Global Existence of Weak Solutions for the Burgers-Hilbert Equation

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February 17, 2014

Abstract

This paper establishes the global existence of weak solutions to the Burgers-Hilbert equation, for general initial data in $\mathbf{L}^2(I\!\!R)$. For positive times, the solution lies in $\mathbf{L}^2 \cap \mathbf{L}^\infty$. A partial uniqueness result is proved for spatially periodic solutions, as long as the total variation remains locally bounded.

1 Introduction

Consider the balance law obtained from Burgers' equation by adding the Hilbert transform as a source term:

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; \mathbf{H}[u]\,. \tag{1.1}$$

Here

$$\mathbf{H}[f](x) \;\doteq\; \lim_{\varepsilon\to 0+} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy$$

denotes the Hilbert transform of a function $f \in \mathbf{L}^2(I\!\!R)$. It is well known [10] that $\mathbf{H}$ is a linear isometry from $\mathbf{L}^2(I\!\!R)$ onto itself. The equation (1.1) was derived in [1] as a model for nonlinear waves with constant frequency. For sufficiently smooth initial data

$$u(0,x) \;=\; \overline{u}(x)\,, \tag{1.2}$$

the local existence and uniqueness of solutions of (1.1) was proved in [7], together with an estimate on the time interval where the solution remains smooth. Here we are mainly concerned with existence and uniqueness of entropy weak solutions globally in time.

Definition 1.1. *By an* **entropy weak solution** *of (1.1)-(1.2) we mean a function* $u \in \mathbf{L}^1_{loc}([0,\infty[\times I\!\!R)$ *with the following properties.*

(i) The map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^2(I\!R)$ and satisfies the initial condition (1.2).

(ii) For any $k \in I\!R$ and every nonnegative test function $\phi \in \mathcal{C}_c^1(]0,\infty[\times I\!R)$ one has

$$\iint \Big[|u-k|\phi_t + \Big(\frac{u^2-k^2}{2}\Big)\mathrm{sign}(u-k)\phi_x + H[u(t)](x)\mathrm{sign}(u-k)\phi\Big]\,dxdt \;\geq\; 0. \quad (1.3)$$

It is well known [3, 8] that Burgers' equation generates a nonlinear contractive semigroup in $\mathbf{L}^1(I\!R)$. Hence, by adding any source term on the right hand side which is Lipschitz continuous as a map from $\mathbf{L}^1$ into itself, one still obtains a continuous flow. The main difficulty here is that the Hilbert transform is a bounded linear operator on $\mathbf{L}^2$, but not on $\mathbf{L}^1$. Our main result provides the global existence of entropy weak solutions.

Theorem 1.2. *Given any initial data $\bar u \in \mathbf{L}^2(I\!R)$, the Cauchy problem (1.1)-(1.2) has an entropy weak solution $u = u(t,x)$ defined for all $(t,x) \in [0,\infty[\times I\!R$. For this solution, the map $t \mapsto \|u(t,\cdot)\|_{\mathbf{L}^2}$ is non-increasing, while $u(t,\cdot) \in \mathbf{L}^\infty(I\!R)$ for every $t > 0$.*

The above solution will be constructed by a flux-splitting method. Relying on the decay properties of the semigroup generated by Burgers' equation [3, 4, 9], we prove that the sequence of approximate solutions is precompact and has a convergent subsequence in $\mathbf{L}^1_{loc}$. Toward a proof of compactness and of $\mathbf{L}^2$-continuity in time, the main technical difficulties stem from the fact that (i) the Hilbert transform is a non-local operator, not bounded w.r.t. the $\mathbf{L}^1$ norm, and (ii) since the initial data can be unbounded, no uniform bound on wave speeds is available. As shown in the following sections, the sequence of approximate solutions satisfies a "tightness" property. According to Lemma 2.1, all characteristics are Hölder continuous. Moreover, the strength with which the values of $u(t,\cdot)$ near two points x and y affect each other (through the Hilbert transform) decays as the distance $|x-y|$ gets larger. A precise estimate in this direction is given in Lemma 3.1.

The $\mathbf{L}^\infty$ bound for solutions of (1.1) is a special case of the a priori estimate proved in Proposition 2.2, for the general balance law

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; g(t,x).$$

In this case, an $\mathbf{L}^\infty$ bound on $u(t,\cdot)$ holds provided that the source term satisfies $\|g(t,\cdot)\|_{\mathbf{L}^2} \leq C$ for all $t \geq 0$. Example 2.5 shows that the conclusion can fail if one only assumes $\|g(t,\cdot)\|_{\mathbf{L}^1} \leq C$.

Uniqueness is a more subtle issue. Indeed, the semigroup $\{S_t\,;\;\; t \geq 0\}$ generated by Burgers' equation is contractive w.r.t. the $\mathbf{L}^1$ distance, but not w.r.t. the $\mathbf{L}^2$ distance. For any $1 < p \leq \infty$, one has

$$\|S_t \bar u\|_{\mathbf{L}^p} \;\leq\; \|\bar u\|_{\mathbf{L}^p}\,.$$

However, for $\bar u, \bar v \in \mathbf{L}^p$ the inequality

$$\|S_t\bar u - S_t \bar v\|_{\mathbf{L}^p} \;\leq\; \|\bar u - \bar v\|_{\mathbf{L}^p}$$

fails, in general. In the present paper we only prove a uniqueness result for spatially periodic solutions having locally bounded variation. The proof relies on Jensen's inequality and on

Lemma 4.2, providing an estimate on the $\mathbf{L}^1$-norm of the Hilbert transform of a periodic function in terms of its total variation over one period.

The question of uniqueness remains largely open. To appreciate the difficulties involved, in Example 4.4 we consider a balance law of the form

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; G(u(t))(x)$$

where $u \mapsto G(u)$ is a Lipschitz continuous map from $\mathbf{L}^2$ into $\mathbf{L}^2$. For a suitable initial data, we prove that the Cauchy problem has multiple solutions, all with bounded variation and uniformly compact support. This shows that, if uniqueness were to hold for solutions of (1.1), the proof cannot be based simply on $\mathbf{L}^2$-Lipschitz continuity combined with BV regularity properties; rather, it must rely on specific properties of the Hilbert transform.

The remainder of the paper is organized as follows. In Section 2 we construct a sequence of approximate solutions of (1.1) by a flux-splitting method and derive a priori $\mathbf{L}^\infty$-bounds. Section 3 is devoted to the proof of global existence of an entropy weak solution. Finally, in Section 4 we prove a result on the uniqueness of spatially periodic solutions of (1.1), and discuss an example where an $\mathbf{L}^2$-Lipschitz perturbation of Burgers' equation yields multiple solutions.

2 Approximate solutions by a flux-splitting method

We shall construct a solution for $t \in [0,1]$. By repeating the procedure, the solution can then be prolonged to any time interval $[0,T]$.

1. A sequence of approximate solutions will be constructed by a flux-splitting method. Let S^B be the semigroup generated by Burgers' operator. More precisely we denote by $t \mapsto S^B_t \bar u$ the solution to

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; 0 \qquad\qquad u(0) \;=\; \bar u \;\in\; \mathbf{L}^1(I\!\!R) \cup \mathbf{L}^\infty(I\!\!R). \tag{2.1}$$

Since $\eta(u) = u^2$ is a convex entropy for the conservation law in (2.1), every admissible solution satisfies

$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x \;\leq\; 0 \tag{2.2}$$

in distributional sense. For every $\bar u \in \mathbf{L}^2(I\!\!R)$ and $t \geq 0$, we thus have the bound

$$\|S^B_t \bar u\|_{\mathbf{L}^2} \;\leq\; \|\bar u\|_{\mathbf{L}^2}\,. \tag{2.3}$$

We also recall that the Hilbert transform satisfies

$$\|\mathbf{H}[u]\|_{\mathbf{L}^2} \;=\; \|u\|_{\mathbf{L}^2}\,, \qquad\qquad \langle \mathbf{H}[u]\,,\; u\rangle \;=\; 0 \qquad\qquad \text{for all } u \in \mathbf{L}^2(I\!\!R)\,. \tag{2.4}$$

Here $\langle\cdot,\cdot\rangle$ denotes the $\mathbf{L}^2$ inner product.

Fix an integer $\nu \geq 1$ and define the times

$$t_i \;\doteq\; i\cdot 2^{-\nu}, \qquad\qquad i = 0,1,2,\ldots$$

The approximate solution u_ν is defined inductively as

$$\begin{cases} u_\nu(0) = \bar{u}, \qquad u_\nu(t_i) \;=\; u_\nu(t_i-) + 2^{-\nu}\mathbf{H}[u_\nu(t_i-)], \qquad i = 1, 2, \ldots \\ u_\nu(t) \;=\; S^B_{t-t_i} u_\nu(t_i) \qquad t \in [t_i,\, t_{i+1}[\,, \quad i = 0, 1, 2, \ldots \end{cases} \tag{2.5}$$

The inequality in (2.3) and the identities in (2.4) yield

$$\|u_\nu(t_i-)\|_{\mathbf{L}^2} \;=\; \|S_{2^{-\nu}} u_\nu(t_{i-1})\|_{\mathbf{L}^2} \;\leq\; \|u_\nu(t_{i-1})\|_{\mathbf{L}^2}\,, \tag{2.6}$$

$$\|u_\nu(t_i)\|_{\mathbf{L}^2} \;\leq\; \|u_\nu(t_i-)\|_{\mathbf{L}^2} \cdot \sqrt{1 + 2^{-2\nu}} \;\leq\; \|u_\nu(t_i-)\|_{\mathbf{L}^2} \cdot \exp\{2^{-\nu}\}\,. \tag{2.7}$$

The second inequality in (2.7) yields the easy estimate

$$\|u_\nu(t)\|_{\mathbf{L}^2} \;\leq\; e^t\, \|\bar{u}\|_{\mathbf{L}^2}\,. \tag{2.8}$$

Using the first inequality in (2.7), by an inductive argument one obtains

$$\|u_\nu(t)\|_{\mathbf{L}^2} \;\leq\; (1 + 2^{-2\nu})^{2^\nu t/2} \cdot \|\bar{u}\|_{\mathbf{L}^2}\,.$$

Taking logarithms of both sides and letting $\nu \to \infty$ we obtain

$$\limsup_{\nu \to \infty} \|u_\nu(t)\|_{\mathbf{L}^2} \;\leq\; \|\bar{u}\|_{\mathbf{L}^2} \qquad \text{for all } t \geq 0.$$

In the next steps we will show that the sequence of flux-splitting approximations $(u_\nu)_{\nu\geq 1}$ is precompact and has a convergent subsequence in $\mathbf{L}^1_{loc}$. This will follow from the decay properties of the semigroup generated by Burgers' equation.

