ON DECAY OF PERIODIC RENORMALIZED SOLUTIONS TO SCALAR CONSERVATION LAWS*

E. YU. PANOV†

Abstract. We establish a necessary and sufficient condition for decay of periodic renormalized solutions to a multidimensional conservation law with merely continuous flux vector.

Key words. periodic renormalized solutions, decay property, H-measures

AMS subject classification. 35L65 35B10 35B35

1. Introduction. In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$, we consider the Cauchy problem for a first order multidimensional conservation law

$$u_t + \mathrm{div}_x \varphi(u) = 0 \tag{1.1}$$

with initial data

$$u(0, x) = u_0(x). \tag{1.2}$$

The flux vector $\varphi(u)$ is supposed to be only continuous:

$$\varphi(u) = (\varphi_1(u), \ldots, \varphi_n(u)) \in C(\mathbb{R}, \mathbb{R}^n).$$

We assume that initial function $u_0(x)$ is periodic, that is, $u_0(x+e_i) = u_0(x)$ for almost all $x \in \mathbb{R}^n$ and all $i = 1, \ldots, n$, where $\{e_i\}_{i=1}^n$ is a basis of periods in $\mathbb{R}^n$. Denote by P the corresponding fundamental parallelepiped

$$P = \{ \ x = \sum_{i=1}^n \alpha_i e_i \mid \alpha_i \in [0, 1), \ i = 1, \ldots, n \ \}.$$

If $u_0(x) \in L^\infty(\mathbb{R}^n)$ then the notion of entropy solution of (1.1), (1.2) in the sense of S.N. Kruzhkov [5] is well-defined.

Definition 1.1. *A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1.1), (1.2) if for all $k \in \mathbb{R}$*

$$|u - k|_t + \mathrm{div}_x[\mathrm{sign}(u - k)(\varphi(u) - \varphi(k))] \le 0 \tag{1.3}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$);

$$\operatorname*{ess\,lim}_{t \to 0} u(t, \cdot) = u_0 \quad in\ L^1_{loc}(\mathbb{R}^n).$$

Condition (1.3) means that for all non-negative test functions $f = f(t, x) \in C^1_0(\Pi)$

$$\int_\Pi [|u - k| f_t + \mathrm{sign}(u - k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] dt dx \ge 0$$

*This research was carried out with the financial support of the Russian Foundation for Basic Research (grant 12-01-00230).

†Novgorod State University, B. Sankt-Peterburgskaya, 41, 173003 Veliky Novgorod, Russia (Eugeny.Panov@novsu.ru).

(here $\cdot$ denotes the inner product in $\mathbb{R}^n$).

In the case under consideration when the flux functions are merely continuous, the effect of infinite speed of propagation for initial perturbations appears, which leads even to the nonuniqueness of e.s. to problem (1.1), (1.2) if $n > 1$ (see examples in [6, 7]). But, if initial function is periodic (at least in $n-1$ independent directions), the uniqueness holds: an e.s. of (1.1), (1.2) is unique and space-periodic, see the proof in [14, 15].

In the present paper we assume that the initial function $u_0 \in L^1(P)$ and may be unbounded. In this general situation even the natural requirement $\varphi(u) \in L^1_{loc}(\Pi, \mathbb{R}^n)$ turns out to be too restrictive. However, if we reject these assumption, we cannot consider entropy conditions (and even the equation itself) within the framework of the theory of distributions. To define such solutions $u = u(t,x)$ (called renormalized), one uses entropy conditions for superpositions $s(u)$, where s are bounded functions of special form (cut-off functions). Renormalized entropy solutions to the problem (1.1), (1.2) with summable initial data were first introduced in [1], where the existence and uniqueness of such solutions were also established. The results of [1] were generalized in [8] to the case of arbitrary measurable initial data. The notion of renormalized entropy solution was later modified in [9] for the periodic case.

Recall the corresponding definition. We denote by $s_{a,b}(u) = \max(a, \min(b,u))$ the cut-off function at levels a and b, where $a, b \in \mathbb{R}$, $a \le b$.

Definition 1.2. *A x-periodic measurable function $u = u(t,x)$ is called a renormalized entropy solution (r.e.s. for short) of (1.1), (1.2) if for all $a, b \in \mathbb{R}$, $a \le b$*

$$(s_{a,b}(u))_t + \operatorname{div}_x(\varphi(s_{a,b}(u))) = \mu_b - \mu_a \quad \text{in } \mathcal{D}'(\Pi), \tag{1.4}$$

where μ_p, $p \in \mathbb{R}$, is a family of x-periodic nonnegative locally finite measures on Π ($\mu_p \in M_{loc}(\Pi)$, $\mu_p \ge 0$) such that $\lim\limits_{p\to\infty} \mu_p((0,T)\times P) = 0$ for all $T > 0$, and

$$\operatorname*{ess\,lim}_{t\to 0} |s_{a,b}(u(t,\cdot)) - s_{a,b}(u_0)| = 0 \quad \text{in } L^1(P).$$

In the case of bounded u, u_0 the notions of r.e.s. and e.s. coincide. Moreover, in this case the defect measures μ_p satisfy the representation

$$\mu_p = -\frac{1}{2}\left\{|u-p|_t + \operatorname{div}_x[\operatorname{sign}(u-p)(\varphi(u)-\varphi(p))]\right\}.$$

As was shown in [9], for each $u_0 \in L^1(P)$ there exists a unique r.e.s. $u = u(t,x)$ of problem (1.1), (1.2). Moreover, the following contraction property holds in $L^1(P)$ (see [9, Corollary 3.3]):

Proposition 1.3. *Let $u(t,x)$ and $v(t,x)$ be r.e.s. to the problem (1.1), (1.2) with the initial data $u_0(x)$ and $v_0(x)$ (which are supposed to be merely measurable functions), respectively. Then for almost all $t > 0$*

$$\int_P (u(t,x) - v(t,x))^+ dx \le \int_P (u_0(x) - v_0(x))^+ dx, \tag{1.5}$$

where we use the notation $r^+ = \max(r, 0)$.

Changing the places of u and v in (1.5), we obtain the inequality

$$\int_P (v(t,x) - u(t,x))^+ dx \le \int_P (v_0(x) - u_0(x))^+ dx. \tag{1.6}$$

Putting inequalities (1.5), (1.6) together, we derive the following L^1 contraction property: for almost all $t > 0$

$$\int_P |u(t,x) - v(t,x)|dx \le \int_P |u_0(x) - v_0(x)|dx. \tag{1.7}$$

As was established by G.-Q. Chen and H. Frid [2], under the conditions $\varphi(u) \in C^2(\mathbb{R}, \mathbb{R}^n)$ and

$$\forall (\tau, \xi) \in \mathbb{R}^{n+1}, (\tau, \xi) \ne 0, \quad \operatorname{meas} \{ \ u \in \mathbb{R} \mid \tau + \varphi'(u) \cdot \xi = 0 \ \} = 0, \tag{1.8}$$

the following decay property holds for bounded space-periodic entropy solutions $u(t,x)$ of (1.1), (1.2):

$$\operatorname*{ess\,lim}_{t\to\infty} u(t,\cdot) = \text{const} = \frac{1}{|P|}\int_P u_0(x)dx \ \text{ in } L^1(P). \tag{1.9}$$

Here $|P|$ denotes the Lebesgue measure of P.

In the present paper we generalize this result to the case of renormalized entropy solutions of (1.1), (1.2) and propose the following necessary and sufficient condition for the decay property

$$\forall \xi \in L', \xi \ne 0, \ \textit{the function } u \to \varphi(u)\cdot\xi \textit{ is not affine on non-empty intervals,} \tag{1.10}$$

where $L' = \{ \ \xi \in \mathbb{R}^n \mid \xi \cdot e_i \in \mathbb{Z} \ \forall i = 1,\dots,n \ \}$ is the dual lattice to the lattice of periods $L = \{ \ x = \sum_{i=1}^n k_i e_i \mid k_i \in \mathbb{Z}, \ i = 1,\dots,n \ \}$, $\mathbb{Z}$ being the set of integers. Thus, our main result is the following theorem.

THEOREM 1.4. *Every r.e.s. of equation (1.1) satisfies the decay property (1.9) if and only if condition (1.10) holds.*

In the case when the basis of periods is not fixed and may depend on a solution, the statement of Theorem 1.4 remains valid after replacement of condition (1.10) by the following stronger one:

$$\forall \xi \in \mathbb{R}^n, \xi \ne 0, \ \textit{the function } u \to \varphi(u)\cdot\xi \textit{ is not affine on non-empty intervals.} \tag{1.11}$$

Obviously, condition (1.11) is strictly weaker than (1.8) even in the case of smooth flux $\varphi(u)$.

