COUPLING CONSTANTS AND THE GENERALIZED RIEMANN PROBLEM FOR ISOTHERMAL JUNCTION FLOW

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Abstract. We consider gas flow in pipe networks governed by the isothermal Euler equations. A set of coupling conditions are required to completely specify the Riemann problem at the junction. The momentum related condition has no obvious expression and different approaches have been used in previous work. For the condition of equal momentum flux, Colombo and Garavello [*Netw. Heterog. Media*, 1 (2006), pp. 495-511] proved existence and uniqueness of solutions globally in time and locally in the subsonic region of the state space.

If the entropy constraint is not considered, we are able to prove existence and uniqueness globally in the subsonic region for any coupling constant satisfying a monotonicity requirement. The previously suggested conditions of equal pressure and equal momentum flux satisfy this requirement, but in general they both fail to fulfil the entropy constraint.

The classical Bernoulli invariant is a natural scalar formulation of momentum conservation under ideal flow conditions. Our analysis shows that this invariant is monotone and unconditionally leads to solutions satisfying the entropy constraint. Of the coupling constants considered, this is therefore the only choice that guarantees the unique existence of *entropic* solutions to the N-junction Riemann problem for all initial data in the subsonic region.

Key words. gas flow, networks, junctions

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1. Introduction

This paper is concerned with a particular instance of a more general question; how to properly define global weak solutions for hyperbolic conservation laws defined on N segments of the real line, connected by a junction. Such conservation laws are given by

$$\frac{\partial \boldsymbol{U}_i}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}\left(\boldsymbol{U}_i\right) = 0, \qquad i \in \{1, \dots, N\}, \tag{1}$$

where in each segment i, we seek the solution $\boldsymbol{U}_i(x,t)$ for

$$t \in \mathbb{R}^+, \tag{2}$$

$$x \in \mathbb{R}^+. \tag{3}$$

The segments are assumed to be connected at the origin, as schematically illustrated in Figure 1.

Herein, for any segment i we may instead of (3) consider a finite interval $x \in (0, b_i)$ if proper boundary conditions may be supplied at $x = b_i$.

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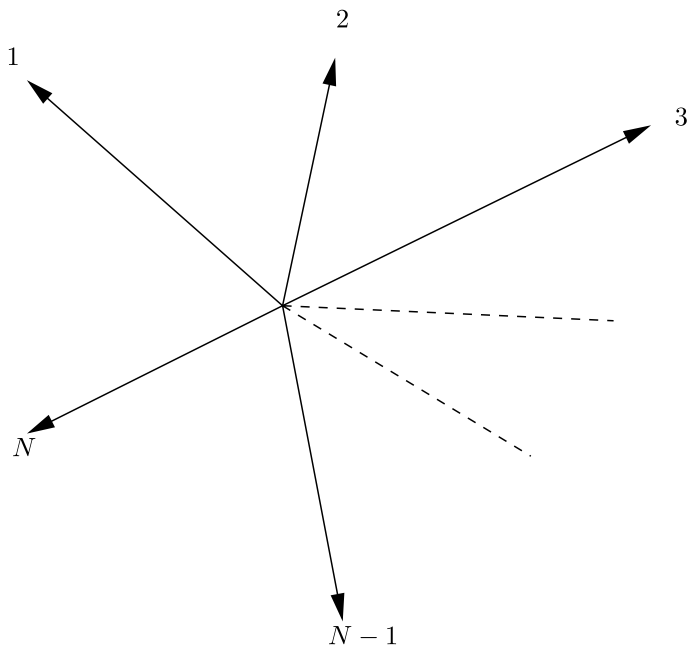

Figure 1. An N-junction. The different segments are joined at a vertex, with the positive x-direction always pointing away from the junction.

We observe that even in the scalar case, the initial value problem for (1) given by

$$\boldsymbol{U}_i(x,0) = \boldsymbol{U}_{i,0}(x) \qquad \forall i \in \{1,\dots,N\} \tag{4}$$

is in general incompletely specified; boundary conditions, or *coupling conditions*, must be provided at the point $x = 0$ for all segments. The specification of such coupling conditions for the isothermal Euler equations of gas dynamics is the topic to be addressed in this paper.

1.1. **The Generalized Riemann Problem.** Problems in the form (1)–(4) naturally arise in the study of traffic flow [6, 11] and fluid flow in pipe networks [2, 3, 4, 7, 10, 13]. Central to the study of the well-posedness of any such model formulation is the concept of the *generalized Riemann problem* [7, 10], which may be stated as follows: The equations (1)–(3) are to be solved given constant initial data in each segment:

$$\boldsymbol{U}_i(x,0) = \bar{\boldsymbol{U}}_i \qquad \forall i \in \{1,\dots,N\}. \tag{5}$$

In general, one must expect that the evolved solutions $\boldsymbol{U}_i(x,t)$ depend on *all* initial states, $\bar{\boldsymbol{U}}_i$, through their interaction in the junction. One may however introduce a natural condition: in each segment, the solution should be compatible with a *standard* Riemann problem at the segment-junction interface [7, 10, 11]. This condition may be precisely stated as follows.

C1: For all $i \in \{1,\dots,N\}$, there exists a state

$$\boldsymbol{U}_i^*\left(\bar{\boldsymbol{U}}_1,\dots,\bar{\boldsymbol{U}}_N\right) = \lim_{x\to 0^+} \boldsymbol{U}_i(x,t) \tag{6}$$

such that $\boldsymbol{U}_i(x,t)$ is given by the restriction to $x \in \mathbb{R}^+$ of the Lax solution to the standard Riemann problem for $x \in \mathbb{R}$:

$$\begin{aligned} \frac{\partial \boldsymbol{U}_i}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}_i) = 0, \\ \boldsymbol{U}_i(x,0) = \begin{cases} \bar{\boldsymbol{U}}_i & \text{if } x > 0 \\ \boldsymbol{U}_i^* & \text{if } x < 0. \end{cases} \end{aligned} \tag{7}$$

In other words, $\boldsymbol{U}_i^*$ is the *similarity solution* $\boldsymbol{w}(x/t)$ to the Riemann problem (7) evaluated at $x/t = 0$.

To close the system, a number of additional *coupling conditions* are needed to relate the various vectors $\boldsymbol{U}_i^*$. These conditions should respect the following somewhat related considerations.

(i) The conditions should adequately represent the underlying physics we seek to describe by the model.

(ii) The conditions should, in conjunction with C1, lead to a well-posed initial value problem.

Arguably, (ii) could be considered a necessary requirement for (i).

1.2. **The isothermal Euler equations.** The general Euler equations in M dimensions may be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{8}$$

$$\frac{\partial}{\partial t}(\rho v_j) + \sum_{i=1}^{M} \frac{\partial}{\partial x_i}(\rho v_i v_j) + \frac{\partial p}{\partial x_j} = 0 \qquad \forall j \in [1, \ldots, M], \tag{9}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (\boldsymbol{v}(E + p)) = 0. \tag{10}$$

Here, ρv is the mass flux, ρ is the fluid density and v is the fluid velocity. The total energy is defined as:

$$E = \frac{1}{2}\rho v^2 + \rho e, \tag{11}$$

where the internal energy fulfils the differential:

$$de = T ds + \frac{p}{\rho^2} d\rho. \tag{12}$$

In this work, we follow the approach of [2, 3, 7, 14, 17, 18] and consider one dimensional pipe flow governed by the *isothermal* Euler equations. These consist of the isentropic Euler equations together with the specific pressure law:

$$p(\rho) = a^2 \rho, \tag{13}$$

where a is the fluid speed of sound. A more general formulation of (13) was considered in [7]. The one dimensional equations may now be written as:

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho v \\ \rho v^2 + p(\rho) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{14}$$

Here, the momentum equations (9) reduce to a scalar equation. In addition, the isentropic assumption eliminates the energy conservation equation (10).

