FROM POLYNOMIAL INTEGRALS OF HAMILTONIAN FLOWS TO A MODEL OF NON-LINEAR ELASTICITY

MISHA BIALY AND ANDREY E. MIRONOV

ABSTRACT. We prove non existence of smooth solutions of a quasi-linear system suggested by Ericksen in a model of Nonlinear Elasticity. This system is of mixed elliptic-hyperbolic type. We discuss also a relation of such a system to polynomial integrals of Classical Hamiltonian systems.

1. INTRODUCTION AND MAIN RESULTS

In this paper we study smooth periodic solutions of the following equation

$$(1) \qquad u_{tt} + (\sigma(u))_{xx} = 0.$$

Throughout this paper the function $u(t,x)$ is assumed to be periodic in x, $u(t, x+1) = u(t,x)$ and C^2-smooth on the whole cylinder or on the half-cylinder. Equation (1) is a compatibility condition of the quasi-linear 2×2 system usually called p-system (here we shall use σ instead of p since p is reserved for momentum which appears below).

$$(2) \qquad \begin{cases} u_t = -v_x \\ v_t = (\sigma(u))_x. \end{cases}$$

One can easily see that the periodicity of u leads to

$$(3) \qquad v(t, x+1) = v(t,x) + C,$$

C is a constant. Therefore v is a sum of a linear function and a function periodic in x.

The system (2) and the equation (1) appear in many applications. The purpose of this paper is to prove that there are no smooth periodic solution of these equations. It is widely known, starting from [11], that smooth solutions for Hyperbolic quasi-linear system *generically* do not exist after a finite time. However rigorous proof of this general belief are not immediate and usually is not that simple. It is important that our conditions on σ are such that system (2) is of mixed elliptic-hyperbolic type and therefore the analysis of elliptic and hyperbolic zones as well as the boundary between them is required (we refer to [12] for a recent survey on mixed problems).

In this paper we shall discuss two appearances of the equation (1) (2): the first is related to Classical mechanics and the second is a model suggested by Ericksen from Nonlinear Elasticity.

In Classical mechanics one is looking for conserved quantities or integrals of Hamiltonian flow

$$(4) \qquad \frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}.$$

In this context, assume $H = \frac{1}{2}p^2 + u(t,x)$ be a Hamiltonian function with potential function $u(t,x)$. Write $F = \frac{1}{3}p^3 + up + v$. Then the condition that F has constant values along the flow lines of H reads:

$$\frac{dF}{dt} = F_t + pF_x - u_x F_p = 0$$

and leads to the system (2) with the function $\sigma(u) = \frac{u^2}{2}$. In this case (1) turns out to be dispersionless Boussinesq equation. This fact was first noticed by V.V. Kozlov in [10] where trigonometric polynomial solutions were considered. Doubly periodic solution were further studied in [2],[3](other quasi-linear systems arising in Classical mechanics are discussed in [4],[5]). Our method in this paper is a continuation of [1] where the case of quadratic-like function σ is studied.

Another appearance of (1) and (2) comes from a model of non-linear elasticity suggested by Ericksen [8]. In this model σ appears to be a cubic-like function (of type II below). We refer to the paper [14], where this model is discussed.

Motivated by these two applications we shall consider two types of σ:

I. Quadratic-like type.
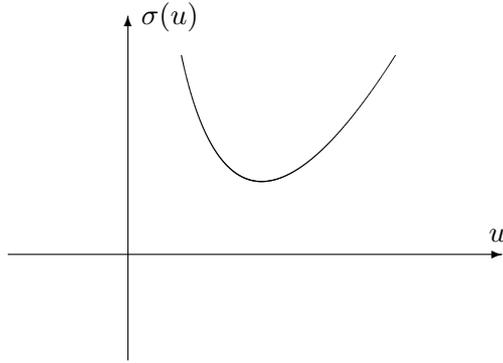The function σ is strictly convex function having a minimum (see Fig. 1).

Fig. 1

II. Cubic-like type.

The function σ behaves like a cubic polynomial in u (see Fig. 2), i.e.

$$\sigma'(u) < 0, \quad for \quad u < \alpha \quad and \quad u > \beta,$$

$$\sigma'(u) > 0, \quad for \quad u \in (\alpha, \beta),$$

$$\sigma''(u) > 0, \quad for \quad u \le \alpha \quad and \quad \sigma'' < 0, \quad for \quad u \ge \beta.$$
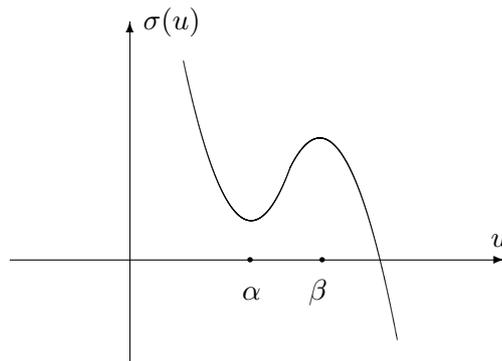

Fig. 2

Our main result is given in the following two theorems where the case of quadratic like function σ was proved in [1] and is included here for completeness.