2. Preliminaries. The following technical lemma is rather well-known (cf. [9, Lemma 3.3]):

LEMMA 2.1. *Let μ be locally finite space-periodic Borel measure on Π; $q(t) \in C_0((0,+\infty))$, $p(y) \in C_0(\mathbb{R}^n)$. Then, as $\nu \to \infty$*

$$\nu^{-n}\int q(t)p(x/\nu)d\mu(t,x) \to \int_{(0,+\infty)\times P} q(t)d\mu(t,x)\int_{\mathbb{R}^n} p(y)dy.$$

Proof. For the sake of completeness we put below the proof. Let us define locally finite Borel measure $m(x)$ on $\mathbb{R}^n$, setting:

$$\langle m, p(x)\rangle = \int q(t)p(x)d\mu(t,x), \quad p(x) \in C_0(\mathbb{R}^n).$$

It is clear that m is periodic and

$$\nu^{-n}\int q(t)p(x/\nu)d\mu(t,x)=\int p(y)dm_\nu(y), \tag{2.1}$$

where $m_\nu=\nu^{-n}g_\nu^*m$ while g_ν^*m is the image of m under the linear map $y=g_\nu(x)=x/\nu$. In other words, for each Borel set $B\subset\mathbb{R}^n$ we have $m_\nu(B)=\nu^{-n}m(\nu B)$. It is well-known (see, for example, [23]) that the sequence $m_\nu(y)$ weakly converges as $\nu\to\infty$ to the measure Cdy, proportional to the Lebesgue measure dy on $\mathbb{R}^n$ with the constant

$$C=m(P)=\int_{(0,+\infty)\times P}q(t)d\mu(t,x).$$

Therefore, as $\nu\to\infty$

$$\nu^{-n}\int q(t)p(x/\nu)d\mu(t,x)=\int p(y)dm_\nu(y)\to$$
$$C\int_{\mathbb{R}^n}p(y)dy=\int_{(0,+\infty)\times P}q(t)d\mu(t,x)\int_{\mathbb{R}^n}p(y)dy,$$

as was to be proved. □

We will need some further properties of r.e.s.

Lemma 2.2. *If $u(t,x)$ is a r.e.s. of (1.1), (1.2) with initial function $u_0\in L^1(P)$ then after possible correction on a set of null measure,*

$$u(t,\cdot)\in C([0,+\infty),L^1(P)) \tag{2.2}$$

and for every $t>0$

$$\forall R\ge 0\ \int_P(|u(t,x)|-R)^+dx\le\int_P(|u_0(x)|-R)^+dx, \tag{2.3}$$

$$\int_P u(t,x)dx=\int_P u_0(x)dx. \tag{2.4}$$

Proof. Evidently, the constants $\pm R$ are e.s. of (1.1). Therefore, they are r.e.s. of (1.1) as well, and by Proposition 1.3 the following inequalities hold for almost all $t>0$:

$$\int_P(u(t,x)-R)^+dx\le\int_P(u_0(x)-R)^+dx,\quad \int_P(-R-u(t,x))^+dx\le\int_P(-R-u_0(x))^+dx.$$

Putting these inequalities together, we obtain that relation (2.3) holds for almost all $t>0$. Taking $R=0$ in this relation, we derive that $u(t,\cdot)\in L^1(P)$ for almost all $t>0$ and

$$\int_P|u(t,x)|dx\le\int_P|u_0(x)|dx. \tag{2.5}$$

We define the sequence $v_k(t,x)=s_{-k,k}(u(t,x))$, $k\in\mathbb{N}$. If $a,b\in\mathbb{R}$, $a\le b$, then $s_{a,b}(v_k)=s_{a',b'}(u)$ whenever $a'=\max(a,-k)\le b'=\min(b,k)$ (that is, $[a,b]\cap[-k,k]\ne\emptyset$) while $s_{a,b}(v_k)=c=\begin{cases} a & , \quad a>k,\\ b & , \quad b<-k,\end{cases}$ otherwise. Therefore,

$$(s_{a,b}(v_k))_t+\operatorname{div}_x\varphi(s_{a,b}(v_k))=\gamma_{a,b}^k\in M_{loc}(\Pi)\ \text{ in } \mathcal{D}'(\Pi), \tag{2.6}$$

where the measure $\gamma^k_{a,b} = \begin{cases} \mu_{b'} - \mu_{a'} & , \quad a' < b', \\ 0 & , \quad a' \ge b'. \end{cases}$

Relation (2.6) means that $v_k(t,x)$ is a quasi-solution of (1.1) in the sense of [16, 17]. We fix $t_0 \ge 0$. Then by [16, Theorem 1.2, Corolary 7.1] there exist strong traces $v_k(t_0+,x) = \operatorname*{ess\,lim}_{t\to t_0+} v_k(t,x)$ and $v_k(t_0-,x) = \operatorname*{ess\,lim}_{t\to t_0-} v_k(t,x)$ (in the case $t_0 > 0$) in $L^1_{loc}(\mathbb{R}^n)$ (and therefore in $L^1(P)$ as well). By (2.3) with $R = k$, we see that for almost all $t > 0$ and all $k, l \in \mathbb{N}$, $l > k$

$$\int_P |v_l(t,x) - v_k(t,x)|dx \le \int_P |u(t,x) - v_k(t,x)|dx =$$
$$\int_P (|u(t,x)| - k)^+ dx \le I_k \doteq \int_P (|u_0(x)| - k)^+ dx.$$

This implies that

$$\int_P |v_l(t_0\pm,x) - v_k(t_0\pm,x)|dx \le I_k. \tag{2.7}$$

Since $I_k \to 0$ as $k \to \infty$, it follows from the above estimate that $\{v_k(t_0\pm,x)\}_{k\in\mathbb{N}}$ are Cauchy sequences in $L^1(P)$. Therefore there exist functions $u(t_0\pm,x) \in L^1(P)$ such that $v_k(t_0\pm,x) \to u(t_0\pm,x)$ in $L^1(P)$ as $k \to \infty$. Passing in (2.7) to the limit as $l \to \infty$ we find that

$$\int_P |u(t_0\pm,x) - v_k(t_0\pm,x)|dx \le I_k. \tag{2.8}$$

Recall that also for almost every $t > 0$

$$\int_P |u(t,x) - v_k(t,x)|dx \le I_k. \tag{2.9}$$

In view of (2.8), (2.9)

$$\operatorname*{ess\,limsup}_{t\to t_0\pm} \int_P |u(t,x) - u(t_0\pm,x)|dx \le \operatorname*{ess\,limsup}_{t\to t_0\pm} \left(\int_P |v_k(t,x) - v_k(t_0\pm,x)|dx + \right.$$
$$\left. \int_P |u(t,x) - v_k(t,x)|dx + \int_P |u(t_0\pm,x) - v_k(t_0\pm,x)|dx \right) \le 2I_k$$

and since $I_k \to 0$ as $k \to \infty$, we conclude that

$$\operatorname*{ess\,lim}_{t\to t_0\pm} \int_P |u(t,x) - u(t_0\pm,x)|dx = 0. \tag{2.10}$$

Now, we will demonstrate that $u(t_0+,x) = u(t_0-,x)$ for each $t_0 > 0$. Since $v_k(t,x) = s_{-k,k}(u(t,x))$ then

$$(v_k)_t + \operatorname{div}_x \varphi(v_k) = \gamma_k = \mu_k - \mu_{-k}$$

and by [16, Corolary 7.1] for each $f(x) \in C_0(P)$

$$\int_P (v_k(t_0+,x) - v_k(t_0-,x))f(x)dx = \int_{\{t_0\}\times P} f(x)d\gamma_k(t,x) \le \|f\|_\infty \varepsilon_k, \tag{2.11}$$

where $\varepsilon_k = \mu_k(\{t_0\} \times P) + \mu_{-k}(\{t_0\} \times P) \to 0$ as $k \to \infty$. It follows from relations (2.8), (2.11) that

$$\pm \int_P (u(t_0+, x) - u(t_0-, x))f(x)dx = \pm \int_P (u(t_0+, x) - v_k(t_0+, x))f(x)dx$$
$$\pm \int_P (v_k(t_0+, x) - v_k(t_0-, x))f(x)dx \pm \int_P (v_k(t_0-, x) - u(t_0-, x))f(x)dx$$
$$\le \|f\|_\infty (2I_k + \varepsilon_k).$$

Passing in this inequality to the limit as $k \to \infty$, we arrive at the equality

$$\int_P (u(t_0+, x) - u(t_0-, x))f(x)dx = 0.$$

Since the function $f(x) \in C_0(P)$ is arbitrary, we conclude that $u(t_0+, x) = u(t_0-, x)$ almost everywhere on P. Hence, for $t_0 > 0$ there exists the essential limit

$$\operatorname*{ess\,lim}_{t \to t_0} u(t, x) = u(t_0+, x) \quad \text{in } L^1(P).$$

From this relation it follows that for each $T > 0$

$$\frac{1}{h} \int_0^h \left(\int_{[0,T] \times P} |u(t + \tau, x) - u(t, x)| dt dx \right) d\tau \underset{h \to 0}{\to} \int_{[0,T] \times P} |u(t+, x) - u(t, x)| dt dx.$$

On the other hand, by the known property of integrable functions,

$$\frac{1}{h} \int_0^h \left(\int_{[0,T] \times P} |u(t + \tau, x) - u(t, x)| dt dx \right) d\tau \underset{h \to 0}{\to} 0$$

and we claim that for all $T > 0$

$$\int_{[0,T] \times P} |u(t+, x) - u(t, x)| dt dx = 0.$$

Therefore, $u(t, x) = u(t+, x)$ almost everywhere on Π and, evidently, the function $u(t+, x) \in C([0, +\infty), L^1(P))$.

Hence, without lost of generality we may initially assume that $u(t, \cdot) \in C([0, +\infty), L^1(P))$. Then inequalities (1.5), (1.7), (2.3) hold for all $t > 0$ (without exemption of a set of null measure).

