A Semigroup Approach to an Integro-Differential Equation Modeling Slow Erosion

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Abstract

The paper is concerned with a scalar conservation law with nonlocal flux, providing a model for granular flow with slow erosion and deposition. While the solution $u = u(t,x)$ can have jumps, the inverse function $x = x(t,u)$ is always Lipschitz continuous; its derivative has bounded variation and satisfies a balance law with measure-valued sources. Using a backward Euler approximation scheme combined with a nonlinear projection operator, we construct a continuous semigroup whose trajectories are the unique entropy weak solutions to this balance law. Going back to the original variables, this yields the global well-posedness of the Cauchy problem for the granular flow model.

1 Introduction

In this paper we study the scalar conservation law with nonlocal flux

$$u_t(t,x) - \left(\exp\int_x^\infty f\left(u_x(t,y)\right)\, dy\right)_x \;=\; 0\,, \qquad\qquad u(0,x) = \bar u(x). \tag{1.1}$$

Here $x \in I\!R$ is the space variable, and one can think of $u(t,\cdot)$ the height of a standing profile of sand (or some other granular material). We assume that $x \mapsto u(t,x)$ is strictly increasing, with $u_x \to 1$ as $x \to \pm\infty$. In this model, the variable t should not be thought as the usual time on the clock. Rather, t measures the total amount of sand poured from the top, i.e. at $x = +\infty$. As it slides downward, this thin moving layer of sand will put further sand into motion, at points where the slope is $u_x > 1$. On the other hand, if the slope is $u_x < 1$, part of the moving layer will be deposited and become part of the standing profile.

To understand the meaning of the flux in (1.1), consider a unit amount of sand poured down at $x = +\infty$. Let $\sigma(t,x)$ be the amount of sand which crosses the point x, from right to left. For a fixed t, this is determined by solving the linear ODE

$$-\frac{d}{dx}\sigma(t,x) \;=\; f(u_x(t,x))\cdot\sigma(t,x), \qquad\qquad \sigma(+\infty) \;=\; 1. \tag{1.2}$$

Here f is called the *erosion function*, since it describes the amount of erosion as a function of the slope, per unit distance travelled in space and per unit mass passing through. We shall always assume that f is an increasing function with $f(1) = 0$. Solving (1.2) one obtains

$$\sigma(t,x) \;=\; \exp\int_x^\infty f(u_x(t,y))\,dy.$$

The rate at which sand is deposited inside any given interval $[a,b]$ is thus computed by

$$\frac{d}{dt}\int_a^b u(t,x)\,dx \;=\; \sigma(t,b)-\sigma(t,a) \;=\; \int_a^b\left(\exp\int_x^\infty f(u_x(t,y))\,dy\right)_x dx.$$

Since $a < b$ are arbitrary, this yields the conservation law (1.1).

Equation (1.1) was first derived in [1] as the slow erosion limit for the two-layer model of granular flow by Hadeler and Kuttler [13], with the specific erosion function $f(p) = (p-1)/p$. In this paper, more general increasing functions f will be considered.

Differentiating (1.1) w.r.t. x, and denoting by $p = u_x > 0$ the slope, one obtains the additional conservation law

$$p_t(t,x) + \left(f(p(t,x))\cdot\exp\int_x^\infty f(p(t,y))\,dy\right)_x \;=\; 0\,, \qquad p(0,x) = \bar p(x)\,. \tag{1.3}$$

If the function f satisfies

$$f(1)=0,\quad f'>0,\quad f''<0,\quad \lim_{p\to 0+} f(p) = -\infty,\quad \lim_{p\to+\infty}\frac{f(p)}{p} = 0, \tag{1.4}$$

then one can show that solutions $p(t,x)$ of (1.3) remain bounded for all $t \geq 0$. In particular, this is the case when $f(p) = (p-1)/p$, as for the limit of Hadeler-Kuttler model, studied in [1]. Under suitable assumptions on the initial data, the existence and uniqueness of BV solutions for (1.3) has been established in [2, 3], using front tracking and operator splitting techniques.

If the erosion function f is allowed to have asymptotically linear growth, then it is known that the slope $p = u_x$ can blow up in finite time. Throughout this paper, instead of (1.4) we shall use the following assumptions on the erosion function:

(A1) The function $f : I\!R_+ \mapsto I\!R$ is twice continuously differentiable and satisfies

$$\begin{aligned} &f(1)=0, \qquad f''<0, \qquad \eta \;\doteq\; \lim_{p\to+\infty} f'(p) \;>\; 0,\\ &\lim_{p\to 0+} f(p) \;=\; -\infty, \qquad \lim_{p\to+\infty} f(p)-pf'(p) \;<\; \infty. \end{aligned} \tag{1.5}$$

These conditions imply that, as $p \to +\infty$, the graph of f approaches a linear asymptote with slope $\eta > 0$.

When the slope $p = u_x$ becomes infinite and the function u becomes discontinuous, the equation (1.3) is no longer appropriate and one must study the original equation (1.1). As shown in [15], solutions can have three types of singularities. These are *kinks* (where u_x has jumps but u is continuous), *shocks* (where u has jumps), and *hyperkinks* (where u is continuous but u_x approaches $+\infty$). With the presence of the jumps in u, the distributional derivative $\partial_x u$ contains point masses, causing technical difficulties in the analysis. For a suitable family of initial data, the global existence of entropy admissible solutions was proved in [15], by means of piecewise affine approximations generated by an adapted front tracking algorithm. However, the uniqueness of these solutions has remained an open problem.

We observe that, as long as $u_x(t,x) \geq c_0 > 0$, the inverse function $x = X(t,u)$ is always well-defined and globally Lipschitz continuous. Whenever $u(t,x)$ has a jump, with left and right states $u^- < u^+$, the map $u \mapsto X(t,u)$ remains constant over the interval $[u^-, u^+]$. If $u = u(t,x)$ is a smooth solution of (1.1), a straightforward computation shows that $X = X(t,u)$ satisfies the conservation law

$$X_t(t,u) + \left(\exp \int_u^{+\infty} g\left(X_u(t,v)\right)\, dv\right)_u \;=\; 0\,, \qquad X(0,u) = \overline{X}(u)\,. \tag{1.6}$$

Here the function g is recovered from f according to

$$g(z) \doteq z\, f\left(\frac{1}{z}\right)\,. \tag{1.7}$$

A straightforward computation yields

$$g'(z) \;=\; f\left(\frac{1}{z}\right) - \frac{1}{z} f'\left(\frac{1}{z}\right), \qquad g''(z) \;=\; \frac{1}{z^3} f''\left(\frac{1}{z}\right). \tag{1.8}$$

From the assumptions (1.5) on f it thus follows

$$g(1) = 0, \qquad g'' < 0, \qquad \lim_{z \to +\infty} g(z) = -\infty, \quad g(0) > 0, \quad g'(0) < \infty. \tag{1.9}$$

Differentiating (1.6) w.r.t. u, and writing $z(t,u) \doteq X_u(t,u)$, one obtains

$$z_t(t,u) - \left(g(z(t,u)) \cdot \exp \int_u^{+\infty} g(z(t,v))\, dv\right)_u \;=\; 0\,, \qquad z(0,u) = \bar{z}(u)\,. \tag{1.10}$$

The advantage of this alternative formulation is that, while u in (1.1) can be discontinuous and p in (1.3) can become a distribution with point masses, the variable X in (1.6) is always Lipschitz continuous and z in (1.10) remains a globally bounded function. However, this comes at a price, because a solution of (1.10) may well become negative. In this case, the map $u \mapsto X(t,u)$ is no longer invertible and the connection with the original equation (1.3) is lost.

To preserve its physical meaning, the equation (1.10) must be supplemented by the pointwise constraint $z \geq 0$. This leads to

$$z_t(t,u) - \left(g(z(t,u)) \cdot \exp \int_u^{+\infty} g(z(t,v))\, dv\right)_u \;=\; \mu^{(t)}\,, \qquad z(0,u) = \bar{z}(u)\,, \tag{1.11}$$

where, for each $t \geq 0$, $\mu^{(t)}$ is a suitable measure supported on the set where $z = 0$. Throughout this paper we shall consider solutions of (1.11) which are nonnegative, lower semicontinuous, and such that $z(t,\cdot) \in BV$ for every $t \geq 0$. In this case, a precise set of conditions on the measures $\mu^{(t)}$ can be stated as follows.

(C) There exists a jointly measurable function $\Theta = \Theta(t,u)$ such that $\Theta(t,\cdot) \in BV$ and $\mu^{(t)} = \partial_u \Theta(t,\cdot)$ is the derivative in distributional sense, for a.e. $t \geq 0$. Moreover

$$z(t,u) \neq 0 \qquad \Longrightarrow \qquad \mu^{(t)}(\,]-\infty, u]) \;=\; 0, \tag{1.12}$$

$$z(t,a) \neq 0, \quad z(t,b) \neq 0 \qquad \Longrightarrow \qquad \int_a^b \mu^{(t)}(\,]-\infty, u])\, du \;=\; 0. \tag{1.13}$$

A semigroup of solutions to (1.11) was first constructed in [9]. The analysis in [9] shows that the limits of front tracking approximations yield entropy weak solutions which depend continuously on the initial data as well as on the erosion function g.

The purpose of the present paper is three-fold. First, we provide an entirely different construction of the flow generated by (1.11). Solutions are here obtained by a flux-splitting method, alternating backward Euler steps for (1.10) with a nonlinear projection operator on the cone of positive functions. This approach is much in the spirit of nonlinear semigroup theory, as in [10]. We then prove the uniqueness of entropy weak solutions of (1.11) by a classical Kruzhkov-type argument. Finally, we prove the equivalence between entropy solutions of (1.11) and entropy solutions of the original equation (1.1). As a consequence, this yields the global existence and uniqueness of entropy admissible solutions to (1.1), and their continuous dependence on the initial data.

The remainder of the paper is organized as follows. In Section 2 we define a backward Euler step for (1.10) and establish several estimates. In Section 3 we study a nonlinear projection operator from a subset of $\mathbf{L}^1_{loc}$ into the cone of non-negative functions. By combining these two steps, approximate solutions to (1.11) are constructed in Section 4. Letting the time step approach zero, a compactness argument derived from Helly's theorem yields a continuous semigroup of entropy weak solutions. See Definition 5.1 and Theorem 5.4 in Section 5 for a precise result.

The uniqueness of entropy weak solutions to (1.11) is proved in Section 6, by adapting the classical variable doubling technique [14]. Finally, Theorem 7.4 in Section 7 shows that the entropy weak solutions to (1.11) correspond to entropy admissible solutions for the original problem (1.1). This equivalence heavily relies on the fact that our solutions are BV functions and the flux function is convex. In this case, the Kruzhkov entropy conditions are satisfied if and only if the Lax admissibility conditions hold at every point of approximate jump. From the existence and uniqueness of solutions to (1.11), thanks to this equivalence result we eventually obtain the well-posedness of the Cauchy problem for (1.1).

For the basic theory of conservation laws we refer to [5, 16, 17]. The admissibility conditions and the variable-doubling technique to establish uniqueness of entropy weak solutions were introduced in the classical papers [18] and [14]. The semigroup approach to a scalar conservation law, based on backward Euler approximations, is originally due to Crandall [10].

It is interesting to compare the equation (1.1) with similar conservation laws with nonlocal flux. In the models considered in [6, 7, 8], the structure of the equation provides uniform a priori bounds on the integral $\int u_x^2\, dx$. As a consequence, solutions remain uniformly Hölder continuous and no shock is ever formed. On the other hand, solutions to (1.1) can become discontinuous in finite time, and display various types of singularities.

As a related result, we mention that the existence and local stability of traveling wave solutions for (1.11) have been recently established in [12].

2 Backward Euler approximations

In this section we study the backward Euler step for the slow erosion model without the constraint $z \geq 0$. Note that in this case the solution of (1.10) could become negative. We thus need to extend the definition of the erosion function $g(z)$ also for negative values of z. For convenience, we extend the domain of g by setting

$$g(z) \;=\; g(0) + g'(0)z, \qquad z \in [-1, 0]. \tag{2.1}$$

and further extend g in a smooth way for $z \leq -1$. Recalling (1.9), we can assume that this extended function g satisfies the following assumptions:

(A2) The function $g : I\!R \mapsto I\!R$ is continuously differentiable, vanishes for $s \leq -2$, is affine for $s \in [-1, 0]$, and is twice continuously differentiable for $s > 0$. Moreover it satisfies

$$g(0) \geq 0, \qquad g(1) = 0, \qquad g''(s) < 0 \quad \text{for all } s > 0. \tag{2.2}$$

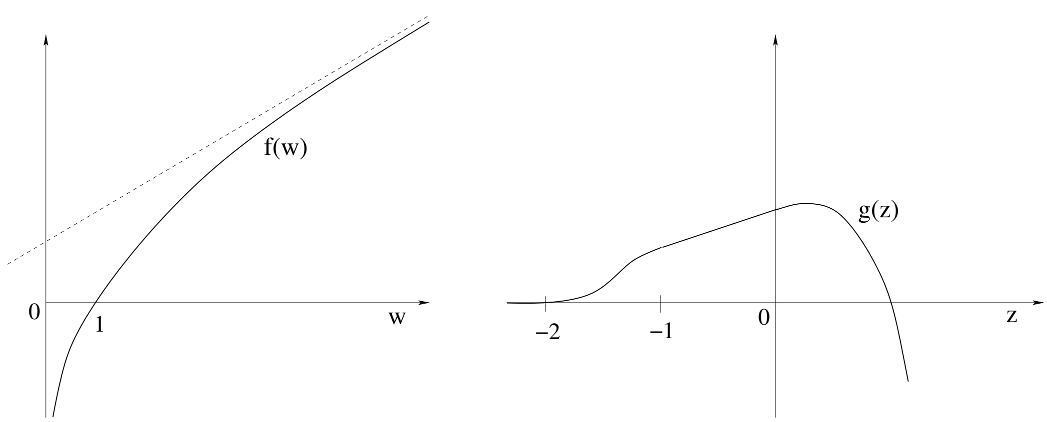

Figure 1: A function f and the corresponding function g in (1.7), extended to negative values according to (A2).

In the rest of the paper, we denote TV$\{\cdot\}$ the total variation of a function. Our approximate solutions will take values inside the domain

$$\mathcal{D}_0 \;\doteq\; \Big\{ z : I\!R \mapsto I\!R\,; \quad z \text{ is absolutely continuous and there exist constants } M, U_0 > 0 \text{ such that } \|z(\cdot) - 1\|_{\mathbf{L}^1} \leq M, \quad \text{TV}\{z(\cdot)\} \leq M, \quad z(u) = 1 \quad \text{for all } u \geq U_0 \Big\}. \tag{2.3}$$

The set of nonnegative functions in $\mathcal{D}_0$ will be written as

$$\mathcal{D}_0^+ \;\doteq\; \Big\{ z \in \mathcal{D}_0\,; \quad z(u) \geq 0 \quad \text{for all } u \in I\!R \Big\}. \tag{2.4}$$

For $z \in \mathcal{D}_0$, we define

$$G(u;z) \doteq \exp \int_u^\infty g(z(y))\,dy\,. \tag{2.5}$$

Using the assumptions (**A2**) one obtains

$$G_u(u;z) = -\,g(z)\cdot G(u;z), \qquad \lim_{u\to+\infty} G(u;z) = 1\,, \tag{2.6}$$

and

$$0 < G(u;z) \le \exp \int_{z\in[-2,1]} g(z(y))\,dy \le \exp\left\{ \max_{-2\le z\le 1} |g'(z)| \cdot \|z-1\|_{\mathbf{L}^1} \right\}. \tag{2.7}$$

We shall construct approximate solutions using a backward Euler scheme. For convenience, instead of (1.10) we consider the equivalent equation

$$z_t(t,u) - \big(g(z)\cdot G(u;z)\big)_u - \lambda z_u = 0\,, \tag{2.8}$$

where the constant $\lambda > 0$ is chosen large enough (depending on the initial condition) such that all the characteristic speeds for (2.8) become ≤ -1. By (2.6), this is the case provided that

$$\sup_{t,u} \big|g'(z(t,u))\big| \cdot \sup_{t,u} G(u;z(t,u)) + \lambda \ \ge\ 1. \tag{2.9}$$

It is clear that these two problems are entirely equivalent: $z(t,u)$ is a solution of (2.8) if and only if $z(t,u-\lambda t)$ is a solution of (1.10).

Definition 2.1 (Backward Euler operator). *Consider a function $z \in \mathcal{D}_0^+$ and let $\varepsilon > 0$ be given. We define the backward Euler operator $E_\varepsilon^- : \mathcal{D}_0^+ \mapsto \mathcal{D}_0$ by setting*

$$E_\varepsilon^- z = w\,, \tag{2.10}$$

where $w \in \mathcal{D}_0$ is the unique function satisfying the implicit ODE

$$w(u) = z(u) + \varepsilon\Big(g(w(u))G(u;w)\Big)_u + \varepsilon\lambda w_u\,. \tag{2.11}$$

Notice that the condition $w \in \mathcal{D}_0$ singles out the unique solution of (2.11) such that

$$w(u) = 1 \qquad \text{for all } u \text{ sufficiently large.} \tag{2.12}$$

The next lemma shows that the backward Euler operator is well defined, and establishes some of its properties.