1.3. **Coupling conditions used with the isothermal Euler equations.** Two different coupling conditions [2, 3, 14] together with an entropy constraint [7] may be used to completely specify the problem. The first coupling condition is related to equation (8) and accounts for the conservation of mass at the junction. As remarked in [7], this is an obvious requirement. For N pipes of equal cross-section connected at a junction this may be stated as:

$$\sum_{i=1}^{N} \rho_i^*(x,t) v_i^*(x,t) = 0 \quad \text{for all} \quad t > 0, \tag{15}$$

where in the context of (6) we have

$$\boldsymbol{U}_i^* = \begin{bmatrix} \rho_i^* \\ \rho_i^* v_i^* \end{bmatrix}. \tag{16}$$

Colombo and Garavello [7] proposed that an *entropy selection* principle should apply to solutions through the junction, analogous to the standard admissibility theory for weak solutions to conservation laws. A number of viable entropy–entropy flux pairs may be constructed for the 1-dimensional equations (14) [16]. Garavello and Piccoli [12] note that for junctions, different entropies do not necessarily select the same solutions.

For isothermal flow, Colombo and Garavello [7] suggested using the *mechanical energy* as the entropy function. We will follow this approach as described in Section 3. In Section 4, this choice will be mathematically justified from the underlying multidimensional equations (8)–(10).

The final coupling condition is related to the momentum equations, (9), and does not seem to have an obvious expression. Colombo and Mauri [10] observe that a system described by the full set of Euler equations can in general not conserve linear momentum at the junction. On the contrary, the total momentum vector is constrained by the relative position of the pipes. For various flow models, momentum conservation has been replaced with the condition that some scalar flow parameter, $\tilde{\mathcal{H}}$, remains constant through the junction [3, 4, 5, 7, 10, 13, 14]. In the recent literature, two approaches are seen to be the most common. These are the conditions of equal pressure [3, 7, 13, 14]:

$$p(\rho_i^*(x,t)) = \tilde{\mathcal{H}}_p \quad \text{for all} \quad i \quad \text{and} \quad t > 0, \tag{17}$$

and equal momentum flux [4, 5, 7, 10]:

$$\left(\rho_i^* v_i^{*2} + p(\rho_i^*)\right)(x,t) = \tilde{\mathcal{H}}_{\mathrm{MF}} \quad \text{for all} \quad i \quad \text{and} \quad t > 0. \tag{18}$$

The choice of equal pressure is made primarily as it is a simple model that is widely used in the engineering community [3, 15, 17, 18]. The model is expected to be a fair approximation for low Mach number flows.

Colombo and Garavello [7] introduced (18) as a coupling condition. This was motivated primarily from continuity considerations; the authors wanted to ensure that a stationary shock infinitesimally close to the junction would remain stationary if perturbed. This is essential for the problem to be *well posed* in the strict sense that the solution should depend continuously on the initial data. The equal pressure condition (17) does not have this property [7].

However, one should note that for pipe networks, the junction itself represents a discontinuity in the local topology of the problem; hence the physical relevance of this requirement may be open for debate. In this paper, we will not discuss this issue. Instead, we focus only on the *existence* and *uniqueness* of solutions of the pure generalized Riemann problem with constant initial data in each pipe. In this respect, a main result of our current paper is that both the conditions (17) and (18) fail to provide global existence of solutions if the entropy constraint is taken into account. Furthermore, we propose an alternative coupling condition where unique global existence of entropic solutions is guaranteed.

For the generalized Riemann problem for (14), Colombo and Garavello [7] prove the existence and uniqueness of some stationary solutions and their perturbations when (18) is used as coupling condition. The results are shown to be global in space-time and local in the subsonic region of the state space $(\rho, \rho v)$. These results were extended to non-uniform initial data in [8].

Similar local results were achieved by Banda et al. [2, 3] for the coupling condition (17). Herein, the authors did not consider the entropy constraint through the junction. A unified framework was presented in [9], providing local existence and uniqueness of solutions to the Cauchy problem for general coupling conditions.

In [14], numerical simulations were performed in order to evaluate the coupling condition of equal pressure at the junction (eq. (17)). The simulations were performed on a tee-shaped junction, as the analytical solution for piecewise constant initial data in this kind of geometry was available from earlier work. As prerequisite for this solution it is stated that for the given geometry and initial data, the equations (7), (15) and (17) form a well-posed mathematical problem.

Two different flow configurations were considered in the two dimensional simulations. The first configuration consisted of one ingoing and two outgoing flows, the second of two ingoing and one outgoing flow. The simulation results were averaged and compared to the analytical results. A clear deviance between the simulations and the analytical results was found for the second configuration. Thus, for this configuration the use of geometry and flow dependent empirical pressure loss coefficients was recommended.

In the present work we propose a momentum related coupling condition by using the idea of ideal, reversible flow as starting point. Combined with the observation that conservation of energy is strongly related to conservation of momentum, we suggest to use the Bernoulli invariant, an energy invariant with constant value along streamlines. This allows us to prove global existence both in time and in the subsonic region of state space.

1.4. **Outline of the paper.** In section 2, we present the conditions defining the Riemann problem at a junction for the isothermal Euler equations. Further we investigate solutions where the entropy condition is not taken into account. The main result is presented in Proposition 4; such solutions exist and are unique whenever the coupling condition $\tilde{\mathcal{H}}$ satisfies a monotonicity property. In particular, the conditions (17) and (18) have this property.

Section 3 deals with the entropy condition. Results are derived for a three-pipe junction when equal pressure (17) and equal momentum flux (18) are used as coupling condition. Proposition 5 summarises the findings, that both conditions have solutions violating the entropy condition in certain ranges of pipe flow rates. Interestingly, there is a perfect duality between these two conditions; for any given velocity distribution, the entropy productions associated with the two different coupling conditions will be of opposite sign.

In Section 4, we present a classical derivation of the Bernoulli invariant as a scalar quantity that is conserved along streamlines. This invariant is related to the flux of the mechanical energy, which motivates the use of the mechanical energy as our entropy function. For the multidimensional equations, the Bernoulli invariant incorporates information from momentum conservation into a scalar quantity. Hence we propose and analyse a new coupling condition; momentum conservation should be replaced with a unique value of the Bernoulli invariant in the junction.

Propositions 9 and 10 contain our main result; among the three investigated momentum related coupling conditions, only equal Bernoulli invariant leads to unique existence of entropic solutions for the entire subsonic region of state space.

2. The Riemann problem at a junction of N pipes

We consider a system of N pipes of equal cross-sectional area, connected at a junction as illustrated in Figure 1. In each segment, the flow is governed by the conservation law (1) given by the isothermal Euler equations (14). Following [7], we define the generalized Riemann problem as follows.

Definition 1. *A solution to the Riemann problem* (5) *is a set of self-similar functions* $\boldsymbol{U}_i(x,t)$ *such that*

RP0: *For all* $i \in \{1,\ldots,N\}$*, there exists a state*

$$\boldsymbol{U}_i^*\left(\bar{\boldsymbol{U}}_1,\ldots,\bar{\boldsymbol{U}}_N\right) = \lim_{x\to 0^+} \boldsymbol{U}_i(x,t) \tag{19}$$

such that $\boldsymbol{U}_i(x,t)$ *is given by the restriction to* $x\in\mathbb{R}^+$ *of the Lax solution to the standard Riemann problem for* $x\in\mathbb{R}$*:*

$$\begin{aligned} &\frac{\partial \boldsymbol{U}_i}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}_i) = 0, \\ &\boldsymbol{U}_i(x,0) = \begin{cases} \bar{\boldsymbol{U}}_i & \text{if } x>0 \\ \boldsymbol{U}_i^* & \text{if } x<0. \end{cases} \end{aligned} \tag{20}$$

RP1: *Mass is conserved at the junction:*

$$\sum_{i=1}^{N} \rho_i^* v_i^* = 0. \tag{21}$$

RP2: *There is a unique, scalar momentum related coupling constant at the junction:*

$$\mathcal{H}(\rho_i^*, v_i^*) = \tilde{\mathcal{H}} \qquad \forall i \in \{1,\ldots,N\}. \tag{22}$$