2. We begin by proving a bound on the speed of *generalized characteristics*. By definition, these are absolutely continuous functions $t \mapsto x(t)$ that satisfy the differential inclusion

$$\dot{x}(t) \;\in\; \Big[u_\nu(t, x(t)+),\, u_\nu(t, x(t)-)\Big] \qquad \text{for a.e. } t \in [0, 1]\,. \tag{2.9}$$

Here and in the sequel, an upper dot denotes a derivative w.r.t. time. Notice that $u_\nu(t, \cdot) \in BV$ for every $t \notin \{t_i\,;\;\; i \geq 1\}$, hence the right and left limits in (2.9) are well defined. We say that a characteristic is *genuine* if $u_\nu(t, x(t)+) = u_\nu(t, x(t)-)$ for a.e. t. This happens if the characteristic does not trace a shock. As proved by Dafermos [6], the minimal and maximal backward characteristics through any given point are always genuine.

Lemma 2.1. *For any $\nu \geq 1$, let $t \mapsto x(t)$ be any characteristic for the approximate solution u_ν. Then*

$$|x(t) - x(\tau)| \;\leq\; C_1(\tau - t)^{2/3} \qquad \textit{for all } \; 0 \leq t < \tau \leq 1\,, \tag{2.10}$$

with $C_1 \doteq \; \big(12e^2\, \|\bar{u}\|^2_{\mathbf{L}^2}\big)^{1/3}$.

Proof. It will be convenient to consider the positive and negative part of the initial data

$$\bar{u}(x) \;=\; \max\{\bar{u}(x), 0\} + \min\{\bar{u}(x), 0\} \;\doteq\; \bar{u}^+(x) + \bar{u}^-(x),$$

Similarly, we split the source term into its positive and negative part:

$$g_\nu(t_i, x) \;\doteq\; \mathbf{H}[u_\nu(t_i-)](x) \;=\; g^+_\nu(t_i, x) + g^-_\nu(t_i, x).$$

We then define the functions u_ν^+, u_ν^- inductively by setting

$$
\begin{aligned}
u_\nu^\pm(t) &= S^B_{t-t_i} u_\nu^\pm(t_i) \qquad\qquad t\in [t_i,\, t_{i+1}[\,,\\
u_\nu^\pm(0)=\bar u^\pm, \qquad\qquad u_\nu^\pm(t_i,\cdot) &= u_\nu^\pm(t_i-,\cdot)+2^{-\nu} g_\nu^\pm(t_i,\cdot)
\end{aligned}
$$

A standard comparison theorem for solutions of Burgers' equation yields

$$
\begin{cases}
u_\nu^-(t,x) \;\leq\; 0 \;\leq\; u_\nu^+(t,x), \qquad\qquad \|u_\nu^\pm(t)\|_{\mathbf{L}^2} \;\leq\; e^t\,\|\bar u\|_{\mathbf{L}^2}\,,\\
u_\nu^-(t,x) \;\leq\; u_\nu(t,x) \;\leq\; u_\nu^+(t,x),\\
\|g_\nu^\pm(t_i,\cdot)\|_{\mathbf{L}^2} \;\leq\; \|u_\nu(t_i)\|_{\mathbf{L}^2} \;\leq\; e^{t_i}\,\|\bar u\|_{\mathbf{L}^2}\,.
\end{cases}
\tag{2.11}
$$

Call $t\mapsto y(t)$ the minimal backward characteristic for the positive solution u_ν^+ through the point $(\tau, x(\tau))$. By (2.11), a comparison argument yields $y(t)\leq x(t)$ for all $t\leq\tau$. To estimate the difference $x(\tau)-y(t)$ we shall use the divergence theorem for the conservation law (2.2) on the domain $\{(s,x)\,;\quad s\in[t,\tau],\quad x\geq y(t)\}$. Taking into account the source terms $g_\nu^+(t_i,\cdot)$, we find

$$
\begin{aligned}
0 \;&\leq\; \int_{y(\tau)}^{+\infty} (u_\nu^+(\tau,x))^2\,dx\\
&\leq\; \int_{y(t)}^{+\infty} (u_\nu^+(t,x))^2\,dx + \sum_{t<t_i\leq\tau}\int_{y(t_i)}^{+\infty} (u_\nu^+(t_i,x))^2-(u_\nu^+(t_i-,x))^2 dx\\
&\qquad + \int_t^\tau \Big[\frac{2}{3}(u_\nu^+(s,y(s)))^3-(u_\nu^+(s,y(s)))^2\,\dot y(s)\Big]ds\\
&\leq\; e^{2\tau}\|\bar u\|^2_{\mathbf{L}^2} + \sum_{t<t_i\leq\tau}\int_{y(t_i)}^{+\infty}\big(u_\nu^+(t_i-,x)+2^{-\nu}g_\nu^+(t_i,x)\big)^2-(u_\nu^+(t_i-,x))^2dx - \frac{1}{3}\int_t^\tau \dot y^3(s)ds\\
&\leq e^{2\tau}\|\bar u\|^2_{\mathbf{L}^2} + \sum_{t<t_i\leq\tau}\Big[2^{1-\nu}\|u_\nu^+(t_i,\cdot)\|_{\mathbf{L}^2}\cdot\|g_\nu^+(t_i,\cdot)\|_{\mathbf{L}^2}+2^{-2\nu}\|g_\nu^+(t_i,\cdot)\|^2_{\mathbf{L}^2}\Big] - \frac{1}{3}\int_t^\tau \dot y^3(s)ds\\
&\leq e^{2\tau}\|\bar u\|^2_{\mathbf{L}^2} + \sum_{t<t_i\leq\tau}\Big[2^{1-\nu}\cdot e^{2t_i}\|\bar u\|^2_{\mathbf{L}^2}+2^{-2\nu}\cdot e^{2t_i}\|\bar u\|^2_{\mathbf{L}^2}\Big] - \frac{1}{3}\int_t^\tau \dot y^3(s)ds\\
&\leq\; 4\tau e^{2\tau}\|\bar u\|_{\mathbf{L}^2} - \frac{1}{3}\int_t^\tau \dot y^3(s)ds \leq\; 4e^2\|\bar u\|_{\mathbf{L}^2} - \frac{1}{3}\int_t^\tau \dot y^3(s)ds.
\end{aligned}
$$

Indeed, $\dot y(s)=u_\nu^+(s,y(s))$ because y is a genuine characteristic. Applying Hölder's inequality with $p=3$, $q=3/2$ we obtain

$$
y(\tau)-y(t) \;=\; \int_t^\tau \dot y(s)\,ds \;\leq\; (\tau-t)^{2/3}\left(\int_t^\tau \dot y^3(s)ds\right)^{1/3} \;\leq\; (\tau-t)^{2/3}\cdot\big(12e^2\,\|\bar u\|^2_{\mathbf{L}^2}\big)^{1/3}.
$$

This establishes the bound $x(t)\geq y(t)\geq x(\tau)-C_1(\tau-t)^{2/3}$. The other inequality $x(t)\leq x(\tau)+C_1(\tau-t)^{2/3}$ is proved in the same way, considering the maximal backward characteristic for u_ν^- through the point $(\tau, x(\tau))$. □

The next lemma provides an a priori L^∞-bound, for an entropy solution of Burgers' equation with source term bounded in $\mathbf{L}^2$.

Lemma 2.2. *Let $u = u(t,x)$ be an entropy weak solution to the balance law*

$$u_t + \left(\frac{u^2}{2}\right)_x = g(t,x), \qquad u(0,\cdot) = \bar{u} \in \mathbf{L}^2(I\!\!R). \tag{2.12}$$

Assume that the map $t \mapsto g(t,\cdot)$ is continuous with values in $\mathbf{L}^2(I\!\!R)$. Then $u(t,\cdot) \in \mathbf{L}^\infty(I\!\!R)$ for every $t > 0$.