Lemma 2.2. *Let g satisfy the assumptions (A1). Let $z \in \mathcal{D}_0$ and let M, U_0 be the corresponding constants in (2.3). We introduce the constants*

$$\begin{cases} \kappa = M\ \|g'\|_{\mathbf{L}^\infty([-2,\,M+1])}\,, \\ \lambda = 1 + e^\kappa\ \|g'\|_{\mathbf{L}^\infty([-2,\,M+1])}\,, \end{cases} \qquad \begin{cases} C_0 = e^\kappa \cdot \sup_{s\le 0} g^2(s)\,, \\ C_1 = \|g'\|^2_{\mathbf{L}^\infty([-2,M+1])} \cdot 2M\,e^\kappa. \end{cases} \tag{2.13}$$

Then for every $\varepsilon > 0$ the problem (2.11)-(2.12) admits a unique solution $w = E_\varepsilon^- z \in D_0$. Moreover, the following properties hold.

(i) $\sup_u w(u) \leq \sup_u z(u) \leq M+1.$

(ii) $\|w(\cdot)-1\|_{\mathbf{L}^1} \leq \|z(\cdot)-1\|_{\mathbf{L}^1} \leq M.$

(iii) $e^{-\kappa} \leq G(u;w) \leq e^{\kappa}.$

(vi) $\inf_u w(u) \geq -C_0\varepsilon.$

(v) $TV\{w\} \leq (1-\varepsilon C_1)^{-1} TV\{z\},$ *provided that* $\varepsilon < 1/C_1$.

(vi) $\|w-z\|_{\mathbf{L}^1} \leq \varepsilon\,(4\lambda + 2\kappa e^{\kappa})\cdot TV\{z\},$ *provided that* $\varepsilon \leq 1/(2C_1)$.

(vii) *If* $z^* \in \mathcal{D}_0$ *and* $w^* = E_\varepsilon^- z^*$ *is the corresponding solution to (2.11)-(2.12), then*

$$(1-\varepsilon C)\,\|w-w^*\|_{\mathbf{L}^1} \leq \|z-z^*\|_{\mathbf{L}^1}\,, \tag{2.14}$$

for some constant C *depending only on the function* g *and on* M.

(viii) *For any constant* $c>0$ *and any positive test function* ψ, *one has*

$$\begin{aligned}\int \frac{|w-c|-|z-c|}{\varepsilon}\psi\,du + \int \Big|(g(w)-g(c))G(u;w)+\lambda(w-c)\Big|\psi_u\,du \\ \leq -g(c)\int \mathrm{sign}(w-c)g(w)G(u;w)\psi\,du\,.\end{aligned} \tag{2.15}$$

Proof. The implicit ODE (2.11) can be rewritten as

$$\Big(\lambda + g'(w)\,G(u;w)\Big)w_u = \frac{w-z}{\varepsilon} + g^2(w)G(u;w). \tag{2.16}$$

By assumption $z \in \mathcal{D}_0$, hence there exists U_0 such that $z(u)=1$ for all $u \geq U_0$. Since $g(1)=0$, it is clear that $w = E_\varepsilon^- z$ is the unique absolutely continuous function that solves the ODE (2.16) for $u \leq U_0$ and such that

$$w(u) = 1 \qquad \text{for all } u \geq U_0\,. \tag{2.17}$$

Because of the regularity of the coefficients, this ODE has a unique local solution. It thus remains to check that this solution can be prolonged backwards for $u \in\,]-\infty, U_0]$. This requires to prove a priori estimates showing that $w(u)$ remains uniformly bounded, while the coefficient $\lambda + g'(w)G(u;w)$ in (2.16) remains uniformly positive.

(i) - Upper and lower bounds on w. We begin by showing that, on any domain $[u_0,\infty[$ where the solution of (2.11) is defined, one has the a priori bounds

$$-2 \leq w(u) \leq \sup_u z(u)\,. \tag{2.18}$$

Indeed, consider any $w^\sharp > \sup_u z(u) \geq 1$. If $w(u) \geq w^\sharp$ for some u, a contradiction is obtained as follows. Define

$$u^\sharp \doteq \sup\Big\{u \in \mathbb{R}\,;\;\; w(s) > w^\sharp\Big\}.$$

We then have

$$w\left(u^\sharp\right) = w^\sharp, \qquad w(u) \;\leq\; w^\sharp \;\text{ for all }\; u > u^\sharp\,, \quad \text{so} \quad w_u\left(u^\sharp\right) \leq 0\,.$$

However, this is impossible because

$$w_u\left(u^\sharp\right) \;=\; \left[\lambda + g'\left(w^\sharp\right) G\left(u^\sharp; w\right)\right]^{-1} \left(\frac{w^\sharp - z\left(u^\sharp\right)}{\varepsilon} + g^2\left(w^\sharp\right) G\left(u^\sharp; w\right)\right) \;>\; 0.$$

Next, if $w(u) < -2$ for some u, define

$$u^\flat \;\doteq\; \sup\{u\,;\;\; w(u) < -2\}\,.$$

We then have

$$w\left(u^\flat\right) = -2, \qquad w(u) \;\geq\; -2 \;\text{ for all }\; u > u^\flat, \quad \text{so} \quad w_u\left(u^\flat\right) \geq 0\,.$$

However, this is impossible because by using $g(-2) = 0$ and $g'(-2) = 0$ we have

$$w_u\left(u^\flat\right) \;=\; \left[\lambda + g'(-2)\, G\left(u^\flat; w\right)\right]^{-1} \left(\frac{-2 - z\left(u^\flat\right)}{\varepsilon} + g^2(-2) G\left(u^\flat; w\right)\right) \;<\; 0.$$

(ii)-(iii) Bounds on $\|w-1\|_{\mathbf{L}^1}$ and on G. Let the solution of (2.11) be defined on $[u_0, \infty[$. We rewrite (2.11) as

$$w(u) - 1 \;=\; z(u) - 1 + \varepsilon\Big[g(w)G(u;w) + \lambda(w-1)\Big]_u. \tag{2.19}$$

Multiplying by $\operatorname{sign}(w-1)$ and integrating in u, for any $u^* \geq u_0$ one obtains

$$\begin{aligned}
&\int_{u^*}^{\infty} |w(u) - 1|\, du \\
&\leq \int_{u^*}^{\infty} |z(u) - 1|\, du + \varepsilon \int_{u^*}^{\infty} \operatorname{sign}(w-1)\Big[g(w)G(u;w) + \lambda(w(u)-1)\Big]_u du \\
&\leq \|z-1\|_{\mathbf{L}^1} - \varepsilon \operatorname{sign}(w(u^*)-1)\Big[g\left(w\left(u^*\right)\right) G\left(u^*; w\right) + \lambda\left(w\left(u^*\right) - 1\right)\Big], \quad (2.20)
\end{aligned}$$

because w is absolutely continuous and $g(w)G(u;w) + \lambda(w-1) = 0$ whenever $w = 1$. We claim that the last term on the right hand side of (2.20) is non-positive. Indeed, by (2.18) it follows

$$\sup_u \left|g'(w(u))\right| \;\leq\; \sup_{-2 \leq s \leq M+1} \left|g'(s)\right| \;=\; \left\|g'\right\|_{\mathbf{L}^\infty([-2,\,M+1])}\,.$$

Observing that

$$|g(w(u^*))| \;\leq\; \left\|g'\right\|_{\mathbf{L}^\infty([-2,\,M+1])} \cdot \left|w\left(u^*\right) - 1\right|\,,$$

to prove that

$$-\operatorname{sign}\left(w\left(u^*\right) - 1\right) \cdot \Big[g\left(w\left(u^*\right)\right) G\left(u^*; w\right) + \lambda\left(w\left(u^*\right) - 1\right)\Big] \;\leq\; 0\,, \tag{2.21}$$

by (2.13) it suffices to show that $G(u^*;w) \le e^\kappa$. If this inequality fails, a contradiction is obtained as follows. Define

$$u^\sharp \doteq \sup\{u; \quad G(u;w) > e^\kappa\}.$$

By continuity, $G\left(u^\sharp;w\right) = e^\kappa$. Hence by (2.13) there exists $\delta > 0$ such that

$$\|g'\|_{\mathbf{L}^\infty([-2,\,M+1])} \cdot G(u^*;w) \;<\; \lambda \qquad\qquad \text{for all } u^* \in [u^\sharp - \delta,\, \infty[\,. \tag{2.22}$$

Using (2.22) in (2.20), for every $u^* > u^\sharp - \delta$ we obtain

$$\int_{u^*}^\infty |w(u) - 1|\, du \;\le\; \|z-1\|_{\mathbf{L}^1}\,,$$

so

$$\left|\int_{u^*}^\infty g(w(y))\, dy\right| \le \|g'\|_{\mathbf{L}^\infty([-2,\,M+1])} \cdot \|w-1\|_{\mathbf{L}^1} \le \|g'\|_{\mathbf{L}^\infty([-2,\,M+1])} \cdot \|z-1\|_{\mathbf{L}^1} \;\le\; \kappa\,.$$

Hence $G(u^*;w) \le e^\kappa$ for all $u^* > u^\sharp - \delta$, against the assumption.

The previous analysis shows that, if a solution of (2.11)-(2.12) is defined on $[u_0, \infty[$ for some u_0, then the bounds (2.18) hold, together with

$$\lambda + g'(w(u))G(u,w) \;\ge\; 1 \qquad\qquad \text{for all } \; u \ge u_0\,. \tag{2.23}$$

We thus conclude that the solution w of (2.11)-(2.12) can be extended backwards to the entire real line, and satisfies (i)–(iii).

(iv) Lower bound on w. We now refine the lower bound in (2.18), deriving an ε-dependent estimate. Recalling (2.13), consider any value $w^\flat < -C_0\varepsilon$. If $w(u) < w^\flat$ for some u, a contradiction is obtained as follows. Define

$$u^\flat \;\doteq\; \sup\left\{u \in I\!R; \quad w(u) < w^\flat\right\}.$$

We then have

$$w\left(u^\flat\right) \;=\; w^\flat, \qquad\qquad w(u) \;\ge\; w^\flat \;\; \text{for all } \; u > u^\flat.$$

However, this is impossible because the inequalities $G\left(u^\flat;w\right) \le e^\kappa$ and $z\left(u^\flat\right) \ge 0$ yield

$$\begin{aligned} w_u\left(u^\flat\right) \;&=\; \Big(\lambda + g'\left(w^\flat\right) G\left(u^\flat;w\right)\Big)^{-1} \left(\frac{w^\flat - z\left(u^\flat\right)}{\varepsilon} + g^2\left(w^\flat\right) G\left(u^\flat;w\right)\right) \\ &\le\; \Big(\lambda + g'\left(w^\flat\right) G\left(u^\flat;w\right)\Big)^{-1} \left(\frac{w^\flat}{\varepsilon} + C_0\right) \;<\; 0\,. \end{aligned}$$

(v) Bound on the total variation. Fix any $h > 0$. Then

$$\begin{aligned} w(u) \;&=\; z(u) + \varepsilon\Big[g(w(u))G(u;w) + \lambda w(u)\Big]_u\,, \\ w(u-h) \;&=\; z(u-h) + \varepsilon\Big[g(w(u-h))G(u-h;w) + \lambda w(u-h)\Big]_u\,. \end{aligned}$$

Writing $\sigma(u) = \text{sign}(w(u) - w(u-h))$, we obtain

$$\begin{aligned}
\text{TV}\{w\} - \text{TV}\{z\} &= \lim_{h\to 0+} \frac{1}{h} \int \Big\{ |w(u) - w(u-h)| - |z(u) - z(u-h)| \Big\}\, du \\
&\leq \varepsilon \lim_{h\to 0+} \frac{1}{h} \int \sigma(u) \cdot \Big[g(w(u))G(u;w) - g(w(u-h))G(u-h;w) \\
&\qquad\qquad + \lambda(w(u) - w(u-h)) \Big]_u\, du \\
&= \varepsilon \lim_{h\to 0+} \frac{1}{h} \int \sigma(u) \cdot \Big[(g(w(u)) - g(w(u-h)))G(u;w) + \lambda(w(u) - w(u-h)) \Big]_u\, du \\
&\quad + \varepsilon \lim_{h\to 0+} \frac{1}{h} \int \sigma(u) \cdot \Big[g(w(u-h))(G(u;w) - G(u-h;w)) \Big]_u\, du\,.
\end{aligned}$$

Here the first term vanishes because $w(u)$ is absolutely continuous and

$$\text{sign}(w(u) - w(u-h)) = \text{sign}\left[(g(w(u)) - g(w(u-h)))G(u;w) + \lambda(w(u) - w(u-h)) \right].$$

Thus, we have

$$\text{TV}\{w\} - \text{TV}\{z\} \leq \varepsilon \lim_{h\to 0+} \frac{1}{h} \int \sigma(u) \cdot \Big[g(w(u-h))(G(u;w) - G(u-h;w)) \Big]_u\, du\,. \tag{2.24}$$

To simplify notation, call

$$\mathcal{G}(u;w) \doteq \frac{G(u;w) - G(u-h;w)}{h} = \frac{1}{h} \int_{u-h}^{u} G_u(s;w)\, ds = (J_h * G_u)(u),$$

where the right hand side denotes the convolution of the derivative $G_u(u;w)$ with the step function

$$J_h(s) \doteq \begin{cases} h^{-1} & \text{if } s \in [0,h]\,, \\ 0 & \text{otherwise.} \end{cases}$$

By (i) and (iv), and by choosing $\varepsilon > 0$ sufficiently small we can assume that

$$w(u) \in [-1,\, M+1], \qquad \text{so} \qquad |g(w)| \leq M \cdot \|g'\|_{\mathbf{L}^\infty([-2,M+1])}\,. \tag{2.25}$$

By standard properties of convolutions one obtains

$$\begin{aligned}
\text{TV}\{\mathcal{G}(\cdot;w)\} &\leq \text{TV}\{G_u(\cdot;w)\} = \text{TV}\{g(w) \cdot G(\cdot;w)\} \\
&\leq \text{TV}\{g(w)\} \cdot \|G(\cdot;w)\|_{\mathbf{L}^\infty} + \|g(w)\|_{\mathbf{L}^\infty} \cdot \text{TV}\{G(\cdot;w)\}\,.
\end{aligned} \tag{2.26}$$

We now have the following estimates:

$$\begin{aligned}
\|g(w)\|_{\mathbf{L}^\infty} &\leq \|g'\|_{\mathbf{L}^\infty([-2,M+1])} \cdot \|w-1\|_{\mathbf{L}^\infty} \\
&\leq \|g'\|_{\mathbf{L}^\infty([-2,M+1])} \cdot \text{TV}\{w\}\,, \qquad (2.27) \\
\text{TV}\{G(\cdot;w)\} &= \|G_u(\cdot;w)\|_{\mathbf{L}^1} = \|g(w)\, G(\cdot;w)\|_{\mathbf{L}^1} = \|G(\cdot;w)\|_{\mathbf{L}^\infty} \cdot \|g(w)\|_{\mathbf{L}^1} \\
&\leq \|G(\cdot;w)\|_{\mathbf{L}^\infty} \cdot \|g'\|_{\mathbf{L}^\infty([-2,M+1])} \cdot \|w-1\|_{\mathbf{L}^1} \\
&\leq e^\kappa \|g'\|_{\mathbf{L}^\infty([-2,M+1])} \cdot M, \qquad (2.28) \\
\|\mathcal{G}(\cdot,w)\|_{\mathbf{L}^\infty} &\leq \|G_u(\cdot,w)\|_{\mathbf{L}^\infty} \leq \|g(w)\|_{\mathbf{L}^\infty} \cdot \|G(\cdot,w)\|_{\mathbf{L}^\infty} \\
&\leq M \cdot \|g'\|_{\mathbf{L}^\infty([-2,M+1])}\, e^\kappa. \qquad (2.29)
\end{aligned}$$

We can represent the open set

$$\{u\,;\ w(u)\neq w(u-h)\}\ =\ \{I_k\,;\quad k\geq 1\}$$

as a disjoint union of open intervals $I_k =]a_k, b_k[$. Using (2.26) and (2.27)–(2.29) we thus obtain

$$\begin{aligned}
&\int \sigma(u)\cdot\left[g(w(u-h))\frac{G(u;w)-G(u-h;w)}{h}\right]_u du\\
&\leq\ \sum_k \Big|g\left(w\left(b_k-h\right)\right)\cdot\mathcal{G}\left(b_k\right)-g\left(w\left(a_k-h\right)\right)\cdot\mathcal{G}\left(a_k\right)\Big|\\
&\leq\ \|g(w)\|_{\mathbf{L}^\infty}\cdot \mathrm{TV}\{\mathcal{G}(\cdot,w)\}+\mathrm{TV}\{g(w)\}\cdot\|\mathcal{G}(\cdot,w)\|_{\mathbf{L}^\infty}\\
&\leq\ \left\|g'\right\|^2_{\mathbf{L}^\infty([-2,M+1])}\cdot M\,e^\kappa\cdot\mathrm{TV}\{w\}+\left\|g'\right\|^2_{\mathbf{L}^\infty([-2,M+1])}\cdot M\,e^\kappa\cdot\mathrm{TV}\{w\}\\
&=\ \left\|g'\right\|^2_{\mathbf{L}^\infty([-2,M+1])}\cdot 2M\,e^\kappa\cdot\mathrm{TV}\{w\}\,.
\end{aligned}\tag{2.30}$$

Together with (2.24), this yields

$$\mathrm{TV}\{w\}-\mathrm{TV}\{z\}\ \leq\ \varepsilon C_1\,\mathrm{TV}\{w\}\,.$$

Therefore, assuming $\varepsilon C_1 < 1$, we conclude

$$\mathrm{TV}\{w\}\ \leq\ \frac{\mathrm{TV}\{z\}}{1-\varepsilon(2M+2)e^\kappa\ \|g'\|^2_{\mathbf{L}^\infty([-2,M+1])}}\ =\ \frac{\mathrm{TV}\{z\}}{1-\varepsilon C_1}\,.\tag{2.31}$$

(vi) - $\mathbf{L}^1$ continuity in time. If $w=E^-_\varepsilon z$, recalling (2.23) and (2.27)-(2.28) we obtain

$$\begin{aligned}
\|w-z\|_{\mathbf{L}^1}\ &\leq\ \varepsilon\int\Big|\left(g'(w(u))G(u;w)+\lambda\right)w_u\Big|\,du+\varepsilon\int\Big|g(w(u))G_u(u;w)\Big|\,du\\
&\leq\ \varepsilon\cdot\int 2\lambda\,|w_u|\,du+\varepsilon\|g(w)\|_{\mathbf{L}^\infty}\cdot\mathrm{TV}\{G(\cdot;w)\}\\
&\leq\ \varepsilon\Big(2\lambda+Me^\kappa\ \left\|g'\right\|_{\mathbf{L}^\infty([-2,M+1])}\Big)\cdot\mathrm{TV}\{w\}\\
&\leq\ 2\varepsilon\left(2\lambda+\kappa e^\kappa\right)\cdot\mathrm{TV}\{z\}\,.
\end{aligned}\tag{2.32}$$

Here the last inequality follows from (2.31), provided that $\varepsilon C_1\leq 1/2$.

