Continuous Riemann Solvers for Traffic Flow at a Junction

Alberto Bressan and Fang Yu

Department of Mathematics, Penn State University
University Park, PA. 16802, U.S.A.

e-mails: bressan@math.psu.edu , fuy3@psu.edu

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Abstract

The paper studies a class of conservation law models for traffic flow on a family of roads, near a junction. A Riemann Solver is constructed, where the incoming and outgoing fluxes depend Hölder continuously on the traffic density and on the drivers' turning preferences. However, various examples show that, if junction conditions are assigned in terms of Riemann Solvers, then the Cauchy problem on a network of roads can be ill posed, even for initial data having small total variation.

1 Introduction

Conservation laws have become a popular tool in the modeling of traffic flow on a network of roads. For a general introduction and a survey of recent literature we refer to [12, 2, 9]. We recall that the basic conservation law describing traffic density on a single road were first studied in the classic papers [17, 18]. On the other hand, models of traffic flow at road junctions or on an entire network are lively topics of current research.

Due to finite propagation speed, the well-posedness of the Cauchy problem can be studied in the neighborhood of one single intersection. The dynamics is determined by assigning a scalar conservation law on each road, together with a family of boundary conditions describing the flow of cars at the junction. These are determined by (i) drivers' turning preferences and (ii) priorities assigned to different roads. As shown in [7, 12, 13] a set of boundary conditions can be assigned in terms of a *Riemann Solver*. As soon as the solution of the initial value problem with piecewise constant initial data is determined, by front tracking approximations one can then construct solutions to Cauchy problems with general initial data.

The present study was originally motivated by the aim of extending the results in [3], proving the existence of globally optimal solutions and of Nash equilibrium solutions for traffic flow on general networks of roads. We recall that the existence theorems proved in [3] refer to a network where a buffer of infinite capacity is present at the beginning of each outgoing road.

A more realistic model, such as the ones proposed in [7, 12, 13], would allow for queues to propagate backward, along roads leading to a crowded intersection.

In order to be of practical use, the models of traffic flow at intersections should yield unique solutions, continuously depending on the data. Here the initial data comprise:

(i) The initial density $\rho_k^\diamondsuit(x)$ of traffic, on each incoming and on each outgoing road.

(ii) The fraction $\theta_{ij}^\diamondsuit(x)$ of drivers, initially located at the point x of the i-th incoming road, who eventually wish to turn into the j-th outgoing road.

It is easy to see that the Riemann solver $\mathcal{RS}1$ proposed in [13] does not exhibit continuous dependence on the initial data $\theta_{ij}^\diamondsuit$. A natural question is whether there can be other Riemann solvers, satisfying the same natural modeling assumptions, which yield a well posed Cauchy problem.

The findings reported in this paper are largely negative, and motivate the introduction of a different approach to boundary conditions at a junction, described in the forthcoming paper [4]. In summary, we prove:

(i) There exists a class of Riemann solvers $\mathcal{RS}^\sharp$, depending Hölder continuously on the data $\rho_k^\diamondsuit, \theta_{ij}^\diamondsuit$, with Hölder exponent $\gamma = 1/2$.

(ii) For a single intersection with two incoming and two outgoing roads, one can construct initial data $\rho_k^\diamondsuit, \theta_{ij}^\diamondsuit$ yielding two distinct solutions. Here the initial densities $\rho_k^\diamondsuit$ are all constant but the drivers' turning preferences $\theta_{ij}^\diamondsuit$ have unbounded variation.

(iii) For a network with three intersections, one can construct initial data $\rho_k^\diamondsuit, \theta_{ij}^\diamondsuit$ on each road, having arbitrarily small total variation, so that the Cauchy problem after a finite time develops two distinct solutions.

(iv) For a simple junction with one incoming road and two outgoing roads, the solution to the Cauchy problem cannot be continuous w.r.t. the topology of weak convergence.

The remainder of this paper is organized as follows. Section 2 collects the basic modeling assumptions, and reviews the main properties of the flux functions and of a Riemann solver. In Section 3 we construct a Hölder continuous Riemann solver $\mathcal{RS}^\sharp$. This is the same as $\mathcal{RS}1$ in [13], except for the fact that here we maximize a weighted product of the incoming fluxes instead of their sum.

In Section 4 we construct an example of initial data near an intersection, where the Cauchy problem has two distinct solutions. We emphasize that here the non-uniqueness has nothing to do with the lack of Lipschitz continuity. Indeed, multiple solutions are found for the Riemann solver $\mathcal{RS}1$ in [13] as well as for our Hölder continuous Riemann solver $\mathcal{RS}^\sharp$ constructed in Section 3. The real cause of non-uniqueness is the unbounded variation of the measurable coefficients $\theta_{ij}^\diamondsuit$. This should be compared with the example in [6], where a strictly hyperbolic system of three equations in one space dimension was shown to have infinitely many entropy admissible solutions, for a very similar type of initial data having unbounded variation.