Furthermore, *entropic* solutions are defined as:

Definition 2. *An* **entropic** *solution to the Riemann problem* (5) *is a solution satisfying the conditions RP0–RP2 as well as*

RP3: *Energy does not increase at the junction, i. e.*

$$\sum_{i=1}^{N} \rho_i^* v_i^* \left(\frac{1}{2}(v_i^*)^2 + a^2 \ln\frac{\rho_i^*}{\rho_0}\right) \le 0, \tag{23}$$

where ρ_0 *is some reference density.*

Remark 1. *The condition RP3 is the isothermal version of the entropy condition proposed in* [7]*. This will be derived in section 3.1. In Section 4, we will further justify this condition by showing that the multidimensional isothermal equations conserve the energy for smooth solutions.*

2.1. Uniqueness of solutions.

Given subsonic initial data $\bar{\boldsymbol{U}}_i$, and subsonic states $\boldsymbol{U}_i^*$, the two states in the pipe are connected by a wave of the second family [7]. ρ_i^* is therefore related to v_i^* through an explicit equation [7]. If they are connected by a rarefaction wave, they are related by

$$\ln\frac{\rho_i^*}{\bar{\rho}_i} = M_i^* - \bar{M}_i, \quad \rho_i^* \le \bar{\rho}_i, \tag{24}$$

where we for convenience use the Mach number, $M = {}^{v}\!/_{a}$, instead of velocity. Two states connected by a 2-shock curve are related by

$$M_i^* = \bar{M}_i + \left(\sqrt{\frac{\rho_i^*}{\bar{\rho}_i}} - \sqrt{\frac{\bar{\rho}_i}{\rho_i^*}}\right), \quad \rho_i^* > \bar{\rho}_i. \tag{25}$$

Using the appropriate relation, we can express the coupling constant $\mathcal{H}(\rho_i^*, v_i^*)$ as a function of one unknown state variable and the initial data. For example we may use the function $\mathcal{H}_i^*(\rho_i^*, \bar{\rho}_i, \bar{v}_i)$, or written in short form, $\mathcal{H}_i^*(\rho_i^*)$. Before we show results on the uniqueness of solutions, we define a monotonicity property on $\mathcal{H}_i^*$.

Definition 3. *A coupling condition, $\mathcal{H}_i^*$, is said to be* **monotone** *if the following conditions are satisfied:*

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{R2}} > 0, \tag{26}$$

and

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{S2}} > 0. \tag{27}$$

Herein, the subscript R2 *denotes differentiation along the 2-rarefaction curve* (24) *and* S2 *denotes differentiation along the 2-shock curve* (25).

The choice of the variable ρ_i^* is here somewhat arbitrary, as demonstrated in the following lemma.

Lemma 1. *Monotonicity in ρ_i^* is equivalent to monotonicity in M_i^*. More precisely,*

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{R2}} > 0 \tag{28}$$

if and only if

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}M_i^*}\right|_{\mathrm{R2}} > 0. \tag{29}$$

Furthermore,

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{S2}} > 0 \tag{30}$$

if and only if

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}M_i^*}\right|_{\mathrm{S2}} > 0. \tag{31}$$

Proof. The relation between M_i^* and ρ_i^* along a 2-rarefaction curve in Equation (24) may be differentiated to give

$$\frac{\mathrm{d}M_i^*}{\mathrm{d}\rho_i^*} = \frac{1}{\rho_i^*} > 0. \tag{32}$$

Similarly, the relation along a 2-shock curve in Equation (25) may be differentiated to give the relation

$$\frac{\mathrm{d}M_i^*}{\mathrm{d}\rho_i^*} = \frac{1}{2\sqrt{\rho_i^*\rho_i}}\left(1 + \frac{\rho_i}{\rho_i^*}\right) > 0. \tag{33}$$

The chain rule may then be used to write

$$\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}M_i^*} = \frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\frac{\mathrm{d}\rho_i^*}{\mathrm{d}M_i^*}. \tag{34}$$

□

The following may then be stated:

Lemma 2. *Assume that the state $\bar{\boldsymbol{U}}_i$ and a monotone coupling constant $\mathcal{H}_i^*$ with value $\tilde{\mathcal{H}}$ are given. Then there is a unique state $\boldsymbol{U}_i^*$ with the following properties:*

1. *$\mathcal{H}(\boldsymbol{U}_i^*) = \mathcal{H}_i^*(\rho_i^*) = \tilde{\mathcal{H}}$;*
2. *$\boldsymbol{U}_i^*$ is connected to $\bar{\boldsymbol{U}}_i$ with a 2-rarefaction curve if $\mathcal{H}(\bar{\boldsymbol{U}}_i) \geq \tilde{\mathcal{H}}$.*

(3) $\boldsymbol{U}_i^*$ *is connected to* $\bar{\boldsymbol{U}}_i$ *with a 2-shock curve if* $\mathcal{H}(\bar{\boldsymbol{U}}_i) < \tilde{\mathcal{H}}$.

Proof. The monotone coupling condition in the sense of Definition 3 guarantees that $\mathcal{H}_i^*(\rho_i^*)$ is a monotone function. Hence the uniqueness of $\boldsymbol{U}_i^*$ is proved. The monotonicity also enables the selection of the kind of curve connecting the two states, which is then determined by the Lax-condition. If $\bar{\rho}_i \geq \rho_i^*$ they are connected by a rarefaction wave. Otherwise, if $\bar{\rho}_i < \rho_i^*$ they are connected by a shock wave. □

Remark 2. *The monotonicity properties assumed in Lemma 2 provide the opportunity to express the unknown state variables by inverted functions. If* $\bar{\boldsymbol{U}}_i$ *and* $\boldsymbol{U}_i^*$ *are connected by a 2-rarefaction curve, the functions are denoted by the subscript* $\mathcal{R}$*:*

$$\rho_i^* = \rho_{\mathcal{R}}\left(\mathcal{H}_{i,\mathrm{R2}}^*(\rho_i^*) = \tilde{\mathcal{H}}\right), \tag{35}$$

$$M_i^* = M_{\mathcal{R}}\left(\mathcal{H}_{i,\mathrm{R2}}^*(\rho_i^*) = \tilde{\mathcal{H}}\right). \tag{36}$$

Similarly, if connected by a 2-shock curve the inverted functions are denoted by the subscript $\mathcal{S}$*:*

$$\rho_i^* = \rho_{\mathcal{S}}\left(\mathcal{H}_{i,\mathrm{S2}}^*(\rho_i^*) = \tilde{\mathcal{H}}\right), \tag{37}$$

$$M_i^* = M_{\mathcal{S}}\left(\mathcal{H}_{i,\mathrm{S2}}^*(\rho_i^*) = \tilde{\mathcal{H}}\right). \tag{38}$$

A stronger result of Lemma 2 may be stated when both the initial state, $\bar{\boldsymbol{U}}_i$, and the coupling constant, $\tilde{\mathcal{H}}$ are subsonic.