Theorem 1.1. *Let $\sigma(u)$ be of type I,II. Then any C^2-solution of (2) defined on the half-cylinder $[t_0, +\infty) \times \mathbf{S}^1$:*

$$u(t, x+1) = u(t, x), \ v(t, x+1) = v(t, x), t \ge t_0,$$

which has initial values in the Hyperbolic region $U_h = \{u < \alpha\} \cup \{u > \beta\}$ must be constant.

To get the result on the whole infinite cylinder one can remove the initial Hyperbolicity assumption:

Theorem 1.2. *If the function σ is of type I or type II, then any C^2-solution $(u(t, x), v(t, x))$ of the system (2) defined on the whole cylinder $\mathbf{R} \times \mathbf{S}^1$ so that,*

$$u(t, x+1) = u(t, x), \ v(t, x+1) = v(t, x)$$

must be constant.

Let us remark that in these two theorems v is assumed to be periodic together with u, that is the constant C in formula (3) is assumed to be zero. Moreover, the pair $u = -Ct, v = Cx$ is obviously a solution of (2) for any C. In this example u is obviously periodic in x, but v is not. It is an open question if there are other smooth global solutions of (2) periodic in x for $C \neq 0$. Our next result says that there are no, if u is assumed to be periodic in both t and x.

Theorem 1.3. *If $\sigma = \frac{u^2}{2}$ then any C^2-solution $(u(t,x), v(t,x))$ of (2) with u being doubly periodic*

$$u(t+1, x) = u(t, x+1) = u(t, x)$$

must be constant.

Remarkably in the proof below one shows by means of dynamical systems theory that the function v must be periodic either. This fact gives a reduction of Theorem 1.3 to Theorem 1.2.

As a simple corollary of the analysis given in the proofs of Theorems 1.1, 1.2 we can strengthen the result of Theorem 1.3 as follows:

Theorem 1.4. *Let $(u(t,x), v(t,x))$ be a C^2-solution of (2) with $\sigma(u)$ being either of type I or type II, then if in addition $u(t,x)$ is bounded and periodic*

$$u(t, x+1) = u(t, x)$$

then (u, v) must be constant.

Several remarks and questions are in order:

1. It is still not clear to us how to classify all solutions of (1) periodic in x with no assumptions on periodicity of v or boundedness of u.

2. It was crucial for the proof of our main theorems that within the Hyperbolic regions the eigenvalues are genuinely non-linear. It would be interesting to understand the case when the function σ behaves like in van der Waals model (see [9]) where the genuine non-linearity condition is violated.

3. Another classical tool [6] for the system (2) which could be used near the points where the mapping $(t,x) \to (u,v)$ is local diffeomorphism is the Hodograph method. However, we don't know how this method can be used globally, taking care on the singularities, in order to get another proof of our results. Let us also remark that it would be very interesting to find a connection with a recent theory of normal forms of the singularities and the so called Universality conjecture developed in [7].

The paper is organized as follows. In the next section we show the Dynamical systems argument reducing Theorem 1.3 to Theorem 1.2. Then in Section 4 we prove two key Lemmas. In Section 5 we treat Hyperbolic part of the Theorems 1.1, 1.2 and get Theorem 1.4 as a corollary. Section 6 provides a convexity argument for the Elliptic zones.

2. Reduction of Theorem 1.3 to Theorem 1.2

In this section we give a reduction of Theorem 1.3 to Theorem 1.2 based on Dynamical Systems ideas.

Proof. In this theorem $\sigma(u) = \frac{u^2}{2}$. Let $u(t,x)$ be a C^2-solution of (1) which is periodic both in t and x. Then let $v(t,x)$ be a function such that the pair (u, v) solves (2). The function v is defined up to a constant and is not necessarily periodic. It can be written

$$v(t, x) = At + Bx + \tilde{v}(t, x),$$

where A, B are some constants and $\tilde{v}$ is periodic in both t, x. As we mentioned above system (2) is equivalent to the fact that the function

$$F = \frac{1}{3}p^3 + u(t,x)p + v(t,x)$$

has constant values along the Hamiltonian flow of

$$H = \frac{1}{2}p^2 + u(t,x).$$

It is a standard result of calculus variations, that for any positive integer m and any integer n there exists periodic orbit of the Hamiltonian flow of type (m, n). This means that there is a solution of the Hamilton equations (4) $(x(t), p(t))$ with the property