(vii) $\mathbf{L}^1$ stability. Assume $z^*(u)\in\mathcal{D}_0$ and let $w^*=E^-_\varepsilon z^*$, so that

$$w^*(u)\ =\ z^*(u)+\varepsilon\Big[g\left(w^*(u)\right)G\left(u;w^*\right)\Big]_u+\varepsilon\lambda w^*_u\,.\tag{2.33}$$

By possibly increasing the values of M, U_0 we can assume that both $z(\cdot)$ and $z^*(\cdot)$ satisfy the inequalities in (2.3), with these constants. We then have

$$\begin{aligned}
\|w^*-w\|_{\mathbf{L}^1}\ \leq\ &\|z^*-z\|_{\mathbf{L}^1}+\varepsilon\int\ \mathrm{sign}\left(w^*-w\right)\Big[g\left(w^*\right)G\left(u;w^*\right)-g(w)G(u;w)\Big]_u du\\
&+\varepsilon\,\lambda\int\ \mathrm{sign}\left(w^*-w\right)\left(w^*-w\right)_u\,du\,,
\end{aligned}$$

where the last term vanishes. Therefore, we have

$$
\begin{aligned}
&\|w^* - w\|_{\mathbf{L}^1} - \|z^* - z\|_{\mathbf{L}^1} \\
&\quad \leq \ \varepsilon \int \operatorname{sign}(w^* - w) \Big[g(w^*)\, G(u; w^*) - g(w) G(u; w)\Big]_u du \\
&\quad = \ \varepsilon \int \operatorname{sign}(w^* - w) \Big[\{g(w^*) - g(w)\} G(u; w^*)\Big]_u du \\
&\qquad\qquad + \varepsilon \int \operatorname{sign}(w^* - w) \Big[g(w)\, \{G(u; w^*) - G(u; w)\}\Big]_u du \\
&\quad = \ \varepsilon \int \operatorname{sign}(w^* - w) \Big[g(w)\, \{G(u; w^*) - G(u; w)\}\Big]_u du \\
&\quad \leq \ \varepsilon \int \Big| g'(w) w_u \{G(u; w^*) - G(u; w)\} \Big| du + \varepsilon \int \Big| g(w)\, \{G(u; w^*)_u - G(u; w)_u\} \Big| du \\
&\quad \leq \ \varepsilon \|g'\|_{\mathbf{L}^\infty([-2, M+1])} \cdot \mathrm{TV}\{w\} \cdot \|G(\cdot; w^*) - G(\cdot; w)\|_{\mathbf{L}^\infty} \\
&\qquad\qquad + \varepsilon \|g(w(\cdot))\|_{\mathbf{L}^\infty} \cdot \mathrm{TV}\{G(\cdot; w^*) - G(\cdot; w)\}.
\end{aligned} \tag{2.34}
$$

The definition of G at (2.5) and the bound $G \leq e^\kappa$ imply

$$
\|G(\cdot; w) - G(\cdot; w^*)\|_{\mathbf{L}^\infty} \ \leq \ e^\kappa \cdot \big\|g'\big\|_{\mathbf{L}^\infty([-2, M+1])} \|w - w^*\|_{\mathbf{L}^1}. \tag{2.35}
$$

Moreover,

$$
\begin{aligned}
\mathrm{TV}\Big\{G(u; w) - G(u; w^*)\Big\} \ &\leq \ \int \Big| G(u; w) g(w) - G(u; w^*)\, g(w^*) \Big| du \\
&\leq \ \int G(u; w) \Big| g(w) - g(w^*) \Big| du + \int \Big| G(u; w) - G(u; w^*) \Big| \cdot |g(w^*)|\, du \\
&\leq \ \|G(\cdot; w)\|_{\mathbf{L}^\infty} \ \big\|g'\big\|_{\mathbf{L}^\infty([-2,\, M+1])} \|w - w^*\|_{\mathbf{L}^1} \\
&\qquad + \|g(w^*(\cdot))\|_{\mathbf{L}^1} \cdot \|G(\cdot; w) - G(\cdot; w^*)\|_{\mathbf{L}^\infty} \\
&\leq \ e^\kappa \ \big\|g'\big\|_{\mathbf{L}^\infty([-2,\, M+1])} \|w - w^*\|_{\mathbf{L}^1} \\
&\qquad + \big\|g'\big\|_{\mathbf{L}^\infty([-2,\, M+1])} \|w^* - 1\|_{\mathbf{L}^1} \cdot e^\kappa \ \big\|g'\big\|_{\mathbf{L}^\infty([-2, M+1])} \|w - w^*\|_{\mathbf{L}^1} \\
&= \ e^\kappa \ \big\|g'\big\|_{\mathbf{L}^\infty([-2, M+1])} (1 + \kappa) \cdot \|w - w^*\|_{\mathbf{L}^1}.
\end{aligned} \tag{2.36}
$$

Using (2.35) and (2.36) in (2.34) we obtain an estimate of the form

$$
(1 - \varepsilon C)\ \|w - w^*\|_{\mathbf{L}^1} \ \leq \ \|z - z^*\|_{\mathbf{L}^1}
$$

for a suitable constant C, depending on g and on the constant M, but not on ε.

(viii) - Entropy inequality. Let $c > 0$ be an arbitrary constant, and let $\psi \in \mathcal{C}_c^\infty$ be a positive test function. From (2.11) it follows

$$
w - c \ = \ z - c + \varepsilon \Big\{ \big[g(w) - g(c)\big] G(u; w) + \lambda(w - c) \Big\}_u - \varepsilon g(c) g(w) G(u; w). \tag{2.37}
$$

Note that the condition (2.9) implies

$$
\operatorname{sign}(w - c) \ = \ \operatorname{sign}\big[(g(w) - g(c)) G(u; w) + \lambda(w - c)\big]. \tag{2.38}
$$

Multiplying (2.37) by $\operatorname{sign}(w-c)\cdot\psi$ and integrating in u, we obtain

$$\int \frac{|w-c|-|z-c|}{\varepsilon}\,\psi\,du \;\;\leq\;\; \int \operatorname{sign}(w-c)\Big[(g(w)-g(c))G(u;w)+\lambda(w-c)\Big]_u\psi\,du$$
$$-\int \operatorname{sign}(w-c)g(c)g(w)G(u;w)\psi\,du\,.$$

By using integration by parts on the first integral on the right-hand side, and then applying (2.38), we obtain (2.15). This completes the proof. □

3 A projection operator

The backward Euler step maps a positive function $z\in\mathcal{D}_0^+$ to a function $w=E_\varepsilon^- z\in\mathcal{D}_0$ which may also take negative values. We now introduce a projection operator π, mapping D_0 back into D_0^+, and determine some of its properties. For notational convenience, in this section by $f\in\mathbf{L}^1_{loc}$ we denote a generic function, not to be confused with the erosion function.

Consider the sets

$$X \;\doteq\; \Big\{f\in\mathbf{L}^1_{loc}(I\!\!R)\,;\quad \lim_{|x|\to\infty} f(x)=1,\quad \|f(\cdot)-1\|_{\mathbf{L}^1}\leq M\Big\} \tag{3.1}$$

and

$$X^+ \;\doteq\; \{f\in X\,;\quad f(x)\geq 0\}\,. \tag{3.2}$$

For a given $f\in X$, define

$$F(x) \;\doteq\; \int_0^x\int_0^y f(s)\,ds\,dy\,. \tag{3.3}$$

Notice that this implies

$$F'(x) \;\doteq\; \int_0^x f(s)\,ds\,, \tag{3.4}$$

hence F' is absolutely continuous and

$$F'' \;=\; f(x) \qquad \text{for a.e. } x\,. \tag{3.5}$$

Let F_* be the lower convex envelope of F, namely

$$F_*(x) \;\doteq\; \min\,\Big\{\theta f(a)+(1-\theta)f(b)\,;\quad \theta\in[0,1]\,,\;\; x=\theta a+(1-\theta)b\Big\}\,. \tag{3.6}$$

For $f\in X$, we denote by

$$K_f \;\doteq\; \Big\{x\in I\!\!R\,;\;\; F_*(x)=F(x)\Big\} \tag{3.7}$$

the closed set where F coincides with its lower convex envelope. We observe that

$$\begin{aligned} K_f \;&=\; \Big\{x\in I\!\!R\,;\;\; F(y)-F(x)-(y-x)F'(x)\geq 0 \quad\text{for all}\;\; y\in I\!\!R\Big\}\\ &=\; \Big\{x\in I\!\!R\,;\;\; \int_x^y\int_x^s f(r)\,dr\,ds\geq 0 \quad\text{for all}\;\; y\in I\!\!R\Big\}\,. \end{aligned} \tag{3.8}$$

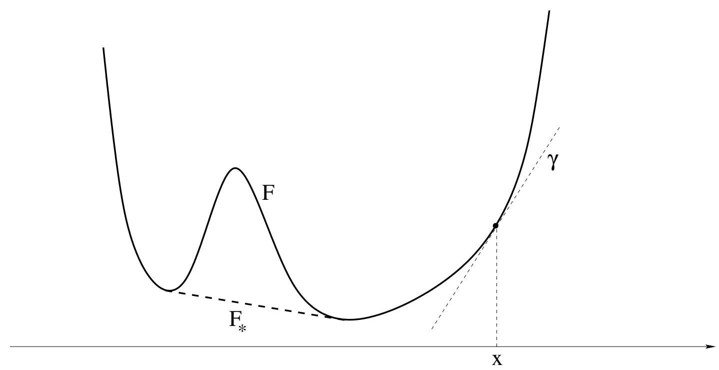

Figure 2: A function F and its lower convex envelope F_*. One has $F_*(x) = F(x)$ if and only there exists a line γ supporting the graph of F at the point x. This is the case if and only if $x \in K_f$.

Moreover, the assumption $\lim_{x\to\pm\infty} f(x) = 1$ implies $F''(x) = f(x) > 1/2$ whenever $|x|$ is sufficiently large. Hence the complement of K_f is a bounded open set, possibly empty.

The projection operator $\pi : X \mapsto X^+$ is now defined by setting

$$\pi f(x) \doteq F_*''(x) = \begin{cases} f(x) & \text{if } x \in K_f\,, \\ 0, & \text{if } x \notin K_f\,. \end{cases} \tag{3.9}$$

Since F_* is convex, its second derivative is non-negative. Hence $\pi f \in X^+$.

The next lemma collects the main properties of this operator.

Lemma 3.1. *Let $\pi : X \mapsto X^+$ be the operator defined at (3.9). Then the following holds.*

(i) $\pi f = f$ *for every* $f \in X^+$.

(ii) For any $a, b \in K_f$ *one has*

$$\int_a^b \pi f(x)\,dx = \int_a^b f(x)\,dx\,, \tag{3.10}$$

$$\int_a^b \int_a^x \pi f(y)\,dy\,dx = \int_a^b \int_a^x f(y)\,dy\,dx\,. \tag{3.11}$$

Moreover

$$\int_a^\xi \int_a^x \pi f(y)\,dy\,dx \le \int_a^\xi \int_a^x f(y)\,dy\,dx \qquad \textit{for all } \xi \in I\!R\,. \tag{3.12}$$

(iii) (monotonicity) If $f, g \in X$ *and* $f(x) \le g(x)$ *for a.e.* x*, then*

$$\pi f(x) \le \pi g(x) \qquad \textit{for a.e. } x, \tag{3.13}$$

and

$$\|\pi g - \pi f\|_{\mathbf{L}^1} = \|g - f\|_{\mathbf{L}^1}\,. \tag{3.14}$$

(iv) *($\mathbf{L}^1$-contractivity) For any $f, g \in X$, we have*

$$\|\pi f - \pi g\|_{\mathbf{L}^1} \;\leq\; \|f - g\|_{\mathbf{L}^1}\,. \tag{3.15}$$

In particular,

$$\|\pi f - 1\|_{\mathbf{L}^1} \;\leq\; \|f - 1\|_{\mathbf{L}^1}\,. \tag{3.16}$$

(v) *(BV stability) For any $f \in X$ having bounded total variation, one has*

$$TV\{\pi f\} \;\leq\; TV\{f\}. \tag{3.17}$$

Proof. *(i)* If $f \in X^+$, then F is convex. Hence $F_* = F$ and $\pi f = f$.
(ii). The assumption $a, b \in K_f$ implies

$$F(a) = F_*(a), \qquad F(b) = F_*(b), \qquad F'(a) = F'_*(a), \qquad F'(b) = F'_*(b).$$

Therefore

$$\int_a^b \pi f(x)\,dx \;=\; \int_a^b F''_*(x)\,dx \;=\; F'_*(b) - F'_*(a) \;=\; F'(b) - F'(a) \;=\; \int_a^b f(x)\,dx\,,$$

proving (3.10). Next, still for $a, b \in K_f$ we have

$$\begin{aligned}
\int_a^b\int_a^x \pi f(y)\,dy\,dx \;&=\; \int_a^b\int_a^x F''_*(y)\,dy\,dx \;=\; \int_a^b \big(F'_*(x) - F'_*(a)\big)\,dx\\
&=\; F_*(b) - F_*(a) - (b-a)F'_*(a) \;=\; F(b) - F(a) - (b-a)F'(a)\\
&=\; \int_a^b\int_a^x f(y)\,dy\,dx\,.
\end{aligned}$$

This proves (3.11). Finally, for any $\xi \in \mathbb{R}$ the inequality (3.12) follows from

$$\begin{aligned}
&\int_a^\xi\int_a^x (\pi f(y) - f(y))\,dy\,dx \;=\; \int_a^\xi\int_a^x \big(F''_*(y) - F''(y)\big)\,dy\,dx\\
&=\; \int_a^\xi \big[\big(F'_*(x) - F'(x)\big) - \big(F'_*(a) - F'(a)\big)\big]\,dx \;=\; \int_a^\xi \big(F'_*(x) - F'(x)\big)\,dx\\
&=\; (F_*(\xi) - F(\xi)) - (F_*(a) - F(a)) \;=\; F_*(\xi) - F(\xi) \;\leq\; 0.
\end{aligned}$$

(iii). If $f \leq g$ a.e., then

$$\int_x^y\int_x^s f(r)\,dr\,ds \leq \int_x^y\int_x^s g(r)\,dr\,ds, \qquad \text{for all} \;\; x, y \in \mathbb{R}.$$

Hence, by the characterization (3.8), the corresponding sets satisfy $K_f \subseteq K_g$.

To prove (3.13) we consider two cases. If $x \in K_f$, then $x \in K_g$, hence $\pi f(x) = f(x) \leq g(x) = \pi g(x)$. Otherwise, if $x \notin K_f$, then $\pi f(x) = 0 \leq \pi g(x)$.

To prove (3.14), we choose an interval $[a,b]$ so large that $[a,b]\cup K_f = [a,b]\cup K_g = I\!\!R$. Since $\pi f(x) \le \pi g(x)$ for all x, using (3.10) we obtain

$$\begin{aligned}\|\pi g - \pi f\|_{\mathbf{L}^1} &= \int_{I\!\!R\setminus[a,b]} [\pi g(x) - \pi f(x)]\,dx + \int_a^b [\pi g(x) - \pi f(x)]\,dx \\ &= \int_{I\!\!R\setminus[a,b]} [g(x) - f(x)]\,dx + \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \;=\; \|g-f\|_{\mathbf{L}^1}\,.\end{aligned}$$

(iv) Let $f,g\in X$, and denote

$$f\vee g \doteq \max\{f,g\}, \qquad f\wedge g \doteq \min\{f,g\}\,.$$

Since the operator π preserves the ordering, for every $x\in I\!\!R$ we have

$$|\pi f(x) - \pi g(x)| \;\le\; \pi(f\vee g)(x) - \pi(f\wedge g)(x).$$

Then, (3.15) follows because

$$\begin{aligned}\|\pi f - \pi g\|_{\mathbf{L}^1} &\le \int_{I\!\!R} \Big[\pi(f\vee g)(x) - \pi(f\wedge g)(x)\Big]\,dx \\ &= \int_{I\!\!R} \Big[(f\vee g)(x) - (f\wedge g)(x)\Big]\,dx \;=\; \|f-g\|_{\mathbf{L}^1}\,.\end{aligned}$$

Finally, by taking $g\equiv 1$ in (3.15), we obtain (3.16).
(v). Since the projection π commutes with translations, using the contractivity property (3.15) one obtains the estimate

$$\begin{aligned}\mathrm{TV}\{\pi f\} &= \limsup_{h\to 0+} \frac{1}{h}\int_{-\infty}^{\infty} |\pi f(x+h) - \pi f(x)|\,dx \\ &\le \limsup_{h\to 0+} \frac{1}{h}\int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx \;=\; \mathrm{TV}\{f\}\,,\end{aligned}$$

completing the proof. □

We now study how the projection operator behaves in connection with a family of convex entropies. For $f\in X$, define the function

$$\Theta^f(x) \doteq \int_{-\infty}^{x} [\pi f(y) - f(y)]\,dy, \tag{3.18}$$

so that Θ^f is an absolutely continuous function which vanishes for $|x|$ large and satisfies

$$\Theta^f_x(x) \;=\; \pi f(x) - f(x)\,. \tag{3.19}$$

The following properties of Θ^f follow immediately from Lemma 3.1.