Proposition 1. *Assume that the state* $\bar{\boldsymbol{U}}_i$ *and the coupling constant* $\tilde{\mathcal{H}}$ *are given, where* $\bar{\boldsymbol{U}}_i$ *is subsonic and* $\tilde{\mathcal{H}}$ *satisfies the inequality*

$$\mathcal{H}_i^*|_{\mathrm{R2}}(M_i^* = -1) < \tilde{\mathcal{H}} < \mathcal{H}_i^*|_{\mathrm{S2}}(M_i^* = 1). \tag{39}$$

Further, assume that the coupling condition is monotone in the sense of Definition 3. Then, a state $\boldsymbol{U}_i^*$ *satisfying RP0 is uniquely defined.*

Proof. The results in Lemma 2 enables the construction of the functions

$$\rho_i^*(\tilde{\mathcal{H}}) = \begin{cases} \rho_{\mathcal{R}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} < \mathcal{H}(\bar{\boldsymbol{U}}_i) \\ \rho_i & \text{if } \tilde{\mathcal{H}} = \mathcal{H}(\bar{\boldsymbol{U}}_i) \\ \rho_{\mathcal{S}}(\mathcal{H}_i^*) & \text{if } \tilde{\mathcal{H}} > \mathcal{H}(\bar{\boldsymbol{U}}_i), \end{cases} \tag{40}$$

$$M_i^*(\tilde{\mathcal{H}}) = \begin{cases} M_{\mathcal{R}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} < \mathcal{H}(\bar{\boldsymbol{U}}_i) \\ M_i & \text{if } \tilde{\mathcal{H}} = \mathcal{H}(\bar{\boldsymbol{U}}_i) \\ M_{\mathcal{S}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} > \mathcal{H}(\bar{\boldsymbol{U}}_i). \end{cases} \tag{41}$$

Note that (40) and (41) are continuous, monotonically increasing functions, and that the range of M_i^* is $(-1, 1)$ in the interval (39). Furthermore, the range of ρ_i^* is

$$\rho_i^* \in \left(\rho_{\mathcal{R}}(\mathcal{H}_i^*(M_i^* = -1)), \rho_{\mathcal{S}}(\mathcal{H}_i^*(M_i^* = 1))\right). \tag{42}$$

Due to the monotonicity property shown in equations (32) and (33), this range may be expressed by equations (24) and (25). Note that along a 2-shock curve, equation (25) may be rearranged to give

$$\rho_i^* = \frac{\bar{\rho}_i}{4}\left(M_i^* - \bar{M}_i + \sqrt{\left(M_i^* - \bar{M}_i\right)^2 + 4}\right)^2. \tag{43}$$

Thus equation (42) can be rewritten as:

$$\rho_i^* \in \left(\bar{\rho}_i \exp\left(-1 - \bar{M}_i\right), \frac{\bar{\rho}_i}{4}\left(1 - \bar{M}_i + \sqrt{(1 - \bar{M}_i)^2 + 4}\right)^2\right). \tag{44}$$

□

The following statement about the solution to the generalized Riemann problem at a junction may then be made.

Proposition 2. *Assume that a solution $\boldsymbol{U}_i^*$ exists that satisfies RP0–RP2 and that the coupling condition is monotone in the sense of Definition 3. Then $M_i^* \in (-1,1)$ if and only if*

$$\max_i \mathcal{H}_i^*|_{\mathrm{R2}}(M_i^*=-1) < \tilde{\mathcal{H}} < \min_i \mathcal{H}_i^*|_{\mathrm{S2}}(M_i^*=1). \tag{45}$$

Proof. First observe that

$$M_i^* \in (-1,1) \qquad \forall i \tag{46}$$

implies that $\mathcal{H}_i^*(\rho_i^*) = \tilde{\mathcal{H}}$ must lie in the interval (39) for all i. And conversely, if it does, it follows from Proposition 1 that the solutions $\boldsymbol{U}_i^*$ are subsonic. □

The *uniqueness* of solutions may now be established.

Proposition 3. *Assume that subsonic initial states $\bar{\boldsymbol{U}}_i$ are given in each pipe segment $i \in \{1,\dots,N\}$ and that the coupling condition is monotone in the sense of Definition 3. If there is a set of subsonic solutions $\boldsymbol{U}_i^*$ satisfying RP0–RP2, this set is unique.*

Proof. Consider the mass flux as a function of $\tilde{\mathcal{H}}$:

$$(\rho M)_i^*(\tilde{\mathcal{H}}) = \rho_i^*(\tilde{\mathcal{H}}) M_i^*(\tilde{\mathcal{H}}) = \begin{cases} \rho_{\mathcal{R}}(\tilde{\mathcal{H}}) M_{\mathcal{R}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} < \mathcal{H}(\bar{\boldsymbol{U}}_i) \\ \rho_i M_i & \text{if } \tilde{\mathcal{H}} = \mathcal{H}(\bar{\boldsymbol{U}}_i) \\ \rho_{\mathcal{S}}(\tilde{\mathcal{H}}) M_{\mathcal{S}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} > \mathcal{H}(\bar{\boldsymbol{U}}_i). \end{cases} \tag{47}$$

Along a 2-rarefaction curve, equation (32) may be inserted to give

$$\mathrm{d}(\rho M)_i^* = (1 + M_i^*)\,\mathrm{d}\rho_i^*. \tag{48}$$

Similarly, along a 2-shock curve equation (33) inserted gives

$$\mathrm{d}(\rho M)_i^* = \left(1 + M_i^* + \frac{\left(\sqrt{\rho_i^*} - \sqrt{\rho_i}\right)^2}{2\sqrt{\rho_i^* \rho_i}}\right)\mathrm{d}\rho_i^*. \tag{49}$$

It then follows from (26) and (27) that (47) is a monotonically increasing function, and in particular the *total* mass flux

$$\mathcal{J}(\tilde{\mathcal{H}}) = \sum_{i=1}^{N} (\rho M)_i^*(\tilde{\mathcal{H}}) \tag{50}$$

is a monotonically increasing function of $\tilde{\mathcal{H}}$ in the subsonic region. This guarantees that there is at most one valid solution to RP1:

$$\mathcal{J}(\tilde{\mathcal{H}}) = 0. \tag{51}$$

□

Although (45) is a necessary condition for subsonic solutions to exist, it is not sufficient. We define the *subsonic region* of the initial data as follows.

Definition 4. *Assume that a set $\{\bar{\boldsymbol{U}}_i\}$ of initial data is given. Assume that this set satisfies the conditions*

(1) $\bar{M}_i \in (-1,1) \quad \forall i$;

(2) $\mathcal{J}(\mathcal{H}^-) < 0$, *where*

$$\mathcal{H}^- = \max_i \mathcal{H}_i^*|_{\mathrm{R2}}(M_i^* = -1); \tag{52}$$

(3) $\mathcal{J}(\mathcal{H}^+) > 0$, *where*

$$\mathcal{H}^+ = \min_i \mathcal{H}_i^*|_{\mathrm{S2}}(M_i^* = 1). \tag{53}$$

Such a set of initial data is said to belong to the **subsonic region***.*

Remark 3. *Condition (2) and (3) in Definition 4 are important when defining the subsonic region as there exists states that satisfy* (45) *where*

$$\mathcal{J}(\mathcal{H}^-) > 0, \tag{54}$$

as well as states that satisfy (45) *where*

$$\mathcal{J}(\mathcal{H}^+) < 0. \tag{55}$$

Hence Definition 4 describes precisely the region where both the initial data and the resulting junction states are subsonic.

The results of this section may be summed up by the following proposition.

Proposition 4. *Assume that the initial data $\bar{\boldsymbol{U}}_i$ belongs to the subsonic region in the sense of Definition 4 and that the coupling condition is monotone in the sense of Definition 3. Then there exists a unique set of subsonic solutions satisfying RP0–RP2.*

Proof. Proposition 1 proves the uniqueness of a state $\boldsymbol{U}_i^*$ satisfying RP0 given subsonic initial state, $\bar{\boldsymbol{U}}_i$, and coupling constant, $\tilde{\mathcal{H}}$. Proposition 3 proves the uniqueness of the set of solutions $\boldsymbol{U}_i^*$ that satisfies RP0–RP2, given that such a set of solutions exist. Finally, the definition of the subsonic region in Definition 4 guarantees the existence of the unique set of solutions. □

Remark 4. *The analysis so far has not taken into account the entropy condition, (RP3, eq.* (23)*). According to Proposition 4, a set of initial conditions satisfying RP0–RP2 (eq.* (20)*,* (21) *and* (22)*) has the unique solution*

$$\boldsymbol{U}_i^* = \bar{\boldsymbol{U}}_i. \tag{56}$$

If this solution does not satisfy the entropy condition, it is impossible *to construct an entropic Lax solution to the generalized Riemann problem defined by the initial condition. The relation between the solution to RP0–RP2 and the entropy condition (RP3) is found in section 3.*