(5) $x(t + m) = x(t) + n, \ p(t + m) = p(t), \ m, n \in \mathbb{Z}, \ m > 0.$

Along this orbit F has a constant value. Therefore

(6) $F(t + m, x(t + m), p(t + m)) = F(t, x(t), p(t)).$

Left hand side of (6) with the help of (5) and periodicity of u equals

$$F(t + m, x(t + m), p(t + m))$$

$$= \frac{1}{3}p^3(t) + u(x(t) + n, t + m)p(t) + v(x(t) + n, t + m)$$

$$= \frac{1}{3}p^3(t) + u(x(t), t)p(t) + v(x(t) + n, t + m)$$

$$= \frac{1}{3}p^3(t) + u(x(t), t)p(t) + Am + Bn + At + Bx(t) + \tilde{v}(x(t), t).$$

The right hand side of (6) is the following

$$F(t, x(t), p(t)) = \frac{1}{3}p^3(t) + u(x(t), t)p(t) + At + Bx(t) + \tilde{v}(x(t), t).$$

Equating the expressions of the right and the left hand side we get the identity:

$$Am + Bn = 0$$

for any m, n. So $A = B = 0$. Thus v is a periodic function so Theorem 1.2 applies and yields the result.

□

3. Preliminaries on the Hyperbolic regions

Here we shall collect the needed facts on the p-systems (see [13] and also [9]). The system (2) can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + A(u, v)\begin{pmatrix} u \\ v \end{pmatrix}_x = 0, \quad A\begin{pmatrix} 0 & 1 \\ -\sigma'(u) & 0 \end{pmatrix}.$$

Let $(u(t, x), v(t, x))$ be a solution periodic in x. Denote by

$$U_e = \{(t, x) : u(t, x) \in (\alpha, \beta)\}$$

the elliptic region where the matrix A has complex eigenvalues. Denote by U_h the hyperbolic region consisting of two disjoint domains

$$U_h = U_\alpha \cup U_\beta,$$

where
$$U_\alpha = \{(t, x) : u(t, x) < \alpha\}, \ U_\beta = \{(t, x) : u(t, x) > \beta\}.$$
On U_h the matrix A has two real distinct eigenvalues
$$\lambda_1 = \sqrt{-\sigma'(u)}, \ \lambda_2 = -\sqrt{-\sigma'(u)}.$$
The boundaries of U_e and U_h belong to
$$U_0 = \{(t, x) : u(t, x) = \alpha \quad or \quad u(t, x) = \beta\}.$$

It is important for the sequel that U_α, U_β have disjoint closure on the cylinder. Therefore the characteristics of the system cannot jump from one of the domains U_α or U_β to the other. We analyze the behavior of the characteristics in U_α, and for the U_β all conclusions are analogous with the obvious changes.

In U_α there are Riemann invariants
$$r_1 = v - \int_u^\alpha \sqrt{-\sigma'(u)}du, \quad r_2 = v + \int_u^\alpha \sqrt{-\sigma'(u)}du,$$
$$(r_i)_t + \lambda_i(r_i)_x = 0, \ i = 1, 2,$$
and so u, v can be recovered from the Riemann invariants by the formulas:
$$(7) \qquad v = \frac{r_1 + r_2}{2}, \quad u = q^{-1}\left(\frac{r_2 - r_1}{2}\right),$$
where by definition
$$q(u) := \int_u^\alpha \sqrt{-\sigma'(s)}ds,$$
is a positive monotone decreasing function for $u < \alpha$ with
$$q(\alpha) = 0, \quad q'(u) = -\sqrt{-\sigma'(u)}, \quad q''(u) = \frac{\sigma''(u)}{2\sqrt{-\sigma'(u)}}.$$

It is crucial fact that both eigenvalues are genuinely non-linear in U_α by the formulas:
$$(\lambda_1)_{r_1} = (\lambda_2)_{r_2} = \frac{\sigma''(u)}{4\sigma'(u)} \neq 0.$$
Notice that near the boundary ∂U_α the non-linearity becomes infinite. Moreover, verifying literarily the Lax method [11] for our p-system one arrives to the following Riccati equations along characteristics of the first and the second eigenvalues:
$$(8) \qquad L_{v_1}(z_1) + kz_1^2 = 0, \quad L_{v_2}(z_2) + kz_2^2 = 0$$
where
$$z_1 := (r_1)_x(-\sigma'(u))^{\frac{1}{4}}, \quad z_2 := (r_2)_x(-\sigma'(u))^{\frac{1}{4}}, \quad k := -\frac{\sigma''(u)}{4(-\sigma'(u))^{\frac{5}{4}}},$$
and
$$L_{v_1} = \partial_t + \lambda_1\partial_x, \quad L_{v_2} = \partial_t + \lambda_2\partial_x$$
stand for derivatives along the first and the second characteristic fields respectively.