To prove (2.4), we choose nonnegative functions $p(y) \in C_0^\infty(\mathbb{R}^n)$, $q(t) \in C_0^\infty((0, +\infty))$ such that $\int_{\mathbb{R}^n} p(y)dy = 1$ and set $p_\nu(x) = \nu^{-n} p(x/\nu)$, $\nu \in \mathbb{N}$. Applying relation (1.4) to the test function $f(t, x) = p_\nu(x)q(t)$, we arrive at the relation

$$\int_\Pi s_{a,b}(u) q'(t) p_\nu(x) dt dx + \nu^{-n-1} \int_\Pi \varphi(s_{a,b}(u)) \cdot \nabla_y p(x/\nu) q(t) dt dx =$$
$$\int q(t) p_\nu(x) d\mu_a(t, x) - \int q(t) p_\nu(x) d\mu_b(t, x).$$

Passing to the limit as $\nu \to \infty$ in the above equality with the help of Lemma 2.1, we obtain that

$$\int_{\mathbb{R}_+ \times P} s_{a,b}(u) q'(t) dt dx = \int_{\mathbb{R}_+ \times P} q(t) d\mu_a(t, x) - \int_{\mathbb{R}_+ \times P} q(t) d\mu_b(t, x). \tag{2.12}$$

We chose $T>0$ such that $\operatorname{supp} q(t)\subset(0,T)$. Then for $k\in\mathbb{N}$

$$0\le\int_{\mathbb{R}_+\times P} q(t)d\mu_{\pm k}(t,x)\le\|q\|_\infty\mu_{\pm k}((0,T)\times P)\underset{k\to\infty}{\to}0 \tag{2.13}$$

Since $|s_{-k,k}(u)|\le|u|$ while, in view of (2.5),

$$\int_{\mathbb{R}_+\times P}|u(t,x)||q'(t)|dtdx\le\|q'\|_\infty\int_{(0,T)\times P}|u(t,x)|dtdx$$
$$\le T\|q'\|_\infty\int_P|u_0(x)|dx<+\infty,$$

then by the Lebesgue dominated convergence theorem and relation (2.13) we derive from (2.12) with $-a=b=k$ in the limit as $k\to+\infty$ that

$$\int_{\mathbb{R}_+\times P}u(t,x)q'(t)dtdx=0,$$

that is, for each $q(t)\in C_0^\infty(\mathbb{R}_+)$, $q(t)\ge0$

$$\int_0^{+\infty}I(t)q'(t)dt=0,\quad\text{where } I(t)=\int_P u(t,x)dx.$$

Since every function $q(t)\in C_0^\infty(\mathbb{R}_+)$ is the difference of two nonnegative functions from $C_0^\infty(\mathbb{R}_+)$, then the above relation remains valid for all $q(t)\in C_0^\infty(\mathbb{R}_+)$, that is, $I'(t)=0$ in $\mathcal{D}'(\mathbb{R}_+)$. This implies that for all $t>0$

$$I(t)=\text{const}=I(0)=\int_P u_0(x)dx$$

and completes the proof. □

LEMMA 2.3. *Let $u=u(t,x)$ be a r.e.s. of (1.1), (1.2), and $\mu_p\in M_{loc}(\Pi)$, $p\in\mathbb{R}$, be the family of defect measures from condition (1.4). Then*

$$\mu_p(\mathbb{R}_+\times P)\le\int_P(|u_0(x)|-|p|)^+dx.$$

Proof. We choose a function $\rho(s)\in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\rho(s)\subset[0,1]$, $\rho(s)\ge0$, $\int_{-\infty}^{+\infty}\rho(s)ds=1$ and set for $\nu\in\mathbb{N}$ $\delta_\nu(s)=\nu\rho(\nu s)$. Obviously, the sequence $\delta_\nu(s)$ converges as $\nu\to\infty$ to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$. Let

$$\theta_\nu(t)=\int_{-\infty}^t\delta_\nu(s)ds=\int_{-\infty}^{\nu t}\rho(s)ds.$$

It is clear that the sequence $\theta_\nu(t)$ converges pointwise as $\nu\to\infty$ to the Heaviside function $\theta(t)=(\operatorname{sign} t)^+=\begin{cases}1, & t>0,\\ 0, & t\le0.\end{cases}$ We choose $t_0,t_1\in\mathbb{R}_+$, $t_1>t_0$ and set

$$q_\nu(t)=\theta_\nu(t-t_0)-\theta_\nu(t-t_1).$$

Obviously, $0\le q_\nu(t)\le1$ and $q_\nu(t)$ converges pointwise as $\nu\to\infty$ to the indicator function $\chi_{(t_0,t_1]}(t)$ of the segment $(t_0,t_1]$. Taking $q=q_\nu(t)$ in relation (2.12), we obtain the equality

$$\int_0^{+\infty}I_{a,b}(t)(\delta_\nu(t-t_0)-\delta_\nu(t-t_1))dt=\int_{\mathbb{R}_+\times P}q_\nu(t)d\mu_a(t,x)-\int_{\mathbb{R}_+\times P}q_\nu(t)d\mu_b(t,x),$$

where $I_{a,b}(t)=\int_P s_{a,b}(u(t,x))dx\in C([0,+\infty))$. Passing in this relation to the limit as $\nu\to\infty$ and taking into account that $\chi_\nu(t)\underset{\nu\to\infty}{\to}\chi_{(t_0,t_1]}(t)$ pointwise, we arrive at the equality

$$I_{a,b}(t_0)-I_{a,b}(t_1)=\mu_a((t_0,t_1]\times P)-\mu_b((t_0,t_1]\times P). \tag{2.14}$$

In view of the initial requirement in Definition 1.2 (or relation (2.2)), as $t_0\to 0$

$$I_{a,b}(t_0)\to I_{a,b}(0)=\int_P s_{a,b}(u_0(x))dx$$

and it follows from (2.14) in the limit as $t_0\to 0$ that for all $t=t_1>0$

$$\int_P s_{a,b}(u(t,x))dx-\int_P s_{a,b}(u_0(x))dx=I_{a,b}(t)-I_{a,b}(0)=\\ \mu_b((0,t]\times P)-\mu_a((0,t]\times P). \tag{2.15}$$

Let us assume first that $p\ge 0$. Taking in (2.15) $a=p$, $b>p$, and passing to the limit as $b\to+\infty$, we find that

$$\int_P \max(u(t,x),p)dx-\int_P \max(u_0(x),p)dx=-\mu_p((0,t]\times P)$$

(we take here into account that $\mu_b((0,t]\times P)\to 0$ as $b\to\infty$ by the definition of r.e.s.). Therefore,

$$\mu_p((0,t]\times P)=\int_P[\max(u_0(x),p)-\max(u(t,x),p)]dx\le\\ \int_P[\max(u_0(x),p)-p]dx=\int_P(u_0(x)-p)^+dx\le\int_P(|u_0(x)|-|p|)^+dx. \tag{2.16}$$

In the case $p<0$, we take in (2.15) $b=p$, $a<b$ and pass to the limit as $a\to-\infty$, deriving the relation

$$\int_P \min(u(t,x),p)dx-\int_P \min(u_0(x),p)dx=\mu_p((0,t]\times P),$$

which implies

$$\mu_p((0,t]\times P)=\int_P[\min(u(t,x),p)-\min(u_0(x),p)]dx\le\\ \int_P[p-\min(u_0(x),p)]dx=\int_P(p-u_0(x))^+dx\le\int_P(|u_0(x)|-|p|)^+dx. \tag{2.17}$$

To conclude the proof, we only need to pass to the limit in relations (2.16), (2.17) as $t\to+\infty$. $\square$

If $u(t,x)$ is a r.e.s. of (1.1), (1.2) then for each $h\in\mathbb{R}^n$ the function $u(t,x+h)$ is a r.e.s. of (1.1), (1.2) with initial function $u_0(x+h)$. By (1.7) and (2.2) for each $t>0$ and all $h\in\mathbb{R}^n$

$$\int_P|u(t,x+h)-u(t,x)|dx\le\int_P|u_0(x+h)-u_0(x)|dx, \tag{2.18}$$

which implies that the family of functions $u(t,\cdot)$, $t>0$, is precompact in $L^1(P)$. This allows to derive the following result.