2.2. **Monotonicity of specific coupling conditions.** Let $\mathcal{H}_{\mathrm{MF}}$ denote the momentum related coupling condition of equal momentum flux (18), which for the isothermal Euler equations (14) is equivalent to:

$$\mathcal{H}_{\mathrm{MF}} = \rho(M^2 + 1). \tag{57}$$

Similarly, let $\mathcal{H}_p$ denote the condition of equal pressure:

$$\mathcal{H}_{\mathrm{p}} = \rho. \tag{58}$$

The following results may then be stated:

Lemma 3. *The coupling condition of equal pressure is monotone in the sense of Definition 3.*

Proof. The coupling condition of equal pressure leads to a trivial result:

$$\mathcal{H}_{i,p}^*(\rho_i^*) = \rho_i^* \tag{59}$$

and accordingly

$$\frac{\mathrm{d}\mathcal{H}_{i,p}^*}{\mathrm{d}\rho_i^*} = 1. \tag{60}$$

Thus the coupling condition is monotone. □

Lemma 4. *In the subsonic region, the coupling condition of equal momentum flux is monotone in the sense of Definition 3.*

Proof. Along a 2-rarefaction curve, equation (24) may be inserted to give

$$\mathcal{H}^*_{i,\mathrm{MF}}\left(\rho^*_i\right) = \rho^*_i \left(1 + \left(\ln \frac{\rho^*_i}{\bar{\rho}_i} + \bar{M}_i\right)^2\right), \tag{61}$$

with corresponding derivative

$$\left.\frac{\mathrm{d}\mathcal{H}^*_{i,\mathrm{MF}}}{\mathrm{d}\rho^*_i}\right|_{\mathrm{R2}} = \left(\left(1 + \ln \frac{\rho^*_i}{\bar{\rho}_i}\right) + \bar{M}_i\right)^2 > 0. \tag{62}$$

Along a 2-shock curve, equation (25) may be inserted to give

$$\mathcal{H}^*_{i,\mathrm{MF}}\left(\rho^*_i\right) = \rho^*_i \left(1 + \left(\bar{M}_i + \left(\sqrt{\frac{\rho^*_i}{\bar{\rho}_i}} - \sqrt{\frac{\bar{\rho}_i}{\rho^*_i}}\right)\right)^2\right). \tag{63}$$

The derivative is thus

$$\left.\frac{\mathrm{d}\mathcal{H}^*_{i,\mathrm{MF}}}{\mathrm{d}\rho^*_i}\right|_{\mathrm{S2}} = \left(\frac{\rho^*_i}{\bar{\rho}_i} - 1\right)\left(1 + \bar{M}_i\sqrt{\frac{\bar{\rho}_i}{\rho^*_i}}\right) + \left(\bar{M}_i + \sqrt{\frac{\rho^*_i}{\bar{\rho}_i}}\right)^2 + \frac{\rho^*_i}{\bar{\rho}_i} > 0, \tag{64}$$

and consequently the coupling condition is monotone. □

3. Energy conservation in a junction

3.1. **The entropy condition.** In the previous section, the monotonicity of the two momentum related coupling conditions (17) and (18) was established to verify the uniqueness of solutions to RP0–RP2. In this section we will investigate if the coupling conditions obey the entropy condition (RP3, eq. (23)). The investigation will use the case of a junction with three connected pipes.

The entropy condition originates from the energy flux in the general Euler equations (10). Due to the isentropic assumption and the pressure law (eq. (13)), the differential in equation (12) is simplified to

$$de = \frac{a^2}{\rho} d\rho. \tag{65}$$

Integrating this equation yields:

$$e = a^2 \ln\left(\frac{\rho}{\rho_0}\right). \tag{66}$$

Inserting (66) in (10) and using (11), we may express the energy flux as:

$$v(E+p) = v\rho(\frac{1}{2}v^2 + a^2 \ln\left(\frac{\rho}{\rho_0}\right) + a^2). \tag{67}$$

For an N-junction, the total energy flux thus becomes:

$$\begin{aligned} Q &= \sum_{i=1}^{N}\left(v_i\rho_i\left(\frac{1}{2}v_i^2 + a^2 \ln\left(\frac{\rho_i}{\rho_0}\right) + a^2\right)\right) \\ &= \sum_{i=1}^{N}\left(v_i\rho_i\left(\frac{1}{2}v_i^2 + a^2 \ln\left(\rho_i\right)\right)\right), \end{aligned} \tag{68}$$

where the terms a^2 and $a^2 \ln(\rho_0)$ in (68) cancel due to the conservation of mass (21).

3.2. **Coupling condition: equal pressure.** By the assumptions $N = 3$ and equal pressure as coupling condition, the equations (13), (21) and (68) become:

$$\rho_i^* = \tilde{\rho}, \tag{69}$$

$$\sum_{i=1}^{3} v_i^* = 0 \tag{70}$$

and

$$\begin{aligned} Q =& \tilde{\rho} \sum_{i=1}^{3} \left(v_i^* \left(\frac{1}{2}(v_i^*)^2 + a^2 \ln(\tilde{\rho}) \right) \right) \\ =& \frac{1}{2}\tilde{\rho} \sum_{i=1}^{3} (v_i^*)^3 + a^2 \tilde{\rho} \ln(\tilde{\rho}) \sum_{i=1}^{3} v_i^* \\ =& \frac{1}{2}\tilde{\rho} \sum_{i=1}^{3} (v_i^*)^3. \end{aligned} \tag{71}$$

Equation (71) may be expanded to give

$$Q = \frac{1}{2}\tilde{\rho} \left(\left(\sum_{i=1}^{3} v_i^* \right)^3 - 3(v_1^* + v_2^*)(v_2^* + v_3^*)(v_1^* + v_3^*) \right). \tag{72}$$

Inserting (70) into (72) results in the expression

$$Q = \frac{3}{2}\tilde{\rho} v_1^* v_2^* v_3^*. \tag{73}$$

Hence, the entropy condition is only fulfilled for one ingoing and two outgoing flows, or for cases with zero flow-rate in one of the pipes.

3.3. **Coupling condition: equal momentum flux.** The assumption of equal momentum flux at the junction, $\rho_i^*(1 + (M_i^*)^2) = \tilde{\mathcal{H}}$, results in the following set of equations:

$$\rho_i^* = \frac{\tilde{\mathcal{H}}}{1 + (M_i^*)^2}, \tag{74}$$

$$\sum_{i=1}^{3} \rho_i^* v_i^* = \tilde{\mathcal{H}} a \sum_{i=1}^{3} \frac{M_i^*}{1 + (M_i^*)^2} = 0 \tag{75}$$

and

$$\begin{aligned} Q =& \sum_{i=1}^{3} \tilde{\mathcal{H}} a \frac{M_i^*}{1 + (M_i^*)^2} \frac{a^2 \left((M_i^*)^2 + 2 \ln \left(\frac{1}{1+(M_i^*)^2} \right) \right)}{2} \\ =& \tilde{\mathcal{H}} a^3 \sum_{i=1}^{3} \frac{M_i^*}{1 + (M_i^*)^2} \frac{\left((M_i^*)^2 - 2 \ln \left(1 + (M_i^*)^2 \right) \right)}{2}. \end{aligned} \tag{76}$$

As shown in appendix A, the function Q takes the value of zero only when one of the flow velocities is zero. Further, the function is positive for a certain range of flow velocities.

As seen, both coupling conditions results in unphysical solutions at certain ranges of flow velocities. In addition it should be noted that the range of flow velocities yielding physical solutions for one condition, has unphysical solutions for the other condition.

Proposition 5. *In the case of a three-pipe junction, the energy flux functions for coupling conditions of equal pressure (eq.* (73)*) and equal momentum flux (eq.* (76)*) takes values of opposite sign for all cases with non-zero flow velocities. In particular, for the equal pressure condition, whenever there are two incoming and one outgoing flow the entropy constraint is violated. For the equal momentum flux condition, the entropy constraint is violated whenever there are one incoming and two outgoing flows.*

Proof. We use the derivation in appendix A as starting point, with $z_2 > 0$ and $M_2 > 0$. For non-positive flows in the second pipe, the procedure is similar, but with opposite signs.

