)}{(3a-1)(1-2a)}} \|\theta\|_\infty^{\frac{1-b-\delta}{2}} \|\theta\|_\infty^{\frac{\delta}{2}} \\ &\leq C + C\|\theta\|_\infty^{\frac{(1+5a-6a^2)(1-b)}{(3a-1)(1-2a)}}. \end{aligned} \tag{2.89}$$

Hence under the assumption $\dfrac{2}{1+5a-6a^2} < b < 1$, we deduce the upper bound on $\theta(x,t)$ from (2.89).

With this, the lower and upper bound on $v(x,t)$ can be obtained from (2.68)–(2.69) and (2.58) implies that we can deduce the lower bound on $\theta(x,t)$ immediately. This completes the proof of the corollary.

With the above results in hand, Theorem 1.1 follows immediately from the continuation argument and we omit the details for brevity.

3 Proof of Theorem 1.2

The main purpose of this section is to prove Theorem 1.2 by the continuation argument. Since the local solvability of the initial-boundary value problem (1.1), (1.2), (1.4) is well-established (cf. [14, 25]), if we suppose that the local solution $(v(x,t), u(x,t), \theta(x,t))$ to the initial-boundary value problem (1.1), (1.2), (1.4) has been extended to the time step $t = T > 0$ for some

$T > 0$, then to extend such a solution $(v(x,t), u(x,t), \theta(x,t))$ step by step to a global one, one only need to deduce certain a priori estimates on $(v(x,t), u(x,t), \theta(x,t))$ based on the a priori assumption (H) given in Section 2. Note that, as in Section 2, among these a priori estimates, it suffices to deduce the lower and upper bounds on the specific volume and the absolute temperature which are independent of $\underline{V}'$, $\overline{V}'$, $\underline{\Theta}'$, and $\overline{\Theta}'$, but may depend on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$.

Before we turn to derive the desired a priori estimates, we must point out that due to the change of the boundary condition, some estimates valid in Section 2 may not be true any more and we need to pay particular attention to the boundary terms appeared when performing the energy type estimates.

Our first result is concerned with the estimate on the total energy. For this purpose, we obtain from $(1.1)_3$ and (1.6) that

Lemma 3.1 (Estimate on the total energy). *Let the conditions stated in Theorem 1.2 hold and suppose that $(v(x,t), u(x,t), \theta(x,t))$ is a solution to the initial-boundary value problem (1.1), (1.2), (1.4) defined on $I\times[0,T]$ for some $T > 0$. If we assume further that $(v(x,t), u(x,t), \theta(x,t))$ satisfies the a priori assumption (H), then we have for $0 \leq t \leq T$ that*

$$\int_0^1 \left(C_v\theta + \frac{u^2}{2} \right) dx = \int_0^1 \left(C_v\theta_0 + \frac{u_0^2}{2} \right) dx. \tag{3.1}$$

First we consider the case when the transport coefficients μ and κ satisfy (1.8) and (1.11).

Lemma 3.2 *Under the conditions listed in Lemma 3.1 and assume that the transport coefficients μ and κ satisfy (1.8) and (1.11), there exist positive constants $\underline{V}_3$, $\overline{V}_3$, and $\underline{\Theta}_3$ depending only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$ such that*

$$\underline{V}_3 \leq v(x,t) \leq \overline{V}_3, \quad \forall (x,t) \in I\times[0,T] \tag{3.2}$$

and

$$\theta(x,t) \geq \underline{\Theta}_3, \quad \forall (x,t) \in I\times[0,T]. \tag{3.3}$$

Proof: Set $y = 0$ in (2.16), then involving the boundary condition (1.4), we have

$$-\mu_0 \log v(x,t) + \int_0^t p(x,s)ds = \int_0^x (u_0(z) - u(z,t))dz - \mu_0 \log v_0(x). \tag{3.4}$$

(3.4) together with the fact that $p(x,t) > 0$ and the estimate (3.1), we can easily get the lower bound of $v(x,t)$ and the lower bound on $\theta(x,t)$ can be obtained by combining the lower bound estimate on $v(x,t)$ with (2.8). That is,

$$v(x,t) \geq \underline{V}_3, \quad \theta(x,t) \geq \underline{\Theta}_3, \quad \forall (x,t) \in I\times[0,T]. \tag{3.5}$$

Consequently (2.31) holds for some positive constant K for all v and θ under our consideration. Here K depends on $\underline{V}_3$ and $\underline{\Theta}_3$.