LEMMA 2.4. *Let $s(u)$ be a bounded Lipschitz function, $v(t,x)=s(u(t,x))$, and*

$$v(t,x)=\sum_{\kappa\in L'} a_\kappa(t)e^{2\pi i\kappa\cdot x}$$

be the Fourier series of $v(t,\cdot)$ in $L^2(\mathbb{R}^n)$, so that

$$a_\kappa(t)=|P|^{-1}\int_P e^{-2\pi i\kappa\cdot x}v(t,x)dx.$$

Then this series converges to $v(t,\cdot)$ in $L^2(P)$ uniformly with respect to t, that is, for each $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that

$$|P|\sum_{\kappa\in L',|\kappa|>N}|a_\kappa(t)|^2<\varepsilon^2 \quad \forall t>0. \tag{2.19}$$

Proof. Taking (2.18) into account, we find that for all $t>0$

$$\begin{aligned}\int_P |v(t,x+h)-v(t,x)|^2dx \le \\ \|v(t,x+h)-v(t,x)\|_\infty \int_P |v(t,x+h)-v(t,x)|^2dx \le \\ 2L\max|s(u)|\int_P |u(t,x+h)-u(t,x)|dx \le \\ 2L\max|s(u)|\int_P |u_0(x+h)-u_0(x)|dx,\end{aligned} \tag{2.20}$$

where L is a Lipschitz constant of $s(u)$. In view of (2.20), the set of functions $F=\{\ v(t,\cdot)\ \mid\ t>0\ \}$ is precompact in $L^2(P)$. By Hausdorff's compactness criterion there exists a finite $\varepsilon/2$-net $\{g_k(x)\}_{k=1}^m$ for F in $L^2(P)$. Let $b_{\kappa,k}=|P|^{-1}\int_P e^{-2\pi i\kappa\cdot x}g_k(x)dx$, $\kappa\in L'$, be Fourier coefficients of $g_k(x)$. Observe that

$$|P|\sum_{\kappa\in L'}|b_{\kappa,k}|^2=\|g_k\|^2_{L^2(P)}<+\infty.$$

Therefore, there exists an integer N such that

$$|P|\sum_{\kappa\in L',|\kappa|>N}|b_{\kappa,k}|^2<\varepsilon^2/4 \tag{2.21}$$

for all $k=1,\dots,m$. Since $\{g_k(x)\}_{k=1}^m$ is a $\varepsilon/2$-net for F then for each $t>0$ one can find such $k\in\{1,\dots,m\}$ that

$$|P|\sum_{\kappa\in L'}|a_\kappa(t)-b_{\kappa,k}|^2=\|v(t,\cdot)-g_k\|^2_{L^2(P)}<\varepsilon^2/4. \tag{2.22}$$

In view of (2.21), (2.22) and Minkowski inequality we find

$$\left(|P|\sum_{\kappa\in L',|\kappa|>N}|a_\kappa(t)|^2\right)^{1/2} \le \left(|P|\sum_{\kappa\in L',|\kappa|>N}|a_\kappa(t)-b_{\kappa,k}|^2\right)^{1/2} + \left(|P|\sum_{\kappa\in L',|\kappa|>N}|b_{\kappa,k}|^2\right)^{1/2} < \varepsilon,$$

and (2.19) follows. □

To prove Theorem 1.4, we use, as in [2], the strong pre-compactness property for the self-similar scaling sequence $u_k = u(kt, kx)$, $k \in \mathbb{N}$. This pre-compactness property will be obtained under condition (1.10) with the help of localization principles for H-measures with "continuous indexes", introduced in [11, 12].

First, we recall the original concept of H-measure invented by L. Tartar [22] and, independently, by P. Gerárd [4]. Let

$$F(u)(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} u(x) dx, \quad \xi \in \mathbb{R}^N,$$

be the Fourier transform extended as an unitary operator on the Hilbert space of functions $u(x) \in L^2(\mathbb{R}^N)$, $S = S^{N-1} = \{ \ \xi \in \mathbb{R}^N \ \mid \ |\xi| = 1 \ \}$ be the unit sphere in $\mathbb{R}^N$. Denote by $u \to \overline{u}$, $u \in \mathbb{C}$ the complex conjugation.

Let Ω be an open domain in $\mathbb{R}^N$, and let $U_k(x) \in L^2_{loc}(\Omega)$ be a sequence weakly convergent to the zero function.

Proposition 2.5 (see Theorem 1.1 in [22]). *There exists a nonnegative Borel measure μ in $\Omega \times S$ and a subsequence $U_r(x) = U_k(x)$, $k = k_r$, such that*

$$\langle \mu, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^N} F(\Phi_1 U_r)(\xi)\overline{F(\Phi_2 U_r)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi \tag{2.23}$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

The measure μ is called the Tartar H-measure corresponding to $U_r(x)$.

Remark 2.6. *In the case when the sequence $U_k(x)$ is bounded in $L^\infty(\Omega)$ it follows from (2.23) and the Plancherel identity that* $\mathrm{pr}_x|\mu^{pq}| \le C\,\mathrm{meas}$, *and that (2.23) remains valid for all $\Phi_1(x), \Phi_2(x) \in L^2(\Omega)$, cf. [18, Remark 2(a)]. Here we denote by $|\mu|$ the variation of measure μ (it is a nonnegative measure), and by* meas *the Lebesgue measure on Ω.*

We need also the concept of measure valued functions (Young measures). Recall (see [3, 21]) that a measure-valued function on a domain $\Omega \subset \mathbb{R}^N$ is a weakly measurable map $x \to \nu_x$ of Ω into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of ν_x means that for each continuous function $g(\lambda)$ the function $x \to \langle \nu_x, g(\lambda)\rangle = \int g(\lambda) d\nu_x(\lambda)$ is measurable on Ω.

Measure-valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $u(x) \in L^\infty(\Omega)$ and $\delta(\lambda - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are called *regular.* We identify these measure-valued functions and the corresponding functions $u(x)$, so that there is a natural embedding of $L^\infty(\Omega)$ into the set $\mathrm{MV}(\Omega)$ of measure-valued functions on Ω.

Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Omega)$ in the sense of the following theorem by L. Tartar [21].

Theorem 2.7. *Let $u_k(x) \in L^\infty(\Omega)$, $k \in \mathbb{N}$, be a bounded sequence. Then there exist a subsequence (we keep the notation $u_k(x)$ for this subsequence) and a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that*

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_k) \underset{k\to\infty}{\to} \langle \nu_x, g(\lambda)\rangle \quad \textit{weakly-}* \textit{ in } L^\infty(\Omega). \tag{2.24}$$

Besides, ν_x is regular, i.e., $\nu_x(\lambda) = \delta(\lambda - u(x))$ if and only if $u_k(x) \underset{k\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ (strongly).

In [11] the new concept of H-measures with "continuous indexes" was introduced, corresponding to sequences of measure valued functions. We describe this concept in the particular case of "usual" sequences in $L^\infty(\Omega)$. Let $u_k(x)$ be a bounded sequence in $L^\infty(\Omega)$. Passing to a subsequence if necessary, we can suppose that this sequence converges to a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ in the sense of relation (2.24). We introduce the measures $\gamma_x^k(\lambda) = \delta(\lambda - u_k(x)) - \nu_x(\lambda)$ and the corresponding distribution functions $U_k(x,p) = \gamma_x^k((p,+\infty))$, $u_0(x,p) = \nu_x((p,+\infty))$ on $\Omega \times \mathbb{R}$. Observe that $U_k(x,p), u_0(x,p) \in L^\infty(\Omega)$ for all $p \in \mathbb{R}$, see [11, Lemma 2]. We define the set

$$E = E(\nu_x) = \left\{ p_0 \in \mathbb{R} \ \mid \ u_0(x,p) \underset{p\to p_0}{\to} u_0(x,p_0) \ \text{ in } L^1_{loc}(\Omega) \right\}.$$

As was shown in [11, Lemma 4], the complement $\mathbb{R} \setminus E$ is at most countable and if $p \in E$ then $U_k(x,p) \underset{k\to\infty}{\rightharpoonup} 0$ weakly-$*$ in $L^\infty(\Omega)$.

The next result, similar to Proposition 2.5, has been established in [11, Theorem 3], [13, Proposition 2, Lemma 2].

Proposition 2.8. *1) There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Omega \times S$ and a subsequence $U_r(x,p) = U_{k_r}(x,p)$ such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$*

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^N} F(\Phi_1 U_r(\cdot,p))(\xi)\overline{F(\Phi_2 U_r(\cdot,q))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi; \tag{2.25}$$

2) The correspondence $(p,q) \to \mu^{pq}$ is a continuous map from $E \times E$ into the space $M_{loc}(\Omega \times S)$ of locally finite Borel measures on $\Omega \times S$ (with the standard locally convex topology);

3) For any $p_1, \dots, p_l \in E$ the matrix $\{\mu^{p_ip_j}\}_{i,j=1}^l$ is Hermitian and positive semidefinite, that is, for all $\zeta_1, \dots, \zeta_l \in \mathbb{C}$ the measure

$$\sum_{i,j=1}^l \mu^{p_ip_j}\zeta_i\overline{\zeta_j} \ge 0.$$

Notice that assertion 3) readily follows from relation (2.25).

We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the H-measure corresponding to the subsequence $u_r(x) = u_{k_r}(x)$.

As was demonstrated in [11], the H-measure $\mu^{pq} = 0$ for all $p, q \in E$ if and only if the subsequence $u_r(x)$ converges as $r \to \infty$ strongly (in $L^1_{loc}(\Omega)$). Observe also that assertion 3) in Proposition 2.8 implies that measures $\mu^{pp} \ge 0$ for all $p \in E$, and that

$$|\mu^{pq}(A)| \le \sqrt{\mu^{pp}(A)\mu^{qq}(A)} \tag{2.26}$$

for any Borel set $A \subset \Omega \times S$ and all $p, q \in E$.

3. Main results. We fix a periodic r.e.s. $u = u(t,x)$ of (1.1), (1.2).

Let $s(u)$ be a bounded Lipschitz function, $v(t,x) = s(u(t,x))$, and

$$v(t,x) = \sum_{\kappa\in L'} a_\kappa(t)e^{2\pi i\kappa\cdot x} \tag{3.1}$$

be the Fourier series of $v(t,\cdot)$ in $L^2(P)$. Then

$$v_k(t,x) = v(kt,kx) = \sum_{\kappa\in L'} a_\kappa(kt)e^{2\pi ik\kappa\cdot x},$$

which implies that, may be after extraction of a subsequence, $v_k \rightharpoonup v^*$ as $k \to \infty$ weakly-$*$ in $L^\infty(\Pi)$, where $v^* = v^*(t)$ being the weak limit of the coefficient $a_0(kt)$. Let $\hat\mu$ be the Tartar's H-measure corresponding to the sequence $v_r - v^*$, where $v_r = v_{k_r}(t,x)$ is a subsequence of v_k.