The derivation showed that for a coupling condition of equal momentum flux, the energy flux is non-positive only in the range $z_1 \in \langle -z_2, 0 \rangle$. From equation (105) it may be deduced that for $z_1 = -z_2$, $M_1 = -M_2$. Hence, from equation (73) it may be found that the energy flux for the coupling condition of equal pressure is non-negative only in the range $z_1 \in \langle -z_2, 0 \rangle$. □

4. Proposal for coupling condition: equal Bernoulli invariant

As pointed out in [10], for a system modelled by the full set of Euler equations, the linear momentum of the fluid may not be conserved at the junction. Hence there is a dependence on the relative position of the pipes. A scalar conserved quantity derived from the vector momentum conservation would therefore be desirable. In classical mechanics, this scalar quantity is the Hamiltionian energy function, and its conservation follows from the underlying symmetries of the equations of motion. The theory is extendable to fluid mechanics [1, 19]; the Euler equations give rise to constants of motion known as *Bernoulli invariants*.

These invariants are constant along a streamline, thus if used as coupling constant ideal subsonic conditions are assumed. At these conditions uninterrupted streamlines should flow into and out of the junction. Since we are dealing with a one dimensional model, the coupling constant would represent the cross sectional average value of the streamline invariant.

We will now briefly review the underlying theory as applied to the isentropic Euler equations. Noticing that streamlines are everywhere tangent to the local velocity field, we may state the following.

Definition 5. *Assume that for steady flows, a flow parameter B is constant along streamlines, i. e. it satisfies equation* (77)*:*

$$\boldsymbol{v} \cdot \nabla B = 0. \tag{77}$$

Then B is denoted a streamline invariant.

There is a streamline invariant associated with the flux of a conserved scalar quantity. In particular, we have the following proposition.

Proposition 6. *Assume that we for some variable ξ have the conservation equation*

$$\frac{\partial \xi}{\partial t} + \nabla \cdot (\rho B \boldsymbol{v}) = 0. \tag{78}$$

Then B is a streamline invariant for steady flows.

Proof. It follows from (14) that for steady flows we have

$$\nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{79}$$

Furthermore,

$$\nabla \cdot (\rho B \boldsymbol{v}) = \rho \boldsymbol{v} \cdot \nabla B + B \nabla \cdot (\rho \boldsymbol{v}) = \rho \boldsymbol{v} \cdot \nabla B = 0, \tag{80}$$

and we recover (77) by dividing by ρ. □

We are looking for an energy related streamline invariant that is based on vector momentum conservation. We therefore use the isentropic Euler equations in N space dimensions as the starting point (Equations (8) and (9)). Pressure is assumed to follow equation (13).

Based on the equations, we introduce the total mechanical energy, $\mathcal{M}$, as a function of compression, $\mathcal{C}$, and kinetic energy, E_{K}. The compression energy is defined through the differential

$$\mathrm{d}\mathcal{C} = \frac{p}{\rho^2}\,\mathrm{d}\rho, \tag{81}$$

which may be exactly integrated given that $p = p(\rho)$. The kinetic energy is defined as

$$E_{\mathrm{K}} = \frac{1}{2}\rho \boldsymbol{v}^2, \tag{82}$$

and the total mechanical energy is defined by the relation:

$$\mathcal{M} = \rho\mathcal{C} + \frac{1}{2}\rho\boldsymbol{v}^2. \tag{83}$$

The following then holds:

Proposition 7. *The total mechanical energy, $\mathcal{M}$ is conserved.*

Proof. The differential defining the compression energy, (81), and the mass equation, (8), may be used to write

$$\frac{\partial}{\partial t}(\rho\mathcal{C}) + \nabla\cdot(\rho\mathcal{C}\boldsymbol{v}) = \rho\frac{\partial\mathcal{C}}{\partial t} + \rho\boldsymbol{v}\cdot\nabla\mathcal{C} = \frac{p}{\rho}\left(\frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\nabla\rho\right), \tag{84}$$

and by the mass equation we simplify to

$$\frac{\partial}{\partial t}(\rho\mathcal{C}) + \nabla\cdot(\rho\mathcal{C}\boldsymbol{v}) + p\nabla\cdot\boldsymbol{v} = 0. \tag{85}$$

The kinetic energy has the differential

$$\mathrm{d}\left(\frac{1}{2}\rho\boldsymbol{v}^2\right) = \boldsymbol{v}\cdot\mathrm{d}(\rho\boldsymbol{v}) - \frac{1}{2}\boldsymbol{v}^2\,\mathrm{d}\rho. \tag{86}$$

Hence by (8) and (9) we may write

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\boldsymbol{v}^2\right) + \sum_{j=1}^{N} v_j \sum_{i=1}^{N}\frac{\partial}{\partial x_i}(\rho v_i v_j) - \frac{1}{2}\boldsymbol{v}^2\sum_{j=1}^{N}\frac{\partial}{\partial x_j}(\rho v_j) + \boldsymbol{v}\cdot\nabla p = 0, \tag{87}$$

or

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\boldsymbol{v}^2\right) + \nabla\cdot\left(\boldsymbol{v}\left(\frac{1}{2}\rho\boldsymbol{v}^2\right)\right) + \boldsymbol{v}\cdot\nabla p = 0. \tag{88}$$

It then follows from (85), (88) and the divergence theorem that $\mathcal{M}$ is conserved:

$$\frac{\partial\mathcal{M}}{\partial t} + \nabla\cdot(\boldsymbol{v}(\mathcal{M}+p)) = 0. \tag{89}$$

□

Comparing the equations (78) and (89), it is clear that $(\mathcal{M}+p)/\rho$ is a streamline invariant. Using the pressure law, Equation (13), the expression for the compression energy may be integrated. Thus, the invariant becomes:

$$B = a^2\left(\ln\frac{\rho}{\rho_0} + 1\right) + \frac{1}{2}\boldsymbol{v}^2. \tag{90}$$

Since a is a constant, an equivalent invariant is:

$$B = a^2\ln\frac{\rho}{\rho_0} + \frac{1}{2}\boldsymbol{v}^2. \tag{91}$$

This may be recognised as the invariant in Bernoulli's equation for steady frictionless flow along a streamline when the gravity force has been neglected [20, eq. (3.76)]. Hence the invariant in Equation (91) will be denoted the Bernoulli invariant in the following sections.

Remark 5. *The Bernoulli invariant arises from the equation for conservation of total mechanical energy. This implies that the model does not allow any mechanical energy to be transformed into heat. As a consequence, the Bernoulli invariant is related to* reversible *flows.*

4.1. **Existence and uniqueness of solutions when using equal Bernoulli invariant as coupling condition.** Noticing that ρ_0 is a constant, we may simplify the coupling constant of equal Bernoulli invariant to:

$$\mathcal{H}_{\mathrm{BI}} = \ln(\rho) + \frac{1}{2}M^2. \tag{92}$$

Lemma 5. *The Riemann problem at a junction with RP2 expressed by Equation* (92) *has a unique solution satisfying RP0–RP2 given that the initial data belongs to the subsonic region in the sense of Definition 4.*

Proof. To prove the uniqueness of solutions to the Riemann problem at the junction it is sufficient to prove that the coupling condition of equal Bernoulli invariant is monotone in the sense of Definition 3 and Lemma 1. Existence and uniqueness is then guaranteed by Proposition 4.