To deduce an upper bound on $v(x,t)$ by exploiting the argument used in Lemma 2.3, we only need to recover the dissipative estimates $\int_0^t\int_0^1 \left(\frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2}\right) dxds$. For this purpose, multiplying

$(2.3)_3$ by θ^{-1} and integrating the resulting identity with respect to x and t over $I \times [0,t]$, one has

$$\begin{aligned}
&\int_0^t\int_0^1 \frac{\mu_0 u_x^2}{v\theta}dxds + \int_0^t\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dxds\\
&= C_v\int_0^1 \log\theta dx - C_v\int_0^1 \log\theta_0 dx + R\int_0^1 \log v dx - R\int_0^1 \log v_0 dx \\
&\leq C + R\int_0^1 \log v dx,
\end{aligned} \tag{3.6}$$

where (3.1) and (3.5) are used.

As for the last term on the right-hand side of (3.6), we have by integrating (3.4) with respect to x over $[0,1]$ that

$$\int_0^1 \mu_0 \log v dx \leq C + \int_0^t\int_0^1 p(x,s)dxds \leq C, \tag{3.7}$$

which together with (3.6) implies that

$$\int_0^t\int_0^1 \frac{\mu_0 u_x^2}{v\theta}dxds + \int_0^t\int_0^1 \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dxds \leq C. \tag{3.8}$$

Having obtained (3.8), we can deduce the upper bound on $v(x,t)$ by repeating the argument used in Lemma 2.3. This completes the proof of Lemma 3.2.

Now we turn to deduce the upper bound on $\theta(x,t)$ for the case when the transport coefficients μ and κ satisfy (1.8) and (1.11).

First notice that once we have obtained Lemma 3.2, since the analysis leading to Corollary 2.1, Lemma 2.4–Lemma 2.7, and Lemma 2.9 in Section 2 involves only the boundary condition $\sigma(0,t) = \sigma(1,t) = 0$, we can deduce that the estimates (2.34), (2.35), (2.36), (2.38), (2.42), (2.44), and (2.51) obtained there hold true. Now we turn to estimate $\|v_x(t)\|_{L^2(I)}$ which is the main content of Lemma 2.8. To this end, multiplying the identity (2.47) by $\frac{\mu v_x}{v}$, we get

$$\left(\frac{\mu^2 v_x^2}{2v^2}\right)_t = \left(\frac{\mu u v_x}{v}\right)_t - (u\sigma)_x + \frac{\mu u_x^2}{v} - (up)_x + \frac{\mu p_x v_x}{v}. \tag{3.9}$$

Integrating (3.9) with respect to x and t over $I\times[0,t]$, with the help of (3.1), Cauchy's inequality, and the fact $\sigma(0,t) = \sigma(1,t) = 0$, we have

$$\begin{aligned}
\int_0^1 v_x^2 dx + \int_0^t\int_0^1 \theta v_x^2 dxds &\leq C + C\int_0^t\int_0^1 \left(u_x^2 + u^2\theta + \frac{\theta_x^2}{\theta} + \theta^2\right)dxds \\
&\leq C + C\int_0^t\int_0^1 \frac{\theta_x^2}{\theta^{1-r}}dxds,
\end{aligned} \tag{3.10}$$

where (2.35)–(2.36) are used. Then by (2.42), we can easily get (2.46).

By employing the arguments used in [6, 16, 14], we can control $\int_0^t\int_0^1 u_x^4 dxds$ as in the following lemma

Lemma 3.3 *Under the conditions listed in Lemma 3.2, we have for* $0 \leq t \leq T$ *that*

$$\int_0^t\int_0^1 u_x^4 dxds \leq C\left(1 + \|\theta\|_\infty^2\right). \tag{3.11}$$

Proof: Set
$$U(x,t)=\int_0^x u(y,t)dy. \tag{3.12}$$
Under the boundary condition
$$\sigma(0,t)=\sigma(1,t)=0, \tag{3.13}$$
we can get by integrating $(2.3)_2$ over $(0,x)$ and by using (3.13) that
$$\begin{cases} U_t-\frac{\mu}{v}U_{xx}=-p(x,t),\\ \quad U(x,0)=\int_0^x u_0(y)dy,\\ \quad U(0,t)=0,\\ \quad U(1,t)=\int_0^1 u_0(x)dx. \end{cases} \tag{3.14}$$
Hence the standard L^p-estimates for solutions to the linear problem (3.14), cf. [16], yields
$$\int_0^t\int_0^1 U_{xx}^4dxds\le C\left(\|u_0\|_{L^2(I)}\right)+C\int_0^t\int_0^1 p^4dxds\le C+C\int_0^t\int_0^1\theta^4dxds. \tag{3.15}$$
Thus by (2.35), we get (3.11) and the proof of Lemma 3.3 is complete.