4. The Key Lemmas

The following two lemmas are crucial for the proofs. Recall again that we shall state them for U_α so that for U_β the statements go with obvious replacements.

Lemma 4.1. *Along the characteristic curves we have:*

1) If a characteristic curve of the first or of the second eigenvalue starting from the initial time t_0 reaches the boundary ∂U_α in a finite time $t_+ > t_0$ (respectively $t_- < t_0$), then the corresponding Riemann invariant satisfies $r_x \leq 0$ (resp. $r_x \geq 0$) along this characteristic.

2) If a characteristic curve of the first or of the second eigenvalue extends to a semi-infinite interval ($[t_0, +\infty)$ (resp. $(-\infty, t_0]$), then for the corresponding Riemann invariant either $(r)_x \leq 0$ (resp. $r_x \geq 0$) or $-u$ and $-\sigma'(u)$ tend to $+\infty$ along this characteristic curve when $t \to +\infty$ (resp. $t \to -\infty$).

Proof. To prove the Lemma we use the exact formula for the solutions of equation (8):

$$z(t) = \frac{z(t_0)}{1 + z(t_0) \int_{t_0}^{t} k(s)ds}.$$

Let us prove the first part of the Lemma. Suppose that characteristic extends to the maximal interval $[t_0; t_+)$. Recall that the characteristics are solutions of the equation

$$(9) \qquad \dot{x} = \pm\sqrt{-\sigma'(u)}.$$

It is a standard fact in ODE theory, that if a characteristic curve approaches the boundary of U_α, so that u tends to α, then the characteristic curve must converge to a limit point say (t_+, x_+) on the boundary ∂U_α (see Fig. 3).
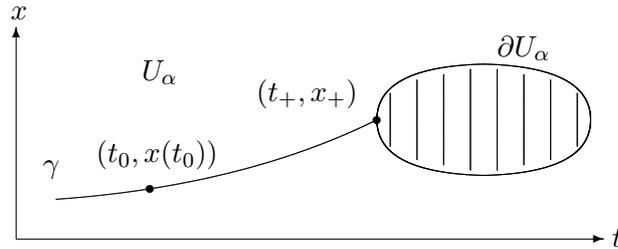

Fig. 3

Moreover, it follows then that the integral

$$(10) \qquad \int_{t_0}^{t_+} k(s)ds = -\int_{t_0}^{t_+} \frac{\sigma''(u(s, x(s)))}{4(-\sigma'(u(s, x(s))))^{\frac{5}{4}}} ds$$

diverges to $-\infty$. Indeed, for $t \to t_+$ the function $u(t, x(t)) \to \alpha$ and can be estimated from above by

$$|u(t, x(t)) - \alpha| \leq C_1 |t - t_+|,$$

also for u close to α one can estimate:

$$\left|\sigma'(u)\right| = \left|\int_u^\alpha \sigma''(u)du\right| \le C_2\left|u-\alpha\right|, \ where \ C_2 = \max_{u\in[\alpha-1,\alpha]} \sigma''(u).$$

So the nominator in of the integrand of (10) is bounded away from zero and the denominator is less or equal then $C_1 C_2 \left|t-t_+\right|^{\frac{5}{4}}$, and

$$-\int_{t_0}^{t_+} \frac{\sigma''(u)}{4(-\sigma'(u))^{\frac{5}{4}}} ds < -\int_{t_0}^{t_+} \frac{C_0}{C_1 C_2 \left|t-t_+\right|^{\frac{5}{4}}} ds \to -\infty,$$

thus the integral (10) diverges. This proves the first part of the lemma.
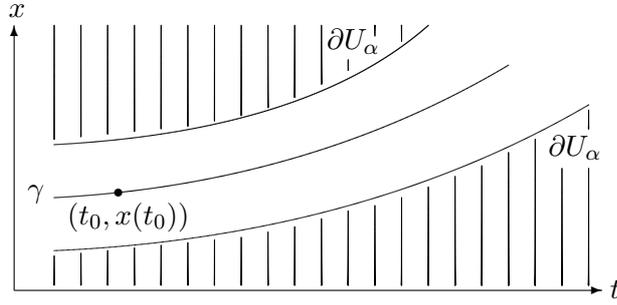
The second part of the lemma is as follows.