Along a 2-rarefaction curve, the coupling constant expressed as a function of Mach number is

$$\mathcal{H}^*_{i,\mathrm{BI}}\left(M^*_i\right) = M^*_i - \bar{M}_i + \frac{1}{2}(M^*_i)^2 + \ln(\bar{\rho}_i), \tag{93}$$

with corresponding derivative:

$$\left.\frac{\mathrm{d}\mathcal{H}^*_{i,\mathrm{BI}}}{\mathrm{d}M^*_i}\right|_{\mathrm{R2}} = 1 + M^*_i \geq 0 \quad \text{for} \quad M^*_i \in [-1,1]. \tag{94}$$

Along a 2-shock curve, the coupling condition is

$$\mathcal{H}^*_{i,\mathrm{BI}}\left(M^*_i\right) = \ln\left(\frac{\bar{\rho}_i}{4}\left(M^*_i - \bar{M}_i + \sqrt{\left(M^*_i - \bar{M}_i\right)^2 + 4}\right)^2\right) + \frac{1}{2}\left(M^*_i\right)^2. \tag{95}$$

The derivative is

$$\left.\frac{\mathrm{d}\mathcal{H}^*_{i,\mathrm{BI}}}{\mathrm{d}M^*_i}\right|_{\mathrm{R2}} = M^*_i + \frac{2}{\sqrt{\left(M^*_i - \bar{M}_i\right)^2 + 4}}. \tag{96}$$

The Lax entropy condition for a 2-shock wave is $\bar{M}_i < M^*_i$. Equation (96) may only be negative for negative values of M^*_i and thus only for negative values of $\bar{M}_i$. It is therefore necessary to prove that equation (96) is zero or positive for all values of $\bar{M}_i \in [-1,0]$, $M^*_i \in [-1,0]$ where $M^*_i - \bar{M}_i > 0$. We apply the notation:

$$f(M^*, \bar{M}) = M^* + \frac{2}{\sqrt{\left(M^* - \bar{M}\right)^2 + 4}}. \tag{97}$$

The end-points for f as a function of M^* are:

$$f(M^* = \bar{M}, \bar{M}) = \bar{M} + 1 \geq 0 \tag{98}$$

and

$$f(M^* = 0, \bar{M}) = \frac{2}{\sqrt{\bar{M}^2 + 4}} > 0. \tag{99}$$

If f is a monotone function of $M^* \in [\bar{M}, 0]$, then the function cannot be negative in this interval. To this end, we find the derivative

$$\frac{\partial f}{\partial M^*} = 1 - \frac{2(M^* - \bar{M})}{((M^* - \bar{M})^2 + 4)^{3/2}}. \tag{100}$$

Observing that we now have a function only of $(M^* - \bar{M})$, we replace this by $z \in [0, 1]$. We want to show that

$$1 - \frac{2z}{(z^2 + 4)^{3/2}} > 0, \tag{101}$$

which results in the calculation:

$$\begin{aligned} 1 >& \frac{2z}{(z^2 + 4)^{3/2}}, \\ 2z <& (z^2 + 4)^{3/2}, \\ 4z^2 <& (z^2 + 4)^3. \end{aligned} \tag{102}$$

This is easily seen to be true given the possible values of z. □

Unlike the two earlier proposed coupling conditions, the equal Bernoulli invariant assumption fulfils the entropy condition in equation (68).

Proposition 8. *When using the Bernoulli invariant as coupling condition, the entropy condition (eq.* (68)*) is satisfied for all flow conditions in the general case of N pipes connected at a junction.*

Proof. Inserting $\tilde{\mathcal{H}}$ defined by Equation (92) into the entropy condition and using equation (21) leads to:

$$\begin{aligned} Q =& a^2 \sum_{i=1}^{N} \rho_i^* v_i^* \left(\frac{1}{2} (M_i^*)^2 + \left(\tilde{\mathcal{H}} - \frac{1}{2} (M_i^*)^2 \right) \right) \\ =& a^2 \tilde{\mathcal{H}} \sum_{i=1}^{N} \rho_i^* v_i^* = 0. \end{aligned} \tag{103}$$

□

Finally, the main results may be summed up by the following propositions:

Proposition 9. *For the Riemann problem at a junction with equal pressure or equal momentum flux as coupling condition (RP2) there exists a unique solution satisfying RP0–RP2 provided that the initial data belongs to the subsonic region in the sense of Definition 4. There does not exist solutions that satisfy RP3 (entropic solutions) for all initial data in the subsonic region given by Definition 4.*

Proof. Existence and uniqueness is given by Proposition 4 together with Lemmas 3 and 4. Proposition 5 shows the lack of entropic solutions for certain intervals of flow rates for $N = 3$. We can extend this negative result to arbitrary N simply by imposing a zero flow velocity in the remaining $N - 3$ pipes. □

Proposition 10. *For the Riemann problem at a junction with equal Bernoulli invariant as coupling condition (RP2) there exists a unique entropic solution satisfying RP0–RP3 provided that the initial data belongs to the subsonic region in the sense of Definition 4.*

Proof. Existence and uniqueness is given by Proposition 4 and Lemma 5. The result in Proposition 8 proves that the solution is entropic. □

5. Summary

In this paper, we have investigated solutions to the isothermal Euler equations modelling flow conditions in junctions. Herein, we have defined the generalized Riemann problem in accordance with the established literature on the topic. Unlike regular Riemann problems, the initial data must be supplemented with a set of *coupling conditions* for the problem at the junction to be completely specified. Furthermore, an entropy constraint is needed in order to select physical solutions.

The coupling condition related to conservation of momentum does not have an obvious expression, and different options have been investigated in previous work. We have evaluated two of these proposed conditions for the *subsonic* region of state space, which has been precisely defined in our paper. When the entropy constraint is not taken into consideration, the existence and uniqueness of solutions to Riemann problems is proved. The proof is valid for momentum related coupling constants that are monotone with respect to the state variables of density and fluid velocity. The two different coupling conditions from earlier work, equal pressure and equal momentum flux, are seen to fulfil this requirement.

In order to distinguish between physical and unphysical solutions, the energy flux from the general Euler equations has been used as entropy constraint. For the case of a 3-pipe junction, solutions using the two proposed momentum related coupling conditions have been investigated. Both conditions give unphysical solutions for certain flow configurations and velocities. Herein, there is a duality; whenever there are non-zero flow velocities in all pipes, one condition gives physical solutions and the other gives unphysical solutions. Hence both these conditions fail to provide global existence of entropic solutions to the generalized Riemann problem.

By considering the flow to be ideal when modelled by the full multi-dimensional Euler equations, we have proposed *equal Bernoulli invariant* as coupling condition. In the case of reversible, subsonic flow, the assumption of uninterrupted streamlines is valid. The Bernoulli invariant, derived from the conservation equation for total mechanical energy, is constant along these lines. This coupling condition has the nice property that all solutions to the Riemann problem at the junction will always be entropic. We have shown that the monotonicity requirement for this condition holds.

Hence, our proposed coupling condition guarantees the existence and uniqueness of entropic solutions to any Riemann problem with initial conditions belonging to the subsonic region.

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Appendix A. Evaluation of the energy flux function for the coupling condition of equal momentum flux

The analysis concerns the expression for the total energy flux in a junction connecting three pipes as given by Equation (76). For the purpose of the investigation, we simplify this expression to

$$\hat{Q}(M_k) = \sum_{k=1}^{3} z_k(M_k) b_k(M_k), \tag{104}$$

where

$$z_k(M_k) = \frac{M_k}{1 + M_k^2}, \tag{105}$$

$$b_k(M_k) = \frac{M_k^2 - 2\ln(1 + M_k^2)}{2}. \tag{106}$$

Conservation of mass (75) may then be expressed as:

$$\sum_{k=1}^{3} z_k(M_k) = 0. \tag{107}$$

The flux function $\hat{Q}$, has two obvious values of z_1 for which it is zero: $z_1 = -z_2$ and $z_1 = 0$. As b_k is a function of M_k^2 only, $b_k(z_k) = b_k(-z_k)$. Thus for $z_1 = -z_2$:

$$z_3 = -(z_1 + z_2) = 0, \tag{108}$$

$$b(z_1) = b_1(-z_1) = b_2(z_2), \tag{109}$$

$$\begin{aligned} \hat{Q} =& z_1 b_1(z_1) + z_2 b_2(z_2) + z_3 b_3(z_3) \\ =& - z_2 b_2(z_2) + z_2 b_2(z_2) = 0. \end{aligned} \tag{110}$$