LEMMA 3.1. *The following inclusion holds:* $\operatorname{supp}\hat\mu \subset \Pi \times S_0$, *where*

$$S_0 = \left\{ \ \hat\xi/|\hat\xi| \in S \mid \hat\xi = (\tau,\xi) \neq 0, \ \tau \in \mathbb{R}, \xi \in L' \ \right\}.$$

Proof. For $m \in \mathbb{N}$ we introduce the sets

$$S_m = \left\{ \ \hat\xi/|\hat\xi| \in S \mid \hat\xi = (\tau,\xi) \neq 0, \ \tau \in \mathbb{R}, \xi \in L', |\xi| \le m \ \right\}.$$

It is clear that S_m is a closed subset of the sphere S (it is the union of the finite set of circles $\{ \ (p, q\xi/|\xi|) \mid p^2 + q^2 = 1 \ \}$, where $\xi \in L'$, $0 < |\xi| \le m$), and $S_0 = \cup_{m=1}^\infty S_m$. Let

$$v(t,x) = s(u(t,x)) = \sum_{\kappa \in L'} a_\kappa(t) e^{2\pi i \kappa \cdot x}$$

be the Fourier series for $v(t,\cdot)$ in $L^2(P)$. Then

$$v_r(t,x) = v(k_r t, k_r x) = \sum_{\kappa \in L'} a_\kappa(k_r t) e^{2\pi i k_r \kappa \cdot x}. \tag{3.2}$$

We denote $b_{0,r} = a_0(k_r t) - v^*(t)$; $b_{\kappa,r} = a_\kappa(k_r t)$, where $\kappa \in L'$, $\kappa \neq 0$. Let $\alpha(t) \in C_0(\mathbb{R}_+)$, and $\beta(x) \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ be such that its Fourier transform is a continuous compactly supported function:

$$\tilde\beta(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \beta(x) dx \in C_0(\mathbb{R}^n). \tag{3.3}$$

We take $R = \max\limits_{\xi \in \operatorname{supp}\tilde\beta} |\xi|$. Let $\Phi(t,x) = \alpha(t)\beta(x)$. By (3.2) we find that

$$(v_r(t,x) - v^*(t))\Phi(t,x) = \sum_{\kappa \in L'} b_{\kappa,r}(t)\alpha(t) e^{2\pi i k_r \kappa \cdot x} \beta(x). \tag{3.4}$$

Observe that the Fourier transform of $e^{2\pi i k_r \kappa \cdot x}\beta(x)$ coincides with $\tilde\beta(\xi - k_r\kappa)$. Let

$$d = \min\{ \ |\kappa| \mid \kappa \in L', \kappa \neq 0 \ \} > 0.$$

Since for $k_r > 2R/d$ the supports of the functions $\tilde\beta(\xi - k_r\kappa)$ do not intersect for different κ, then for such r the series

$$\sum_{\kappa \in L'} b_{\kappa,r}(t)\alpha(t)\tilde\beta(\xi - k_r\kappa) \tag{3.5}$$

is orthogonal in $L^2(\mathbb{R}^n)$ for each $t > 0$. Besides, by the Plancherel equality $\|\tilde\beta(\xi - k_r\kappa)\|_{L^2(\mathbb{R}^n)} = \|\tilde\beta\|_2 = \|\beta\|_2$, and

$$|P| \sum_{\kappa \in L'} |b_{\kappa,r}(t)\alpha(t)|^2 \|\tilde\beta(\xi - k_r\kappa)\|^2_{L^2(\mathbb{R}^n)} =$$

$$|P||\alpha(t)|^2\|\beta\|_2^2 \sum_{\kappa \in L'} |b_{\kappa,r}(t)|^2 = |\alpha(t)|^2\|\beta\|_2^2 \cdot \|v(k_r t,\cdot) - v^*(t)\|^2_{L^2(P)} < +\infty.$$

Therefore, orthogonal series (3.5) converges in $L^2(\mathbb{R}^n)$ for each $t>0$. Moreover, by Lemma 2.4

$$\sum_{\kappa\in L',|\kappa|>N}|b_{\kappa,r}(t)|^2=\sum_{\kappa\in L',|\kappa|>N}|a_\kappa(k_rt)|^2\underset{N\to\infty}{\to}0$$

uniformly with respect to $t>0$. Hence, series (3.5) converges in $L^2(\mathbb{R}^n)$ uniformly with respect to t. Since the Fourier transformation is an isomorphism on $L^2(\mathbb{R}^n)$, we conclude that series (3.4) also converges in $L^2(\mathbb{R}^n)$ (not only in $L^2(P)$) uniformly with respect to t. Since $\alpha(t)\in C_0(\mathbb{R})$, this implies that (3.4) converges in $L^2(\Pi)$, and

$$F((v_r-v)\Phi)(\hat\xi)=\sum_{\kappa\in L'}F^t(\alpha b_{\kappa,r})(\tau)\tilde\beta(\xi-k_r\kappa),\ \hat\xi=(\tau,\xi), \tag{3.6}$$

where $F^t(h)(\tau)=\int_{\mathbb{R}}e^{-2\pi i\tau t}h(t)dt$ denotes the Fourier transform over the time variable (we extend functions $h(t)\in L^2(\mathbb{R}_+)$ on the whole line $\mathbb{R}$, setting $h(t)=0$ for $t<0$). It follows from (3.6) that for $k_r>2R/d$

$$\int_{\mathbb{R}^{n+1}}|F(\Phi(v_r-v^*))(\hat\xi)|^2\psi(\hat\xi/|\hat\xi|)d\hat\xi=$$
$$\sum_{\kappa\in L'}\int_{\mathbb{R}^{n+1}}|F^t(\alpha b_{\kappa,r})(\tau)|^2|\tilde\beta(\xi-k_r\kappa)|^2\psi(\hat\xi/|\hat\xi|)d\hat\xi, \tag{3.7}$$

where the function $\psi(\hat\xi)\in C(S)$ is arbitrary. Now we fix $\varepsilon>0$. Recall that $b_{\kappa,r}=a_\kappa(k_rt)$ for $\kappa\neq0$, and by Lemma 2.4 there exists $m\in\mathbb{N}$ such that

$$\sum_{\kappa\in L',|\kappa|>m}\int_{\mathbb{R}^{n+1}}|F^t(\alpha b_{\kappa,r})(\tau)|^2|\tilde\beta(\xi-k_r\kappa)|^2d\hat\xi=$$
$$\sum_{\kappa\in L',|\kappa|>m}\int_\Pi|\alpha(t)a_\kappa(k_rt)|^2|\beta(x)|^2dtdx\le$$
$$\|\Phi\|_2^2\cdot\sup_{t>0}\sum_{\kappa\in L',|\kappa|>m}|a_\kappa(t)|^2<\varepsilon. \tag{3.8}$$

Now we suppose that $\|\psi\|_\infty\le1$ and $\psi(\hat\xi)=0$ on the set S_m. By (3.8)

$$\sum_{\kappa\in L',|\kappa|>m}\int_{\mathbb{R}^{n+1}}|F^t(\alpha b_{\kappa,r})(\tau)|^2|\tilde\beta(\xi-k_r\kappa)|^2|\psi(\hat\xi/|\hat\xi|)|d\hat\xi\le\varepsilon. \tag{3.9}$$

Since continuous function $\psi(\hat\xi)$ is uniformly continuous on the compact S then we can find such $\delta>0$ that $|\psi(\hat\xi_1)-\psi(\hat\xi_2)|<\varepsilon$ whenever $\hat\xi_1,\hat\xi_2\in S$, $|\hat\xi_1-\hat\xi_2|<\delta$. Suppose that $\kappa\neq0$, $\tilde\beta(\xi-k_r\kappa)\neq0$. Then $|\xi-k_r\kappa|\le R$. For a fixed $\tau\in\mathbb{R}$ we denote $\hat\xi=(\tau,\xi)$, $\hat\eta=(\tau,k_r\kappa)$. As is easy to compute,

$$\left|\frac{\hat\xi}{|\hat\xi|}-\frac{\hat\eta}{|\hat\eta|}\right|\le\frac{2|\hat\xi-\hat\eta|}{|\hat\eta|}=\frac{2|\xi-k_r\kappa|}{|\hat\eta|}\le2R/|\hat\eta|. \tag{3.10}$$

Observe that for each nonzero $\kappa\in L'$ $|\hat\eta|\ge k_rd$. Then, by (3.10) we see that for all $r\in\mathbb{N}$ such that $k_r>2R/(d\delta)$ and all $\kappa\in L'$, $0<|\kappa|\le m$,

$$|\psi(\hat\xi/|\hat\xi|)|=|\psi(\hat\xi/|\hat\xi|)-\psi(\hat\eta/|\hat\eta|)|<\varepsilon. \tag{3.11}$$