Fig. 4

For an infinite characteristic (Fig. 4)

$$\gamma_1(t) = (t, x_1(t)), \ t\in[t_0;t_+), \ t_+ = +\infty,$$

there are two possibilities.

The first is when the integral (10) diverges to $-\infty$, in this case $r_x \le 0$ exactly as in the previous case.
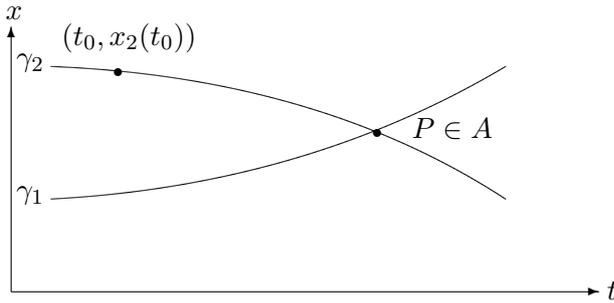

Fig. 5

In the second possibility the integral (10) is converging. In this case we need to prove that $-u$ and $-\sigma'(u)$ must tend to $+\infty$ as $t \to +\infty$. For this we use the periodicity of u, v in x. Let $P = (t_*, x_1(t_*))$ be any point on the characteristic γ_1. Consider characteristic of the second eigenvalue $\gamma_2 = (t, x_2(t))$ passing through this point P. Let us follow γ_2 backwards.

Either the characteristic γ_2 can be extended backwards on $[t_0; t_*)$ (in such a case we shall call point P accessible from t_0, see Fig. 5) or it can be extended backward to the maximum interval of existence $(t_-; t_*), t_0 < t_-$ (then P will be called unaccessible).

Notice that in the last case the possibility $x_2(t) \to +\infty, t \searrow t_-$ is easily excluded because by periodicity in x, the solutions of the equation (9) are bounded on any compact interval of time. Thus in case point P is unaccessible the characteristic γ_2 reaches the boundary ∂U_α in a backward time (Fig. 6).

Denote by A and B the set of all accessible and unaccessible points respectively on the characteristic γ_1. The set A is obviously open in γ_1 and so consists of the union of open intervals. From the periodicity condition it follows that r_2 is bounded on the line $t = t_0$. Moreover, since r_2 is preserved along characteristics of the second family and by the formula (7), it follows that u is uniformly bounded on all intervals of the set A (see Fig. 6).
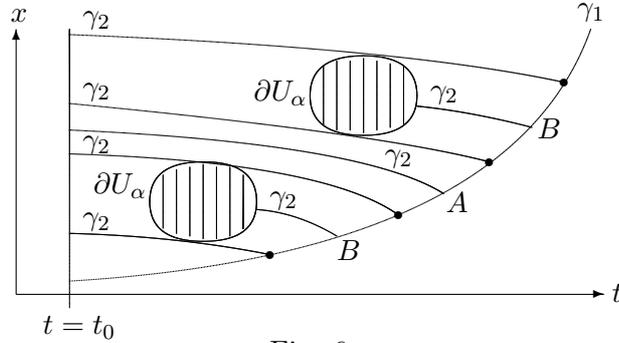

Fig. 6

On the other hand, it follows from the first claim of the lemma that on each interval of the interior of B, $(r_2)_x \geq 0$ and since r_2 is preserved along characteristics of the second family r_2 is an increasing function on each interval of the interior of B and therefore $-u$ is increasing also due to (7).

Summing up these two properties we get for the function $-u$ along γ_1 the following possibilities: either the set A is bounded and then $-u$ is increasing function along γ_1 starting from a certain point, or the set A is unbounded but then $-u$ must be bounded on the whole γ_1, since $-u$ is uniformly bounded on the intervals of A which may alternate with intervals of B where the function $-u$ is monotonic. Notice that the last possibility cannot happen in fact since we are in the case of convergent integral (10). So we have that $-u$ is a monotonic increasing function and the claim follows. Lemma is proved. $\square$

Lemma 4.1 enables us to distinguish between two types of characteristics which start at t_0 in a positive or negative direction of time as follows.

Definition 4.2. Let γ be a characteristic curve lying in the Hyperbolic domain U_α defined on a *maximal* interval $[t_0, t_+)$ (or respectively $(t_-, t_0]$). We shall say that γ is of type B_+ (res. B_-) if $t_+ = +\infty$ (resp. $t_- = -\infty$) and $-u \to +\infty$ when $t \to +\infty$ (resp. $t \to -\infty$).

We shall say that γ is of type A_+ (resp. A_-) in the opposite case. That is if either t_+ (resp. t_-) is finite, or $t_+ = +\infty$ (resp. $t_- = -\infty$) and $-u$ does not tend to $+\infty$ when $t \to +\infty$ (resp. $t \to -\infty$).