Proof. **1.** Fix $T > 0$. By continuity we can assume $\|g(t,\cdot)\|_{\mathbf{L}^2} \leq C_g$ for some constant C_g and all $t \in [0,T]$. By a comparison argument, we can also assume that $\bar{u}(x) \geq 0$ and $g(t,x) \geq 0$ for all t,x. If n is an integer such that $2^{-n} \leq T$, we call

$$u_n(T,\cdot) \doteq S^B_{2^{-n}} u(T - 2^{-n})$$

and observe that $u_n(T,\cdot)$ satisfies Oleinik's inequalities

$$u_n(T,y) - u_n(T,x) \leq 2^n(y-x) \qquad \text{for all } x < y. \tag{2.13}$$

If $u_n(T,z) \geq M$ for some $M > 0$, then

$$\|u_n(t,\cdot)\|^2_{\mathbf{L}^2} \geq \int_{z-2^{-n}M}^z u_n^2(T,x)\,dx \geq \int_{z-2^{-n}M}^z [M + 2^n(x-z)]^2\,dx = \frac{1}{3}2^{-n}M^3.$$

Solving for M we obtain the preliminary bound

$$\|u_n(T,\cdot)\|_{\mathbf{L}^\infty} \leq 2^{n/3}\Big(3\|u_n(T,\cdot)\|_{\mathbf{L}^2}\Big)^{1/3} < \infty. \tag{2.14}$$

2. Next, assuming $2^{1-n} < T$, consider the constant

$$K_n \doteq \|u_n(T,\cdot)\|_{\mathbf{L}^\infty} - \|u_{n-1}(T,\cdot)\|_{\mathbf{L}^\infty} - 2^{-n}\,. \tag{2.15}$$

We can then find a point $\bar{x}$ such that

$$u_n(T,\bar{x}) \geq \|u_{n-1}(T,\cdot)\|_{\mathbf{L}^\infty} + K_n\,.$$

If $K_n > 0$, it follows

$$u_n(T,x) - u_{n-1}(T,x) \geq \frac{K_n}{2} \qquad \text{for all } x \in J_n \doteq \Big[\bar{x} - 2^{-n-1}K_n\,,\ \bar{x}\Big].$$

This implies

$$\|u_n(T,\cdot) - u_{n-1}(T,\cdot)\|_{\mathbf{L}^1(J_n)} \geq K_n^2 \cdot 2^{-2-n}. \tag{2.16}$$

The same arguments as in Lemma 2.1 show that the minimal backward characteristic $x(\cdot)$ for u_n starting from $(T,\bar{x})$ satisfies

$$|x(T-\delta) - x(T)| \leq C \cdot \delta^{2/3}$$

for some uniform constant C. Choosing $\delta = 2^{-n}$ and defining

$$J_n^- \doteq \Big[\bar{x} - 2^{-n-1}K_n - C\cdot 2^{-2n/3}\,,\ \bar{x}\Big],$$

by (2.16) we obtain

$$2^{-2-n}K_n^2 \leq \int_{J_n} \Big|u_n(T,x) - u_{n-1}(T,x)\Big|dx \leq \int_{t_{n-1}}^{t_n}\int_{J_n^-} g(t,x)\,dxdt$$

$$\leq 2^{-n}C_g\cdot\sqrt{\text{meas}(J_n^-)} \leq 2^{-n}C_g\cdot\Big[2^{-(n+1)/2}K_n^{1/2} + C^{1/2}2^{-n/3}\Big].$$

Therefore, for some new constant C_0,

$$K_n^2 \leq C_0\Big[2^{-n/2}K_n^{1/2} + 2^{-n/3}\Big] \leq C_0\,2^{-n/3}(K_n^{1/2}+1).$$

For n large this implies $K_n < 1$ and hence

$$K_n \leq 2C_0^2\cdot 2^{-n/6}. \tag{2.17}$$

3. To estimate the norm $\|u(T,\cdot)\|_{\mathbf{L}^\infty}$, choose an integer $N \geq 1$ such that $2^{-N} \leq T$. Using (2.15) and (2.17) we obtain

$$\|u(T,\cdot)\|_{\mathbf{L}^\infty} - \|u_N(T,\cdot)\|_{\mathbf{L}^\infty} \leq \sum_{i=N+1}^{\infty}\Big(\|u_n(T)\|_{\mathbf{L}^\infty} - \|u_{n-1}(T)\|_{\mathbf{L}^\infty}\Big) \leq \sum_{i=N+1}^{\infty}(K_n + 2^{-n}) < \infty. \tag{2.18}$$

By (2.14) we already know that $\|u_N(T,\cdot)\|_{\mathbf{L}^\infty} < \infty$. Hence $\|u(T,\cdot)\|_{\mathbf{L}^\infty} < \infty$ as well. □

Remark 2.3. The above proof shows that the norm $\|u(t,\cdot)\|_{\mathbf{L}^\infty}$ remains uniformly bounded as t ranges over any compact interval $[a,b]$, with $0 < a < b$. It is interesting to understand the asymptotic decay rate of $\|u(T,\cdot)\|_{\mathbf{L}^\infty}$ as $T \to 0$. Given $T > 0$ small, we can choose N such that $2^{-N} \leq T < 2^{1-N}$. In this case, (2.14) yields

$$\|u_N(T,\cdot)\|_{\mathbf{L}^\infty} \leq \frac{C'}{T^{1/3}},$$

for some constant C' depending only on the $\mathbf{L}^2$ norm of the solution. Moreover, the difference in (2.18) remains uniformly bounded as $N \to \infty$. Therefore, by possibly increasing the constant C, we conclude that the solution of (2.12) satisfies an estimate of the form

$$\|u(t,\cdot)\|_{\mathbf{L}^\infty} \leq \frac{C}{t^{1/3}} \qquad \text{for all } t \in\,]0,1]\,, \tag{2.19}$$

for a suitable constant C.

Remark 2.4. A similar estimate remains valid if one assumes $\|g(t,\cdot)\|_{\mathbf{L}^p} \leq C_g$ for some $p > 1$. On the other hand, the following example shows that a uniform bound on $\|g(t,\cdot)\|_{\mathbf{L}^1}$ does not guarantee the boundedness of $u(T,\cdot)$.

Example 2.5 (finite time blow up)**.** Consider the balance law (2.12). As initial data and source term, take

$$\bar{u}(x) = 0, \qquad\qquad g(t,x) \;=\; \frac{1}{1-t}\cdot\chi_{[a(t),b(t)]}(x),$$

where

$$a(t) \;=\; \int_0^t |\ln(1-s)|ds = t + (1-t)\ln(1-t)\,, \qquad b(t) \;=\; 1+(1-t)\ln(1-t) \qquad t\in[0,1[\,.$$

Since $b(t)-a(t) = 1-t$, it is clear that $\|g(t,\cdot)\|_{\mathbf{L}^1} = 1$ for all $t<1$. For $0\le t<1$, the

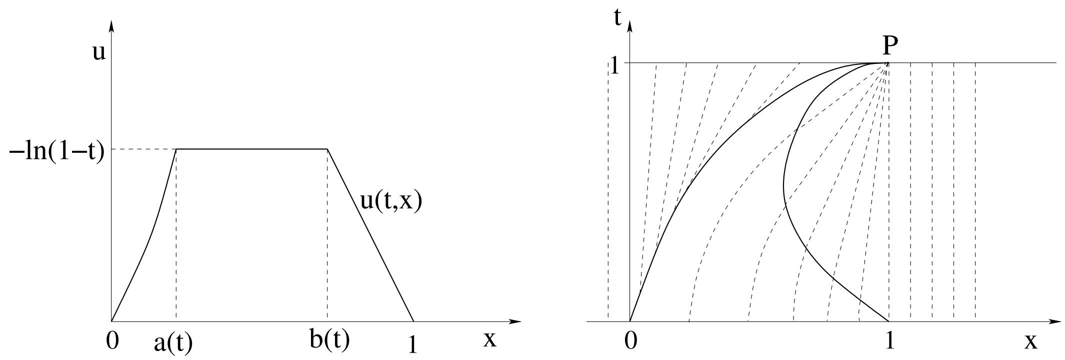

Figure 1: Constructing a solution of Burgers' equation with source, that blows up in finite time. Left: the profile of $u(t,\cdot)$ at some time $0<t<1$. Right: sketch of the characteristics in the t-x plane. Here $P=(1,1)$ is the blow up point.

solution satisfies

$$u(t,x) \;=\; \begin{cases} |\ln(1-t)| & \text{if } \; x\in[a(t),b(t)],\\[2mm] \dfrac{1-x}{1-t}, & \text{if } \; x\in[b(t),1],\\[2mm] 0 & \text{if } \; x\notin[0,1]. \end{cases}$$

Note that, for all $x\in I\!R$ and $t\in[0,1[\,$,

$$u_x(t,x) \;\ge\; -\frac{1}{1-t},$$

hence no shock is formed for $t<1$. The $\mathbf{L}^\infty$ norm of this solution blows up as $t\to 1-$.

3 Global existence of entropy weak solutions

In this section we give a proof of Theorem 1.2, in several steps.

1. Toward a convergence proof, we first estabish a *Tightness Property* for the approximating sequence u_ν in (2.5). Namely:

(TP) *Given $\varepsilon>0$, there exists M so large that*

$$\int_{\{|x|>M\}} |u_\nu(t,x)|^2\,dx \;\le\; \varepsilon \qquad \textit{for every } t\in[0,1],\ \nu\ge 1. \tag{3.1}$$

The key idea toward a proof of (TP) is contained in the following

Lemma 3.1. *Let $H = H_1 \oplus H_2 \oplus \cdots$ be an orthogonal decomposition of a Hilbert space. For each $i \geq 1$, call*

$$K_i^- \doteq H_1 \oplus H_2 \oplus \cdots \oplus H_{i-1}\,, \qquad K_i^+ \doteq H_i \oplus H_{i+1} \oplus \cdots$$

so that $H = K_i^- \oplus K_i^+$, with perpendicular projections

$$\pi_i^- : H \mapsto K_i^-\,, \qquad \pi_i^+ : H \mapsto K_i^+\,.$$

Let $\Lambda : H \mapsto H$ be a bounded linear operator with norm $\|\Lambda\| \leq 1$, such that

$$\left\|\pi_i^+(\Lambda(\pi_{i-1}^-(u)))\right\| \leq 2^{-i}\|u\| \qquad \text{whenever } i \geq 2.$$

Let $t \mapsto u(t) = e^{t\Lambda}\bar{u}$ be the solution to the Cauchy problem

$$\frac{d}{dt}u(t) = \Lambda(u(t)), \qquad u(0) = \bar{u},$$

and assume that

$$\|\pi_i^+\bar{u}\| \leq 2^{-i} \qquad \text{for all } i \geq 1.$$

Then the components of the solution grow slowly in time. Namely

$$\|\pi_i^+ u(t)\| \leq a_i(t)$$

for some nondecreasing functions $a_i(t)$ satisfying

$$\sum_{i\geq 1} a_i(t) \leq (2t+1)e^t. \tag{3.2}$$

In particular, for every $T > 0$ this yields

$$\lim_{i\to\infty} \sup_{t\in[0,T]} \|\pi_i^+ u(t)\| = 0. \tag{3.3}$$

Proof. We first observe that

$$\|u(t)\| = \|\pi_1^+ u(t)\| \leq e^t.$$

Call $p_i(t) \doteq \|\pi_i^+ u(t)\|$. Then the functions $p_i(\cdot)$ satisfy the chain of differential inequalities

$$p_i(0) \leq 2^{-i}, \qquad \frac{d}{dt}p_i(t) \leq p_{i-1}(t) + 2^{-i}e^t. \tag{3.4}$$