In spite of this counterexample, one may still hope to achieve an existence-uniqueness theory within a class of solutions having small total variation. For a network having three junctions, the additional example given in Section 5 shows that this is not the case. Here the initial densities $\rho_k^\diamondsuit$ have arbitrarily small total variation, and the turning preferences $\theta_{ij}^\diamondsuit$ are initially constant along each road. Yet, after a finite time, one reaches the same configuration as in the previous example with multiple solutions. The key point here is that, if the car densities can be close to zero, after one junction the total variation of the functions θ_{jk} can immediately become unbounded.

The final example, given in Section 6, describes an intersection with three roads: road 1 is incoming while road 2 and road 3 are outgoing. We consider weakly convergent sequences of drivers' turning preferences $\theta_{12}^\nu \rightharpoonup \theta_{12}$, $\theta_{13}^\nu \rightharpoonup \theta_{13}$. For $k = 1, 2, 3$, as $\nu \to \infty$ the corresponding car densities converge strongly: $\rho_k^\nu \to \rho_k$ in $\mathbf{L}^1_{loc}$, for some functions $\rho_k(t, x)$. However, these limit functions ρ_k are not the correct densities corresponding to the drivers' preferences θ_{12}, θ_{13}.

This last example is particularly relevant in connection with the problem of existence of Nash equilibria for traffic flow on a network. Indeed, the proof developed in [3] relies on the fact that the travel times depend continuously on the initial data, in the topology of weak convergence. Unfortunately, this crucial property fails whenever the boundary conditions are defined in terms of a Riemann solver. This observation leaves little hope of extending the results in [3] to this class of boundary conditions. In the concluding section, we briefly indicate how all these difficulties can be resolved by the alternative model in [4].

An introduction to scalar conservation laws and shock waves can be found in [1, 10, 19]. We recall that the method of front tracking, for constructing approximate solutions to scalar conservation laws, was introduced in [8]. Additional models for vehicle flow at intersections can be found in [11, 14, 15, 16].

2 Modeling assumptions

Consider a family of $n + m$ roads, joining at a node. Indices $i \in \{1, \ldots, m\} = \mathcal{I}$ denote incoming roads, while $j \in \{m + 1, \ldots, m + n\} = \mathcal{O}$ denote outgoing roads. On the k-th road, the density of cars $\rho_k(t, x)$ is described by the scalar conservation law

$$\rho_t + f_k(\rho)_x \;=\; 0\,. \tag{2.1}$$

Here $t \geq 0$ denotes time, while the space variable is $x \in]-\infty, 0]$ for incoming roads and $x \in [0, \infty[$ for outgoing roads. The flux functions are $f_k(\rho) = \rho\, v_k(\rho)$, where $v_k(\rho)$ is the speed of cars on the k-th road. Following [17, 18] we assume that this speed depends only on the density ρ. Moreover, we assume

$$f_k \in \mathcal{C}^2, \qquad f_k''(\rho) \;<\; 0, \qquad f_k(0) \;=\; f_k(\rho_k^{jam}) \;=\; 0, \tag{2.2}$$

where ρ_k^{jam} is the maximum possible density of cars on the k-th road. This corresponds to bumper-to-bumper packing, so that the speed of cars is zero. For a given road $k \in \{1, \ldots, m+n\}$, we denote by

$$f_k^{max} \;\doteq\; \max_s f_k(s)$$

the maximum flux and

$$\rho_k^{max} \;\doteq\; \operatorname*{argmax}_s f_k(s) \tag{2.3}$$

the traffic density corresponding to this maximum flux (see Fig. 1).

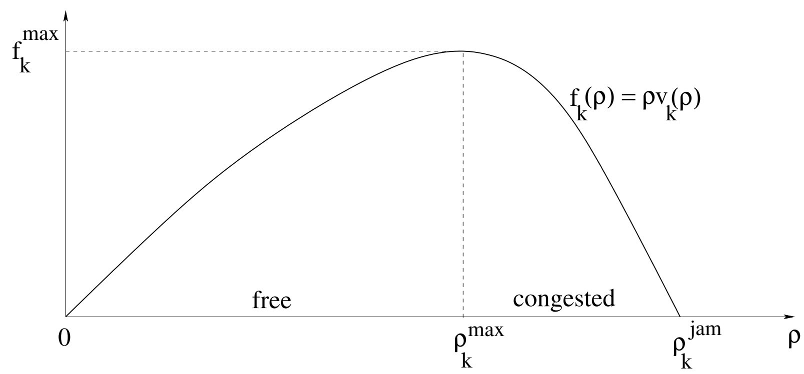

Figure 1: The flux f_k as a function of the density ρ, along the k-th road.