For the estimate on $\int_0^t\|u_{xx}(s)\|^2_{L^2(I)}ds$, we have

Lemma 3.4 *Under the conditions listed in Lemma 3.2, we have for* $0\le t\le T$ *that*
$$\int_0^t\int_0^1 u_{xx}^2dxds\le C+C\|\theta\|_\infty^{\max\{2r,1,c+1\}}. \tag{3.16}$$

Proof: By differentiating $(2.3)_2$ with respect to x and multiplying the resulting equation by $u_x-\dfrac{R\theta}{\mu_0}$, we have
$$\left(\frac{u_x^2}{2}-\frac{R\theta u_x}{\mu_0}\right)_t=-u_x\left(\frac{R\theta}{\mu_0}\right)_t+\left(\frac{v\sigma\sigma_x}{\mu_0}\right)_x-\sigma_x\left(\frac{v\sigma}{\mu_0}\right)_x. \tag{3.17}$$
Integrating (3.17) with respect to x and t over $[0,1]\times[0,t]$, one has
$$\int_0^1\left(\frac{u_x^2}{2}-\frac{R\theta u_x}{\mu_0}\right)dx\le C-\frac{R}{\mu_0}\int_0^t\int_0^1 u_x\theta_tdxds-\int_0^t\int_0^1\sigma_x\left(\frac{v\sigma}{\mu_0}\right)_xdxds. \tag{3.18}$$
Since by (1.11), (2.36), (2.42), (2.44), (2.46), (2.51), and (3.11), we have
$$\begin{aligned} &-\int_0^t\int_0^1\sigma_x\left(\frac{v\sigma}{\mu_0}\right)_xdxds\\ \le&-\frac{\underline{V}_3}{\mu_0}\int_0^t\int_0^1\sigma_x^2dxds-\frac{1}{\mu_0}\int_0^t\int_0^1\sigma\sigma_xv_xdxds\\ \le&-\frac{\underline{V}_3}{2\mu_0}\int_0^t\int_0^1\sigma_x^2dxds+C\int_0^t\int_0^1\sigma^2v_x^2dxds\\ \le&-\frac{\mu_0\underline{V}_3}{4\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2dxds+C\int_0^t\int_0^1\left(u_x^2+\theta^2\right)v_x^2dxds+C\int_0^t\int_0^1\theta_x^2dxds \end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&\le -\frac{\mu_0\underline{V}_3}{4\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2dxds + C\int_0^t\left(\|u_x(s)\|^2_{L^\infty(I)}+\|\theta(s)\|^2_{L^\infty(I)}\right)\|v_x(s)\|^2_{L^2(I)}ds\\
&\quad +C\|\theta\|_\infty^{1-r}\int_0^t\int_0^1\frac{\theta_x^2}{\theta^{1-r}}dxds\\
&\le -\frac{\mu_0\underline{V}_3}{8\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2dxds + C\|\theta\|_\infty^{\max\{2r,r+\frac12,1\}},
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{R}{\mu_0}\int_0^t\int_0^1 u_x\theta_t dxds\\
&= -\frac{R}{\mu_0 C_v}\int_0^t\int_0^1 u_x\left[\left(\frac{\kappa\theta_x}{v}\right)_x+\frac{\mu_0u_x^2}{v}-\frac{R\theta u_x}{v}\right]dxds\\
&\le \frac{\mu_0\underline{V}_3}{16\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2dxds + C\int_0^t\int_0^1\left(\kappa^2(v,\theta)\theta_x^2+u_x^3+\theta u_x^2\right)dxds \qquad (3.20)\\
&\le \frac{\mu_0\underline{V}_3}{16\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2dxds + C\left(\int_0^t\int_0^1 u_x^4dxds\right)^{\frac12}\left(\int_0^t\int_0^1\left(u_x^2+\theta^2\right)dxds\right)^{\frac12}\\
&\quad +C\|\theta\|_\infty^{c+1-r}\int_0^t\int_0^1\frac{\kappa(v,\theta)\theta_x^2}{\theta^{1-r}}dxds\\
&\le \frac{\mu_0\underline{V}_3}{16\overline{V}_3^2}\int_0^t\int_0^1 u_{xx}^2dxds + C\|\theta\|_\infty^{\max\{1,c+1\}}.
\end{aligned}
$$

The above two estimates together with (3.1), (3.18) and Cauchy's inequality, we get (3.16). This completes the proof of Lemma 3.4.