By Lemma 4.1 if γ is of type A_+ then $(r)_x \le 0$ along γ, and if γ is of type A_- then $(r)_x \ge 0$ along γ.

Lemma 4.3. *There cannot exist two semi-infinite characteristics in the same direction*

$$\gamma_1 = (t, x_1(t)), \ \gamma_2 = (t, x_2(t))$$

of the first and the second eigenvalue such that both of them belong to the same class $B_\pm$.

Proof. Assume on the contrary that there exist such $\gamma_1 = (t, x_1(t))$, $\gamma_2 = (t, x_2(t))$ belonging to the same class, say B_+, so that $-u|_{\gamma_1} \to +\infty$ and $-u|_{\gamma_2} \to +\infty$ when $t \to +\infty$.

Then by periodicity we can shift the characteristics to get

$$\gamma_1^{(k)} = (t, x_1(t) + k), \ \gamma_2^{(l)} = (t, x_2(t) + l)$$

which are characteristics of class B_+ also for all $k, l \in \mathbf{Z}$. Since the functions x_1, x_2 are solutions of the ODEs

$$\dot{x} = \pm\sqrt{-\sigma'(u)}$$

respectively, it follows that x_1 (respectively x_2) are strictly monotone increasing (respectively decreasing) function with the derivative bounded away from zero. Therefore for sufficiently large k the characteristics γ_1 and $\gamma_2^{(k)}$ must intersect in a unique point, call it P_k (see Fig. 7).
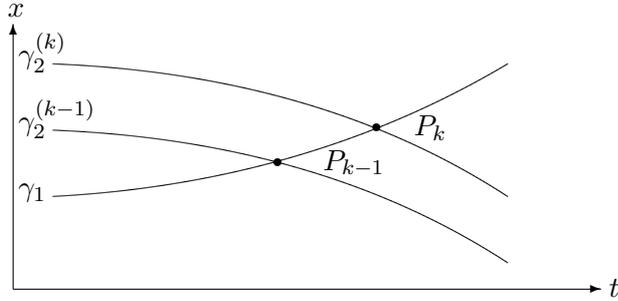

Fig. 7

Denote by t_k the t-coordinates of P_k. One can see that t_k is monotone increasing and must tend to $+\infty$. Indeed in the opposite case there exist limits $t_k \nearrow t_*$ and $P_k \to P_*$ so that the characteristic γ_2 lies in the half plane $t < t_*$ and tends to $-\infty$ when $t \to t_*$. But this contradicts the fact that solutions of the equation (9) are bounded on any compact interval of time.

Therefore we have,

$$-u(t_k, x(t_k)) \to +\infty, \ k \to +\infty,$$

and then by formula (7) also

$$r_2(P_k) - r_1(P_k) \to +\infty, \ k \to +\infty.$$

But this is not possible since by periodicity in x of (u, v) one has that $r_1(t_0, x)$ and $r_2(t_0, x)$ are bounded, and so by conservation of r_1, r_2 along characteristics $r_2(P_k) - r_1(P_k)$ must be bounded also. This contradiction proves the lemma. $\square$

5. Hyperbolic part of the proof of main Theorems 1.1, 1.2

Recall that Lemma 4.3 means that if at least one of the characteristics of the first eigenvalue γ_1 is unbounded on $[t_0, +\infty)$ with the property that $-u \to +\infty$ along it, then along any characteristic of the second eigenvalue one has: $(r_2)_x < 0$.

Let us prove now Theorem 1.1.

Proof. Assume without loss of generality that the initial data $u(t_0, x)$ lies within the Hyperbolic domain U_α. Introduce

$$t' = \sup\{t : [t_0, t] \times \mathbf{S}^1 \subseteq U_\alpha\},$$

this means that t' is the first moment where non-Hyperbolic type appears. In other words u becomes equal to α at some point on the circle $\{t'\} \times \mathbf{S}^1$. Write $U' = [t_0, t') \times \mathbf{S}^1$. We prove that t' equals in fact to $+\infty$. Indeed, it follows from Lemma 4.3 that all characteristics of at least one of the eigenvalues are of class A_+. Without loss of generality let it be the family of the second eigenvalue with this property. Then it follows from Lemma 4.1 that

$$(r_2)_x(t_0, x) \leq 0,$$

holds true for every x. But by periodicity this is possible only when $r_2(t_0, x)$ is in fact constant for the initial moment and so also everywhere on the whole U'. This means that within the domain U' only r_1 can vary. But then u is a function of r_1 only and therefore has constant values along characteristics of the first eigenvalue. By the construction there exists a point, say E, on $\{t'\} \times \mathbf{S}^1$ where $u = \alpha$. It follows from continuous dependence of the solutions of the ODE (9) on the initial data that there exists a characteristic of the first family terminating at the point E, so that $u = \alpha$ also on the whole characteristic. But this is a contradiction, since $u < \alpha$ for all points inside U_α. This implies that the hyperbolic domain U_α coincides with the whole semi-infinite cylinder $[t_0, +\infty) \times \mathbf{S}^1$.