Moreover, we say that

$$\begin{array}{rl} \rho \ \text{ is a } \textbf{free state} & \text{if } \rho \in \ [0,\ \rho_k^{max}[\,, \\ \rho \ \text{ is a } \textbf{congested state} & \text{if } \rho \in \ ]\rho_k^{max},\ \rho_k^{jam}]\,. \end{array}$$

Given initial data on each road

$$\rho_k(0,x) \;=\; \rho_k^{\diamondsuit}(x), \qquad\qquad k = 1,\ldots,m+n, \tag{2.4}$$

in order to determine a (hopefully unique) solution to the Cauchy problem we must supplement the conservation laws (2.1) with a suitable set of boundary conditions. These provide additional constraints on the limiting values of the vehicle densities

$$\bar\rho_k(t) \;\doteq\; \lim_{x\to 0} \rho_k(t,x), \qquad\qquad k = 1,\ldots,m+n, \tag{2.5}$$

near the intersection. In a realistic model, these boundary conditions should depend on

- **(i) Drivers' preferences.** For every $i \in \mathcal{I}$, $j \in \mathcal{O}$, these are modeled by assigning the fraction θ_{ij} of drivers arriving from the i-th road who wish to turn into the j-th road.
- **(ii) Relative priority given to incoming roads.** For example, if the intersection is regulated by a crosslight, this is modeled by assigning the fraction of time η_i when cars arriving from the i-th road get a green light.

Notice that $(\eta_1,\ldots,\eta_m)$ is a constant vector, while $\theta_{ij} = \theta_{ij}(t,x)$ are passive scalars, transported along the flow. The Cauchy problem for traffic flow on a network of roads can thus be formulated as

$$\left\{ \begin{array}{rcll} \rho_{i,t} + f_i(\rho_i)_x &=& 0, & i \in \mathcal{I}\cup\mathcal{O}\,, \\[2ex] \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} &=& 0 & i \in \mathcal{I}, \quad j \in \mathcal{O}\,, \end{array} \right. \tag{2.6}$$

supplemented by the initial conditions

$$\left\{ \begin{array}{rcll} \rho_i(0,x) &=& \rho_i^{\diamondsuit}(x), & i \in \mathcal{I}\cup\mathcal{O}\,, \\[2ex] \theta_{ij}(0,x) &=& \theta_{ij}^{\diamondsuit}(x) & i \in \mathcal{I}, \quad j \in \mathcal{O}\,, \end{array} \right. \tag{2.7}$$

and by suitable boundary conditions, defined in terms of the constants η_i and of the boundary values $\rho_k(t,0)$, $\theta_{ij}(t,0)$. For obvious modeling reasons, we assume that the $m \times n$ coefficients $\theta = (\theta_{ij})$ satisfy

$$\theta_{ij} \in [0,1]\,, \qquad \sum_{j\in\mathcal{O}} \theta_{ij} \;=\; 1 \quad \text{for each } i \in \mathcal{I}\,. \tag{2.8}$$

Following [7, 12, 13], the boundary conditions at the junction will be assigned in terms of a **Riemann Solver**. We recall that a Riemann problem at the junction (Fig. 2) is determined by

(i) A vector of $n+m$ constant densities $\rho_0 = (\rho_{0,1}, \ldots, \rho_{0,m+n})$, with $\rho_{0,k} \in [0, \rho_k^{jam}]$.

(ii) An $m \times n$ matrix $\theta = (\theta_{ij})$, for $i \in \mathcal{I}$ and $j \in \mathcal{O}$, satisfying (2.8). Here the constants θ_{ij} determine the fraction of drivers coming from road i who wish to go on road j.

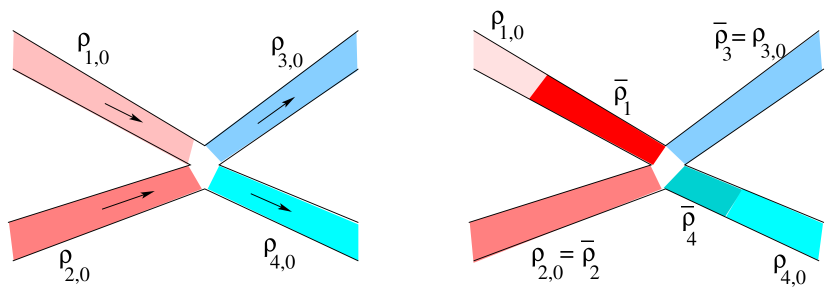

Figure 2: Given constant densities $\rho_{0,k}$ on every incoming and outgoing road at time $t=0$, a Riemann solver determines the self-similar solution for $t>0$. Notice that one may well have $\rho_{0,k} = \bar\rho_k$ for some k.