Let $a_1(t) = e^t$ and let $a_2, a_3, \ldots$ be the solutions to the system of ODEs

$$a_i(0) = 2^{-i}, \qquad \frac{d}{dt}a_i(t) = a_{i-1}(t) + 2^{-i}e^t. \tag{3.5}$$

Calling $A(t) = \sum_{i\geq 1} a_i(t)$ we have

$$\frac{d}{dt}A(t) \leq A(t) + 2e^t\,, \qquad A(0) = 3/2, \qquad \text{hence} \quad A(t) = (2t + 3/2)e^t.$$

This proves (3.2), and hence the uniform convergence in (3.3). □

2. We now use Lemma 3.1 to prove the tightness property (TP). A key observation is that, if $u \in \mathbf{L}^2(I\!R)$ has support contained in the interval $[-b, b]$, then for every $\kappa > 0$ one has

$$\Big(\int_{I\!R\setminus[-b-\kappa,\, b+\kappa]} \Big|\mathbf{H}[u](x)\Big|^2 dx\Big)^{\frac{1}{2}} \;\leq\; \frac{1}{\pi}\sqrt{\frac{4b}{\kappa}}\cdot \|u\|_{\mathbf{L}^2}\,. \tag{3.6}$$

Indeed, consider the function

$$\varphi(x) \;\doteq\; \begin{cases} (\pi x)^{-1} & \text{if } \;|x|\geq\kappa,\\ 0 & \text{otherwise.}\end{cases}$$

Then (3.6) follows from

$$\int_{I\!R\setminus[-b-\kappa,b+\kappa]} \Big|\mathbf{H}[u](x)\Big|^2 dx \;=\; \int_{I\!R\setminus[-b-\kappa,b+\kappa]} \Big|(\varphi * u)(x)\Big|^2 dx \;\leq\; \|\varphi\|^2_{\mathbf{L}^2}\cdot\|u\|^2_{\mathbf{L}^1} \;=\; \frac{4b}{\pi^2\kappa}\cdot\|u\|^2_{\mathbf{L}^2}\,.$$

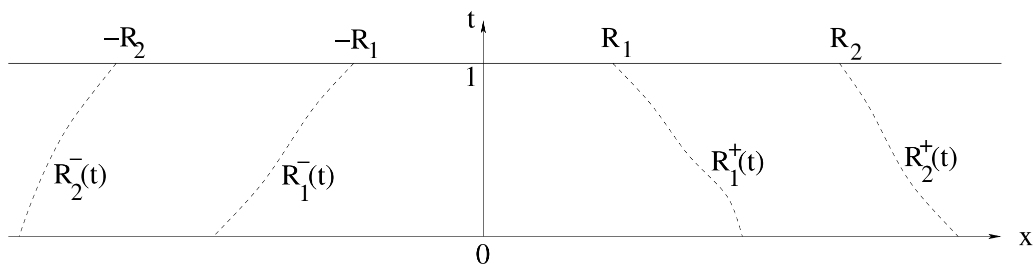

Figure 2: The radii R_i and the backward characteristics $R_i^-(t)$, $R_i^+(t)$.

For a given initial condition $\bar u \in H = \mathbf{L}^2(I\!R)$, let C_1 be the constant in Lemma 2.1 and consider any approximate solution u_ν constructed by the flux splitting method in (2.5). By induction, we define the sequence of radii $(R_i)_{i\geq 1}$ as follows.

(i) The radius R_1 is chosen so that

$$\int_{|x|\geq R_1 - C_1} \bar u^2(x)\,dx \;\leq\; \frac{1}{2}\,. \tag{3.7}$$

(ii) If R_{i-1} is given, we choose R_i large enough so that

$$\int_{|x|\geq R_i - C_1} \bar u^2(x)\,dx \;\leq\; 2^{-i}, \qquad\qquad R_i - R_{i-1} \;\geq\; 2^{i+2}R_{i-1}\|\bar u\|_{\mathbf{L}^2} + 2C_1\,. \tag{3.8}$$

As shown in Fig. 2, given the approximate solution u_ν, we denote by $R_i^+(t)$ the maximal backward characteristic through the point $(t,x) = (1, R_i)$, while $R_i^-(t)$ will denote the minimal backward characteristic through the point $(t,x) = (1,-R_i)$. For each $t \in [0,1]$ we define the spaces

$$H_i(t) \;\doteq\; \Big\{u \in \mathbf{L}^2(I\!R)\,;\quad Supp(u) \;\subseteq\; [R^+_{i-1}(t), R^+_i(t)] \cup [R^-_i(t), R^-_{i-1}(t)]\Big\}.$$

The spaces $K_i^\pm(t)$ and the projections $\pi_i^\pm$ are then defined as in Lemma 3.1.

Let $t \mapsto a_i(t)$ be the functions inductively defined at (3.5). We claim that

$$p_i(t) \doteq \|\pi_i^+ u_\nu(t)\|_{\mathbf{L}^2} \leq a_i(t) \tag{3.9}$$

for every $\nu, i \geq 1$ and $t \in [0,1]$. Indeed, since the curves R_i^-, R_i^+ are characteristics, during each time subinterval $[t_{j-1}, t_j[$ we have

$$\frac{d}{dt} p_i^2(t) = \frac{d}{dt} \int_{I\!R \setminus [R_i^-(t),\, R_i^+(t)]} u_\nu^2(t,x)\, dx \leq 0\,. \tag{3.10}$$

On the other hand, at each time $t_j = j\, 2^{-\nu}$, by (3.6) the source term $\mathbf{H}[u_\nu(t_j-)]$ satisfies

$$\begin{aligned} p_i(t_j) - p_i(t_j-) &\leq 2^{-\nu} \cdot \Big\| \mathbf{H}[u_\nu(t_j-)] \Big\|_{\mathbf{L}^2(I\!R \setminus [R_i^-(t_j),\, R_i^+(t_j)]} \\ &\leq 2^{-\nu} \cdot \Big[p_{i-1}(t_{j-1}) + 2^{-i} \|u_\nu(t_j-)\|_{\mathbf{L}^2} \Big]. \end{aligned} \tag{3.11}$$

By the same argument used in Lemma 3.1, we conclude that, for every $\nu, i \geq 1$ and $t \in [0,1]$, the approximate solution u_ν satisfies

$$\int_{\{x < R_i^-(t)\} \cup \{x > R_i^+(t)\}} u_\nu^2(t,x)\, dx \leq a_i^2(t)\,.$$

Notice that the radii R_i depend only on the initial data $\bar u$, while the characteristics $R_i^\pm(t)$ depend on the particular approximation u_ν. However, Lemma 2.1 yields the uniform bounds

$$|R_i^+(t) - R_i| \leq C_1, \qquad |R_i^-(t) + R_i| \leq C_1, \qquad \text{for all } i, \nu \geq 1. \tag{3.12}$$

Given $\varepsilon > 0$, we choose i such that $a_i^2(t) < \varepsilon$ for all $t \in [0,1]$. By choosing $M > R_i + C_1$, the inequalities (3.1) are then satisfied.

3. Fix an integer $\mu \geq 1$ and define $\delta = 2^{-\mu}$. For any $t_i \doteq i\, 2^{-\nu} \in [\delta, 1]$, consider the approximation

$$u_{\nu,\delta}(t_i) \doteq S_\delta^B u_\nu(t_i - \delta). \tag{3.13}$$

We then extend $u_{\nu,\delta}$ to all times $t \in [\delta, 1]$ in a piecewise affine way, by setting

$$u_{\nu,\delta}(t,x) \doteq (1-\theta) u_{\nu,\delta}(t_i, x) + \theta u_{\nu,\delta}(t_{i+1}, x) \qquad \text{if} \qquad t = (1-\theta)t_i + \theta t_{i+1} \in [\delta, 1]\,. \tag{3.14}$$

For $t \in [\tau - \delta, \tau]$, let $t \mapsto x(t)$ be the minimal backward characteristic for u_ν, through the point $(\tau, -R)$. Similarly, for $t \in [\tau - \delta, \tau]$, call $t \mapsto y(t)$ the maximal backward characteristic for u_ν, through the point (τ, R). By Lemma 2.1 it follows

$$|x(\tau - \delta) + R| \leq C_1 \delta^{2/3}, \qquad |y(\tau - \delta) - R| \leq C_1 \delta^{2/3}.$$

By choosing $\delta > 0$ small enough, we can thus assume

$$-R - 1 \leq x(t) \leq -R, \qquad R \leq y(t) \leq R + 1, \qquad \text{for all } t \in [\tau - \delta, \tau].$$

We claim that, for any fixed $R>0$, there exists constants C_δ, L_δ such that

$$\text{Tot.Var.}\Big\{u_{\nu,\delta}(t)\,;\quad [-R,R]\Big\}\ \leq\ C_\delta \qquad\qquad \text{for all } \nu\geq\mu,\ t\in[\delta,1], \tag{3.15}$$

$$\|u_{\nu,\delta}(t)-u_{\nu,\delta}(s)\|_{\mathbf{L}^1([-R,R])}\ \leq\ L_\delta\,|t-s| \qquad\qquad \text{for all } \nu\geq\mu,\ s,t\in[\delta,1]. \tag{3.16}$$