Having obtained (2.38), (2.44), (2.51), and (3.16), we can obtain the upper bound on $\theta(x,t)$ if the parameter c is chosen such that $c<1$. Here we have used the fact that $r>0$ can be chosen as small as wanted.

Now we consider the case when the transport coefficients μ and κ satisfy (1.7) with $0\le a<\frac15$ and $b\ge 2$. For such a case, (3.4) should be replaced by

$$
-g(v(x,t))+\int_0^t p(x,s)ds=\int_0^x(u_0(z)-u(z,t))dz+g(v_0(x)) \tag{3.21}
$$

with

$$
g(v)=\begin{cases}\frac{1-v^{-a}}{a}, & a>0,\\ \ln v, & a=0.\end{cases}
$$

With (3.21) in hand, we can deduce by repeating the argument used in the proof of Lemma 3.2, especially the way to deduce (3.5)–(3.6), that there exist some positive constants $\underline{V}_3>0$ and $\underline{\Theta}_3>0$ such that

$$
v(x,t)\ge \underline{V}_3,\quad \theta(x,t)\ge\underline{\Theta}_3
$$

hold for all $(x,t)\in I\times[0,T]$. But since the boundary condition (1.4) does not yield any L^p-estimate on v, we can deduce from the fact $|\ln v|\le\|v\|_\infty^\varepsilon$ for any $\varepsilon>0$ that

$$
\int_0^t\int_0^1\frac{\theta^{b-2}\theta_x^2}{v}dxds\le C+C\|v\|_\infty^\varepsilon. \tag{3.22}
$$

To deduce an upper bound on $v(x,t)$, we try to recover the L^1-estimate on $v(x,t)$, which plays an important role in deriving the upper bound on $v(x,t)$ for the case when the transport coefficients μ and κ satisfy (1.7). To do so, integrating $(2.3)_1$ with respect to x and t over $I\times[0,t]$, we get

$$\begin{aligned}\int_0^1 vdx &\leq \int_0^1 v_0dx+\int_0^t\int_0^1 u_xdxds\\ &\leq C+C\|v\|_\infty^{\frac{a}{2}}\left(\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\right)^{\frac12}\left(\int_0^t\int_0^1 vdxds\right)^{\frac12}\\ &\leq C+C\|v\|_\infty^{a}\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds+\int_0^t\int_0^1 vdxds.\end{aligned}\tag{3.23}$$

Then by the Gronwall inequality, we can easily deduce that

$$\int_0^1 vdx\leq C+C\|v\|_\infty^{a}\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds.\tag{3.24}$$

Since $b\geq 2$, we have

$$\begin{aligned}\int_0^t\|\theta(s)\|_{L^\infty(I)}ds &\leq C\int_0^t\|\theta(s)\|_{L^\infty(I)}^{\frac{b}{2}}ds+C\\ &\leq C+C\int_0^t\int_0^1\theta^{\frac{b}{2}-1}|\theta_x|dxds\\ &\leq C+C\left(\int_0^t\int_0^1 vdxds\right)^{\frac12}\left(\int_0^t\int_0^1\frac{\theta^{b-2}\theta_x^2}{v}dxds\right)^{\frac12}\\ &\leq C+C\|v\|_\infty^{\frac{\varepsilon+a}{2}}\left(\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\right)^{\frac12},\end{aligned}\tag{3.25}$$

which implies that

$$\int_0^t\int_0^1\theta^2dxds\leq C+C\|v\|_\infty^{\frac{\varepsilon+a}{2}}\left(\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\right)^{\frac12}.\tag{3.26}$$

Thus with the help of (2.37), we have

$$\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\leq C+C\|v\|_\infty^{\frac{\varepsilon+a}{2}}\left(\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\right)^{\frac12},\tag{3.27}$$

then by Cauchy's inequality and (3.24)–(3.27), we can easily obtain the following results

Lemma 3.5 *Under the conditions listed in Lemma 3.2, we have for* $0\leq t\leq T$ *that*

$$\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxds\leq C+C\|v\|_\infty^{\varepsilon+a},\tag{3.28}$$

$$\int_0^1 vdx\leq C+C\|v\|_\infty^{\varepsilon+2a},\tag{3.29}$$

$$\int_0^t \|\theta(s)\|_{L^\infty(I)} ds \leq C + C\|v\|_\infty^{\varepsilon+a}, \tag{3.30}$$

and

$$\int_0^t \int_0^1 \theta^2 dxds \leq C + C\|v\|_\infty^{\varepsilon+a}. \tag{3.31}$$