A **Riemann Solver** is a map

$$(\rho_{0,1}, \ldots, \rho_{0,m+n}) \;\mapsto\; (\bar\rho_1, \ldots, \bar\rho_{m+n}) \;=\; \mathcal{RS}(\rho_{0,1}, \ldots, \rho_{0,m+n}), \tag{2.9}$$

satisfying the following conditions.

(I) - mass conservation: For every outgoing road $j \in \mathcal{O}$ one has

$$\sum_{i\in\mathcal{I}} f_i(\bar\rho_i)\,\theta_{ij} \;=\; f_j(\bar\rho_j)\,. \tag{2.10}$$

(II) - admissibility:

- For every incoming road $i \in \mathcal{I}$, the Riemann problem

$$\rho_t + f_i(\rho)_x \;=\; 0, \qquad \rho(0,x) \;=\; \begin{cases} \rho_{0,i} & \text{if} \quad x<0, \\[2ex] \bar\rho_i & \text{if} \quad x>0, \end{cases}$$

  is solved by waves with negative speed.

- For every $j \in \mathcal{O}$, the Riemann problem

$$\rho_t + f_j(\rho)_x \;=\; 0, \qquad\qquad \rho(0,x) \;=\; \begin{cases} \bar\rho_j & \text{if} \quad x<0, \\ \\ \rho_{0,j} & \text{if} \quad x>0, \end{cases}$$

is solved by waves with positive speed.

Thanks to the admissibility condition **(II)**, a solution $\rho_i(t,x)$ is well defined for all incoming roads $i \in \mathcal{I}$, $x<0$, and also for all outgoing roads $j \in \mathcal{O}$, $x>0$. In turn, by taking limits of front-tracking approximations, one can obtain solutions to the general Cauchy problem.

Definition. *We say that a family of BV functions $\rho_k(t,x)$, $k \in \mathcal{I} \cup \mathcal{O}$, satisfies the boundary conditions determined by the Riemann Solver (2.9) if, for a.e. $t>0$, the limits*

$$\rho_k(t,0) \;\doteq\; \begin{cases} \lim_{x\to 0-} \rho(t,x) & \text{if } \; k \in \mathcal{I}\,, \\ \\ \lim_{x\to 0+} \rho(t,x) & \text{if } \; k \in \mathcal{O}\,, \end{cases}$$

satisfy

$$\mathcal{RS}\Big(\rho_1(t,0),\ldots,\rho_{m+n}(t,0)\Big) \;=\; \Big(\rho_1(t,0),\ldots,\rho_{m+n}(t,0)\Big). \tag{2.11}$$

With the above choice of boundary conditions, the Cauchy problem should (hopefully) be well posed. Toward this goal, a further property of the Riemann Solver must be imposed.

(III) - consistency: If $\mathcal{RS}(\rho_{0,1},\ldots,\rho_{0,m+n}) \;=\; (\bar\rho_1,\ldots,\bar\rho_{m+n})$, then

$$\mathcal{RS}(\bar\rho_1,\ldots,\bar\rho_{m+n}) \;=\; (\bar\rho_1,\ldots,\bar\rho_{m+n}). \tag{2.12}$$

Moreover, let $\rho_0^* = (\rho_{0,1}^*,\ldots,\rho_{0,m+n}^*)$ be a second vector of Riemann data such that

- For every $i \in \mathcal{I}$, the Riemann problem with left and right data $(\rho_{0,i}^*,\, \bar\rho_i)$ is solved by waves with negative speed.
- For every $j \in \mathcal{O}$, the Riemann problem with left and right data $(\bar\rho_j, \rho_{0,j}^*)$ is solved by waves with positive speed.