To prove (3.15) we observe that, by Oleinik's inequality

$$u_{\nu,\delta}(t,y)-u_{\nu,\delta}(t,x)\ \leq\ \frac{y-x}{\delta} \qquad \text{for all } x<y, \tag{3.17}$$

one has

$$\|u_{\nu,\delta}(t)\|^2_{\mathbf{L}^2}\ \geq\ \frac{\delta}{3}\|u_{\nu,\delta}(t)\|^3_{\mathbf{L}^\infty}\,.$$

Hence

$$\|u_{\nu,\delta}(t)\|_{\mathbf{L}^\infty}\ \leq\ \left(\frac{3}{\delta}\|u_{\nu,\delta}(t)\|^2_{\mathbf{L}^2}\right)^{1/3}\ \leq\ \left(\frac{3}{\delta}e^{2t}\|\bar u\|^2_{\mathbf{L}^2}\right)^{1/3}. \tag{3.18}$$

Using (3.17)-(3.18), for every $t\in[\delta,1]$ the total variation of $u_{\nu,\delta}(t)$ over the set $[-R,R]$ can be bounded by

$$\begin{aligned}\text{Tot.Var.}\Big\{u_{\nu,\delta}(t)\,;\quad [-R,R]\Big\}\ &\leq\ [\text{upward variation}]+[\text{downward variation}]\\ &\leq\ \frac{2R}{\delta}+\frac{2R}{\delta}+u(t,-R)-u(t,R)\ \leq\ \frac{4R}{\delta}+2\left(\frac{3}{\delta}e^2\|\bar u\|^2_{\mathbf{L}^2}\right)^{1/3}\ \doteq\ C_\delta\,.\end{aligned} \tag{3.19}$$

To prove the Lipschitz estimates (3.16), it suffices to consider the case where $t=t_i$, $s=t_{i-1}$. By construction one has

$$\begin{aligned}\Big\|u_{\nu,\delta}(t_i-)-u_{\nu,\delta}(t_{i-1})\Big\|_{\mathbf{L}^1([-R,R])}\ &=\ \Big\|S^B_{2^{-\nu}}u_{\nu,\delta}(t_{i-1})-u_{\nu,\delta}(t_{i-1})\Big\|_{\mathbf{L}^1([-R,R])}\\ &\leq\ 2^{-\nu}\cdot[\text{total variation}]\times[\text{maximum characteristic speed}]\ \leq\ 2^{-\nu}\cdot C_\delta\cdot\sup_t\|u_{\nu,\delta}(t)\|_{\mathbf{L}^\infty}\,.\end{aligned} \tag{3.20}$$

In addition,

$$\begin{aligned}\|u_{\nu,\delta}(t_i)-u_{\nu,\delta}(t_i-)\|_{\mathbf{L}^1([-R,R])}\ &\leq\ \big\|S^B_\delta u_\nu(t_i-\delta)-S^B_\delta u_\nu(t_i-\delta-)\big\|_{\mathbf{L}^1([-R,R])}\\ &\leq\ \|u_\nu(t_i-\delta)-u_\nu(t_i-\delta-)\|_{\mathbf{L}^1([-R-1,R+1])}\ =\ 2^{-\nu}\|\mathbf{H}[u_\nu(t_i-\delta-)]\|_{\mathbf{L}^1([-R-1,R+1])}\\ &\leq\ 2^{-\nu}(2R+2)^{1/2}\ \|\mathbf{H}[u_\nu(t_i-\delta-)]\|_{\mathbf{L}^2(I\!R)}\ \leq\ 2^{-\nu}(2R+2)^{1/2}\cdot e\,\|\bar u\|_{\mathbf{L}^2(I\!R)}\,.\end{aligned} \tag{3.21}$$

Together, (3.18), (3.20), and (3.21) imply (3.16), with Lipschitz constant

$$L_\delta\ \doteq\ C_\delta\cdot\left(\frac{3}{\delta}e^2\|\bar u\|^2_{\mathbf{L}^2}\right)^{1/3}+(2R+2)^{1/2}\cdot e\,\|\bar u\|_{\mathbf{L}^2(I\!R)}\,.$$

4. For any $\delta=2^{-\mu}>0$, thanks to the uniform bounds (3.15)-(3.16) we can apply Helly's compactness theorem (see Thm.2.3 in [2]) to the sequence $u_{\nu,\delta}$ on the domain $Q_R\doteq[\delta,1]\times$

$[-R,R]$. We thus obtain a countable subset of indices $\mathcal{I}_\delta \subset I\!N$ and a limit function $u_\delta : Q_R \mapsto I\!R$ satisfying the estimates (3.15)-(3.16), and such that

$$\lim_{\nu\to\infty,\ \nu\in\mathcal{I}_\delta} \|u_{\nu,\delta}(t) - u_\delta(t)\|_{\mathbf{L}^1([-R,R])} ~=~ 0 \qquad \hbox{for all } t\in [\delta, 1]\,, \tag{3.22}$$

$$\lim_{\nu\to\infty,\ \nu\in\mathcal{I}_\delta} \|u_{\nu,\delta} - u_\delta\|_{\mathbf{L}^1(Q_R)} ~=~ 0\,. \tag{3.23}$$

5. We now claim that for any $\tau\in[\delta,1]$, one has the estimate

$$\|u_\nu(\tau) - u_{\nu,\delta}(\tau)\|_{\mathbf{L}^1([-R,R])} ~\leq~ 2\delta(2R+2)^{1/2}\, e^\tau \|\bar u\|_{\mathbf{L}^2}\,. \tag{3.24}$$

To prove (3.24), let $t\mapsto x(t)$ be the minimal backward characteristic for u_ν^+, through the point $(\tau,-R)$. Similarly, call $t\mapsto y(t)$ the maximal backward characteristic for u_ν^-, through the point (τ,R). For every subinterval $[t_i, t_{i+1}]\subset[\tau-\delta,\tau]$, we have

$$\int_{x(t_{i+1})}^{y(t_{i+1})} \Big| u_\nu(t_{i+1}-,x) - S^B_{t_{i+1}-(\tau-\delta)} u_\nu(\tau-\delta,x)\Big|\,dx ~\leq~ \int_{x(t_i)}^{y(t_i)} \Big| u_\nu(t_i-,x) - S^B_{t_i-(\tau-\delta)} u_\nu(\tau-\delta,x)\Big|\,dx\,.$$

Therefore, an inductive argument yields

$$\begin{aligned}
\|u_\nu(\tau) - u_{\nu,\delta}(\tau)\|_{\mathbf{L}^1([-R,R])} ~&\leq~ 2^{-\nu} \sum_{\tau-\delta<t_i\leq\tau} \int_{x(t_i)}^{y(t_i)} \Big|\mathbf{H}[u_\nu(t_i)](x)\Big| dx\\
&\leq~ 2^{-\nu} \sum_{\tau-\delta<t_i\leq\tau} \int_{-R-1}^{R+1} \Big|\mathbf{H}[u_\nu(t_i)](x)\Big| dx ~\leq~ 2^{-\nu} \sum_{\tau-\delta<t_i\leq\tau} (2R+2)^{1/2} \Big\|\mathbf{H}[u_\nu(t_i)](x)\Big\|_{\mathbf{L}^2(I\!R)}\\
&\leq~ 2\delta(2R+2)^{1/2}\, e^\tau \|\bar u\|_{\mathbf{L}^2}\,.
\end{aligned} \tag{3.25}$$

Since the above construction can be repeated for every integer $R\geq 1$ and every $\delta = 2^{-\mu}$, we can select countable set of indices $\mathcal{I}\subset I\!N$ such that

$$\lim_{\nu\to\infty,\ \nu\in\mathcal{I}} \|u_{\nu,\delta}(t) - u_\delta(t)\|_{\mathbf{L}^1([-R,R])} ~=~ 0 \qquad \hbox{for all } t\in [\delta, 1]\,. \tag{3.26}$$

$$\lim_{\nu\to\infty,\ \nu\in\mathcal{I}} \|u_{\nu,\delta} - u_\delta\|_{\mathbf{L}^1(Q_R)} ~=~ 0\,. \tag{3.27}$$

for every $R\geq 1$ and $\delta>0$. We claim that, with this same set $\mathcal{I}$ of indices, one has

$$\lim_{\nu\to\infty,\ \nu\in\mathcal{I}} \|u_\nu(t) - u(t)\|_{\mathbf{L}^1([-R,R])} ~=~ 0 \qquad \hbox{for all } t\in [\delta, 1]\,. \tag{3.28}$$

$$\lim_{\nu\to\infty,\ \nu\in\mathcal{I}} \|u_\nu - u\|_{\mathbf{L}^1(Q_R)} ~=~ 0\,, \tag{3.29}$$

for some limit function $u=u(t,x)$. Indeed, for every fixed $R>0$ and $t\in\,]\delta,1]$, by (3.25) and (3.26) one has

$$\begin{aligned}
\limsup_{\nu,\nu'\to\infty,\,\nu,\nu'\in\mathcal{I}} \|u_\nu(t) - u_{\nu'}(t)\|_{\mathbf{L}^1([-R,R])} ~&\leq~ \limsup_{\nu\to\infty,\,\nu\in\mathcal{I}} \|u_\nu(t) - u_{\nu,\delta}(t)\|_{\mathbf{L}^1([-R,R])}\\
&\quad + \limsup_{\nu,\nu'\to\infty,\,\nu,\nu'\in\mathcal{I}} \|u_{\nu,\delta}(t) - u_{\nu',\delta}(t)\|_{\mathbf{L}^1([-R,R])} + \limsup_{\nu'\to\infty,\,\nu'\in\mathcal{I}} \|u_{\nu',\delta}(t) - u_{\nu'}(t)\|_{\mathbf{L}^1([-R,R])}\\
&\leq~ 2\delta(2R+2)^{1/2}\, e^\tau \|\bar u\|_{\mathbf{L}^2} + 0 + 2\delta(2R+2)^{1/2}\, e^\tau \|\bar u\|_{\mathbf{L}^2}\,.
\end{aligned} \tag{3.30}$$

This proves that the sequence $u_\nu(t,\cdot)$ is Cauchy and hence has a limit in $\mathbf{L}^1([-R,R])$. The proof of (3.29) is entirely similar.