Lemma 3.2. *Let $f\in X$, and assume $a,b\in K_f$. Then*

$$\Theta^f(a) \;=\; \Theta^f(b) \;=\; 0, \qquad \int_a^b \Theta^f(y)\,dy \;=\; 0, \tag{3.20}$$

and

$$\int_a^x \Theta^f(y)\,dy \;\le\; 0 \qquad \textit{for all} \quad x\in I\!\!R. \tag{3.21}$$

The next lemma shows that the projection operator is dissipative w.r.t. a family of convex entropies.

Lemma 3.3. *Let $f \in X$. For any constant $c > 0$ and any non-negative test function $\psi \in \mathcal{C}_c^\infty(I\!\!R)$ one has*

$$\begin{aligned}\int_{I\!\!R} |\pi f(x) - c|\,\psi(x)\,dx \;&\leq\; \int_{I\!\!R} |f(x) - c|\,\psi(x)\,dx \\ &\quad - \int_{I\!\!R} \mathrm{sign}(\pi f(x) - c)\Theta^f(x)\,\psi_x(x)\,dx.\end{aligned} \tag{3.22}$$

Proof. By (3.19) we have

$$\pi f(x) - c \;=\; f(x) - c + \Theta_x^f(x).$$

Multiplying both sides by $\mathrm{sign}(\pi f(x) - c)\cdot\psi$ and integrating over $I\!\!R$, we obtain

$$\int_{I\!\!R} |\pi f(x) - c|\,\psi(x)\,dx \;\leq\; \int_{I\!\!R} |f(x) - c|\,\psi(x)\,dx + \int_{I\!\!R} \mathrm{sign}(\pi f(x) - c)\,\Theta_x^f\psi(x)\,dx\,,$$

To handle the last term, we observe that Θ^f is supported on the region where $\pi f = 0$, hence $\mathrm{sign}(\pi f(x) - c) = -1$. An integration by parts yields

$$\begin{aligned}\int_{I\!\!R} \mathrm{sign}(\pi f(x) - c)\Theta_x^f\psi(x)\,dx \;&=\; -\int_{I\!\!R} \Theta_x^f(x)\,\psi(x)\,dx \\ &=\; -\int_{I\!\!R} \mathrm{sign}(\pi f(x) - c)\Theta^f(x)\,\psi_x(x)\,dx\,,\end{aligned}$$

completing the proof. □

4 Approximate solutions by a flux-splitting algorithm

Combining the backward Euler operator and the projection operator introduced in the previous sections, we now construct a family of approximate solutions. Let an initial data $z(0,\cdot) = \bar{z} \in \mathcal{D}_0$ be given. Fix a time step $\varepsilon > 0$ and let $z_\varepsilon : I\!\!R \mapsto \mathcal{D}_0$ be the unique function such that

$$\begin{cases} z_\varepsilon(t) \;=\; \bar{z} & \text{if } t \leq 0\,, \\ z_\varepsilon(t) \;=\; \pi\left(E_\varepsilon^- z_\varepsilon(t - \varepsilon)\right) & \text{if } t > 0\,. \end{cases} \tag{4.1}$$

Here E_ε^- is the backward Euler step introduced in Definition 1, and π is the projection operator defined at (3.9). It is understood that, in the construction of E_ε^-, we choose $\lambda > 0$ sufficiently large so that (2.9) holds. Thanks to Lemma 2.2, the constant λ can be chosen as in (2.13), depending on the initial data $\bar{z}$ but not on ε.

Notice that $z_\varepsilon(\cdot)$ can be constructed through discrete time iterations. Consider the times

$$t_k \;\doteq\; k\varepsilon\,, \qquad\qquad k = 0, 1, 2, \ldots$$

We begin by setting

$$z^0(u) \;=\; \bar z(u). \tag{4.2}$$

Next, given $z^k(\cdot) = z_\varepsilon(t_k,\cdot)$, the function $z^{k+1} = z_\varepsilon(t_{k+1},\cdot)$ is computed by setting

$$w^{k+1} \;=\; E_\varepsilon^- z^k\,, \qquad\qquad z^{k+1} \;=\; \pi w^{k+1}\,, \tag{4.3}$$

The solution of (4.1) is then

$$z_\varepsilon(t,u) \;=\; z^k(u), \qquad \text{if} \;\; t\in [t_k, t_{k+1}[\,. \tag{4.4}$$

To study the projection operator at every time step t_k, it is convenient to introduce the functions

$$\Theta^k(u) \doteq \frac{1}{\varepsilon}\int_{-\infty}^u \Big[z^k(u) - w^k(u)\Big]\, du\,, \qquad \Theta^k_u(u) = \frac{1}{\varepsilon}\Big[z^k(u) - w^k(u)\Big]\,, \tag{4.5}$$

and

$$\Theta^\varepsilon(t,u) \;=\; \Theta^k(u), \qquad \Theta^\varepsilon_u(t,u) \;=\; \Theta^k_u(u) \qquad \text{if} \;\; t\in [t_k, t_{k+1}[\,. \tag{4.6}$$

Combining the properties of the backward Euler operator proved in Lemma 2.2 and the properties of the projection operator in Lemma 3.1 and Lemma 3.3, we obtain similar estimates for z_ε.

Lemma 4.1. *Consider initial data $\bar z, \bar z^* \in \mathcal{D}_0$ where $\mathcal{D}_0$ is defined in* (2.3)*, and let*

$$M \;\doteq\; \max\,\Big\{\,\|\bar z - 1\|_{\mathbf{L}^1}\,,\;\; \|\bar z^* - 1\|_{\mathbf{L}^1}\,,\;\; TV\{\bar z\}\,,\;\; TV\{\bar z^*\}\,\Big\}.$$

Let $z_\varepsilon(t,u)$, $z^*_\varepsilon(t,u)$ be the corresponding solutions of (4.1), with λ chosen as in (2.13). Then, for every $\varepsilon > 0$ sufficiently small and every $t \geq 0$, the following estimates hold.

(i) $\sup_u\{z_\varepsilon(t,u)\} \leq \sup_u\{\bar z(u)\}$.

(ii) $\|z_\varepsilon(t,\cdot) - 1\|_{\mathbf{L}^1} \leq \|\bar z - 1\|_{\mathbf{L}^1} \leq M$.

(iii) $0 < C^{-1} \leq G\,(u; z_\varepsilon(t)) \leq C$.

(iv) $TV\{z_\varepsilon(t,\cdot)\} \leq e^{Ct}\, TV\{\bar z\}$.

(v) $\|z_\varepsilon(t,\cdot) - z^*_\varepsilon(t,\cdot)\|_{\mathbf{L}^1} \leq e^{Ct}\cdot \|\bar z - \bar z^*\|_{\mathbf{L}^1}$.

(vi) $\|z_\varepsilon(t_j,\cdot) - z_\varepsilon(t_i,\cdot)\|_{\mathbf{L}^1} \;\leq\; C\, e^{Ct_j}\, |t_j - t_i|$ *for any integers* $0 \leq i < j$.

(vii) (Kruzhkov entropy inequality) For any $c > 0$ and any positive test function $\psi \in \mathcal{C}^\infty_c(]0,T[\times I\!R)$, we have

$$\begin{aligned}\iint |z_\varepsilon - c|\,\psi_t\, du\, dt \;\;\geq\;\; & \iint |(g(z_\varepsilon) - g(c))G(u;z_\varepsilon) + \lambda(z_\varepsilon - c)|\,\psi_u\, du\, dt \\ & + \iint \operatorname{sign}(z_\varepsilon - c)\cdot g(c)g(z_\varepsilon)G(u;z_\varepsilon)\psi\, du\, dt \\ & + \iint \operatorname{sign}(z_\varepsilon - c)\cdot \Theta^\varepsilon \psi_u\, du dt - C\varepsilon. \end{aligned} \tag{4.7}$$

Here C is a suitable constant independent of ε.

(viii) *(Θ^ε has uniformly bounded support) For any given $T, R_0 > 0$ there exist $R, \delta > 0$ such that the following holds. If $\bar z(u) \geq 1-\delta$ for $|u| > R_0$, then for every $\varepsilon > 0$ small enough one has*

$$z^\varepsilon(t,u) > \frac{1}{2} \qquad \textit{for all} \quad t \in [0,T], \quad |u| > R. \tag{4.8}$$

In particular, the support of Θ^ε is contained in $[0,T] \times [-R,R]$.

Here the constant C depends only on M and on the function g, while K depends on M, g, c, T, and on $\|\psi\|_{\mathcal{C}^2}$. The constants R, δ depend on M, g, c, T, and on R_0.

Proof. **1.** The properties *(i)–(iv)* are straightforward consequences of the corresponding properties in Lemma 2.2 for the backward Euler step and in Lemma 3.1 for the projection operator.

2. To prove *(v)*, we observe that the (2.14) and the contraction property of the projection π imply

$$\|z_\varepsilon(t_{k+1}) - z^*_\varepsilon(t_{k+1})\|_{\mathbf{L}^1} \;\leq\; \frac{1}{1-C\varepsilon}\, \|z_\varepsilon(t_k) - z^*_\varepsilon(t_k)\|_{\mathbf{L}^1}\ ,$$

for some constant C depending only on M and for any $\varepsilon < C^{-1}$. By induction on $k = 0,1,2\ldots$ we obtain (v), with a possibly different constant C.

3. To prove *(vi)* we observe that, by (2.32),

$$\left\|E^-_\varepsilon z^k - z^k\right\|_{\mathbf{L}^1} \;\leq\; \varepsilon C_0 \cdot \mathrm{TV}\left\{z^k\right\}. \tag{4.9}$$

Here C_0 is a constant depending only on $\sup_u z^k(u)$ and on $\left\|z^k - 1\right\|_{\mathbf{L}^1}$. In addition, since $z^k = \pi z^k$ and π is a contraction, we have

$$\begin{aligned}\left\|\pi(E^-_\varepsilon z^k) - E^-_\varepsilon z^k\right\|_{\mathbf{L}^1} \;&\leq\; \left\|\pi(E^-_\varepsilon z^k) - \pi z^k\right\|_{\mathbf{L}^1} + \left\|\pi z^k - E^-_\varepsilon z^k\right\|_{\mathbf{L}^1}\\ &\leq\; 2\left\|E^-_\varepsilon z^k - z^k\right\|_{\mathbf{L}^1}.\end{aligned} \tag{4.10}$$

Putting together (4.9)-(4.10) and using the estimate (iv) on the total variation of $z^k = z(t_k)$, we obtain

$$\left\|z^{k+1} - z^k\right\|_{\mathbf{L}^1} = \left\|\pi(E^-_\varepsilon z^k) - z^k\right\|_{\mathbf{L}^1} \leq 3\varepsilon C_0 \cdot \mathrm{TV}\left\{z^k\right\} \leq 3\varepsilon C_0 \cdot e^{Ct_k}\,\mathrm{TV}\{\bar z\}. \tag{4.11}$$

We now write

$$\|z_\varepsilon(t_j,\cdot) - z_\varepsilon(t_i,\cdot)\|_{\mathbf{L}^1} \;\leq\; \sum_{k=i}^{j-1} \left\|z^{k+1} - z^k\right\|_{\mathbf{L}^1}$$

and use (4.11) to estimate each term. This yields *(vi)*, for a suitable constant C and all $\varepsilon > 0$ sufficiently small.

4. To prove the Kruzhkov entropy inequality, we use property *(viii)* in Lemma 2.2 with $z = z^k = z_\varepsilon(t_k,\cdot)$, $w = w^{k+1}$, and then Lemma 3.3 with $f = w^{k+1}$, $\pi f = z^{k+1} = z_\varepsilon(t_{k+1},\cdot)$. Choose N so large that $T < N\varepsilon$. By assumption, the test function ψ

vanishes for $t = 0$ and for $t \geq N\varepsilon$. Summing over $k = 0, \ldots, N-1$, by a standard summation-by-parts technique we obtain

$$
\begin{aligned}
& \sum_{k=0}^{N} \varepsilon \int \left| z^{k+1}(u) - c \right| \cdot \frac{\psi(t_{k+1}, u) - \psi(t_k, u)}{\varepsilon} \, du \\
= & \sum_{k=0}^{N-1} \int \left(\left| z^k(u) - c \right| - \left| z^{k+1}(u) - c \right| \right) \psi(t_k, u) \, du \\
= & \sum_{k=0}^{N} \int \Bigg[\left(\left| z^k(u) - c \right| - \left| w^{k+1}(u) - c \right| \right) \\
& \qquad + \left(\left| w^{k+1}(u) - c \right| - \left| z^{k+1}(u) - c \right| \right) \Bigg] \psi(t_k, u) \, du \\
\geq & \sum_{k=0}^{N} \varepsilon \int \left| \left(g\left(w^{k+1}(u)\right) - g(c) \right) G\left(u; w^{k+1}\right) + \lambda \left(w^{k+1}(u) - c \right) \right| \psi_u(t_k, u) \, du \\
& \qquad + \sum_{k=0}^{N} \varepsilon \int \operatorname{sign}\left(w^{k+1}(u) - c \right) \cdot g(c) g\left(w^{k+1}(u) \right) G\left(u; w^{k+1}\right) \psi(t_k, u) \, du \\
& \qquad + \sum_{k=0}^{N} \varepsilon \int \operatorname{sign}\left(z^{k+1}(u) - c \right) \cdot \Theta^{\varepsilon}(t_{k+1}, u) \, \psi_u(t_k, u) \, du.
\end{aligned}
\tag{4.12}
$$

As $\varepsilon \to 0$, the difference between the left hand side of (4.7) and the left hand side of (4.12) is bounded by a constant multiple of ε. Similarly, comparing the right hand side of (4.7) (with $C = 0$) with the right hand side of (4.12), we see that the difference is again bounded by a constant multiple of ε. Therefore the inequality in (4.7) follows from (4.12), for a suitable constant C.

5. The bound (4.8) will be established by a comparison argument. For every $t = t_k \doteq k\varepsilon$ we will prove by induction that $z(t, \cdot)$ satisfies bounds of the form

$$
z(t,u) \geq \begin{cases} \xi(t) & \text{if } u > R(t), \\ 0 & \text{if } u \in [-R(t),\, R(t)], \\ \eta(t,u) & \text{if } u < -R(t). \end{cases}
\tag{4.13}
$$

See Figure 3. Here the functions $t \mapsto \xi(t)$, $t \mapsto R(t)$ and $\eta(t,u)$ are defined as

$$
\xi(t) \doteq 1 - \delta - 2\delta t\,, \qquad R(t) \doteq R_0 + Mt\,, \qquad \eta(t,u) \doteq 1 - \delta e^{2Ct} - e^{\frac{u+R(t)}{2\epsilon}}.
\tag{4.14}
$$

The value of δ will be chosen sufficiently small, as specified later.

At $t = 0$, the bounds (4.13) hold by assumption. Now assume that, at time $t = t_k$, $z(t, \cdot)$ satisfies the bounds (4.13). We will show that (4.13) holds at $t = t_{k+1} = t + \varepsilon$. On the interval $u \in [-R(t_{k+1})\,,\ R(t_{k+1})]$, (4.13) holds trivially, because $z^{k+1}(u) \geq 0$. Next, consider the half line $\{u > R(t_{k+1})\}$. We claim that

$$
w^{k+1}(u) \geq \xi(t_k) - 2\varepsilon\delta = \xi(t_{k+1}) \qquad \text{for all } u > R(t_k)\,.
\tag{4.15}
$$

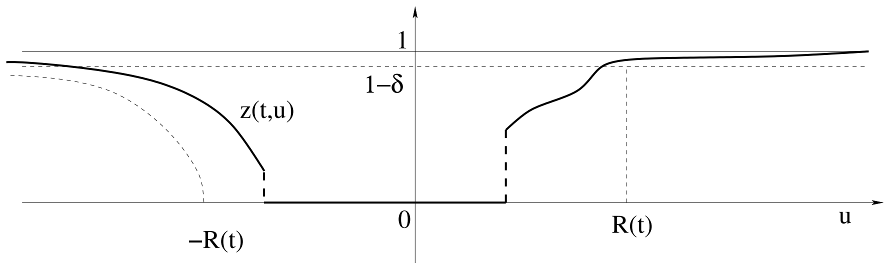

Figure 3: The lower estimates on a flux splitting approximation $z(t,\cdot)$.