To estimate $\|v_x(t)\|_{L^2(I)}$, we have by integrating (3.9) with respect to x and t over $I \times [0,t]$ and with the help of (3.1) and Cauchy's inequality that

$$\begin{aligned}
&\int_0^1 \frac{v_x^2}{v^{2(1+a)}} dx + \int_0^t \int_0^1 \frac{\theta v_x^2}{v^{3+a}} dxds \\
&\leq C + C\int_0^t \int_0^1 \left(\frac{u_x^2}{v^{1+a}} + \frac{u^2\theta}{v^{1-a}} + \frac{\theta_x^2}{v^{1+a}\theta} + \frac{\theta^2}{v^{1-a}} \right) dxds \\
&\leq C + C\|v\|_\infty^{\varepsilon+a} + \int_0^t \int_0^1 \frac{\theta_x^2}{v\theta} dxds.
\end{aligned} \tag{3.32}$$

To control $\int_0^t \int_0^1 \frac{\theta_x^2}{v\theta} dxds$, we have by multiplying $(2.3)_3$ by θ^{-b}, and integrating the resulting identity over $I \times [0,t]$ that

$$\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}\theta^b} dxds + \int_0^t \int_0^1 \frac{\theta_x^2}{v\theta} dxds \leq C + C\int_0^t \int_0^1 \frac{|u_x|}{v} dxds \leq C + C\|v\|_\infty^{\frac{\varepsilon+a}{2}}, \tag{3.33}$$

and the above estimate together with (3.32) imply

$$\int_0^1 \frac{v_x^2}{v^{2(1+a)}} dx \leq C + C\|v\|_\infty^{\varepsilon+a}. \tag{3.34}$$

Since

$$\begin{aligned}
v(y,t) &\leq \int_0^1 v(x,t)dx + \int_0^1 |v_x| dx \\
&\leq C + C\|v\|_\infty^{\varepsilon+2a} + C\|v\|_\infty^{\frac{1}{2}+a} \left(\int_0^1 v dx \right)^{\frac{1}{2}} \left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}} dx \right)^{\frac{1}{2}} \\
&\leq C + C\|v\|_\infty^{\varepsilon+\frac{1}{2}+\frac{5a}{2}},
\end{aligned} \tag{3.35}$$

from which and the assumption $0 \leq a < \frac{1}{5}$, we can deduce that

$$v(x,t) \leq \overline{V}_3, \quad \forall (x,t) \in I \times [0,T] \tag{3.36}$$

holds for some positive constant $\overline{V}_3$ which depends only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$. As a by-produce of the estimate (3.36), we can deduce that the terms on the right-hand side of the inequalities in Lemma 3.5 and (3.34) can all be bounded by some constant C depending only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$.

Now we turn to derive the upper bound on $\theta(x,t)$. For this purpose, we have by multiplying $(2.3)_3$ by $\theta^{-\gamma}$ for some $\gamma \in (0,1)$ and integrating the resulting identity over $I \times [0,t]$ that

$$\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}\theta^\gamma} dxds + \int_0^t \int_0^1 \frac{\theta^{b-1-\gamma}\theta_x^2}{v} dxds \leq C. \tag{3.37}$$

Then by (3.15), we have

$$\begin{aligned}\int_0^t\int_0^1 u_x^4 dxds &\le C + C\int_0^t\int_0^1 \theta^4 dxds\\ &\le C + C\int_0^t \|\theta(s)\|_{L^\infty(I)}^3 ds\\ &\le C + C\int_0^t \left(\int_0^1 \sqrt{\theta}|\theta_x|dx\right)^2 ds\\ &\le C + C\|\theta\|_\infty^{\max\{2+\gamma-b,0\}}\int_0^t\int_0^1 \theta^{b-1-\gamma}\theta_x^2 dxds\\ &\le C + C\|\theta\|_\infty^{\max\{2+\gamma-b,0\}}.\end{aligned}\tag{3.38}$$

Now we set

$$X := \int_0^t\int_0^1 \theta^b\theta_t^2 dxds,\quad Y := \max_t \int_0^1 \theta^{2b}\theta_x^2 dx,\quad Z := \max_t \int_0^1 u_{xx}^2 dx.\tag{3.39}$$