6. In this step we show that the map $t \mapsto u(t)$ is continuous from $[0,T]$ into $\mathbf{L}^2(I\!R)$. To help the reader, we first give a sketch the main argument.

For each $R>0$, the restriction map $t \mapsto u(t,\cdot) \in \mathbf{L}^1([-R,R])$ is continuous. For $t>0$, thanks to the $\mathbf{L}^\infty$ bound proved in Lemma 2.2, this map is also continuous with values in $\mathbf{L}^2([-R,R])$. Thanks to the tightness property (3.1), by choosing R suitably large, the size of the remainder $\|u(t,\cdot)\|_{\mathbf{L}^2(I\!R\setminus[-R,R])}$ can be made arbitrarily small. This shows that the map $t \mapsto u(t) \in \mathbf{L}^2(I\!R)$ can be uniformly approximated by continuous maps. Hence it is continuous at every time $t>0$. An additional argument, based on weak convergence, will establish the continuity also at time $t=0$.

Toward a rigorous proof, we first prove the following Hölder continuity result. For any $R>0$ and $\tau>0$, there exists $0<\delta_0<1$ sufficiently small such that,

$$\|u(t,\cdot)-u(s,\cdot)\|_{\mathbf{L}^1[-R,R]} \;\leq\; L\cdot|t-s|^{3/7}, \tag{3.31}$$

for every $t,s \in [\tau-\delta_0,\tau+\delta_0]$. Here L is a constant depending on R,τ and on $\|\bar u\|_{\mathbf{L}^2}$. Indeed, assume that $t>s$ and set

$$\delta \;\doteq\; (t-s)^{3/7}, \qquad\qquad \delta' \;\doteq\; (t-s)^{3/7}-(t-s).$$

Using (3.24) we obtain the inequalities

$$\|u_\nu(t,\cdot)-u_{\nu,\delta}(t,\cdot)\|_{\mathbf{L}^1[-R,R]} \;\leq\; 2(2R+2)^{1/2}\,e^{\tau+1}\|\bar u\|_{\mathbf{L}^2}\cdot\delta, \tag{3.32}$$

$$\Big\|u_\nu(s,\cdot)-u_{\nu,\delta'}(s,\cdot)\Big\|_{\mathbf{L}^1[-R,R]} \;\leq\; 2(2R+2)^{1/2}\,e^{\tau+1}\|\bar u\|_{\mathbf{L}^2}\cdot\delta. \tag{3.33}$$

On other hand, recalling (3.18) and (3.19), we obtain

$$\Big\|u_{\nu,\delta'}(s,\cdot)\Big\|_{\mathbf{L}^\infty(I\!R)} \;\leq\; 2\left(3e^{2\tau+2}\|\bar u\|^2_{\mathbf{L}^2(I\!R)}\right)^{1/3}\cdot\frac{1}{\delta^{1/3}},$$

$$\text{Tot.Var.}\Big\{u_{\nu,\delta'}(t,\cdot)\,;\;\;[-R,R]\Big\} \;\leq\; 2\Big(4R+2\left(3e^{2\tau+2}\|\bar u\|^2_{\mathbf{L}^2(I\!R)}\right)^{1/3}\Big)\cdot\frac{1}{\delta},$$

for $\delta_0>0$ sufficiently small and $t,s\in[\tau-\delta_0,\tau+\delta_0]$. Therefore,

$$\begin{aligned}
&\Big\|S^B_{t-s}u_{\nu,\delta'}(s,\cdot)-u_{\nu,\delta'}(s,\cdot)\Big\|_{\mathbf{L}^1([-R,R])}\\
&\quad\leq\; (t-s)\cdot[\text{total variation}]\times[\text{maximum characteristic speed}]\\
&\quad\leq\; 4\left(4R+2\left(3e^{2\tau+2}\|\bar u\|^2_{\mathbf{L}^2(I\!R)}\right)^{1/3}\right)\cdot\left(3e^{2\tau+2}\|\bar u\|^2_{\mathbf{L}^2(I\!R)}\right)^{1/3}\cdot\frac{t-s}{\delta^{4/3}}\\
&\quad=\; 4\left(4R+2\left(3e^{2\tau+2}\|\bar u\|^2_{\mathbf{L}^2(I\!R)}\right)^{1/3}\right)\cdot\left(3e^{2\tau+2}\|\bar u\|^2_{\mathbf{L}^2(I\!R)}\right)^{1/3}\cdot\delta\,.
\end{aligned} \tag{3.34}$$

Combining (3.32), (3.33), (3.34) and noting that $S^B_{t-s}u_{\nu,\delta'}(s,\cdot)=u_{\nu,\delta}(t,\cdot)$, we obtain

$$\|u_\nu(t,\cdot)-u_\nu(s,\cdot)\|_{\mathbf{L}^1[-R,R]} \;\leq\; L\cdot\delta,$$

for some constant L depending on R, τ, and on an a priori bound on the $\mathbf{L}^2$ norm of the solution. The Hölder continuity estimate (3.31) is now obtained by letting $\nu \to \infty$ (with $\nu \in \mathcal{I}$).

We are now ready to prove the continuity of the function $t \mapsto u(t) \in \mathbf{L}^2(I\!R)$. Fix any $\tau > 0$. By (3.1), for any $\varepsilon > 0$ there exist $\delta > 0$ and $R > 0$ such that

$$\|u(t,\cdot)\|_{L^2(I\!R\setminus[-R,R])} \;\leq\; \frac{\varepsilon}{4} \qquad \text{for all } t \in [\tau-\delta, \tau+\delta].$$

Therefore

$$\|u(t,\cdot) - u(\tau,\cdot)\|_{L^2(I\!R\setminus[-R,R])} \;\leq\; \frac{\varepsilon}{2}, \quad \text{for all } t \in [\tau-\delta, \tau+\delta].$$

On the other hand, recalling (3.31) and Remark 2.3, one can show that there exists $\delta_\tau > 0$ such that $\|u(t,\cdot)\|_{\mathbf{L}^\infty}$ is uniformly bounded for all $t \in [\tau-\delta_\tau, \tau+\delta_\tau]$. Hence, there exists $0 < \delta_0 < \min\{\delta, \delta_\tau\}$ such that

$$\|u(t,\cdot) - u(\tau,\cdot)\|_{\mathbf{L}^2([-R,R])} \;\leq\; \frac{\varepsilon}{2}, \quad \text{for all } t \in [\tau-\delta_0, \tau+\delta_0].$$

Therefore

$$\|u(t,\cdot) - u(\tau,\cdot)\|_{\mathbf{L}^2(I\!R)} \;\leq\; \varepsilon \qquad \text{for all } t \in [\tau-\delta_0, \tau+\delta_0].$$

This proves the continuity of the map $t \mapsto u(t,\cdot) \in \mathbf{L}^2(I\!R)$ for all $t > 0$.

Finally, we show that continuity also holds at time $t = 0$. Given any $R > 0$, by (3.24) we obtain

$$\|u(t,\cdot) - S_t^B \bar{u}\|_{\mathbf{L}^1([-R,R])} \;\leq\; 2t \cdot (2R+2)^{1/2}\, e^t \|\bar{u}\|_{\mathbf{L}^2}.$$

In particular, $\lim_{t\to 0+} \|u(t,\cdot) - S_t^B \bar{u}\|_{\mathbf{L}^1([-R,R])} = 0$. Moreover, the continuity of $t \mapsto S_t^B \bar{u}$ in $\mathbf{L}^1_{loc}$ implies that

$$\lim_{t\to 0+} \|u(t,\cdot) - \bar{u}\|_{\mathbf{L}^1([-R,R])} \;=\; 0. \tag{3.35}$$

Next, consider any sequence $t_n \downarrow 0$. Thanks to the uniform bound on $\|u(t_n,\cdot)\|_{\mathbf{L}^2}$, by possibly taking a subsequence we can assume the weak convergence $u(t_n,\cdot) \to w$ for some limit function $w \in \mathbf{L}^2$. From (3.35) it now follows that $w = \bar{u}$. By the inequality

$$\limsup_{t_n\to 0} \|u(t_n,\cdot)\|_{\mathbf{L}^2} \;\leq\; \|\bar{u}\|_{\mathbf{L}^2}$$

we deduce the strong convergence $\|u(t_n,\cdot) - \bar{u}\|_{\mathbf{L}^2} \to 0$.

7. In this last step we show that u is an entropy weak solution of (1.1). Let $\eta \in \mathcal{C}^2(I\!R)$ be a convex entropy with flux q, so that $q'(u) = u\eta'(u)$. Define the times $t_i = i \cdot 2^{-\nu}$ and consider the flux-splitting approximations u_ν in (2.5).