Indeed, if $w^{k+1}(u)$ achieves a local min at u^*, one then has $w_u^{k+1}(u^*) = 0$ if $u^* > R(t_k)$, while $w_u^{k+1}(u^*) \geq 0$ if $u^* = R(t_k)$. Then, from (2.16) we obtain

$$w^{k+1}(u^*) \;\geq\; z^k(u^*) - \varepsilon G_{\max} L^2 \left(1 - w^{k+1}(u^*)\right)^2 ,$$

which is equivalent to

$$\left[1 - w^{k+1}(u^*)\right] \leq \left[1 - z^k(u^*)\right] + \varepsilon G_{\max} L^2 \left[1 - w^{k+1}(u^*)\right]^2 , \tag{4.16}$$

where $G_{\max}$ provides an an upper bound on G and L is a constant strictly larger than the Lipschitz constant of g. We compare (4.16) to the problem

$$b = a + \varepsilon M b^2, \quad (0 < 4\varepsilon M a \leq 0.5), \qquad b = \frac{1 - \sqrt{1 - 4\varepsilon M a}}{2\varepsilon M} < a + 2\varepsilon M a^2,$$

where in the last inequality we used the relation $1 - \sqrt{1-x} < \frac{1}{2}x + \frac{1}{4}x^2$ for $0 < x < 0.5$. By a standard comparison argument, choosing ε and δ sufficiently small such that

$$4\varepsilon G_{\max} L \leq \frac{1}{2}, \qquad \delta \cdot G_{\max} L^2 (1 + 2T)^2 \leq \frac{1}{2} \tag{4.17}$$

we have

$$\left[1 - w^{k+1}(u^*)\right] \leq \left[1 - z^k(u^*)\right] + 2\varepsilon G_{\max} L^2 \left[1 - z^k(u^*)\right]^2 .$$

Applying the assumption $\left[1 - z^k(u^*)\right] \leq \delta(1 + 2t)$ and the condition (4.17), we get

$$\left[1 - w^{k+1}(u^*)\right] \leq \delta(1+2t) + 2\varepsilon G_{\max} L^2 \delta^2 (1+2t)^2 \leq \delta(1+2t) + 2\varepsilon\delta = 1 - \xi(t_k + \varepsilon),$$

proving (4.15). The projection operator could move the support of Θ^ε further to the right. Thanks to the properties (ii) and (vi) in Lemma 2.2, we have the estimate

$$R\left(t_{k+1}\right) - R\left(t_k\right) \;\leq\; \frac{M C_0 \varepsilon}{1 - \xi\left(t_{k+1}\right)} \;\leq\; 2 M C_0 \varepsilon.$$

Finally, we consider the half line $u < -R(t_{k+1})$. Suppose $z^k(u) \geq \eta(t_k, u)$, we first show that, for some constant $\widetilde{M}$ it holds

$$w^{k+1}(u) \;\geq\; \tilde{\eta}(u) \doteq 1 - \delta e^{2C(t_k+\varepsilon)} - e^{\left(u + R(t_k) + \widetilde{M}\varepsilon\right)/(2\varepsilon)} . \tag{4.18}$$

From (2.16) and property (vi) in Lemma 2.2, we have

$$\left[w^{k+1}\right]_u \leq \frac{w^{k+1}-z^k}{\varepsilon} + C_3\left(1-w^{k+1}\right), \qquad w^{k+1}\left(R(t_k)\right) \geq -C_0\varepsilon. \tag{4.19}$$

We proceed by contradiction. Assume that (4.18) fails and let $u^\sharp < R(t_k)$ be the right-most point where the equality holds:

$$u^\sharp \doteq \max\left\{u \leq R(t_k):\ w^{k+1} \ = \ \tilde{\eta}(u)\right\}.$$

This yields a contradiction provided that

$$w_u^{k+1}\left(u^\sharp\right) \ < \ \tilde{\eta}'\left(u^\sharp\right) \ = \ -\frac{1}{2\varepsilon}e^{\left[u^\sharp+R(t_k)+M\varepsilon\right]/(2\varepsilon)}. \tag{4.20}$$

Using (4.19), at $u = u^\sharp$ we have

$$\begin{aligned}
w_u^{k+1}\left(u^\sharp\right) &\leq \frac{\tilde{\eta}\left(u^\sharp\right)-\eta(t_k,u^\sharp)}{\varepsilon} + C\left(1-\tilde{\eta}\left(u^\sharp\right)\right)\\
&\leq -\frac{1}{\varepsilon}\left[\delta e^{2Ct_k}\left(e^{2C\varepsilon}-1\right) + e^{\left[u^\sharp+R(t_k)+\widetilde{M}\varepsilon\right]/(2\varepsilon)}\left(1-e^{-M/2}\right)\right]\\
&\quad +C\left(\delta e^{2Ct_{k+1}} + e^{\left[u^\sharp+R(t_k)+\widetilde{M}\varepsilon\right]/(2\varepsilon)}\right)\\
&= -\delta e^{2Ct_k}\left(\frac{e^{2C\varepsilon}-1}{\varepsilon} - Ce^{2C\varepsilon}\right) - \frac{1-e^{-\frac{M}{2}}-C\varepsilon}{\varepsilon}\, e^{\left[u^\sharp+R(t_k)+\widetilde{M}\varepsilon\right]/(2\varepsilon)}.
\end{aligned}$$

Here the first term is negative, and the constant in the second term is bounded by

$$-\frac{1-e^{-M/2}-C\varepsilon}{\varepsilon} < -\frac{1}{2\varepsilon}$$

for ε sufficiently small and $\widetilde{M}$ sufficiently large, thus (4.20) holds, providing the contradiction. We choose δ small such that $\delta e^{2CT} < 1/3$. By the property of the exponential function, there exist constants $\widetilde{C}$ and C', such that

$$\int_{-C'\varepsilon}^{0}\left(2/3 - e^{-u/(2\varepsilon)}\right) du \ = \ C\varepsilon, \qquad \tilde{\eta}(u) \ > \ \frac{1}{3}, \qquad \left(u \leq -R(t_k) - \widetilde{M}\varepsilon - \widetilde{C}\varepsilon\right).$$

The projection step will push $-R(t_k) - \widetilde{M}\varepsilon$ further to the left. Thanks the properties (ii) and (vi) in Lemma 2.2, and the properties of the exponential function, we have, $z^{k+1}(u) \geq \tilde{\eta}(u)$ for $u \leq -R(t_k) - \widetilde{M}\varepsilon - C'\varepsilon$. Finally, letting $M = \widetilde{M} + C'$, we conclude that $z^{k+1} \geq \eta(t_{k+1}, u)$ for $u \leq -R(t_{k+1})$, completing the inductive step. □

5 A semigroup of weak solutions

Taking a sequence of flux-splitting approximations, as the time step $\varepsilon \to 0$, in the limit we expect to recover a semigroup of weak solutions. Before stating the main result in this direction, we give a precise definition of entropy weak solution.

Definition 5.1. *Given a time interval* $[0,T]$*, an entropy weak solution to the Cauchy problem* (1.11) *is a bounded, measurable function* $z = z(t,u) \geq 0$ *with the following properties.*

(P1) The map $t \mapsto z(t,\cdot)$ *is continuous from* $[0,T]$ *into* $\mathbf{L}^1_{loc}(\mathbb{R})$*. Moreover,* $z(0,u) - \bar{z}(u) = 0$*.*

(P2) There exist a measurable function $\Theta = \Theta(t,u)$ *with compact support in* $[0,T] \times \mathbb{R}$*, such that*

$$\begin{array}{rcl} z(t,u) \ > \ 0 & \Longrightarrow & \Theta(t,u) \ = \ 0 \\ z(t,a) > 0, \quad z(t,b) > 0 & \Longrightarrow & \displaystyle\int_a^b \Theta(t,u)\, du = 0\,. \end{array} \tag{5.1}$$

Moreover, for any constant $c \geq 0$ *and every non-negative test function* $\psi \in \mathcal{C}^\infty_c(]0,T[\times\mathbb{R})$*, the following entropy inequality holds:*

$$\begin{aligned} & -\int_0^T \int_{\mathbb{R}} |z-c|\, \psi_t \, du\, dt + \int_0^T \int_{\mathbb{R}} \mathrm{sign}(z-c)\cdot (g(z)-g(c))\, G(u;z(t))\, \psi_u \, du\, dt \\ & \leq \ -\int_0^T \int_{\mathbb{R}} \mathrm{sign}(z-c)\cdot g(c)\, g(z)\, G(u;z(t))\, \psi \, du\, dt \\ & \qquad -\int_0^T \int_{\mathbb{R}} \mathrm{sign}(z-c)\cdot \Theta(t,u)\, \psi_u \, du\, dt\,. \end{aligned} \tag{5.2}$$

Remark 5.2. Since the function Θ is supported on the set where $z = 0$, for $c > 0$ in (5.2) we always have

$$\mathrm{sign}(z-c)\cdot\Theta(t,u) = -\Theta(t,u).$$

Remark 5.3. According to (5.2), for every $c \geq 0$ the function $z = z(t,u)$ satisfies the inequality

$$|z-c|_t - \Big[\mathrm{sign}(z-c)\,(g(z)-g(c))\,G(u;z(t))\Big]_u - \Theta_u \ \leq \ 0 \tag{5.3}$$

in distributional sense, for some measurable function $\Theta(t,u)$ satisfying (5.1).

We now state the main result on the global existence of BV solutions to the Cauchy problem. Consider the domain

$$\begin{aligned} \mathcal{D}^+ \ \doteq \ \Big\{ z : \mathbb{R} \mapsto [0,\infty[\,; \quad & z \text{ is absolutely continuous and} \\ & \|z(\cdot)-1\|_{\mathbf{L}^1} < \infty, \quad \mathrm{TV}\{z(\cdot)\} < \infty \Big\}, \end{aligned} \tag{5.4}$$

and, for any $M > 0$, the subdomain

$$\mathcal{D}^M \ \doteq \ \Big\{ z \in \mathcal{D}^+ ; \quad \|z(\cdot)-1\|_{\mathbf{L}^1} < M \Big\}\,. \tag{5.5}$$

Theorem 5.4. *Let the function* g *satisfy the assumptions* (**A2**)*. Then for any* $M > 0$ *there exists a map* $S : \mathcal{D}^+_M \times [0,\infty[\, \mapsto \mathcal{D}^+$ *with the following properties.*

(i) For every $\bar{z} \in \mathcal{D}^M$, the trajectory $t \mapsto S_t\bar{z}$ is an entropy weak solution to the Cauchy problem (1.11) in the sense of Definition 5.1.

(ii) For any M', one can find a constant C such that, if

$$\bar{z}, \bar{z}^* \in \mathcal{D}^M, \qquad TV\{\bar{z}\} \leq M', \qquad TV\{\bar{z}^*\} \leq M',$$

then for all $t > s \geq 0$ one has

$$\|S_t\bar{z} - S_s\bar{z}\|_{\mathbf{L}^1} \leq Ce^{Ct}(t-s), \tag{5.6}$$
$$\|S_t\bar{z} - S_t\bar{z}^*\|_{\mathbf{L}^1} \leq Ce^{Ct}\|\bar{z} - \bar{z}^*\|_{\mathbf{L}^1}. \tag{5.7}$$

Proof. **1.** The domain $\mathcal{D}^M$ is a separable metric space, with the $\mathbf{L}^1$ distance. In particular, we can select a countable subset $\mathcal{D}^\sharp \subset \mathcal{D}^M \cap \mathcal{D}_0^+$ such that the following holds.

($\mathbf{P}^\sharp$) For every $\bar{z} \in \mathcal{D}^M$, there exists a sequence of elements $\bar{z}_n \in \mathcal{D}^\sharp$ such that

$$\|\bar{z}_n - \bar{z}\|_{\mathbf{L}^1} \to 0, \qquad \limsup_{n\to\infty} \mathrm{TV}\{\bar{z}_n\} \leq \mathrm{TV}\{\bar{z}\}. \tag{5.8}$$

2. Let the constants κ, λ be as in (2.13), depending on g and on the constant M. Let an initial condition $\bar{z} \in \mathcal{D}^\sharp$ be given. Consider a sequence $\varepsilon_n \to 0$. Let $t \mapsto z_{\varepsilon_n}(t)$ be the corresponding solutions to (4.1). Observe that, as t ranges over any compact interval $[0,T]$, the total variation of $z_{\varepsilon_n}(t,\cdot)$ remains uniformly bounded. Next, let $Z_{\varepsilon_n} : [0,\infty[\mapsto \mathcal{D}_0^+$ be the piecewise affine function which coincides with z_{ε_n} at the discrete times $t_k \doteq k\cdot\varepsilon_n$. Then the maps $t \mapsto Z_{\varepsilon_n}(t)$ are uniformly Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, on bounded intervals of time. By Helly's compactness theorem (see for example [5]), we can extract a subsequence $(\varepsilon_\nu)_{\nu\geq 1}$ such that the functions Z_{ε_ν} converge in $\mathbf{L}^1_{loc}$, and hence the same holds for the functions z_{ε_ν}. By a standard diagonalisation argument, we can assume that the same sequence $\varepsilon_\nu \to 0$ achieves convergence for every $\bar{z} \in \mathcal{D}^\sharp$.

If now $z_{\varepsilon_\nu} \to z$ in $\mathbf{L}^1_{loc}$, we define the function $S_t\bar{z}$ by setting

$$(S_t\bar{z})(u) = z(t, u - \lambda t). \tag{5.9}$$

3. For initial data $\bar{z}, \bar{z}^* \in \mathcal{D}^\sharp$, the estimates (5.6)-(5.7) follow from the corresponding estimates *(v)-(vi)* in Lemma 4.1. We can now extend the definition of S from $\mathcal{D}^\sharp \times [0,\infty[$ to the whole domain $\mathcal{D}^M \times [0,\infty[$, by continuity. Indeed, given $\bar{z} \in \mathcal{D}^M$, there exists a sequence of initial data $\bar{z}_n \in \mathcal{D}^\sharp$ such that (5.8) holds. We then define

$$S_t\bar{z} = \lim_{n\to\infty} S_t\bar{z}_n. \tag{5.10}$$

The inequalities (5.6)-(5.7), valid for $\bar{z}, \bar{z}^* \in \mathcal{D}^\sharp$, guarantee that the limit exists and is independent of the approximating sequence.

By continuity, the estimates (5.6)-(5.7) remain valid also for $\bar{z}, \bar{z}^*$ in the larger domain $\mathcal{D}^M$. This already proves part *(ii)* of the theorem.

4. It remains to prove that each trajectory $t \mapsto S_t\bar{z}$ is an entropy weak solution. We fix a $\bar{z} \in \mathcal{D}^M$. By choosing a further subsequence, we can assume the weak convergence

$\Theta^{\varepsilon_n} \rightharpoonup \widetilde{\Theta}$. By construction, the map $t \mapsto \tilde{z}(t)$ is Lipschitz continuous from $[0,T]$ into $\mathbf{L}^1(I\!\!R)$ and satisfies the initial condition $\tilde{z}(0) - \bar{z} = 0$. Moreover, by (vii) in Lemma 4.1, as $\varepsilon \to 0$ we have

$$\begin{aligned} &\int\!\!\int |\tilde{z} - c|\psi_t \, du\, dt - \int\!\!\int \Big|(g(\tilde{z}) - g(c))G(u;\tilde{z}) + \lambda(\tilde{z} - c)\Big|\psi_u \, du\, dt \\ &\geq \int\!\!\int \operatorname{sign}(\tilde{z} - c) \cdot g(c)g(\tilde{z})G(u;\tilde{z})\,\psi\, du\, dt + \int\!\!\int \operatorname{sign}(\tilde{z} - c) \cdot \widetilde{\Theta}\,\psi_u \, du\, dt, \end{aligned} \tag{5.11}$$

for any $c > 0$ and any positive test function $\psi \in \mathcal{C}_c^\infty(]0,T[\times I\!\!R)$. Defining

$$z(t,u) \doteq \tilde{z}(t, u + \lambda t), \quad \Theta(t,u) \doteq \widetilde{\Theta}(t, u + \lambda t) \tag{5.12}$$

we obtain an entropy weak solution of the Cauchy problem (1.11). □

Remark 5.5. We cannot construct the flow generated by the equation (1.11) simultaneously for all initial data $\bar{z} \in \mathcal{D}^+$. This is because the backward Euler approximations are defined for the conservation law (2.8), where the shift λ must be chosen large enough so that the characteristic speed is strictly negative. This choice of λ depends on $\|z(t) - 1\|_{\mathbf{L}^1}$ and on $\sup_u z(t,u)$. As shown in Lemma 2.2, both of these quantities do not increase in time, hence their upper bounds are determined by the initial conditions.

Remark 5.6. If we assume that $g(0) = 0$, then for any $z \geq 0$ the solution $w = E_\varepsilon^- z$ to the backward Cauchy problem (2.16)-(2.17) can never become negative. In this case, in (4.3) one has $z^{k+1} = \pi w^{k+1} = E_\varepsilon^- z^k$. Hence the projections π can be omitted, and trajectories of the semigroup S can be constructed simply as limits of backward Euler approximations.

Under the assumption $g(0) = 0$, a unique solution of the Cauchy problem (1.1) with Lipschitz continuous initial data $\bar{u}$ was constructed in [2, 9]. In particular, it was proved that the u-profile never develops shocks. In terms of the transformed variables, this means that

$$\bar{z}(u) > 0 \quad \text{for all } \ u \in I\!\!R \qquad \Longrightarrow \qquad z(t,u) > 0 \quad \text{for all } \ u \in I\!\!R, \ t \geq 0.$$

6 Uniqueness of entropy weak solutions

We are now ready to state the uniqueness Theorem.