Observe that

$$\begin{aligned}\theta^{2b+2} &\le C + C\int_0^1 \theta^{2b+1}|\theta_x|dx\\ &\le C + C\|\theta\|_{L^\infty(I)}^{b+\frac{1}{2}}\left(\int_0^1 \theta dx\right)^{\frac{1}{2}}\left(\int_0^1 \theta^{2b}\theta_x^2 dx\right)^{\frac{1}{2}}\\ &\le C + C\|\theta\|_{L^\infty(I)}^{b+\frac{1}{2}} Y^{\frac{1}{2}},\end{aligned}\tag{3.40}$$

which implies

$$\|\theta\|_{L^\infty(I)} \le C + CY^{\frac{1}{2b+3}}.\tag{3.41}$$

Combining (2.50), the inequality

$$\int_0^1 u_x^2 dx \le C\int_0^1 u^2 dx + C\left(\int_0^1 u^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{2}},\tag{3.42}$$

and by (3.1), we have

$$\max_t \int_0^1 u_x^2 dx \le C + CZ^{\frac{1}{2}},\tag{3.43}$$

and

$$\|u_x\|_{L^\infty(I)} \le C + CZ^{\frac{3}{8}}.\tag{3.44}$$

Our next result is to show that X and Y can be controlled by Z.

Lemma 3.6 *Under the conditions listed in Lemma 3.2, we have*

$$X + Y \le C + CZ^{\frac{3}{4}}.\tag{3.45}$$

Proof: Multiplying $(2.3)_3$ by $\theta^b\theta_t$, and integrating the resulting identity over $I\times[0,t]$, one has

$$X + Y \le C + C\int_0^t\int_0^1 \left(\theta^{b+1}|u_x||\theta_t| + \theta^b u_x^2|\theta_t| + \theta^{2b}|u_x|\theta_x^2\right) dxds.\tag{3.46}$$

Since by Cauchy's inequality and (3.36), (3.37), (3.38), we can get from (3.41) and (3.44) that

$$\int_0^t\int_0^1 \theta^{b+1}|u_x||\theta_t|dxds \le \frac{X}{4} + C\|\theta\|_\infty^{b+2}\int_0^t\int_0^1 u_x^2dxds \le \frac{X}{4} + CY^{\frac{b+2}{2b+3}}, \tag{3.47}$$

$$\int_0^t\int_0^1 \theta^{b}u_x^2|\theta_t|dxds \le \frac{X}{4} + C\|\theta\|_\infty^{b}\int_0^t\int_0^1 u_x^4dxds \le \frac{X}{4} + CY^{\frac{\max\{b,2+\gamma\}}{2b+3}}, \tag{3.48}$$

and

$$\int_0^t\int_0^1 \theta^{2b}|u_x|\theta_x^2dxds \le \|u_x\|_\infty\|\theta\|_\infty^{b+1+\gamma}\int_0^t\int_0^1 \theta^{b-1-\gamma}\theta_x^2dxds \le CY^{\frac{b+1+\gamma}{2b+3}}\left(1+Z^{\frac{3}{8}}\right). \tag{3.49}$$

Based on the above three estimates and (3.46) and by employing the Cauchy inequality, we can get (3.45) immediately if we choose $\gamma\in(0,\frac{1}{2})$. This completes the proof of Lemma 3.6.

Our last result in this section is to show that Z can be bounded by X and Y.

Lemma 3.7 *Under the conditions listed in Lemma 3.2, we have*

$$Z \le C + CY^{\frac{2+\gamma}{2b+3}} + CX + CZ^{\frac{3}{4}} \tag{3.50}$$

for some $\gamma\in(0,1)$.

Proof: Using $(2.3)_2$, we can easily get the following identity

$$u_{xx} = v^{1+a}\left(u_t + p_x + \frac{(1+a)v_xu_x}{v^{2+a}}\right). \tag{3.51}$$

Integrating (3.51) with respect to x and t over $I\times[0,t]$ yields

$$\begin{aligned}\int_0^t\int_0^1 u_{xx}^2dxds &\le C\int_0^t\int_0^1\left(u_t^2+\theta_x^2+\theta^2v_x^2+v_x^2u_x^2\right)dxds\\ &\le C\int_0^t\int_0^1 u_t^2dxds + C\int_0^t\int_0^1\frac{\theta^{b-1-\gamma}\theta_x^2}{v}dxds\\ &\quad + C\left(\|\theta\|_\infty^2+\|u_x\|_\infty^2\right)\int_0^t\int_0^1 v_x^2dxds\\ &\le C\int_0^t\int_0^1 u_t^2dxds + CY^{\frac{2}{2b+3}} + CZ^{\frac{3}{4}}.\end{aligned} \tag{3.52}$$