We use here that $\hat\eta/|\hat\eta|\in S_m$ and, therefore, $\psi(\hat\eta/|\hat\eta|)=0$. In view of (3.11), for all $k_r>2R/(d\delta)$

$$\sum_{\kappa\in L',0<|\kappa|\le m}\int_{\mathbb{R}^{n+1}}|F^t(\alpha b_{\kappa,r})(\tau)|^2|\tilde\beta(\xi-k_r\kappa)|^2|\psi(\hat\xi/|\hat\xi|)|d\hat\xi\le$$
$$\varepsilon\sum_{\kappa\in L',0<|\kappa|\le m}\int_{\mathbb{R}^{n+1}}|F^t(\alpha b_{\kappa,r})(\tau)|^2|\tilde\beta(\xi-k_r\kappa)|^2 d\hat\xi\le$$
$$\varepsilon\|\beta\|_2^2\sum_{\kappa\in L'}\int_{\mathbb{R}}|\alpha(t)b_{\kappa,r}(t)|^2dt\le\varepsilon\|\Phi\|_2^2\sup_{t>0}\sum_{\kappa\in L'}|b_{\kappa,r}(t)|^2=$$
$$\varepsilon|P|^{-1}\|\Phi\|_2^2\sup_{t>0}\|v(k_rt,\cdot)-v^*(t)\|^2_{L^2(P)}\le C\varepsilon\|\Phi\|_2^2, \tag{3.12}$$

where $C=4\|v\|_\infty^2$. Further, it follows from (3.10) with $\hat\eta=(\tau,0)$ that for $|\xi|\le R$ and $|\tau|>R_1=2R/\delta$

$$|\psi(\hat\xi/|\hat\xi|)|=|\psi(\hat\xi/|\hat\xi|)-\psi(\tau/|\tau|,0)|<\varepsilon.$$

Therefore,

$$\int_{\mathbb{R}^{n+1}}\theta(|\tau|-R_1)|F^t(\alpha b_{0,r})(\tau)|^2|\tilde\beta(\xi)|^2|\psi(\hat\xi/|\hat\xi|)|d\hat\xi\le C\varepsilon\|\Phi\|_2^2. \tag{3.13}$$

Here $\theta(r)$ is the Heaviside function.

For $|\tau|\le R_1$ we are reasoning in the following way. Since $\alpha(t)b_{0,r}(t)=\alpha(t)(a_{0,r}(t)-v^*(t))\rightharpoonup 0$ as $r\to\infty$, and $\|\alpha b_{0,r}\|_1\le C_1=2\|v\|_\infty\|\alpha\|_1$, the Fourier transform $F^t(\alpha b_{0,r})(\tau)\underset{r\to\infty}{\to}0$ for all $\tau\in\mathbb{R}$ and uniformly bounded: $|F^t(\alpha b_{0,r})(\tau)|\le C_1$. By Lebesgue dominated convergence theorem

$$\int_{\mathbb{R}}\theta(R_1-|\tau|)|F^t(\alpha b_{0,r})(\tau)|^2d\tau\underset{r\to\infty}{\to}0.$$

Therefore (recall that $\|\psi\|_\infty\le1$),

$$\int_{\mathbb{R}^{n+1}}\theta(R_1-|\tau|)|F^t(\alpha b_{0,r})(\tau)|^2|\tilde\beta(\xi)|^2|\psi(\hat\xi/|\hat\xi|)|d\hat\xi\le$$
$$\|\beta\|_2\int_{\mathbb{R}}\theta(R_1-|\tau|)|F^t(\alpha b_{0,r})(\tau)|^2d\tau\underset{r\to\infty}{\to}0. \tag{3.14}$$

In view of (3.13), (3.14) we find

$$\limsup_{r\to\infty}\int_{\mathbb{R}^{n+1}}|F^t(\alpha b_{0,r})(\tau)|^2|\tilde\beta(\xi)|^2|\psi(\hat\xi/|\hat\xi|)|d\hat\xi\le C\varepsilon\|\Phi\|_2^2. \tag{3.15}$$

Using (3.7), (3.9), (3.12) and (3.15), we arrive at the relation

$$\limsup_{r\to\infty}\int_{\mathbb{R}^{n+1}}|F(\Phi(v_r-v^*))(\hat\xi)|^2|\psi(\hat\xi/|\hat\xi|)|d\hat\xi\le C_2\varepsilon, \tag{3.16}$$

where C_2 is a constant independent on ψ and m. By the definition of H-measure and Remark 2.6

$$\lim_{r\to\infty}\int_{\mathbb{R}^{n+1}}|F(\Phi(v_r-v^*))(\hat\xi)|^2|\psi(\hat\xi/|\hat\xi|)|d\hat\xi=$$
$$\langle\hat\mu,|\Phi(t,x)|^2|\psi(\hat\xi)|\rangle=\int_{\Pi\times(S\setminus S_m)}|\Phi(t,x)|^2|\psi(\hat\xi)|d\hat\mu(t,x,\hat\xi),$$

and (3.16) implies that

$$\int_{\Pi\times(S\setminus S_m)} |\Phi(t,x)|^2\psi(\hat\xi)d\hat\mu(t,x,\hat\xi)\le C_2\varepsilon$$

for all $\psi(\hat\xi)\in C_0((S\setminus S_m))$ such that $0\le\psi(\hat\xi)\le 1$. Therefore, we can claim that

$$\int_{\Pi\times(S\setminus S_m)} |\Phi(t,x)|^2 d\hat\mu(t,x,\hat\xi)\le C_2\varepsilon,$$

and since $S\setminus S_0\subset S\setminus S_m$, we obtain the relation

$$\int_{\Pi\times(S\setminus S_0)} |\Phi(t,x)|^2 d\hat\mu(t,x,\hat\xi)\le C_2\varepsilon,$$

which holds for arbitrary positive ε. Therefore,

$$\int_{\Pi\times(S\setminus S_0)} |\Phi(t,x)|^2 d\hat\mu(t,x,\hat\xi)= 0. \tag{3.17}$$

Since for every point $(t_0,x_0)\in\Pi$ one can find functions $\alpha(t)$, $\beta(x)$ with the prescribed above properties in such a way that $\Phi(t,x)=\alpha(t)\beta(x)\neq 0$ in a neighborhood of (t_0,x_0), we derive from (3.17) the desired inclusion $\operatorname{supp}\hat\mu\subset\Pi\times S_0$.
□

We fix $l\in\mathbb{N}$ and consider the H-measure $\{\mu^{pq}\}_{p,q\in E}$ corresponding to a subsequence $v_r=v_{k_r}(t,x)$ of the sequence $v_k=s_{-l,l}(u(kt,kx))$, $k\in\mathbb{N}$, defined in accordance with Proposition 2.8.

Theorem 3.2. *For every $p,q\in E$ $\operatorname{supp}\mu^{pq}\subset\Pi\times S_0$.*

Proof. Let $\nu_{t,x}$ be a weak measure valued limit of the sequence v_r. We introduce measures

$$\gamma^r_{t,x}(\lambda)=\delta(\lambda-v_r(t,x))-\nu_{t,x}(\lambda),$$

and set $U_r(t,x,p)=\gamma^r_{t,x}((p,+\infty))$. Let $s(u)\in C^1(\mathbb{R})$, $r\in\mathbb{N}$. Then $s(v_r)\rightharpoonup v^*(t,x)=\int s(\lambda)d\nu_{t,x}(\lambda)$ as $r\to\infty$ weakly-$*$ in $L^\infty(\Pi)$. Integrating by parts, we find that

$$s(v_r)(t,x)-v^*(t,x)=\int s(\lambda)d\gamma^r_{t,x}(\lambda)=\int s'(\lambda)U_r(t,x,\lambda)d\lambda \tag{3.18}$$

(observe that $U_r(t,x,\lambda)=0$ for $|\lambda|>l$). Let $\Phi(t,x)\in C_0(\Pi)$, $\psi(\hat\xi)\in C(S)$. Then, in view of (3.18), we find

$$\int_{\mathbb{R}^{n+1}}|F(\Phi(s(v_r)-v^*))(\hat\xi)|^2\psi(\hat\xi/|\hat\xi|)d\hat\xi=$$
$$\int\int s'(p)s'(q)\left(\int_{\mathbb{R}^{n+1}}F(\Phi U_r(\cdot,p))(\hat\xi)\overline{F(\Phi U_r(\cdot,q))(\hat\xi)}\psi(\hat\xi/|\hat\xi|)d\hat\xi\right)dpdq. \tag{3.19}$$

By the definition of H-measure, for each $p,q\in E$

$$\lim_{r\to\infty}\int_{\mathbb{R}^{n+1}}F(\Phi U_r(\cdot,p))(\hat\xi)\overline{F(\Phi U_r(\cdot,q))(\hat\xi)}\psi(\hat\xi/|\hat\xi|)d\hat\xi=\langle\mu^{pq},|\Phi(t,x)|^2\psi(\hat\xi)\rangle.$$