Theorem 6.1. *For any initial datum $\bar{z} \in \mathcal{D}^+$, the entropy weak solution of the Cauchy problem* (1.11) *is unique.*

Proof. We implement a "doubling of variables" argument to show that the entropy inequality (5.2) implies uniqueness. Let $\hat{z}$ and z be two entropy weak solutions of (1.11) according to Definition 5.1, and let $\widehat{\Theta}, \Theta$ be the corresponding functions in (5.1)-(5.2). Let $\phi \in \mathcal{C}_c^\infty(]0,T[\times I\!\!R \times]0,T[\times I\!\!R)$ be any positive test function, and let $c, c' \geq 0$ be two arbitrary constants. Then $\hat{z} = \hat{z}(t,u)$ satisfies

$$\begin{aligned} &-\iint |\hat{z} - c| \cdot \phi_t \, du\, dt + \iint \operatorname{sign}(\hat{z} - c) \cdot \Big[(g(\hat{z}) - g(c))G(u;\hat{z}) + \widehat{\Theta}(t,u)\Big]\phi_u \ du\, dt \\ &\leq \ -\iint \operatorname{sign}(\hat{z} - c)g(c)g(\hat{z})G(u;\hat{z}) \cdot \phi\, du\, dt \end{aligned} \tag{6.1}$$

Similarly, using (s,v) as independent variables, the solution $z = z(s,v)$ satisfies

$$-\iint |z-c'|\cdot \phi_s\,dv\,ds + \iint \operatorname{sign}(z-c')\cdot\Big[(g(z)-g(c'))G(v;z)+\Theta(s,v)\Big]\,\phi_v\,dv\,ds$$
$$\leq\ -\iint \operatorname{sign}(z-c')g(c')g(z)G(v;z)\cdot\phi\,dv\,ds \tag{6.2}$$

Choosing $c = z(s,v)$ in (6.1) and $c' = \hat z(t,u)$ in (6.2), integrating w.r.t. all variables, and summing the resulting inequalities, we obtain

$$-\iiiint \Big[L_1+L_2+L_3+L_4\Big]\,du\,dt\,dv\,ds\ \leq\ 0 \tag{6.3}$$

where

$$L_1 \;=\; |\hat z - z|\cdot(\phi_t+\phi_s)\,, \tag{6.4}$$
$$L_2 \;=\; -\operatorname{sign}(\hat z - z)(g(\hat z)-g(z))\cdot\Big[G(u;\hat z)\cdot\phi_u + G(v;z)\cdot\phi_v\Big]\,, \tag{6.5}$$
$$L_3 \;=\; -\operatorname{sign}(\hat z - z)\cdot g(z)g(\hat z)\Big[G(u;\hat z)-G(v;z)\Big]\,\phi\,, \tag{6.6}$$
$$L_4 \;=\; \widehat\Theta(t,u)\phi_u + \Theta(s,v)\phi_v\,. \tag{6.7}$$

Here for L_4 we used the fact that $\operatorname{sign}(\hat z - c) = -1$ where ever $\widehat\Theta(t,u)$ is non-zero, and $\operatorname{sign}(z-c') = -1$ where ever $\Theta(s,v)$ is non-zero. See Remark 5.2.

Let $\delta_\rho(\cdot)$ and $\eta_\rho(\cdot)$ be two standard one-dimensional mollifiers, and let

$$\phi \;=\; \phi(t,u,s,v) \;=\; \psi\Big(\frac{t+s}{2},\frac{u+v}{2}\Big)\cdot\delta_\rho\Big(\frac{t-s}{2}\Big)\cdot\eta_\rho\Big(\frac{u-v}{2}\Big). \tag{6.8}$$

To shorten the notation, in the following we write

$$\psi \;=\; \psi\Big(\frac{t+s}{2},\frac{u+v}{2}\Big),\qquad \delta_\rho \;=\; \delta_\rho\Big(\frac{t-s}{2}\Big),\qquad \eta_\rho \;=\; \eta_\rho\Big(\frac{u-v}{2}\Big),$$

etc. One has

$$\begin{aligned} \partial_u\eta_\rho &= -\partial_v\eta_\rho, & \partial_u\psi &= \partial_v\psi \\ \partial_u\phi &= \partial_u\psi\cdot\delta_\rho\eta_\rho + \psi\delta_\rho\cdot\partial_u\eta_\rho, & \partial_v\phi &= \partial_v\psi\cdot\delta_\rho\eta_\rho - \psi\delta_\rho\cdot\partial_u\eta_\rho, \\ \partial_{t+s}\phi &= (\partial_t\psi+\partial_s\psi)\cdot\delta_\rho\cdot\eta_\rho, & \partial_{v+u}\phi &= (\partial_u\psi+\partial_v\psi)\cdot\delta_\rho\cdot\eta_\rho\,. \end{aligned}$$

With the choice (6.8) of the test function ϕ, the above terms L_1, $L_2 = L_{21}+L_{22}$, L_3 and $L_4 = L_{41}+L_{42}$ in (6.4)-(6.7) take the form

$$\begin{aligned} L_1 &= |\hat z - z|\cdot(\psi_t+\psi_s)\cdot\delta_\rho\cdot\eta_\rho, \\ L_{21} &= -\operatorname{sign}(\hat z - z)(g(\hat z)-g(z))\cdot\Big[G(u;\hat z)+G(v;z)\Big]\cdot\partial_u\psi\cdot\delta_\rho\cdot\eta_\rho, \\ L_{22} &= -\operatorname{sign}(\hat z - z)(g(\hat z)-g(z))\cdot\Big[G(u;\hat z)-G(v;z)\Big]\cdot\psi\cdot\delta_\rho\cdot\partial_u\eta_\rho, \\ L_3 &= -\operatorname{sign}(\hat z - z)g(z)g(\hat z)\Big[G(u;\hat z)-G(v;z)\Big]\cdot\psi\cdot\delta_\rho\cdot\eta_\rho, \\ L_{41} &= \Big[\widehat\Theta(t,u)+\Theta(s,v)\Big]\,\partial_u\psi\cdot\delta_\rho\cdot\eta_\rho, \\ L_{42} &= \Big[\widehat\Theta(t,u)-\Theta(s,v)\Big]\cdot\psi\cdot\delta_\rho\cdot\partial_u\eta_\rho. \end{aligned}$$

Taking the limit as $\rho \downarrow 0$ and writing $\psi = \psi(t,u)$, one obtains

$$\begin{aligned}
\iiiint L_1 &\to \iint |\hat z - z|\psi_t \, du\, dt,\\
\iiiint L_{21} &\to -\iint \operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\frac{1}{2}\Big[G(u;\hat z) + G(u;z)\Big]\psi_u \, du\, dt,\\
\iiiint L_3 &\to \iint -\operatorname{sign}(\hat z - z) g(z) g(\hat z)\Big[G(u;\hat z) - G(u;z)\Big]\psi \, du\, dt,\\
\iiiint L_{41} &\to \iint \frac{1}{2}\Big[\widehat\Theta + \Theta\Big]\,\psi_u \, du\, dt.
\end{aligned}$$

Concerning the term L_{22}, an integration by parts yields

$$\begin{aligned}
\iiiint L_{22} &= -\iiiint \operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\psi \cdot g(\hat z) G(u;\hat z) \cdot \delta_\rho \cdot \eta_\rho \, du\, dt \\
&\quad + \iiiint \Big[\operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\psi\Big]_u \cdot \Big[G(u;\hat z) - G(v;z)\Big] \cdot \delta_\rho \cdot \eta_\rho \, du\, dt\,.
\end{aligned}$$

Taking the limit $\rho \downarrow 0$ and the integrating by parts, we get

$$\begin{aligned}
\iiiint L_{22} &\to \iint \Big[\operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\psi\Big]_u \cdot \Big[G(u;\hat z) - G(u;z)\Big]\, du\, dt \\
&\qquad - \iint \operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\psi \cdot g(\hat z) G(u;\hat z)\, du\, dt \\
&= -\iint \operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\psi \cdot \Big[G(u;\hat z) - G(u;z)\Big]_u \, du\, dt \\
&\qquad - \iint \operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\psi \cdot g(\hat z) G(u;\hat z)\, du\, dt \\
&= -\iint \operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\psi \cdot g(z) G(u;z)\, du\, dt\,.
\end{aligned}$$

Therefore, we have

$$\iiiint L_{22} + L_3 \to \iint -\operatorname{sign}(\hat z - z) g(z)\Big[g(\hat z) G(u;\hat z) - g(z) G(u;z)\Big]\, du\, dt.$$

The term L_{42} is treated in a similar way as for L_{22}. An integration by parts yields

$$\iiiint L_{42} = -\iiiint \Big[(\widehat\Theta - \Theta)\psi_u\, \delta_\rho\, \eta_\rho + (\widehat\Theta - \Theta)_u \psi\, \delta_\rho\, \eta_\rho\Big]\, du\, dt,$$

Taking the limit $\rho \downarrow 0$ and use integration-by-parts for the second term, we get

$$\begin{aligned}
\iiiint L_{42} &\to -\iint (\widehat\Theta - \Theta)\psi_u \, du\, dt - \iint (\widehat\Theta - \Theta)_u \psi\, du\, dt \\
&= -\iint (\widehat\Theta - \Theta)\psi_u \, du\, dt + \iint (\widehat\Theta - \Theta)\psi_u \, du\, dt = 0\,.
\end{aligned}$$

Combining the above expressions one obtains

$$\begin{aligned}
&\iint -|\hat z - z|\psi_t + \operatorname{sign}(\hat z - z)(g(\hat z) - g(z))\frac{1}{2}\Big[G(u;\hat z) + G(u;z)\Big]\psi_u \, du\, dt \\
&\le -\iint \operatorname{sign}(\hat z - z)\Big[g(\hat z) G(u;\hat z) - g(z) G(u;z)\Big] g(z)\psi\, du\, dt \\
&\quad - \iint \frac{1}{2}\Big[\widehat\Theta + \Theta\Big]\,\psi_u \, du\, dt\,. \qquad (6.9)
\end{aligned}$$

Since both $\hat{z}$ and z satisfy the conditions in Definition 5.1, we can find a constant M such that

$$\hat{z}(t,u) \in [0,M], \quad z(t,u) \in [0,M], \qquad G(u;\hat{z}(t)) \leq M, \quad G(u;z(t)) \leq M \tag{6.10}$$

for all t,u. Since g is uniformly Lipschitz continuous on the interval $[0,M]$, setting $\lambda \doteq M\ \|g'\|_{\mathbf{L}^\infty([0,M])}$, for all t,u we obtain

$$\Big|(g(\hat{z}) - g(z))\Big| \cdot \frac{1}{2}\Big[G(u;\hat{z}) + G(u;z)\Big] \;\leq\; \lambda|\hat{z} - z|\,. \tag{6.11}$$

Concerning the first term on the right hand side of (6.9), using (2.35)-(2.36), we have the estimate

$$\begin{aligned}
&|g(\hat{z})G\,(u;\hat{z}) - g(z)G(u;z)| \\
&\quad\leq\; |g(\hat{z}) - g(z)|\,G(u;\hat{z}) + |g(z)|\cdot|G(u;\hat{z}) - G(u;z)| \\
&\quad\leq\; e^{\kappa}\,\|g'\|_{L^\infty([-2,M+1])}\,|\hat{z} - z| + |g(z)|e^{k}\,\|g'\|_{L^\infty([-2,M+1])}\,\|\hat{z} - z\|_{\mathbf{L}^1}
\end{aligned}$$

Let $0 < t_1 < t_2 < T$, and consider the domain

$$\Gamma \doteq \{(t,u);\;\; t \in [t_1,t_2], \quad |u| \leq R - \lambda t\}\ ,$$

where R is chosen large enough so that $\widehat{\Theta}(t,u) = \Theta(t,u) = 0$ whenever $|u| \geq R - \lambda t$. This is possible because, according to Definition 5.1, $\widehat{\Theta}$ and Θ have compact support in $[0,T] \times I\!R$. Following a well established technique, we now consider test functions ψ which approximate the characteristic function of the domain Γ. Thanks to the choices of R and λ in (6.10)-(6.11), in the limit one obtains

$$\begin{aligned}
&\int_{-R+\lambda t_2}^{R-\lambda t_2} |\hat{z}(t_2,u) - z(t_2,u)|\ du - \int_{-R+\lambda t_1}^{R-\lambda t_1} |\hat{z}(t_1,u) - z(t_1,u)|\ du \\
&\quad\leq \int_{t_1}^{t_2} M^2\,\|\hat{z}(t,\cdot) - z(t,\cdot)\|_{\mathbf{L}^1}\ dt\,.
\end{aligned}$$

Letting $R \to \infty$, for any $t_2 \geq t_1 > 0$ we obtain

$$\|\hat{z}(t_2,\cdot) - z(t_2,\cdot)\|_{\mathbf{L}^1} - \|\hat{z}(t_1,\cdot) - z(t_1,\cdot)\|_{\mathbf{L}^1} \;\leq\; M^2 \int_{t_1}^{t_2} \|\hat{z}(t,\cdot) - z(t,\cdot)\|_{\mathbf{L}^1}\ dt\,.$$

By Gronwall's lemma, for any $0 < t \leq T$ this implies

$$\|\hat{z}(t,\cdot) - z(t,\cdot)\|_{\mathbf{L}^1} \;\leq\; e^{M^2 t}\cdot\|\hat{z}(0,\cdot) - z(0,\cdot)\|_{\mathbf{L}^1}\,.$$

This shows the continuous dependence on initial data, and thus the uniqueness of entropy weak solutions. □

7 Equivalence with the original problem

By Theorem 6.1, for every nonnegative initial data $\bar z \in \mathcal{D}^+$ defined at (5.4), the entropy solution of (1.11) is unique. In particular, it does not depend on the constant $\lambda > 0$ chosen to construct the backward Euler approximations. Putting together the estimates proved in the previous sections, we thus obtain

Theorem 7.1. *Let the function g satisfy the assumptions (A2). Then there exists a continuous semigroup $S : \mathcal{D}^+ \times [0,\infty[\mapsto \mathcal{D}^+$ such that, for every $\bar z \in \mathcal{D}^+$, the trajectory $t \mapsto S_t \bar z$ is the unique entropy weak solution to the Cauchy problem (1.11), in the sense of Definition 5.1.*

In this final section we study the equivalence between solutions $z = z(t,u)$ of the equation (1.11) and solutions $u = u(t,x)$ of the original problem (1.1). Before stating a precise result, some definitions are needed. Recall in (1.5) we define the parameter

$$\eta \doteq \lim_{p\to\infty} f'(p) = g(0) > 0.$$

Given an increasing function $u : I\!R \mapsto I\!R$, call $\mu = \mu^a + \mu^s$ the decomposition of the measure $\mu = D_x u$ into an absolutely continuous and a singular part (w.r.t. Lebesgue measure). Motivated by (1.1), we introduce the flux function

$$\Phi^u(x) \;\doteq\; \exp\left\{ \int_x^\infty f(u_x(y))\, dy + \mu^s([x,\infty[) \cdot \eta \right\}. \tag{7.1}$$

Definition 7.2. *Consider a measurable function $u = u(t,x)$, such that $u(t,\cdot)$ is strictly increasing for every fixed time t. We say that u is a weak solution of (1.1) if the map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^1_{loc}$ and*

$$\int_0^\infty \int \left[u\varphi_t - \Phi^{u(t)}\varphi_x \right] dxdt + \int \bar u(x)\varphi(0,x)\, dx \;=\; 0 \tag{7.2}$$

for every test function $\varphi \in \mathcal{C}^1_c(I\!R^2)$.

To achieve uniqueness of solution, one clearly needs to impose additional entropy conditions. We shall do this by assuming some additional BV regularity and by imposing the Lax admissibility conditions at each point of approximate jump.

(RC) *There exists a function $w = w(t,x) \geq 0$ such that, for every $t \geq 0$, the map $x \mapsto w(t,x)$ is lower semicontinuous bounded variation (uniformly w.r.t. time), and satisfies*

$$w(t,x) \;=\; \begin{cases} 0 & \text{if } x \in Supp(\mu^s(t)), \\ [u_x(t,x)]^{-1} & \text{for a.e. } x \in I\!R\,. \end{cases} \tag{7.3}$$

Here $\mu^s(t)$ is the singular part of the measure $D_x u(t,\cdot)$.

Remark 7.3. The variable w introduced here is essentially the same as z in Theorem 5.4. However, we prefer to keep different notations to stress the fact that $w = w(t,x)$ while $z = z(t,u)$.

Following [4, 5, 11], we say that (t,x) is a **point of approximate jump** for the function u if there exist u^-, u^+, λ such that, setting

$$U(s,y) \doteq \begin{cases} u^- & \text{if} \quad y-x < \lambda(s-t), \\ u^+ & \text{if} \quad y-x > \lambda(s-t), \end{cases} \tag{7.4}$$

one has

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int\!\!\int_{(s-t)^2+(y-x)^2<\varepsilon^2} |u(s,y)-U(s,y)| \, dyds = 0. \tag{7.5}$$

We denote by J^u, J^w the jump points of u and w, respectively. Since $u, w \in BV$, a classical structure theorem [4, 11] implies that the sets J^u, J^w are rectifiable, i.e. they are contained in the union of countably many Lipschitz continuous curves, together with a set whose one-dimensional Haussdorff measure is zero. We can now impose additional admissibility conditions on the solution u of (1.1).

(AC) *There exists a set of times $\mathcal{N}$ with measure zero such that, at each point of jump of u or w with $t \notin \mathcal{N}$, the following holds.*

(i) Let $(t,x) \in J^u$, so that (7.4)-(7.5) hold. Then the speed of the jump is greater than or equal to the characteristic speed of the right state. Namely

$$\lambda \geq f'(u_x(t,x+))\, \Phi^{u(t)}(x+). \tag{7.6}$$

(ii) Let $(t,x) \in J^w \setminus J^u$. Then the Lax admissibility condition holds:

$$w(t,x-) > w(t,x+). \tag{7.7}$$

The next theorem provides a basic correspondence between solutions of (1.1) and entropy weak solutions to the auxiliary equation (1.11).