Next we need to estimate $\int_0^t\int_0^1 u_t^2dxds$ to complete the proof of this lemma. To this end, we have by differentiating $(2.3)_2$ with respect to t and multiplying the resulting identity by u_t that

$$\left(\frac{u_t^2}{2}\right)_t + \frac{u_{xt}^2}{v^{1+a}} = (\sigma_tu_t)_x + \frac{(1+a)u_x^2u_{xt}}{v^{2+a}} + \frac{R\theta_tu_{xt}}{v} - \frac{R\theta u_xu_{xt}}{v^2}. \tag{3.53}$$

Integrating (3.53) with respect to x and t over $I\times[0,t]$ and with the help of Cauchy's inequality, one has

$$\int_0^1 u_t^2dx + \int_0^t\int_0^1 u_{xt}^2dxds \le C + C\int_0^t\int_0^1\left(u_x^4+\theta_t^2+\theta^2u_x^2\right)dxds \le C + CY^{\frac{2+\gamma}{2b+3}} + CX. \tag{3.54}$$

(3.54) together with (3.52) implies (3.50) and the proof of Lemma 3.7.

Combining (3.45) and (3.50), we can obtain $Y\le C$, then we derive the upper bounds on $\theta(x,t)$ from (3.41).

In summary, we have obtained the desired lower and upper bounds on v and θ provided that the transport coefficients μ and κ satisfy the conditions listed in Theorem 1.2 and then Theorem 1.2 can be proved by employing the continuation argument.

Remark 3.1 *For the case when $\mu(v)$ is a smooth function of v satisfying $\mu(v) > 0$ for $v > 0$ and $\kappa(\theta) = \theta^b$, if the specific volume v is bounded both from below and from above and the absolute temperature θ is bounded from below, i.e., there exist some positive constants $\underline{V}_3 > 0$, $\overline{V}_3 > 0$, and $\underline{\Theta}_3 > 0$ such that*

$$\underline{V}_3 \leq v(x,t) \leq \overline{V}_3, \quad \theta(x,t) \geq \underline{\Theta}_3 > 0$$

hold for $(x,t) \in I \times [0,T]$, then the argument used above can be employed to derive the upper bound on $\theta(x,t)$ provided that $b \geq 0$.

4 Proof of Theorem 1.3

For the outer pressure problem (1.1), (1.2), (1.5), due to the fact that $0 < Q(t) \in C^1(\mathbf{R}_+)$, compared with the initial-boundary value problem (1.1), (1.2), (1.4), its local solvability is simpler. Thus to prove Theorem 1.3 by the continuation argument, it remains to show that if $(v(x,t), u(x,t), \theta(x,t))$ is a solution to the outer pressure problem (1.1), (1.2), (1.5) defined on $I \times [0,T]$ for some $T > 0$ and satisfies the a priori assumption (H), $v(x,t)$ and $\theta(x,t)$ are bounded, both from below and above, by some positive constants depending only on T and the initial data $(v_0(x), u_0(x), \theta_0(x))$.

To this end, we first derive from $(1.1)_3$ that

Lemma 4.1 (Estimate on the total energy). *Let the conditions in Theorem 1.3 hold and suppose that $(v(x,t), u(x,t), \theta(x,t))$ is a solution to the outer pressure problem (1.1), (1.2), (1.5) defined on $I \times [0,T]$ for some $T > 0$ and satisfies the a priori assumption (H), then we have*

$$\int_0^1 \left(\theta + \frac{u^2}{2} + v\right) dx \leq C. \tag{4.1}$$

Proof: Integrating $(1.1)_3$ with respect to x and t over $I \times [0,t]$ and making use of the boundary condition (1.5) yield

$$\int_0^1 \left(C_v\theta + \frac{u^2}{2}\right) dx + Q(t)\int_0^1 v dx = \int_0^1 \left(C_v\theta_0 + \frac{u_0^2}{2}\right) dx + \int_0^t Q'(s)\int_0^1 v dx ds. \tag{4.2}$$

Then by Gronwall inequality and the assumption on $Q(t)$, we get (4.1). This proves Lemma 4.1.

To derive the desired lower bound estimate on v, we integrating (2.15) over $[0,x] \times [0,t]$ to get that

$$-g(v) + \int_0^t p(x,s)ds = \int_0^x (u_0(z) - u(z,t))dz + \int_0^t Q(s)ds + g(v_0(x)), \tag{4.3}$$

where

$$g(v) = \begin{cases} \frac{1-v^{-a}}{a}, & a > 0, \\ \ln v, & a = 0. \end{cases} \tag{4.4}$$

Thus we can easily deduce the upper bound for $-g(v)$. From which and the fact $a \geq 0$, one can obtain the lower bound on $v(x,t)$ immediately. Having obtained the lower bound for $v(x,t)$, we can deduce the lower bound on $\theta(x,t)$ from (2.8).