Furthermore, since we know that r_2 is a constant on the whole half cylinder then u depends only on r_1 and has constant values along characteristics of the first family (in particular $-u$ does not tend to infinity) so Lemma 4.1 implies

$$(r_1)_x \leq 0.$$

Using periodicity again we conclude that r_1 is constant also everywhere on the half-cylinder. Thus (u, v) is a constant solution on the semi-infinite cylinder. We are done. $\square$

For Theorem 1.2 we have to consider the whole infinite cylinder and characteristics which may be infinite in both directions. Also in this case elliptic

domains cannot be excluded as before, to treat them one needs an additional tool. In the next section we treat the Elliptic region. These two steps provide the proof of Theorem 1.2. The first step goes as follows:

Theorem 5.1. *Let (u, v) be a C^2-solution of the system (2) on the whole infinite cylinder. Then either U_α or U_β coincide with the whole cylinder and u, v are constants everywhere, or both U_α and U_β are empty, i.e. $u \in [\alpha, \beta]$ everywhere.*

Proof. To give a proof assume with no loss of generality that U_α does not coincide with the whole cylinder otherwise Theorem 1.1 yields the result.

We show that then U_α must be empty. We prove this by contradiction. Fix a connected component of U_α, denote it U', and take any initial moment t_0 with the property that the intersection of $\{t = t_0\}$ with component U' is not empty. It consists of the disjoint union of open intervals — we call them intervals of Hyperbolicity (the case when it is the whole circle is covered already by Theorem 1.1).

Consider the following two complementary cases

Case 1. *For any initial moment t_0 with the property $\{t = t_0\} \cap U' \neq \emptyset$, for at least one of the eigenvalues (say for the second one) all characteristics started from t_0 in positive and negative direction belong to the classes A_+, A_-.*

In this case it follows from Lemma 4.1 that $r_2(t_0, x)$ is constant on any interval of the intersection $\{t = t_0\} \cap U'$ (since $(r_2)_x \geq 0$ and in the same time $(r_2)_x \leq 0$). Since t_0 is arbitrary and U' is connected, this implies that r_2 is constant on the whole connected component U'. This implies that only r_1 varies on U' and so $u, v, \lambda_1, \lambda_2$ are functions of r_1 only. This means in particular that u keeps constant values along characteristics of the first eigenvalue. Therefore every such characteristic can be extended infinitely in both directions because if it reaches the boundary of the Hyperbolic region U_α, then u must have value α on the whole characteristic, which contradicts Hyperbolicity. Moreover, since λ_1 is a function of r_1 only, so it is a constant along λ_1-characteristics, then these characteristics are necessarily parallel straight lines of the slope λ_1. So U' is an infinite strip of the slope λ_1. Furthermore on the boundary $u = \alpha$ so $\lambda_1 = \sigma'|_{u=\alpha} = 0$, so that $\lambda_1 = 0$ everywhere on U'. Thus these strips are horizontal and u equals α identically on U'. This contradiction finishes the proof.

Case 2. *There exists t_0 such that $\{t = t_0\} \cap U' \neq \emptyset$ and for each eigenvalue λ_1 and λ_2 there exists a characteristic of class $B_\pm$ started at $\{t = t_0\} \cap U'$ in some direction.*

It follows from the Lemma 4.3 that the directions of these two characteristics must be opposite. So assume without loss of generality that γ_1, γ_2 are λ_1, λ_2-characteristics in the classes B_+, B_- respectively. Then it follows from the lemmas that the characteristics γ_1, γ_2 being extended beyond t_0 in the negative and positive direction respectively belong to classes A_-, A_+ respectively. And thus by the Lemma 4.1

$$(r_1)_x(t_0, x) \geq 0, \ (r_2)_x(t_0, x) \leq 0,$$

for all x in the intervals of Hyperbolicity. So in this case $(r_1 - r_2)(t_0, x)$ is a monotone function in x. Then also $u(t_0, x)$ is monotone by the formula (7)

and since u equals α at the ends of the intervals of Hyperbolicity, then $u = \alpha$ on the whole interval of Hyperbolicity, contradiction. This contradiction completes the proof of the theorem. □

Let us complete this section with the proof of Theorem 1.4.