Using Lebesgue dominated convergence theorem, we can pass to the limit as $r \to \infty$ in equality (3.19) and arrive at

$$\langle \hat{\mu}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^{n+1}} |F(\Phi(v_r - v))(\hat{\xi})|^2\psi(\hat{\xi}/|\hat{\xi}|)d\hat{\xi} = \int\int s'(p)s'(q)\langle \mu^{pq}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle dpdq, \tag{3.20}$$

where $\hat{\mu} = \hat{\mu}(t,x,\hat{\xi})$ is the Tartar's H-measure, corresponding to the scalar sequence $s(v_r) - v^*$. Observe that $s(v_r) = \tilde{s}(u(k_r t, k_r x))$, where $\tilde{s}(u) = s(s_{-l,l}(u))$ is a bounded Lipschitz function. By Lemma 3.1 we claim that $\operatorname{supp}\hat{\mu} \subset \Pi \times S_0$. Clearly, the equality

$$\langle \hat{\mu}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle = \int\int s'(p)s'(q)\langle \mu^{pq}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle dpdq$$

remains valid for every Borel function $\psi(\hat{\xi})$. Taking $\psi(\hat{\xi})$ being the indicator function of the set $S \setminus S_0$, we obtain the relation

$$\int\int s'(p)s'(q)\langle \mu^{pq}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle dpdq = 0. \tag{3.21}$$

Now we take in (3.21) $s'(p) = l\omega(l(p-p_0))$, where $p_0 \in E$, $l \in \mathbb{N}$, and $\omega(y) \in C_0((0,1))$ be a non-negative function such that $\int \omega(y)dy = 1$. Since the H-measure μ^{pq} is strongly continuous with respect to (p,q) at point (p_0,p_0), we derive from (3.21) in the limit as $l \to \infty$ that

$$\langle \mu^{p_0p_0}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle = \lim_{l\to\infty} l^2 \int\int \omega(l(p-p_0))\omega(l(q-p_0))\langle \mu^{pq}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle dpdq = 0.$$

Since $\Phi(t,x) \in C_0(\Pi)$ is arbitrary, we conclude that $\mu^{p_0p_0}(\Pi \times (S \setminus S_0)) = 0$ (remark that $\mu^{p_0p_0} \ge 0$). Hence, for every $p = p_0 \in E$ $\operatorname{supp}\mu^{pp} \subset \Pi \times S_0$. Finally, as directly follows from (2.26), for $p,q \in E$ $\operatorname{supp}\mu^{pq} \subset \operatorname{supp}\mu^{pp} \subset \Pi \times S_0$. The proof is complete. $\square$

Observe that for each $p \in \mathbb{R}$

$$(v_r - p)^+ = s_{p',l}(u_k) - \min(p,l),$$
$$\theta(v_k - p)(\varphi(v_k) - \varphi(p)) = \varphi(s_{p',l}(u_k)) - \varphi(\min(p,l)),$$

where $p' = s_{-l,l}(p)$, and $\theta(u) = (\operatorname{sign}(u))^+$ is the Heaviside function. Therefore, in view of (1.4)

$$((v_k - p)^+)_t + \operatorname{div}_x[\theta(v_k - p)(\varphi(v_k) - \varphi(p))] = \mu_l^k - \mu_{p'}^k \ \text{ in } \mathcal{D}'(\Pi),$$

where $\mu_p^k = k\mu_p(kt,kx)$ in $\mathcal{D}'(\Pi)$, that is, $\langle \mu_p^k, f(t,x)\rangle = k^{-n}\langle \mu_p, f(t/k,x/k)\rangle$ for each $f(t,x) \in C_0(\Pi)$, $p \in \mathbb{R}$. By the periodicity of μ_p this implies that

$$\mu_p^k(\mathbb{R}_+ \times P) = k^{-n}\mu_p(\mathbb{R}_+ \times kP) = \mu_p(\mathbb{R}_+ \times P) \le C_p = \int_P (|u_0(x)| - |p|)^+dx,$$

in view of Lemma 2.3. Thus, the sequence of measures $\mu_l^k - \mu_{p'}^k$ is bounded in $M(\mathbb{R}_+ \times P)$. By the Murat interpolation lemma [10] this sequence is precompact in the Sobolev space $H^{-1}_{loc}(\Pi)$. Then, as one can easily derive from [12, Lemma 2 and the proof of Theorem 4] (see also [18, Theorem 4]), the following *second localization principle* holds.

THEOREM 3.3. *Let $X = X(p) \subset \mathbb{R}^{n+1}$ be the minimal linear subspace such that* $\operatorname{supp} \mu^{pp} \subset \Pi \times X$. *Then there exists $\delta > 0$ such that the function $u \to \tau u + \xi \cdot \varphi(u)$ is constant on the interval $(p-\delta, p+\delta)$ for all $\hat{\xi} = (\tau, \xi) \in X$.*

For the sake of completeness, we give below the proof of Theorem 3.3, based on results of [18, Theorem 4].

Proof. Let $D \subset E$ be a countable dense subset such that $p \in D$. By [18, Proposition 3] (see also [13, Proposition 3]) there exists a family of complex finite Borel measures $\mu^{pq}_{t,x} \in M(S)$ on the sphere $S \subset \mathbb{R}^{n+1}$, where $p, q \in D$, $(t,x) \in \Pi$, such that $\mu^{pq} = \mu^{pq}_{t,x} dtdx$, i.e., for all $\Phi(t,x,\hat{\xi}) \in C_0(\mathbb{R}^n \times S)$ the function

$$(t,x) \mapsto \langle \mu^{pq}_{t,x}(\hat{\xi}), \Phi(t,x,\hat{\xi}) \rangle = \int_S \Phi(t,x,\hat{\xi}) d\mu^{pq}_{t,x}(\hat{\xi})$$

is Lebesgue-measurable, bounded, and

$$\langle \mu^{pq}, \Phi(t,x,\hat{\xi}) \rangle = \int_\Pi \langle \mu^{pq}_{t,x}(\hat{\xi}), \Phi(t,x,\hat{\xi}) \rangle dtdx.$$

Observe that $s_{a,b}(v_k) = s_{a',b'}(u_k)$ if $a' = \max(a,-l) \le b' \doteq \min(b,l)$ while $s_{a,b}(v_k) \equiv$ const in the case when $a' > b'$. Therefore,

$$(s_{a,b}(v_k))_t + \operatorname{div}_x \varphi(s_{a,b}(v_k)) = \gamma^k_{a,b} \quad \text{in } \mathcal{D}'(\Pi),$$

where $\gamma^k_{a,b} = \begin{cases} \mu^k_{b'} - \mu^k_{a'} & , \quad a' < b', \\ 0 & , \quad a' \ge b' \end{cases}$ are bounded sequences in $M(\mathbb{R}_+ \times P)$, in view of the uniform estimates $\mu^k_p(\mathbb{R}_+ \times P) \le C_p$. By the Murat interpolation lemma [10] for every $a, b \in \mathbb{R}$, $a \le b$ the sequence of distributions $(s_{a,b}(v_k))_t + \operatorname{div}_x \varphi(s_{a,b}(v_k))$ is precompact in the Sobolev space $H^{-1}_{loc}(\Pi)$.

Then, by [18, Theorem 4] the H-measure μ, corresponding to the subsequence $v_r = v_{k_r}$, satisfies the following localization property: for all $p \in D$ and for almost all $(t,x) \in \Pi$ it holds $\operatorname{supp} \mu^{pp}_{t,x} \subset X_1$, where

$$X_1 = \{\ \hat{\xi} = (\tau,\xi) \in \mathbb{R}^{n+1} \mid \exists \delta > 0\ \forall u \in (p-\delta, p+\delta) \\ (u-p)\tau + (\varphi(u) - \varphi(p)) \cdot \xi = 0\ \}.$$

In view of the representation $\mu^{pp} = \mu^{pp}_{t,x} dtdx$ we derive that

$$\operatorname{supp} \mu^{pp} \subset \Pi \times X_1.$$

In particular, $X \subset X_1$. Let $\hat{\xi}_i = (\tau_i, \xi_i)$, $i = 1, \dots, m = \dim X$, be a basis in X. Since $\hat{\xi}_i \in X_1$, then there exist $\delta_i > 0$ such that the functions

$$(u-p)\tau_i + (\varphi(u) - \varphi(p)) \cdot \xi_i = 0 \tag{3.22}$$

for all $u \in (p - \delta_i, p + \delta_i)$, $i = 1, \dots, m$. Setting $\delta = \min\limits_{i=1,\dots,m} \delta_i$, we find that (3.22) holds on the interval $u \in (p-\delta, p+\delta)$ for all vectors ξ_i, $i = 1, \dots, m$. Since the linear span of these vectors coincides with X, the relation

$$(u-p)\tau + (\varphi(u) - \varphi(p)) \cdot \xi = 0$$

remains valid for $u \in (p-\delta, p+\delta)$ and every $\hat{\xi} \in X$. The proof is complete. $\square$

Now we are ready to prove our main Theorem 1.4. As follows from Theorems 3.2, 3.3, if $\dim X > 0$ then there exists nonzero vector $(\tau, \xi) \in X \cap (\mathbb{R} \times L')$ such that the function $\tau u + \xi \cdot \varphi(u)$ is constant on some interval $(p-\delta, p+\delta)$. Obviously then $\xi \neq 0$ and $\xi \cdot \varphi(u)$ is affine on $(p-\delta, p+\delta)$. But this contradicts to nondegeneracy condition (1.10). We conclude that $X = \{0\}$ and, therefore, $\mu^{pp} = 0$ for all $p \in E$. In view of (2.26) the H-measure μ^{pq} is trivial and the sequence $v_r = s_{-l,l}(u_r)$ converges strongly in $L^1_{loc}(\Pi)$ to some function u_l^*. As was shown above, before the formulation of Lemma 3.1, the limit functions do not depend on x: $u_l^* = u_l^*(t)$. By the standard diagonal extraction we can choose a subsequence $u_r = u_{k_r}(t,x)$ such that

$$v_r = s_{-l,l}(u_r) \underset{r\to\infty}{\to} u_l^* \text{ in } L^1_{loc}(\Pi)\ \forall l \in \mathbb{N}.$$