For every nonnegative test function $\phi \in \mathcal{C}^1_c(]0,\infty[\times I\!R)$, observing that u_ν is an entropy solution to Burgers' equation on $[t_i, t_{i+1}[$, we obtain

$$\begin{aligned}
\iint \phi_t\eta(u_\nu) + \phi_x q(u_\nu)\,dxdt \;&=\; \sum_i \int_{t_i}^{t_{i+1}} \int \phi_t\eta(u_\nu) + \phi_x q(u_\nu)\,dxdt \\
&\geq\; \sum_i \int_{\mathbb{R}} \phi(t_{i+1}, x)\eta(u_\nu(t_{i+1}-, x)) - \phi(t_i, x)\eta(u_\nu(t_i, x))dx \\
&=\; -\sum_i \int_{\mathbb{R}} \phi(t_i, x)\Big[\eta(u_\nu(t_i, x)) - \eta(u_\nu(t_i-, x))\Big]dx \\
&=\; -\sum_i \int_{\mathbb{R}} \phi(t_i, x)\Big[\eta\Big(u_\nu(t_i, x) + 2^{-\nu}\mathbf{H}[u_\nu(t_i-)](x)\Big) - \eta(u_\nu(t_i-, x))\Big]dx\,.
\end{aligned} \tag{3.36}$$

By the continuity of the maps $t \mapsto u(t) \in \mathbf{L}^2$, $t \mapsto \mathbf{H}[u(t)] \in \mathbf{L}^2$, and the convergence $\|u_\nu(t) - u(t)\|_{\mathbf{L}^2} \to 0$ uniformly for t in compact intervals, we conclude that as $\nu \to \infty$ the left hand side of (3.36) converges to

$$\iint \phi_t\eta(u) + \phi_x q(u)\,dxdt\,,$$

while the right hand side converges to

$$-\int_0^\infty \int \phi(t, x)\eta'(u(t, x))\mathbf{H}[u(t)](x)\,dxdt\,.$$

By (3.36) we thus have the inequality

$$\iint \Big\{\phi_t\,\eta(u) + \phi_x\,q(u) + \phi\,\eta'(u)\,\mathbf{H}[u(t)](x)\Big\}\,dxdt \;\geq\; 0\,,$$

for every test function $\phi \geq 0$ and every convex entropy $\eta \in \mathcal{C}^2$ with flux q. By approximating the entropy $\eta_k(u) = |u - k|$ with a sequence of smooth entropies, say $\eta^{(n)}(u) = \sqrt{(u-k)^2 + n^{-1}}$, the inequality (1.3) is achieved in the limit $n \to \infty$.

4 A uniqueness result

In this section we establish a uniqueness result for solutions to the Burgers-Hilbert equation in the spatially periodic case. More precisely, let us consider

$$u_t + \Big(\frac{u^2}{2}\Big)_x \;=\; \mathbf{H}_{per}[u], \qquad u(0,\cdot) \;=\; \bar{u}. \tag{4.1}$$

Here the initial state $\bar{u}$ is periodic with period 2π and $\mathbf{L}^2$ on $[0, 2\pi]$. Moreover, for any $f \in \mathbf{L}^2$ periodic with period 2π, the Hilbert transform of f is defined in terms of a convolution with the cotangent function:

$$\mathbf{H}_{per}[f](x) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \cot\Big(\frac{x-y}{2}\Big)\,f(y)\,dy.$$

Definition 4.1. *A function $u \in \mathbf{L}^1_{loc}([0,\infty[\times I\!R)$ is an **entropy weak solution** of (4.1) if*

(i) For every $t > 0$, $u(t,\cdot)$ is periodic with period 2π.

(ii) The map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^2([0,2\pi])$ and $u(0,\cdot) = \bar{u}$.

(iii) For any $k \in I\!R$ and every nonnegative test function $\phi \in \mathcal{C}^1_c(]0,\infty[\times I\!R)$ one has

$$\iint \Big[|u-k|\phi_t + \Big(\frac{u^2-k^2}{2}\Big)\,\mathrm{sign}(u-k)\phi_x + \mathbf{H}_{per}[u(t)](x)\,\mathrm{sign}(u-k)\phi\Big]\,dxdt \ \geq\ 0. \quad (4.2)$$

One can construct an entropy weak solution of (4.1) by using the a flux-splitting method as in Section 2. To prove our uniqueness result, the following lemma will be needed.

Lemma 4.2. *For some constant C, the following holds. Let w be a periodic function with period 2π, with*

$$\int_0^{2\pi} w(x)\,dx \ =\ 0, \qquad\qquad Tot.Var.\{w\,;\ [0,2\pi]\}\ <\ \infty. \quad (4.3)$$

Then

$$\|\mathbf{H}_{per}[w]\|_{\mathbf{L}^1([0,2\pi])}\ \leq\ C\|w\|_{\mathbf{L}^1([0,2\pi])}\Big(6+\ln\big(\,Tot.Var.\{w\,;\ [0,2\pi]\}\big)-\ln\big(\|w\|_{\mathbf{L}^1([0,2\pi])}\big)\Big). \quad (4.4)$$

Proof. **1.** For any $a < b$, with $b-a < 2\pi$, let $\chi_{[a,b]}$ be the characteristic function of the interval $[a,b]$, extended to the whole real line by 2π-periodicity. We claim that the Hilbert transform of $\chi_{[a,b]}$ satisfies

$$\big\|\mathbf{H}_{per}[\chi_{[a,b]}](x)\big\|_{\mathbf{L}^1([0,2\pi])}\ \leq\ C(b-a)\cdot(6-\ln(b-a)), \quad (4.5)$$

for some constant $C > 0$. Indeed, by performing a translation (and by possibly replacing $\chi_{[a,b]}$ with $1-\chi_{[a,b]}$), it is not restrictive to assume that

$$\frac{\pi}{2} \leq a \leq b \leq \frac{3\pi}{2}\,.$$

For any $x \in [0,2\pi[$, the Hilbert transform of $\chi_{[a,b]}$ is computed by

$$\mathbf{H}_{per}[\chi_{[a,b]}](x)\ =\ \frac{1}{2\pi}\int_a^b \cot\Big(\frac{x-y}{2}\Big)\,dy\ =\ \frac{1}{\pi}\ln\left|\frac{\sin\frac{x-a}{2}}{\sin\frac{x-b}{2}}\right|.$$

Introducing the smooth function $\phi(x)\ =\ \frac{\sin(x/2)}{x/2}$, we estimate

$$\begin{aligned}
\Big\|\mathbf{H}_{per}[\chi_{[a,b]}]\Big\|_{\mathbf{L}^1([0,2\pi])}\ &=\ \frac{1}{\pi}\int_0^{2\pi}\left|\ln\left|\frac{\sin\frac{x-a}{2}}{\sin\frac{x-b}{2}}\right|\right|dx \\
&\leq\ \frac{1}{\pi}\int_0^{2\pi}\left|\ln\left|\frac{x-a}{x-b}\right|\right|dx + \frac{1}{\pi}\int_0^{2\pi}\left|\ln\left|\frac{\phi(x-a)}{\phi(x-b)}\right|\right|dx \\
&\leq\ \frac{1}{\pi}\Big[(2\pi-a)\ln(2\pi-a)-(2\pi-b)\ln(2\pi-b)+b\ln(b)-a\ln(a) \\
&\qquad -2(b-a)\ln(b-a)+2(b-a)\ln(2)\Big] + C'\cdot\max_{x\in[0,2\pi]}\left|\frac{\phi(x-a)}{\phi(x-b)}-1\right| \\
&\leq\ C(b-a)\cdot(5-\ln(b-a)) + C(b-a).
\end{aligned} \quad (4.6)$$

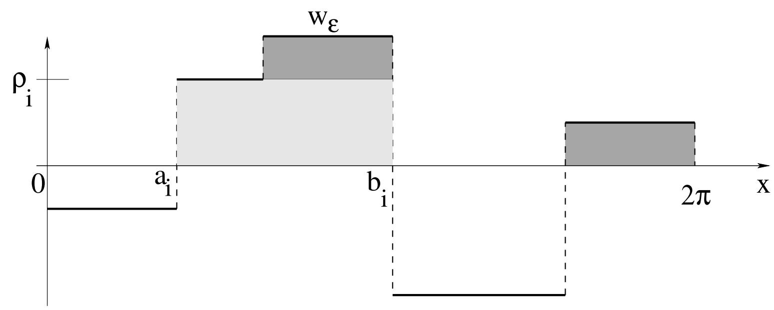

Figure 3: A piecewise constant function w_ε with zero average can be decomposed as a sum of characteristic functions of intervals, satisfying (4.7).

2. Now consider any periodic function w satisfying (4.3). For any $\varepsilon > 0$ we can approximate w with a piecewise constant function w_ε such that

$$\int_0^{2\pi} w_\varepsilon(x)\,dx \;=\; 0, \qquad \|w - w_\varepsilon\|_{\mathbf{L}^2([0,2\pi])} \;\leq\; \varepsilon, \qquad \|w_\varepsilon\|_{\mathbf{L}^1([0,2\pi])} \;\leq\; \|w\|_{\mathbf{L}^1([0,2\pi])}\,,$$

and

$$\text{Tot.Var.}\{w_\varepsilon\,;\ [0,2\pi]\} \;\leq\; \text{Tot.Var.}\{w\,;\ [0,2\pi]\}.$$

By slicing the graph of w_ε horizontally (see Fig. 3), we can write w_ε (restricted to one period) as a sum of characteristic functions:

$$w_\varepsilon \;=\; \sum_{i=1}^N \rho_i \cdot \chi_{[a_i,b_i]},$$

in such a way that

$$\|w_\varepsilon\|_{\mathbf{L}^1([0,2\pi])} \;=\; \sum_{i=1}^N |\rho_i|(b_i - a_i)\,, \qquad \text{Tot.Var.}\{w_\varepsilon\,;\ [0,2\pi]\} \;=\; 2\sum_{i=1}^N |\rho_i|\,. \tag{4.7}$$

By (4.5) it follows

$$\int_0^{2\pi} |H_{per}[w_\varepsilon](x)|\ dx \;\leq\; \sum_{i=1}^N \rho_i \Big\| |H[\chi_{[a_i,b_i]}] \Big\|_{\mathbf{L}^1([0,2\pi])} \;\leq\; \sum_{i=1}^N C\rho_i(b_i - a_i)(6 - \ln(b_i - a_i)). \tag{4.8}$$

We now set $\delta_i \doteq (b_i - a_i)$ and $\rho \doteq \sum_{j=1}^N |\rho_j|$. Applying Jensen's inequality to the concave function $\varphi(s) = -s\ln s$ we obtain

$$\begin{aligned} -\sum_i |\rho_i|\delta_i \ln\delta_i \;&=\; -\,\rho\cdot\sum_{i=1}^N \frac{|\rho_i|}{\rho}\delta_i\ln\delta_i \;\leq\; -\,\rho\left(\sum_{i=1}^N \frac{|\rho_i|\delta_i}{\rho}\right)\cdot\ln\left(\sum_{i=1}^N \frac{|\rho_i|\delta_i}{\rho}\right) \\ &\leq\; \|w_\varepsilon\|_{\mathbf{L}^1([0,2\pi])}\cdot\Big[\ln\left(\frac{1}{2}\,\text{Tot.Var.}\{w_\varepsilon;\ [0,2\pi]\}\right) - \ln\big(\|w_\varepsilon\|_{\mathbf{L}^1([0,2\pi])}\big)\Big]. \end{aligned} \tag{4.9}$$

Hence, from (4.8) it follows

$$\begin{aligned} \int_0^{2\pi} |H_{per}[w_\varepsilon](x)|\,dx \;&\leq\; \sum_{i=1}^N C\,|\rho_i|\delta_i(6 - \ln\delta_i) \\ &\leq\; C\,\|w_\varepsilon\|_{\mathbf{L}^1([0,2\pi])}\Big(6 + \ln\big(\,\text{Tot.Var.}\{w_\varepsilon, [0,2\pi]\}\big) - \ln\big(\|w_\varepsilon\|_{\mathbf{L}^1([0,2\pi])}\big)\Big). \end{aligned} \tag{4.10}$$

The proof is now achieved by letting $\varepsilon \to 0$. $\square$

Relying on the above lemma, we can now prove a uniqueness result in the periodic case.