And for $z_1 = 0$:

$$z_3 = -z_2, \tag{111}$$

$$\hat{Q} = z_2 b_2(z_2) - z_2 b_2(z_2) = 0. \tag{112}$$

The behaviour of $\hat{Q}(z_k)$ may then be found by investigating the derivatives. In the analysis it is assumed that z_2 is a constant, hence only variables related to z_1 and z_3 are included. Now

$$\begin{aligned}\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} &= \frac{\mathrm{d}z_1}{\mathrm{d}z_1} b_1 + z_1 \frac{\mathrm{d}b_1}{\mathrm{d}z_1} + \frac{\mathrm{d}z_3}{\mathrm{d}z_1} b_3 + z_3 \frac{\mathrm{d}b_3}{\mathrm{d}z_1} \\ &= b_1 + z_1 \frac{\mathrm{d}b_1}{\mathrm{d}z_1} - b_3 + z_3 \frac{\mathrm{d}b_3}{\mathrm{d}z_3} \frac{\mathrm{d}z_3}{\mathrm{d}z_1} \\ &= (b_1 - b_3) + \left(z_1 \frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3 \frac{\mathrm{d}b_3}{\mathrm{d}z_3} \right)\end{aligned} \tag{113}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2\hat{Q}}{\mathrm{d}{z_1}^2} &= \frac{\mathrm{d}}{\mathrm{d}z_1} \left[(b_1 - b_3) + \left(z_1 \frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3 \frac{\mathrm{d}b_3}{\mathrm{d}z_3} \right) \right] \\ &= 2 \left(\frac{\mathrm{d}b_1}{\mathrm{d}z_1} + \frac{\mathrm{d}b_3}{\mathrm{d}z_3} \right) + z_1 \frac{\mathrm{d}^2 b_1}{\mathrm{d}{z_1}^2} + z_3 \frac{\mathrm{d}^2 b_3}{\mathrm{d}{z_3}^2}.\end{aligned} \tag{114}$$

For convenience the derivative ${}^{\mathrm{d}b_k}/_{\mathrm{d}z_k}$ is found as a function of M_k.

$$\frac{\mathrm{d}z_k}{\mathrm{d}M_k} = \frac{(1 + M_k^2) - 2M_k^2}{(1 + M_k^2)^2} = \frac{1 - M_k^2}{(1 + M_k^2)^2}, \tag{115}$$

$$\frac{\mathrm{d}M_k}{\mathrm{d}z_k} = \frac{(1 + M_k^2)^2}{1 - M_k^2}, \tag{116}$$

$$\begin{aligned}\frac{\mathrm{d}b_k}{\mathrm{d}z_k} &= \frac{\mathrm{d}b_k}{\mathrm{d}M_k} \frac{\mathrm{d}M_k}{\mathrm{d}z_k} \\ &= \left(M_k - \frac{2M_k}{1 + M_k^2} \right) \left(\frac{(1 + M_k^2)^2}{1 - M_k^2} \right) \\ &= -M_k (1 + M_k^2).\end{aligned} \tag{117}$$

In the subsonic region, $M \in [-1, 1]$ and $z \in [-1/2, 1/2]$. The derivative in Equation (113) may be investigated in three different intervals.

A.1. **Interval 1: $\mathbf{z_1 \in \langle -1/2, -z_2]}$ if $\mathbf{z_2 > 0}$.** If $z_2 < 0$, $z_1 \in [-z_2, 1/2\rangle$. In both cases $|z_1| \geq |z_2|$ and $|z_1| > |z_3|$ due to Equation (107). The symmetry of b_k as a function of z_k and the sign of its derivative (eq. (117)) gives:

$$b_3(z_3) = b_3(-z_3) > b_1(z_1), \tag{118}$$

as well as

$$\left|\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right| < \left|\frac{\mathrm{d}b_1}{\mathrm{d}z_1}\right|. \tag{119}$$

Hence in the first interval

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} = (b_1 - b_3) + \left(z_1 \frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3 \frac{\mathrm{d}b_3}{\mathrm{d}z_3} \right) < 0. \tag{120}$$

A.2. **Interval 2: $\mathbf{z_1} \in \langle -\mathbf{z_2}, \mathbf{0}\rangle$ if $\mathbf{z_2} > \mathbf{0}$.** If $z_2 < 0$, $z_1 \in \langle 0, -z_2\rangle$. In this interval, Equation (113) is equal to zero for $z_1 = z_3 = -{}^{z_2}/2$. Possible additional roots are investigated with the aid of Equation (114). The second derivative needed in the last two terms in the equation is found as:

$$\begin{aligned}\frac{\mathrm{d}^2 b_k}{\mathrm{d}{z_k}^2} &= \frac{\mathrm{d}}{\mathrm{d}M_k}\left[-M(1+M^2)\right]\frac{\mathrm{d}M_k}{\mathrm{d}z_k} = \left(-1-3M_k^2\right)\frac{(1+M_k^2)^2}{1-M_k^2} \\ &= -\frac{(1+3M_k^2)(1+M_k^2)^2}{1-M_k^2} < 0 \quad \text{for} \quad M_k k \in \langle -1, 1\rangle.\end{aligned} \tag{121}$$

The equations (107), (114), (117) and (121) give the following result:
For $z_2 > 0$

$$\frac{\mathrm{d}^2\hat{Q}}{\mathrm{d}{z_1}^2} > 0 \quad \text{for} \quad z_1 \in \langle -z_2, 0\rangle, \tag{122}$$

hence $\hat{Q}(z_1 = -{}^{z_2}/2)$ is the only local minima for $\hat{Q}$ in the range $z_1 \in \langle -z_2, 0\rangle$ and there are no values of z_1 satisfying the equation $\hat{Q}(z_1) = 0$ in the given interval.
For $z_2 < 0$

$$\frac{\mathrm{d}^2\hat{Q}}{\mathrm{d}{z_1}^2} < 0 \quad \text{for} \quad z_1 \in \langle 0, -z_2\rangle, \tag{123}$$

hence $\hat{Q}(z_1 = -{}^{z_2}/2)$ is the only local maxima for $\hat{Q}$ in the range $z_1 \in \langle 0, -z_2\rangle$ and there are no values of z_1 satisfying the equation $\hat{Q}(z_1) = 0$ in the given interval.

A.3. **Interval 3: $\mathbf{z_1} \in [\mathbf{0}, {}^{\mathbf{1}}/\mathbf{2} - \mathbf{z_2}\rangle$ if $\mathbf{z_2} > \mathbf{0}$.** If $z_2 < 0$, $z_1 \in \langle -{}^1/2 - z_2, 0]$. In both cases $|z_1| < |z_3|$ due to Equation (107). The symmetry of b_k as a function of z_k and the sign of its derivative (eq. (117)) gives:

$$b_3(z_3) = b_3(-z_3) < b_1(z_1), \tag{124}$$

as well as

$$\left|\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right| > \left|\frac{\mathrm{d}b_1}{\mathrm{d}z_1}\right|. \tag{125}$$

Hence in region three

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} = (b_1 - b_3) + \left(z_1\frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right) > 0. \tag{126}$$

A.4. **Summary.** It is proved that for $z_2 > 0$:

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} < 0 \quad \text{for} \quad z_1 \in \langle -{}^1/2, -{}^{z_2}/2\rangle, \tag{127}$$

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} > 0 \quad \text{for} \quad z_1 \in \langle -{}^{z_2}/2, {}^1/2 - z_2\rangle. \tag{128}$$

Accordingly, for $z_2 < 0$:

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} > 0 \quad \text{for} \quad z_1 \in \langle -{}^1/2, -{}^{z_2}/2\rangle, \tag{129}$$

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} < 0 \quad \text{for} \quad z_1 \in \langle -{}^{z_2}/2, {}^1/2 - z_2\rangle. \tag{130}$$

Further it is proved that in the interval $z_k \in \langle -{}^1/2, {}^1/2\rangle$, $\hat{Q}(z_1) = 0$ only for $z_1 = -z_2$ and $z_1 = 0$.