Theorem 7.4. *(i) Let $z = z(t,u)$ be an entropy weak solution of (1.11), with $\bar z \in \mathcal{D}^+$. Fix any constant C and define*

$$X(t,u) \doteq u - \int_u^\infty [z(t,\xi)-1]\, d\xi + C. \tag{7.8}$$

Let $x \mapsto u(t,x)$ be the inverse function of $u \mapsto X(t,u)$ and call $\bar u(\cdot)$ the inverse function of $X(0,\cdot)$. Then $u = u(t,x)$ provides a weak solution to the Cauchy problem (1.1) satisfying the regularity condition **(RC)** *and the admissibility conditions* **(AC)**.

(ii) Viceversa, let $u = u(t,x)$ be a solution to a weak solution to (1.1) satisfying the regularity and admissibility conditions **(RC)-(AC)**. *For each $t \geq 0$, let $u \mapsto X(t,u)$ be the inverse function of $x \mapsto u(t,x)$ Then the function $z(t,u) = X_u(t,u)$ provides the unique weak entropy solution of (1.11), with $\bar z = X_u(0,\cdot)$.*

By the uniqueness of entropy solutions to (1.11), the above theorem implies

Corollary 7.5. *Let $\bar{u} : I\!R \mapsto I\!R$ be an increasing function such that the inverse function $\overline{X} : I\!R \mapsto I\!R$ is Lipschitz continuous and its derivative satisfies*

$$\overline{X}_u \in BV, \qquad \|\overline{X}_u - 1\|_{\mathbf{L}^1} \ < \ +\infty. \tag{7.9}$$

Then the Cauchy problem (1.1) has a unique weak solution satisfying the regularity and admissibility conditions **(RC)-(AC)**.

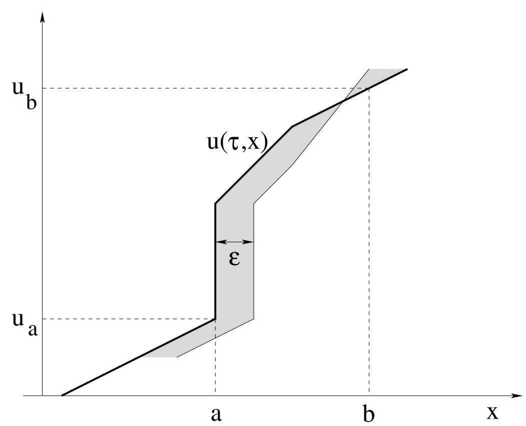

Figure 4: Proving the continuity of the inverse function in $\mathbf{L}^1_{loc}$.

Proof. (of Theorem 7.4) **1.** We start by proving (i). Let $z = z(t,u)$ be a weak entropy solution to (1.11), according to Definition 5.1. Since $z(t,\cdot) - 1 \in \mathbf{L}^1(I\!R)$, the function $X(t,u)$ in (7.8) is well defined. For simplicity we assume $C = 0$, which is not restrictive. Since $0 \le z(t,u) = X_u(t,u) < M$ for some constant M and all t,u, for each $t \ge 0$ the inverse function $u(t,\cdot)$ is well defined and strictly increasing. Since $t \mapsto X_u(t,\cdot)$ is Lipschitz continuous with values in $\mathbf{L}^1(I\!R)$, by integrating w.r.t. u we see that $t \mapsto X(t,\cdot)$ is Lipschitz continuous with values in $\mathbf{L}^\infty(I\!R)$.

Fix $\tau \ge 0$. For any interval $[a,b]$, let

$$u_a \ \doteq \ \sup_{x<a} \ u(\tau,x), \qquad u_b \ \doteq \ \inf_{x>b} \ u(\tau,x)\,.$$

Given $\varepsilon > 0$, choose $\delta > 0$ so that $\|X(t,\cdot) - X(\tau,\cdot)\|_{\mathbf{L}^\infty} \le \varepsilon$ whenever $|t-\tau| \le \delta$. An elementary argument (see Fig. 4) shows that the inverse function satisfies

$$\|u(t,\cdot) - u(\tau,\cdot)\|_{\mathbf{L}^1([a,b])} \ \le \ (u_b - u_a + 2M\varepsilon)\cdot\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, this shows the continuity of the map $t \mapsto u(t,\cdot)$ with values in $\mathbf{L}^1_{loc}$.

2. Let $\varphi \in \mathcal{C}^1_c(I\!R^2)$. Changing the variables of integration from (t,x) to (τ,u) and writing $\phi(\tau,u) \doteq \varphi(\tau, X(\tau,u))$, we compute

$$\left\{ \begin{array}{rcl} t &=& \tau\,, \\ x &=& X(\tau,u)\,, \end{array} \right. \qquad \left\{ \begin{array}{rcl} \phi_\tau &=& \varphi_t + \varphi_x X_\tau\,, \\ \phi_u &=& \varphi_x X_u\,, \end{array} \right. \qquad dx\,dt \ = \ X_u\,du\,d\tau\,.$$

Next, we observe that for a.e. x the flux in (7.1) is equivalently computed by

$$\Phi^{u(t)}(x) \;=\; \exp\left\{\int_{u(t,x)}^{+\infty} g(z(t,u))\,du\right\} \;=\; G(u;z(t))\,, \tag{7.10}$$

where $z = X_u$. Observing that $X = X(\tau, u)$ is locally Lipschitz continuous w.r.t. both variables and the same is true for $(t,u) \mapsto G(u; z(t))$, we compute

$$\begin{aligned}
&\int_0^\infty \int \Big[u(t,x)\varphi_t(t,x) - \Phi^{u(t)}(x)\varphi_x(t,x)\Big]\,dxdt \\
&= \int_0^\infty \int \left[u\phi_\tau X_u - u\phi_u X_\tau - G(u;X_u(\tau))\phi_u\right]\,dud\tau \\
&= \int_0^\infty \int \Big\{X_u(u\phi)_\tau - X_\tau(u\phi)_u + X_\tau\phi - G(u;X_u(\tau))\phi_u\Big\}\,dud\tau \\
&= -\int_0^\infty \int \Big\{X\phi_\tau + G(u;X_u(\tau))\phi_u\Big\}\,dud\tau \\
&= -\int_0^\infty \int \left\{\Big(u - \int_u^\infty [z(\tau,\xi)-1]\,d\xi\Big)\phi_\tau + \Big(G(u;z(\tau))-1\Big)\phi_u\right\}\,dud\tau
\end{aligned} \tag{7.11}$$

Introduce the test function ψ by setting

$$\psi(\tau,u) \;\doteq\; \begin{cases} \displaystyle\int_{-\infty}^{u} \phi(\tau,v)\,dv & \text{if } u \le N, \\ 0 & \text{if } u \ge N+1, \end{cases}$$

and such that $u \mapsto \psi(\tau,u)$ is affine for $u \in [N, N+1]$. Observe that ψ is Lipschitz continuous with compact support, and $\psi_u = \phi$ for $u < N$. Let N be large enough so that the support of Θ is contained on the set where $u < N$. Using (5.2) we then obtain

$$\iint z\psi_\tau\,dud\tau - \iint g(z)G(u;z)\psi_u\,dud\tau \;=\; \iint \Theta\psi_u\,dud\tau \;=\; 0 \tag{7.12}$$

Indeed, $\psi_u(\tau,\cdot) = \phi(\tau,\cdot)$ is constant on every interval $[a,b]$ where $z(\tau,\cdot) = 0$. By (5.1), the integral of Θ_u over this interval is zero.

Integrating by parts, for N sufficiently large, the right hand side of (7.11) can now be estimated by

$$\begin{aligned}
&-\int_0^\infty \int_{-\infty}^N \left\{\Big(u - \int_u^\infty [z(\tau,\xi)-1]\,d\xi\Big)\phi_\tau + \Big(G(u;z(\tau))-1\Big)\phi_u\right\}\,dud\tau \\
&= \int_0^\infty \int_{-\infty}^N z\psi_\tau\,dud\tau - \int_0^\infty \int_{-\infty}^N g(z)G(u;z)\psi_u\,dud\tau \\
&\qquad -\int_0^\infty \left[\Big(u - \int_u^\infty [z(\tau,\xi)-1]\,d\xi\Big)\psi_\tau + \Big(G(u;z(\tau))-1\Big)\psi_u\right]_{u=N} d\tau \\
&\doteq A_N + B_N\,.
\end{aligned} \tag{7.13}$$

Assuming that ϕ, ψ vanish for $t \notin [0,T]$, as $N \to \infty$ one has

$$\begin{aligned}
|A_N| &= \left|\int_0^T \int_N^{N+1} z\psi_\tau \, du d\tau + \int_0^T \int_N^{N+1} g(z)G(u;z)\psi_u \, du d\tau\right| \\
&\leq \|\psi\|_{\mathbf{L}^\infty} \cdot \int_N^{N+1} \mathrm{TV}\Big\{z(\cdot, u)\,;[0,T]\Big\}\, du \\
&\qquad + \|G\|_{\mathbf{L}^\infty} \, \|\phi\|_{\mathbf{L}^\infty} \cdot \int_0^T \int_N^{N+1} |g(u,z)| \, du d\tau \, , \\
|B_N| &= \int_0^T \left[\Big(u - \int_u^\infty [z(\tau,\xi) - 1]\, d\xi\Big)\psi_\tau + \Big(G(u; z(\tau)) - 1\Big)\psi_u\right]_{u=N} d\tau \\
&\leq \|\psi_\tau\|_{\mathbf{L}^\infty} \cdot \sup_{\tau\in[0,T]} \Big|X(\tau, N) - N\Big| + \|\phi\|_{\mathbf{L}^\infty} \cdot \sup_{\tau\in[0,T]} \Big|G(N; z(\tau)) - 1\Big|.
\end{aligned}$$

From the above, it is clear that A_N, $B_N \to 0$ as $N \to \infty$. This shows that the left hand side of (7.11) vanishes. Hence $u = u(t,x)$ provides a weak solution to the Cauchy problem (1.1).

3. Since z has bounded variation, and can be rendered lower semicontinuous by a change on a set of measure zero, the regularity condition (**RC**) is clearly satisfied. It remains to prove that the admissibility conditions (**AC**) are satisfied as well.

Consider first a point $(t,x) \in J^w$ where u is continuous. This means that u_x has a jump, but the limits $u(t,x-) = u(t,x+) \doteq u_0$ coincide. To fix the ideas, assume that (t, u_0) is a point of jump for z, with left and right states z^-, z^+ and speed λ_0. The continuity assumption on u implies that $G(t,u)$ is continuous at (t,u_0). Hence, from (5.3) we deduce (see for example [5], p.84)

$$\begin{aligned}
\lambda_0\Big(\big|z^+ - c\big| - \big|z^- - c\big|\Big) \;\;\geq\;\; -G(u_0, z(t)) \cdot \Big(&\mathrm{sign}(z^+ - c)(g(z^+) - g(c)) \\
&- \mathrm{sign}(z^- - c)(g(z^-) - g(c))\Big). \qquad (7.14)
\end{aligned}$$

Choosing $c = \frac{1}{2}(z^+ + z^-)$ we obtain

$$0 \;\leq\; G(u_0, z(t))\,\mathrm{sign}(z^+ - z^-)\left(g(z^+) + g(z^-) - 2g\Big(\frac{z^+ + z^-}{2}\Big)\right). \qquad (7.15)$$

Since g is concave down, this implies $\mathrm{sign}(z^+ - z^-) < 0$. Recalling that $z = 1/u_x = w$, we conclude that (7.7) holds.

4. Next, assume $(t,x) \in J^u$ and let u^-, u^+ and $\dot{x} = \lambda$ be as in (7.4)-(7.5) (see Fig. 5). Then $z(t,u) = 0$ for $u \in [u^-, u^+]$. By removing a set of times of measure zero, it is not restrictive to assume that (t,u^-) and (t,u^+) are jump points for $z = z(t,u)$, say with speeds $\dot{u}^-, \dot{u}^+$, respectively. For notational convenience, define

$$G^+ \doteq G(u^+; z(t)), \qquad G^- \doteq G(u^-; z(t)) \;=\; e^{(u^+ - u^-)g(0)} G^+ \, . \qquad (7.16)$$

By the projection property, the function Θ is non-zero on the interval $[u^-, u^+]$, with

$$\Theta_u(t,u) \;=\; g^2(0) G(u; z(t)) \;=\; g^2(0) e^{(u^+ - u)g(0)} G^+ > 0, \qquad u \in [u^-, u^+]\, ,$$

while $\Theta(t,\cdot)$ has jumps at u^- and u^+. By using the conditions in (5.1), we can obtain an explicit formula for Θ. In particular the sizes of jumps at u^- and u^+ are given as

$$B^- \doteq \Theta(t,u^-+) - \Theta(t,u^--) = \frac{G^- - G^+}{u^+ - u^-} - g(0)G^-, \tag{7.17}$$

$$B^+ \doteq \Theta(t,u^++) - \Theta(t,u^+-) = -\frac{G^- - G^+}{u^+ - u^-} + g(0)G^+. \tag{7.18}$$

Note that $B^- < 0$ and $B^+ < 0$. Writing $z^- = z(t,u^--)$ and $z^+ = z(t,u^++)$, where $z^- > 0$ and $z^+ > 0$, using (7.17) and the Rankine-Hugoniot condition, we obtain the two jump speeds:

$$\dot u^- = -\frac{g(z^-) - g(0)}{z^-}G^- + \frac{B^-}{z^-}, \tag{7.19}$$

$$\dot u^+ = -\frac{g(z^+) - g(0)}{z^+}G^+ - \frac{B^+}{z^+}. \tag{7.20}$$

Consider the jump in z at (t,u^-), where $z^- \doteq z(t,u^--) > 0$ while $z(t,u^-+) = 0$. A direct computation shows that the Lax condition

$$\dot u^- < -\frac{g(z^-) - g(0)}{z^-}G^- < -g'(z^-)G^- \tag{7.21}$$

is always satisfied, because g is strictly concave.

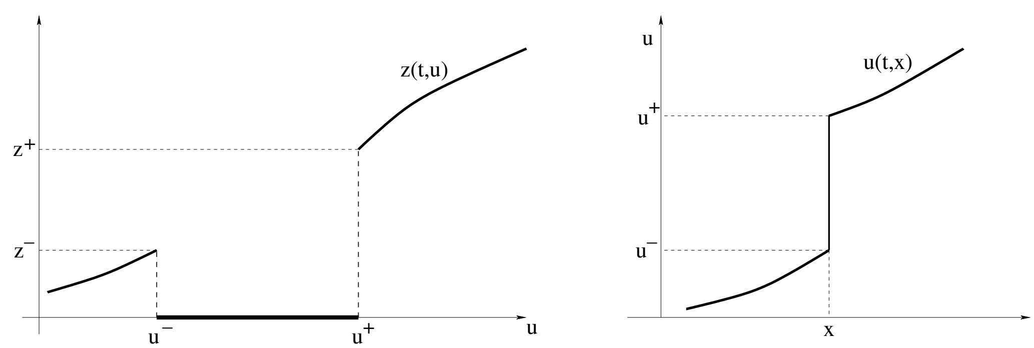

Figure 5: A point where $u(t,\cdot)$ has a jump (right) corresponds to an interval $[u^-,u^+]$ where $z(t,u) \equiv 0$ (left).

Next, consider the jump at u^+, where $z^+ \doteq z(t,u^++) > 0$ while $z(t,u^+-) = 0$. Choosing $0 < c < z^+$, from (5.2) one obtains

$$\Big[(z^+ - c) - (c - 0)\Big]\dot u^+ \geq (g(z^+) - g(c))G^+ + (g(0) - g(c))G^+ + B^+.$$

Adding $z^+\dot u^+$ to both sides and using (7.20) we obtain, for all c with $0 < c < z^+$,

$$\begin{aligned} 2(z^+ - c)\dot u^+ &\geq z^+\dot u^+ + \Big(g(z^+) + g(0) - 2g(c)\Big)G^+ + B^+ \\ &= -\Big(2g(z^+) - 2g(c)\Big)G^+. \end{aligned}$$

Since g is strictly concave, the above condition is equivalent to the constraint

$$\dot u^+ \;\geq\; -\min_{0<c<z^+}\left\{\frac{g(z^+)-g(c)}{z^+-c}\right\}G^+ \;=\; -\,g'(z^+)G^+\,. \tag{7.22}$$

Combining (7.22) with (7.20) and (7.18), then using (7.16), we obtain

$$z^+\dot u^+ \;=\; -\,[g(z^+)-g(0)]G^+ + \left[\frac{G^- - G^+}{u^+-u^-} - g(0)G^+\right] \;\geq\; -\,z^+g'(z^+)G^+, \tag{7.23}$$

and

$$\frac{e^{g(0)(u^+-u^-)}-1}{u^+-u^-} \;\geq\; g(z^+) - z^+g'(z^+)\,. \tag{7.24}$$

We now return to the original (t,x) coordinates. Recalling (7.1), let $\Phi^+ \doteq \Phi^{u(t)}(x+) = G^+$ and $\Phi^- \doteq \Phi^{u(t)}(x-) = G^-$ be the fluxes to the right and to the left of the point of jump. It will be useful to recall the identities

$$f\left(\frac{1}{z}\right) \;=\; \frac{g(z)}{z}, \qquad f'\left(\frac{1}{z}\right) \;=\; g(z)-zg'(z), \qquad \lim_{p\to\infty} f'(p) \;=\; g(0).$$

By the Rankine-Hugoniot condition, using (7.24) we find that the speed of the jump satisfies

$$\begin{aligned}\dot x(t) \;&=\; -\frac{\Phi^+-\Phi^-}{u^+-u^-} \;=\; \frac{e^{f'(+\infty)(u^+-u^-)}-1}{u^+-u^-}\,\Phi^+ \;=\; \frac{e^{g(0)(u^+-u^-)}-1}{u^+-u^-}G^+ \\ &\geq\; g(z^+)-z^+g'(z^+) \;=\; f'\left(u_x(t,x+)\right)\Phi^{u(t)}(x+)\,.\end{aligned}$$

This proves the admissibility condition (7.6), completing the proof of part (i) of the Theorem.