By Lemma 2.2 for all $m, l, r \in \mathbb{N}$, $m > l$

$$\int_P |s_{-m,m}(u_r(t,x)) - s_{-l,l}(u_r(t,x))|dx \le \int_P (|u(k_r t, k_r x)| - l)^+ dx = \int_P (|u(k_r t, y)| - l)^+ dy \le C_l = \int_P (|u_0(y)| - l)^+ dy. \quad (3.23)$$

We use here the change of variables $y = k_r x$ and the space periodicity of $u(t,x)$. It follows from (3.23) in the limit as $r \to \infty$ that for almost $t > 0$

$$|u_m^*(t) - u_l^*(t)| \le |P|^{-1} C_l \underset{l\to\infty}{\to} 0. \quad (3.24)$$

We see that the sequence $u_l^*(t)$ is fundamental in $L^\infty(\mathbb{R}^+)$ and, therefore, this sequence converges in $L^\infty(\mathbb{R}^+)$ to some function $u^* = u^*(t)$. Passing to the limit as $m \to \infty$ in relations (3.23), (3.24), we obtain the inequalities

$$\int_P |u_r(t,x) - s_{-l,l}(u_r(t,x))|dx \le C_l, \quad |P||u^*(t) - u_l^*(t)| \le C_l, \quad (3.25)$$

which hold for almost all $t > 0$. By these inequalities we find that for each $T > 0$

$$\int_{(0,T)\times P} |u_r(t,x) - u^*(t)|dtdx \le \int_{(0,T)\times P} |s_{-l,l}(u_r(t,x)) - u_l^*(t)|dtdx + 2TC_l$$

and, therefore, for every $l \in \mathbb{N}$

$$\limsup_{r\to\infty} \int_{(0,T)\times P} |u_r(t,x) - u^*(t)|dtdx \le 2TC_l.$$

Since $C_l \to 0$ as $l \to \infty$, we claim that for each $T > 0$

$$\lim_{r\to\infty} \int_{(0,T)\times P} |u_r(t,x) - u^*(t)|dtdx = 0,$$

that is, $u_r \underset{r\to\infty}{\to} u^*$ in $L^1_{loc}(\Pi)$. Further, by Lemma 2.2 for all $t > 0$

$$\int_P u_r(t,x)dx = \int_P u(k_r t, k_r x)dx = \int_P u(k_r t, y)dy = \int_P u_0(y)dy,$$

and, in particular, for almost every $t > 0$

$$|P|u^*(t) = \lim_{r\to\infty} \int_P u_r(t,x)dx = \int_P u_0(x)dx.$$

Hence $u^* = c = \dfrac{1}{|P|}\displaystyle\int_P u_0(x)dx$. The relation $u_r(t,x) \underset{r\to\infty}{\to} c$ in $L^1_{loc}(\Pi)$ implies that (after possible extraction of a subsequence) for a.e. $t > 0$ $u_r(t,x) \underset{r\to\infty}{\to} c$ in $L^1_{loc}(\mathbb{R}^n)$. By the periodicity, this reads

$$\int_P |u(k_r t, k_r x) - c|dx \underset{r\to\infty}{\to} 0.$$

Making again the change of variables $y = k_r x$, we find that for almost every $t > 0$

$$\int_P |u(k_r t, y) - c|dy = \int_P |u(k_r t, k_r x) - c|dx \underset{r\to\infty}{\to} 0. \tag{3.26}$$

We fix such $t = t_0 > 0$. Then, by inequality (1.7) together with continuity property (2.2), for each $t > k_r t_0$

$$\int_P |u(t,y) - c|dy \le \int_P |u(k_r t_0, y) - c|dy. \tag{3.27}$$

In view of (3.26) it follows from (3.27) that $\lim\limits_{t\to\infty} u(t,x) = c$ in $L^1(P)$. Hence the decay property holds for every r.e.s. $u(t,x)$.

Conversely, assume that condition (1.10) fails. Then we can find the segment $[a,b]$, $a < b$, and a nonzero point $(\tau,\xi) \in \mathbb{R}\times L'$ such that the function $u \to \tau u + \xi\cdot\varphi(u)$ is constant on the segment $[a,b]$. Then, as is easy to verify, the function

$$u(t,x) = \frac{a+b}{2} + \frac{b-a}{2}\sin(2\pi(\tau t + \xi\cdot x))$$

is a periodic bounded e.s. of (1.1), which does not satisfy the decay property. The obtained contradiction shows that condition (1.10) is also necessary for the decay property. This completes the proof of our main Theorem 1.4.

The proof of Theorem 1.4 for bounded entropy solutions of equation (1.1) can be found in paper [20], see also preprint [19].

REFERENCES

[1] Ph. Bénilan, J. Carrillo and P. Wittbold, *Renormalized entropy solutions of scalar conservation laws*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **29** (2000) 313-327.

[2] G.-Q. Chen, H. Frid, *Decay of entropy solutions of nonlinear conservation laws*, Arch. Rational Mech. Anal., **146 (2)** (1999) 95–127.

[3] R. J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal., **88** (1985) 223–270.

[4] P. Gerárd, *Microlocal defect measures*, Comm. Partial Diff. Equat., *16* (1991) 1761–1794.

[5] S. N. Kruzhkov, *First order quasilinear equations in several independent variables*, Mat. Sb. **81** (1970) 228–255. English transl. in Math. USSR Sb., **10** (1970) 217–243.

[6] S. N. Kruzhkov, E. Yu. Panov, *First-order conservative quasilinear laws with an infinite domain of dependence on the initial data*, Dokl. Akad. Nauk SSSR, **314** (1990) 79–84. English transl. in Soviet Math. Dokl. **42** (1991) 316–321.

[7] S. N. Kruzhkov, E. Yu. Panov, *Osgood's type conditions for uniqueness of entropy solutions to Cauchy problem for quasilinear conservation laws of the first order*, Ann. Univ. Ferrara Sez. VII (N.S.), **40** (1994) 31–54.

[8] P. V. Lysuho, E. Yu. Panov, *Renormalized entropy solutions to the Cauchy problem for a quasilinear first order equation*, J. Math. Sci., **172** (2011) 1–23.

[9] P. V. Lysuho, E. Yu. Panov, *Renormalized entropy solutions to the Cauchy problem for first order quasilinear conservation laws in the class of periodic functions*, J. Math. Sci., **177** (2011) 27–49.

[10] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa, **5** (1978) 489–507.

[11] E. Yu. Panov, *On sequences of measure-valued solutions of first-order quasilinear equations*, Mat. Sb., **185** (1994) 87–106. English transl. in Russian Acad. Sci. Sb. Math., **81** (1995) 211–227.

[12] E. Yu. Panov, *On strong precompactness of bounded sets of measure valued solutions for a first order quasilinear equation*, Mat. Sb., **186** (1995) 103–114. English transl. in Russian Acad. Sci. Sb. Math., **186** (1995) 729–740.

[13] E. Yu. Panov, *Property of strong precompactness for bounded sets of measure valued solutions of a first-order quasilinear equation*, Mat. Sb., **190** (1999) 109–128. English transl. in Russian Acad. Sci. Sb. Math., **190** (1999) 427–446.

[14] E. Yu. Panov, *A remark on the theory of generalized entropy sub- and supersolutions of the Cauchy problem for a first-order quasilinear equation*, Differ. Uravn., **37** (2001) 252–259. English transl. in Differ. Equ., **37** (2001) 272–280.

[15] E. Yu. Panov, *Maximum and minimum generalized entropy solutions to the Cauchy problem for a first-order quasilinear equation*, Mat. Sb., **193** (2002) 95–112. English transl. in Russian Acad. Sci. Sb. Math., **193** (2002) 727-743.

[16] E. Yu. Panov, *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws*, J. Hyperbolic Differ. Equ., **2** (2005) 885–908.

[17] E. Yu. Panov, *Existence of strong traces for quasi-solutions of multidimensional conservation laws*, J. Hyperbolic Differ. Equ. **4** (2007) 729–770.

[18] E. Yu. Panov, *Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux*, Arch. Rational Mech. Anal., **195** (2010) 643–673.

[19] E. Yu. Panov, *On decay of periodic entropy solutions to a scalar conservation law*, preprint, http://www.math.ntnu.no/conservation/2011/026.html

[20] E. Yu. Panov, *On decay of periodic entropy solutions to a scalar conservation law*, To appear in Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire, in press, DOI: 10.1016/j.anihpc.2012.12.009 (2013).

[21] L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear analysis and mechanics: Heriot. Watt Symposium, vol. 4 (Edinburgh 1979), Res. Notes Math., **39** (1979) 136–212.

[22] L. Tartar, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh. Sect. A., **115** (1990) 193–230.

[23] V. V. Zhikov, *On two-scale convergence*, Tr. Semin. Im. I. G. Petrovskogo, **23** (2003) 149-187. English transl. in J. Math. Sci., **120** (2004) 1328-1352.