Theorem 4.3. *Let u, v be entropy weak solutions of the spatially periodic Cauchy problem (4.1), with the same initial data. Assume that the total variation of $u(t, \cdot)$ and $v(t, \cdot)$ over $[0, 2\pi]$ remains uniformly bounded for $t \in [0, T]$. Then u and v coincide for all $t \in [0, T]$.*

Proof. Set

$$w(t,x) \doteq u(t,x) - v(t,x), \qquad Z(t) \doteq \|u(t,\cdot) - v(t,\cdot)\|_{\mathbf{L}^1([0,2\pi])}. \tag{4.11}$$

The uniform BV bounds on u, v imply that the maps $t \mapsto u(t)$, $t \mapsto v(t)$ are both Lipschitz continuous with values in $\mathbf{L}^1([0, 2\pi])$. Therefore, the scalar function $t \mapsto Z(t)$ is also Lipschitz continuous, hence a.e. differentiable. In addition, since u, v are both weak solutions, their average value remains constant in time:

$$\int_0^{2\pi} u(t,x)\,dx = \int_0^{2\pi} u(0,x)\,dx = \int_0^{2\pi} v(0,x)\,dx = \int_0^{2\pi} v(t,x)\,dx.$$

Therefore, the function $w(t, \cdot)$ defined at (4.11) has zero average for every $t \geq 0$.

Since Burgers' equation generates a contractive semigroup, using Lemma 4.2 we obtain

$$\frac{d}{dt}Z(t) \leq \Big\|\mathbf{H}_{per}[w(t)]\Big\|_{\mathbf{L}^1([0,2\pi])} \leq \alpha\, Z(t)\,[\beta - \ln Z(t)]\,,$$

for some constants α, β depending on an upper bound on the total variation of $u(t, \cdot)$ and $v(t, \cdot)$. By Osgood's criterion, $Z(0) = 0$ implies $Z(t) = 0$ for every $t > 0$. This establishes the uniqueness of BV solutions in the spatially periodic case. $\square$

In general, the question of uniqueness of entropy weak solutions remains open. To appreciate the subtlety of the problem, consider the Cauchy problem

$$u_t + \left(\frac{u^2}{2}\right)_x = G[u], \qquad u(0,\cdot) = \bar{u} \in \mathbf{L}^2(I\!R), \tag{4.12}$$

where $G : \mathbf{L}^2(I\!R) \mapsto \mathbf{L}^2(I\!R)$ is a Lipschitz continuous map. As shown by the following example, even if every function $G[u]$ has uniformly bounded total variation and bounded support, the above problem can have multiple solutions.

Example 4.4 (nonuniqueness)**.** Consider the initial data (Fig. 4)

$$\bar{u}(x) \doteq \begin{cases} 0 & \text{if } x < -1 \text{ or } x > 3, \\ x+1 & \text{if } x \in [-1, 0], \\ 1 & \text{if } x \in [0, 1], \\ -1 & \text{if } x \in]1, 2], \\ x-3 & \text{if } x \in]2, 3]. \end{cases} \tag{4.13}$$

In addition, consider the function

$$u(t,x) \doteq \begin{cases} \bar{u}(x) & \text{if } x \leq 0 \text{ or } x > 1 + t^6, \\ 1 + h(t)x & \text{if } x \in [0, 1 + t^6]. \end{cases} \tag{4.14}$$

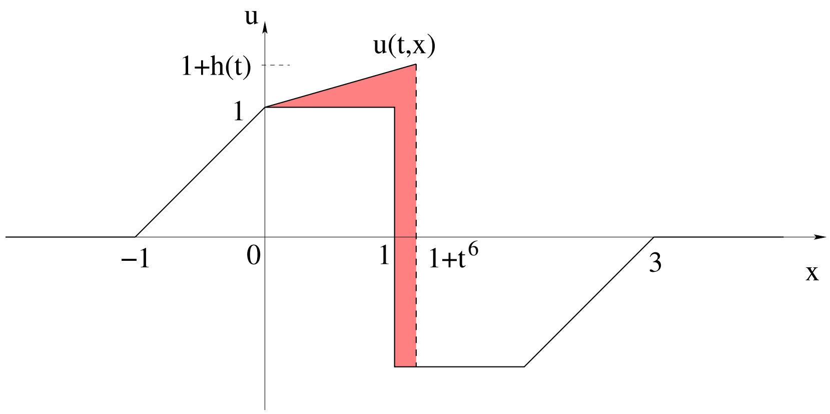

Figure 4: A Cauchy problem for Burgers' equation with $\mathbf{L}^2$-Lipschitz continuous source term but multiple solutions.

Here the function $h(t)$ is chosen so that the shock located at $x(t) = 1 + t^6$ satisfies the Rankine-Hugoniot conditions. More precisely,

$$\dot{x}(t) \;=\; 6t^5 \;=\; \frac{u^-(t) + u^+(t)}{2} \;=\; \frac{1 + h(t)(1+t^6) - 1}{2}\,.$$

This yields

$$h(t) \;=\; \frac{12t^5}{1+t^6} \;\approx\; 12t^5\,. \tag{4.15}$$

For $t \in [0,\, 1/2]$, the first two derivatives of h satisfy

$$0 \;\leq\; \dot{h}(t) \;\leq\; C_0 t^4\,, \qquad\qquad |\ddot{h}(t)| \;\leq\; C_0 t^3, \tag{4.16}$$

for some constant C_0.

We seek a Lipschitz continuous map G such that the above function $u = u(t,x)$ is a solution to the Cauchy problem (4.12). This is the case if

$$G[u(t)](x) \;=\; u_t(t,x) + u(t,x)u_x(t,x).$$

In other words,

$$G[u(t)](x) \;=\; \begin{cases} \bar{u}(x)\bar{u}_x(x) & \text{if} \;\; x < -1 \text{ or } x > 1+t^6, \\ \dot{h}(t)x + (1+h(t)x)h(t) & \text{if} \;\; x \in [0,\, 1+t^6]. \end{cases} \tag{4.17}$$

Notice that (4.17) determines the values of G on the domain

$$\mathcal{D} \;\doteq\; \{u(t)\,; \;\; t \in [0, 1/2]\} \;\subset\; \mathbf{L}^2(I\!\!R)\,.$$

The map $G : \mathcal{D} \mapsto \mathbf{L}^2$ is Lipschitz continuous provided that

$$\|G[u(t)] - G[u(s)]\|_{\mathbf{L}^2} \;\leq\; C\,\|u(t) - u(s)\|_{\mathbf{L}^2} \tag{4.18}$$

for some constant C and any $0 \leq s < t \leq \frac{1}{2}$. To prove (4.18), we observe that

$$\|u(t) - u(s)\|^2_{\mathbf{L}^2} \;\geq\; \int_{1+s^6}^{1+t^6} 2^2\,dx \;=\; 4(t^6 - s^6). \tag{4.19}$$

On the other hand, by (4.16) we have

$$
\begin{aligned}
\|G[u(t)]-G[u(s)]\|^2_{\mathbf{L}^2} \;&=\; \int_0^{1+s^6}\Big[\dot h(t)x+(1+h(t)x)h(t)-\dot h(s)x+(1+h(s)x)h(s)\Big]^2\,dx\\
&\qquad + \int_{1+s^6}^{1+t^6}\Big[\dot h(t)x+(1+h(t)x)h(t)\Big]^2\,dx\\
&\le\; C_1\int_0^2\Big[(\dot h(t)-\dot h(s))^2x^2+(h(t)-h(s))^2+(h^2(t)-h^2(s))^2x^2\Big]\,dx+C_1(t^6-s^6)\\
&\le\; C_2\Big[(t^3(t-s))^2+(t^4(t-s))^2+(t^5\,t^4(t-s))^2+(t^6-s^6)\Big]\\
&\le\; C_3(t^6-s^6),
\end{aligned}
\tag{4.20}
$$

for some constants C_1, C_2, C_3. Comparing (4.20) with (4.19) we conclude that (4.18) holds.

By the Kirszbraun-Valentine extension theorem for Lipschitz continuous maps between Hilbert spaces (see [5]), we can extend G to a globally Lipschitz continuous map $\widetilde G : \mathbf{L}^2(I\!\!R) \mapsto \mathbf{L}^2(I\!\!R)$, whose range is contained in the convex closure of the range of G. In particular, for every $u \in \mathbf{L}^2(I\!\!R)$, the image $\widetilde G(u)$ will be a function with bounded variation and compact support.

Consider now the Cauchy problem (4.12), with G replaced by $\widetilde G$ and with initial data $\bar u$ as in (4.13). This problem has two solutions: one is the function in (4.14), the other is the constant function $u(t,x) = \bar u(x)$.

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