5. From now on, we work on part (ii). Let $u = u(t,x)$ be a solution to (1.1) which satisfies the regularity and admissibility conditions (**RC**)-(**AC**). For each fixed time $t \geq 0$, let $u \mapsto X(t,u)$ be the inverse function. By (**RC**), the derivative $z(t,u) \doteq X_u(t,u)$ is well defined a.e. and has bounded variation. Up to a modification on a set of measure zero, we can assume that z is lower semicontinuous. We claim that the function z provides an entropy solution to (1.11), according to Definition 5.1.

As a first step, we show that the map $t\mapsto z(t,\cdot)$ is continuous with values in $\mathbf{L}^1_{loc}(I\!R)$. By the continuity of the map $t\mapsto u(t,\cdot)$ with values in $\mathbf{L}^1_{loc}$, it is clear that the inverse function $t\mapsto X(t,\cdot)$ is also continuous with values in $\mathbf{L}^1_{loc}$. Consider any convergent sequence of times $t_j\to\tau$. By the regularity condition (**RC**), the functions $z(t_j,\cdot)$ are uniformly bounded and have uniformly bounded total variation. Hence, by extracting a subsequence, we can assume the convergence $z(t_j,\cdot)\to\tilde z$ in $\mathbf{L}^1_{loc}$, for some BV function $\tilde z$. From the convergence $X(t_j,\cdot)\to X(\tau,\cdot)$ it now follows the identity $\tilde z(u) = X_u(\tau,u) = z(\tau,u)$. The above argument shows that from every sequence $t_j\to\tau$ one can extract a subsequence $(t_{j_k})_{k\geq 1}$ such that $z(t_{j_k},\cdot)\to z(\tau,\cdot)$ in $\mathbf{L}^1_{loc}$. This proves continuity property (P1) in Definition 5.1.

6. In the remaining steps we will prove that (P2) in Definition 5.1 also holds. For each $t\geq 0$, the measure $\mu^{(t)}$ on the right hand side of (1.11) is determined as follows. If

$u(t,\cdot)$ is continuous, then $\mu^{(t)} = 0$. In general, let $u(t,\cdot)$ have jumps at countably many points x_i and call

$$u_i^+ \doteq u(t, x_i+), \qquad u_i^- \doteq u(t, x_i-),$$

$$\begin{cases} G_i^+ = \Phi_i^+ \doteq \Phi^{(u(t))}(x_i+), \\ G_i^- = \Phi_i^- \doteq \Phi^{(u(t))}(x_i-) = \exp\{(u_i^+ - u_i^-)f'(+\infty)\}\Phi_i^+ , \end{cases}$$

$$G(u) = e^{(u_i^+ - u)g(0)} G_i^+ , \qquad u \in [u_i^-, u_i^+].$$

Restricted to each interval $[u_i^-, u_i^+]$, the measure $\mu^{(t)}$ is the sum of an absolutely continuous measure with density $G(u)$, plus two point masses at u_i^+, u_i^-. The sizes of these masses are given by B_i^+, B_i^-, where

$$B_i^+ = -\frac{G_i^- - G_i^+}{u_i^+ - u_i^-} + g(0)G_i^+ , \qquad B_i^- = \frac{G_i^- - G_i^+}{u_i^+ - u_i^-} - g(0)G_i^- . \tag{7.25}$$

Notice that these masses are chosen so that $\mu^{(t)}([u_i^-, u_i^+]) = 0$, while the barycenter of the positive part of $\mu^{(t)}$ coincides with the barycenter of the negative part. Equivalently, one can define $\mu^{(t)} = D_u\Theta(t,\cdot)$, where

$$\Theta(t,u) = \sum_i \Theta_i(u), \qquad \Theta_i(u) = \begin{cases} B_i^- + \int_{u_i^-}^{u} g^2(0)\, G(\xi)\, d\xi & \text{if } u_i^- < u < u_i^+, \\ 0 & \text{otherwise.} \end{cases}$$

With the above definitions, one easily checks that the properties in (5.1) hold.

7. For each fixed time $t \geq 0$, let $u \mapsto X(t,u)$ be the inverse of the map $x \mapsto u(t,x)$. By the regularity assumption (**RC**), the derivative u_x is positive and uniformly bounded away from zero. Since the flux function in (1.1) remains uniformly bounded, we conclude that the map $(t,u) \mapsto X(t,u)$ is uniformly continuous and therefore has partial derivatives X_t, X_u defined pointwise for a.e. t, x. Since u provides a solution to (1.1), it follows that X satisfies the PDE

$$X_t(t,u) - g\left(X_u(t,u)\right) \cdot \left(\exp \int_u^\infty g\left(X_u(t,v)\right)\, dv\right) - \Theta(t,u) = 0 \tag{7.26}$$

pointwise almost everywhere. In turn, the BV function $z = X_u$ provides a distributional solution to the equation

$$z_t - \left(g(z) \cdot \exp \int_u^\infty g(z(t,v))\, dv\right)_u - \Theta_u(t,u) = 0. \tag{7.27}$$

In particular, (5.2) is satisfied as an equality in the special case where $c = 0$.

To prove that the inequality (5.2) holds for every non-negative test function $\psi \in \mathcal{C}_c^1(]0,T[\,\times I\!R)$ and every constant c, we recall that z is a BV function of the two variables t, u. By a well known structure theorem [4, 11], for almost every $c \in I\!R$ the sets

$$\Omega_c^+ \doteq \{(t,u)\,;\; z(t,u) > c\}, \qquad \Omega_c^- \doteq \{(t,u)\,;\; z(t,u) < c\},$$

have a common, rectifiable boundary $\Gamma_c = \partial\Omega_c^+ = \partial\Omega_c^-$. Moreover, a.e. point $(t,u) \in \Gamma_c$ is a point of continuity or of approximate jump for z. We can now write

$$\begin{aligned}
I_c \;&\doteq\; \int_0^T \int_{I\!R} \Big\{ |z-c|\,\psi_t - \,\mathrm{sign}(z-c)\cdot \big[(g(z)-g(c))\,G(u;z(t)) + \Theta(t,u)\big]\;\psi_u \Big\}\,du\,dt \\
&\quad - \int_0^T \int_{I\!R} \mathrm{sign}(z-c)\cdot g(c)\,g(z)\,G(u;z(t))\,\psi\,du\,dt \\
&= \iint_{\Omega_c^+} \Big[(z-c)\psi_t - \,(g(z)-g(c))\,G\,\psi_u - g(c)\,g(z)\,G\,\psi - \Theta(t,u)\,\psi_u\Big]\,du\,dt \\
&\quad - \iint_{\Omega_c^-} \Big[(z-c)\psi_t - \,(g(z)-g(c))\,G\,\psi_u - g(c)\,g(z)\,G\,\psi - \Theta(t,u)\,\psi_u\Big]\,du\,dt.
\end{aligned} \tag{7.28}$$

Consider the vector-valued function

$$\mathcal{F}(t,u) \doteq \begin{pmatrix} z-c \\ -(g(z)-g(c))G - \Theta \end{pmatrix} \psi. \tag{7.29}$$

By the assumption $z \in BV$ and the explicit definition of Θ, it follows that the traces

$$\mathcal{F}^-(t,u)\,, \quad \mathcal{F}^+(t,u) \qquad\qquad (t,u) \in \partial\Omega_c^- = \partial\Omega_c^+ = \Gamma_c\,,$$

on the boundaries of Ω_c^- and Ω_c^+ are well defined. In the (t,u) plane, a formal computation of the divergence of $\mathcal{F}$ yields

$$\begin{aligned}
\nabla\cdot\mathcal{F} =\;& ((z-c)\psi)_t - [(g(z)-g(c))G\psi]_u - (\Theta\psi)_u \\
=\;& (z-c)_t\psi + (z-c)\psi_t - ((g(z)-g(c))G)_u\psi - (g(z)-g(c))G\psi_u - \Theta_u\psi - \Theta\psi_u \\
=\;& [(z-c)_t - ((g(z)-g(c))G)_u - \Theta_u]\,\psi + (z-c)\psi_t - (g(z)-g(c))G\psi_u - \Theta\psi_u \\
=\;& g(c)G_u\psi + (z-c)\psi_t - (g(z)-g(c))G\psi_u - \Theta\psi_u \\
=\;& (z-c)\psi_t - (g(z)-g(c))G\psi_u - g(c)g(z)G\psi - \Theta\psi_u\,.
\end{aligned}$$

Using the divergence theorem, the integral in (7.28) can be computed as

$$I_c \;=\; \iint_{\Omega_c^+} \nabla\cdot\mathcal{F}\,du\,dt - \iint_{\Omega_c^-} \nabla\cdot\mathcal{F}\,du\,dt \;=\; \int_{\Gamma_c} \big(\mathcal{F}^+ + \mathcal{F}^-\big)\cdot\mathbf{n}\,d\sigma\,. \tag{7.30}$$

Here $\mathbf{n}$ is the unit outer normal vector to the boundary $\partial\Omega_c^+$ and $d\sigma$ denotes arc-length.

8. To prove that $I_c \geq 0$, it suffices to show that, for a.e. point in Γ_c, one has

$$\big(\mathcal{F}^+ + \mathcal{F}^-\big)\cdot\mathbf{n} \;\geq\; 0. \tag{7.31}$$

Let $(\bar t, \bar u) \in \Gamma_c$. If z is continuous at this point, then

$$z(\bar t,\bar u) \;=\; c, \qquad\qquad \mathcal{F}^+(\bar t,\bar u) \;=\; \mathcal{F}^-(\bar t,\bar u) \;=\; 0\,,$$

and the conclusion is immediate. On the other hand, if z has a jump at $(\bar t,\bar u)$, call $z^\pm, \Theta^\pm$ the left and right values of z, Θ across the jump, and let λ be the jump speed.

We observe that, by the definition of Ω_c^+, Ω_c^-, either $z^- \le c \le z^+$ or else $z^+ \le c \le z^-$. We shall consider three cases.

CASE 1: $z^- > 0$ and $z^+ > 0$. In this case we have $\Theta^+ = \Theta^- = 0$. The admissibility condition (7.7) now implies

$$0 \;<\; z^+ \;<\; z^-.$$

In a neighborhood of $(\bar{t}, \bar{u})$, the set Ω_c^+ lies on the left of Γ_c while Ω_c^- lies on the right of Γ_c. Hence the normal vector $\mathbf{n}$ pointing toward Ω_c^- is

$$\mathbf{n} \;=\; \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} -\lambda \\ 1 \end{pmatrix}. \tag{7.32}$$

We compute

$$\mathcal{F}^+ \;=\; \begin{pmatrix} z^+ - c \\ -(g(z^+) - g(c))G \end{pmatrix} \psi \;=\; \begin{pmatrix} -|z^+ - c| \\ \mathrm{sign}(z^+ - c)(g(z^+) - g(c))G \end{pmatrix} \psi\,,$$

$$\mathcal{F}^- \;=\; \begin{pmatrix} z^- - c \\ -(g(z^-) - g(c))G \end{pmatrix} \psi \;=\; \begin{pmatrix} c|z^- - c| \\ -\,\mathrm{sign}(z^- - c)(g(z^-) - g(c))G \end{pmatrix} \psi\,,$$

and so

$$\begin{aligned}\big(\mathcal{F}^+ + \mathcal{F}^-\big)\cdot \begin{pmatrix} -\lambda \\ 1 \end{pmatrix} \;&=\; \Big\{\lambda\cdot\big(|z^+ - c| - |z^- - c|\big) + \mathrm{sign}(z^+ - c)(g(z^+) - g(c))G \\ &\qquad - \mathrm{sign}(z^- - c)(g(z^-) - g(c))G\Big\}\psi\,. \end{aligned}\tag{7.33}$$

Since g is strictly concave, for any $c \in [z^+, z^-]$ and $z^- > z^+$, the Rankine-Hugoniot speed λ satisfies

$$\lambda\cdot\big(|z^+ - c| - |z^- - c|\big) \ge -\,\mathrm{sign}(z^+ - c)(g(z^+) - g(c))G + \mathrm{sign}(z^- - c)(g(z^-) - g(c))G.$$

Since $\psi \ge 0$ we conclude that, at the point $(\bar{t}, \bar{u})$, the quantity in (7.33) is non-negative. Hence (7.31) holds.

CASE 2: $z^- > 0$, $z^+ = 0$. We then have $\Theta^+ = 0$, $\Theta^- < 0$. As in the previous case, Ω_c^+ lies on the left of Γ_c and Ω_c^- lies on the right of Γ_c, hence $\mathbf{n}$ has the same expression (7.32) as in Case 1. Observing now

$$\mathcal{F}^- \;=\; \begin{pmatrix} -c \\ -(g(0) - g(c))G - \Theta^- \end{pmatrix} \psi, \qquad \mathcal{F}^+ \;=\; \begin{pmatrix} z^- - c \\ -(g(z^-) - g(c))G \end{pmatrix} \psi\,,$$

we thus have

$$\big(\mathcal{F}^+ + \mathcal{F}^-\big)\cdot \begin{pmatrix} -\lambda \\ 1 \end{pmatrix} = \Big\{\lambda\cdot(2c - z^-) - (g(0) - g(c))G - \Theta^- - (g(z^-) - g(c))G\Big\}\,\psi\,. \tag{7.34}$$

The speed λ of the jump, given at (7.19), with the present notation takes the form

$$\lambda \;=\; -\,\frac{g(z^-) - g(0)}{z^-}G + \frac{\Theta^-}{z^-}\,.$$

By (7.21) and the concavity of g, it follows that

$$(z^- - c)\lambda \ \leq \ -(g(z^-) - g(c))G\,.$$

The term between brackets on the right hand side of (7.34) can now be written as

$$\begin{aligned} c\lambda - (g(0) - g(c))G - (z^- - c)\lambda - (g(z^-) - g(c))G - \Theta^- \\ \geq \ c\lambda - (g(0) - g(c))G - \Theta^- \\ = \ -c\,\frac{g(z^-) - g(0)}{z^-}G + c\frac{\Theta^-}{z^-} - (g(0) - g(c))G - \Theta^- \\ = \ -c\left[\frac{g(z^-) - g(0)}{z^-} - \frac{g(c) - g(0)}{c}\right]G + \frac{c - z^-}{z^-}\Theta^- \ \geq \ 0. \end{aligned}$$

Here the last inequality follows since both terms are positive. Since $\psi \geq 0$, we again conclude that at the point $(\bar{t}, \bar{u}) \in \Gamma_c$ the quantity in (7.34) is nonnegative. Hence (7.31) holds.

CASE 3: $z^- = 0$, $z^+ > 0$. In this case $\Theta^+ = 0$, $\Theta^- > 0$, and Ω_c^+ lies on the right of Γ_c while Ω_c^- lies on the left of Γ_c. The normal vector pointing toward Ω_c^- now has the form

$$\mathbf{n} \ = \ \frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix}\lambda \\ -1\end{pmatrix}.$$

We have

$$\mathcal{F}^- \ = \ \begin{pmatrix} -c \\ -(g(0) - g(c))G - \Theta^- \end{pmatrix}\psi, \qquad \mathcal{F}^+ \ = \ \begin{pmatrix} z^+ - c \\ -(g(z^+) - g(c))G \end{pmatrix}\psi.$$

Hence

$$\big(\mathcal{F}^+ + \mathcal{F}^-\big)\cdot\begin{pmatrix}\lambda \\ -1\end{pmatrix} \ = \ \Big\{\lambda\cdot(z^+ - 2c) + (g(0) - g(c))G + \Theta^- + (g(z^+) - g(c))G\Big\}\,\psi\,. \quad (7.35)$$

The speed λ of the jump, given at (7.20), with the present notation takes the form

$$\lambda \ = \ -\frac{g(z^+) - g(0)}{z^+}G + \frac{\Theta^-}{z^+}\,.$$

This yields the relation

$$\Theta^- \ = \ \lambda z^+ + (g(z^+) - g(0))G\,.$$

The admissibility condition (7.6) and the concavity of g imply

$$\lambda \ \geq \ -g'(z^+)G \ \geq \ -\frac{g(z^+) - g(c)}{z^+ - c}G\,. \qquad (7.36)$$

The term between brackets on the right hand side of (7.35) can now be written as

$$\begin{aligned} \lambda\,(z^+ - 2c) + (g(0) - g(c))G + \lambda z^+ + (g(z^+) - g(0))G + (g(z^+) - g(c))G \\ = \ 2\lambda(z^+ - c) + 2(g(z^+) - g(c))G \ \geq \ 0\,, \end{aligned}$$

because of (7.36). Once again, we conclude that, at the point $(\bar{t}, \bar{u}) \in \Gamma_c$, the quantity in (7.35) is nonnegative. Hence (7.31) holds.

The above arguments show that, for any positive test function ψ, the inequality (5.2) holds for a.e. constant c. By continuity, it remains true for all $c \in I\!R$. This completes the proof of part (ii) of the theorem. □

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