Proof of Theorem 1.4. It is now very easy. The assumption that u is bounded implies that in the Hyperbolic region all characteristics belong to class A. Therefore we can conclude exactly as in the Case 1 of the previous theorem that either one of the Hyperbolic domains U_α, U_β coincides with the whole cylinder and the solution (u, v) is constant everywhere, or U_α, U_β are empty and the solution satisfies everywhere $u \in [\alpha, \beta]$. This case is treated in the next section where periodicity in x only of the function u is needed. □

6. Elliptic case

In this section we treat the Elliptic region as follows.

Theorem 6.1. *Suppose (u, v) is a C^2-solution of the system (2) on the whole cylinder which satisfies $u \in [\alpha, \beta]$ everywhere, and such that u is periodic in x. Then u, v are constants.*

Proof. Take any function $f : [\alpha, \beta] \to \mathbf{R}$ satisfying

$$f(\alpha) = f(\beta) = 0,$$

$$f(u) > 0 \ for \ all \ u \in (\alpha, \beta) \ and$$

$$f''(u) < 0 \ for \ all \ u \in [\alpha, \beta].$$

Introduce

$$E(t) = \int_{S^1} f(u(t, x)) dx,$$

which by the construction is a positive function of $t \in \mathbf{R}$ unless u equals identically α or β. Compute the second derivative of E using the system (2) and integration by parts. Notice that for the integration by parts periodicity of u only is essential and not of v. We have

$$\ddot{E} = \int_{S_1} f''(u) \left((v_x)^2 + \sigma'(u)(u_x)^2\right) dx \leq 0.$$

Since $u \in [\alpha, \beta]$ then $\sigma'(u)$ is non-negative, by the assumptions on σ. So we get that E is a positive concave function and thus must be constant. Then obviously u, v are constants everywhere. □

References

[1] M. Bialy. Smooth solutions for a p-system of mixed type // Israel Math. Journal (to appear).
[2] M. Bialy. Polynomial integrals for a Hamiltonian system and breakdown of smooth solutions for quasi-linear equations // Nonlinearity 1994. V. 7. N. 4, P. 1169–1174.
[3] M. Bialy. On periodic solutions for a reduction of Benney chain // Nonlinear Differ. Equ. Appl. 2009. V. 16. P. 731–743.
[4] M. Bialy, A. Mironov. Rich quasi-linear system for integrable geodesic flows on 2-torus // Discrete and Continuous Dynamical Systems - Series A. 2011. V. 29. N. 1. P. 81–90.
[5] M. Bialy, A. Mironov. Qubic and Quartic integrals for geodesic flow on 2-torus via system of Hydrodynamic type // Nonlinearity 2011. V. 24. N. 12, P. 3541-3554.

[6] R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience Publishers, Inc., New York, N. Y., 1948.

[7] B. Dubrovin. On universality of critical behaviour in Hamiltonian PDEs. Geometry, topology, and mathematical physics, 59-109, Amer. Math. Soc. Transl. Ser. 2, 224, Amer. Math. Soc., Providence, RI, 2008.

[8] J.L. Ericksen, Equilibrium of bars, J. Elasticity, 5 (1975), pp. 191–201.

[9] H. Fan and M. Slemrod. Dynamic flows with liquid/vapor phase transitions, 373–420. in S. Friedlander , D. Serre, editors, Handbook of Mathematical Fluid Dynamics v. 1, Elsevier Science, 2002.

[10] V.V. Kozlov. Polynomial integrals of dynamical systems with one-and-a-half degrees of freedom. Math. Notes (translation) // 1989. V. 45. N. 3-4. P. 296–300.

[11] P. D. Lax. Hyperbolic systems of conservation laws and the mathematical theory of shock waves. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973.

[12] T. H. Otway. The Dirichlet problem for elliptic-hyperbolic equations of Keldysh type. Lecture Notes in Math. 2043. Berlin: Springer. ix, 214 p.

[13] D. Serre. Systems of Conservation Laws. Vol.1,2. Cambridge University Press, 1999.

[14] D. Serre, R.L. Pego. Instabilities in Glimms scheme for two systems of mixed type // Siam J. Numer. Anal. 1988. V. 25. N. 5. P. 965–989.

M. BIALY, SCHOOL OF MATHEMATICAL SCIENCES, RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCES, TEL AVIV UNIVERSITY, ISRAEL

E-mail address: bialy@post.tau.ac.il

A.E. MIRONOV, SOBOLEV INSTITUTE OF MATHEMATICS AND LABORATORY OF GEOMETRIC METHODS IN MATHEMATICAL PHYSICS, MOSCOW STATE UNIVERSITY

E-mail address: mironov@math.nsc.ru