A direct consequence of (3.6) and (4.1) is

$$\int_0^t \int_0^1 \frac{\theta^{b-2}\theta_x^2}{v} dx ds \leq C. \tag{4.5}$$

To derive the upper bound on $v(x,t)$, we shall get the following estimates

Lemma 4.2 *Under the conditions listed in Lemma 4.1, we have for* $0 \le t \le T$ *that*

$$\int_0^t \|\theta(s)\|_{L^\infty(I)} ds \le C, \tag{4.6}$$

and

$$\int_0^t \int_0^1 \theta^2 dx ds \le C. \tag{4.7}$$

Proof: By (4.1) and (4.5), we have

$$\begin{aligned} \int_0^t \|\theta(s)\|^b_{L^\infty(I)} ds &\le C + C\int_0^t \left(\int_0^1 \theta^{\frac{b}{2}-1}|\theta_x| dx\right)^2 ds \\ &\le C + C\int_0^t \left(\int_0^1 v dx\right)\left(\int_0^1 \frac{\theta^{b-2}\theta_x^2}{v} dx\right) ds \\ &\le C. \end{aligned} \tag{4.8}$$

If $b \ge 1$, we get (4.6) immediately.

Now we deal with the case for $\frac{1}{2} \le b < 1$. By (4.8), we have

$$\int_0^t \int_0^1 \theta^{b+1} dx ds \le C. \tag{4.9}$$

Multiplying $(2.3)_3$ by θ^{-s} for some $s > 0$ to be determined and integrating the resulting identity with respect to x and t over $I \times [0, t]$, one has

$$\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}\theta^s} dx ds + \int_0^t \int_0^1 \frac{\theta^{b-1-s}\theta_x^2}{v} dx ds \le C + C\int_0^t \int_0^1 \theta^{2-s} dx ds. \tag{4.10}$$

Hence by (4.9) we get

$$\int_0^t \int_0^1 \frac{\theta^{b-1-s}\theta_x^2}{v} dx ds \le C, \quad \forall s \ge 1 - b > 0. \tag{4.11}$$

Letting $s = b$ in (4.11), it reduces to

$$\int_0^t \int_0^1 \frac{\theta_x^2}{v\theta} dx ds \le C. \tag{4.12}$$

Then

$$\begin{aligned} \int_0^t \|\theta(s)\|_{L^\infty(I)} ds &\le C + C\int_0^t \left(\int_0^1 \frac{|\theta_x|}{\sqrt{\theta}} dx\right)^2 ds \\ &\le C + C\int_0^t \left(\int_0^1 v dx\right)\left(\int_0^1 \frac{\theta_x^2}{v\theta} dx\right) ds \\ &\le C, \end{aligned} \tag{4.13}$$

which implies that (4.6) holds for all $b \ge \frac{1}{2}$. And (4.7) can be obtained directly. This completes the proof of Lemma 4.2.

(2.37) together with (4.7) imply

$$\int_0^t \int_0^1 \frac{u_x^2}{v^{1+a}} dx ds \le C + C\int_0^t \int_0^1 \frac{\theta^2}{v^{1-a}} dx ds \le C. \tag{4.14}$$

Integrating (3.9) with respect to x and t over $I\times[0,t]$ and with the help of (3.1) and Cauchy's inequality, we have

$$\begin{aligned}
&\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx+\int_0^t\int_0^1 \frac{\theta v_x^2}{v^{3+a}}dxds\\
&\leq C+C\int_0^t\int_0^1\left(\frac{u_x^2}{v^{1+a}}+\frac{\theta^2}{v^{1-a}}+\frac{u^2\theta}{v^{1-a}}+\frac{\theta_x^2}{v^{1+a}\theta}\right)dxds\\
&\leq C,
\end{aligned}\tag{4.15}$$

where (4.6), (4.7), (4.12), and (4.14) are used.

Hence as in (3.35), we get the upper bound on $v(x,t)$.

Note that from (4.10) and (4.14) we have (3.37) with $\gamma\in(0,1)$. On the other hand, as in (3.15) and with the aid of $Q(t)\in C^1(\mathbf{R}_+)$, we can obtain the inequality (3.38). Thus, as pointed out in Remark 3.1, the upper bound on $\theta(x,t)$ can be obtained by employing the argument used in Section 3. This completes the proof of Theorem 1.3.

Acknowledgment

The research Huijiang Zhao was supported by the grant from the National Natural Science Foundation of China under contract 10925103. This work is also supported by "the Fundamental Research Funds for the Central Universities".

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