Then

$$\mathcal{RS}(\rho_{0,1}^*,\ldots,\rho_{0,m+n}^*) = (\bar\rho_1,\ldots,\bar\rho_{m+n}). \tag{2.13}$$

The motivation behind (2.12) is obvious. We shall explain (2.13) with the aid of Figure 3. Let $i \in \mathcal{I}$ and consider an initial data such that

$$\rho_i(0,x) \;=\; \begin{cases} \rho_{i,0}^* & \text{if} \quad x<-\varepsilon, \\ \\ \rho_{i,0} & \text{if} \quad -\varepsilon<x<0\,. \end{cases}$$

If the states $\rho^*_{0,i}$ and $\bar\rho_i$ are connected by backward moving waves, then as $\varepsilon \to 0$ the solution $\rho_i(t,x)$ approaches a self-similar limit, which should coincide with the solution of the Riemann problem with data $\rho^*_{0,i}$. Continuous dependence on the initial data thus implies (2.13). A similar argument is valid for outgoing roads $j \in \mathcal{O}$.

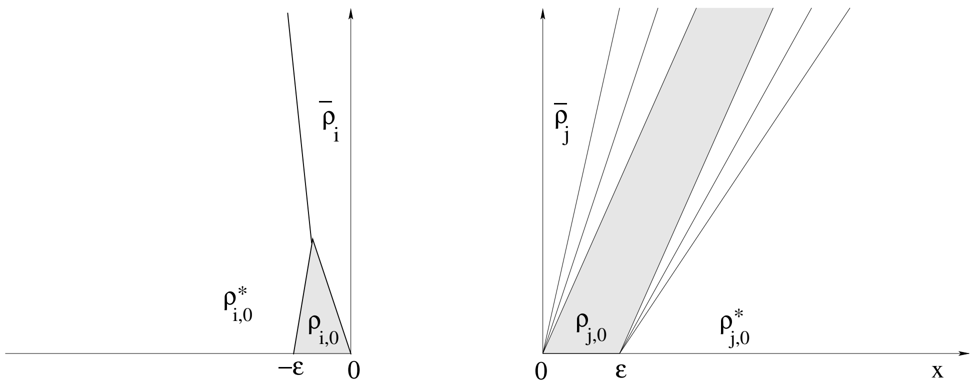

Figure 3: Two configurations motivating the consistency condition (2.13).

Two additional highly desirable properties of a Riemann solver will be considered:

(IV) - continuity: The map $(\rho_0, \theta) \;\mapsto\; (f_1(\bar\rho_1), \ldots, f_{m+n}(\bar\rho_{m+n}))$ is continuous on the domain

$$\mathcal{D} \doteq \Big\{(\rho_{0,1}, \ldots, \rho_{0,m+n})\,; \qquad \rho_{0,j} < \rho_j^{jam} \quad \text{for all} \quad j \in \mathcal{O}\Big\}.$$

In other words, in the solution to the Riemann problem all the incoming and outgoing fluxes should depend continuously on the densities $(\rho_{0,1}, \ldots, \rho_{0,m+n})$ as well as on the drivers' choices θ_{ij}, as long as none of the outgoing roads is completely jammed.

(V) - no jam: If $\rho_{0,j} < \rho_j^{jam}$ for all $j \in \mathcal{O}$, then $\bar\rho_i < \rho_i^{jam}$ for all $i \in \mathcal{I}$.

Namely, if none of the outgoing roads is completely jammed, then cars from all incoming roads should move with positive speed. This is equivalent to the implication

$$\min_{j\in\mathcal{O}} v_j(\rho_{0,j}) \;>\; 0 \qquad \Longrightarrow \qquad \min_{i\in\mathcal{I}} \; v_i(\bar\rho_i) \;>\; 0. \tag{2.14}$$

To construct a meaningful Riemann solver with the above properties, we start with an important observation (Figures 4, 5).

Consider a datum $\rho_{0,i}$ on an incoming road $i \in \mathcal{I}$. As $\bar\rho_i$ varies among all states $\rho \in [0, \rho_i^{jam}]$ such that the Riemann problem with data $(\rho_{0,i},\, \bar\rho_i)$ is solved by waves with negative speed, the corresponding flux $f_i(\bar\rho_i)$ ranges over the interval

$$\Omega_i \;=\; [0, \omega_i], \qquad\qquad \omega_i \;=\; \begin{cases} f(\rho_{0,i}) & \text{if} \quad \rho_{0,i} \le \rho_i^{max} \quad (\rho_{0,i} \text{ is a free state})\,, \\[2ex] f_i^{max} & \text{if} \quad \rho_{0,i} > \rho_i^{max} \quad (\rho_{0,i} \text{ is a congested state})\,. \end{cases} \tag{2.15}$$

Similarly, consider a datum $\rho_{0,j}$ on an outgoing road $j \in \mathcal{O}$. As $\bar\rho_j$ varies among all states $\rho \in [0, \rho_j^{jam}]$ such that the Riemann problem with data $(\bar\rho_j,\, \rho_{0,j})$ is solved by waves with positive speed, the corresponding flux $f_i(\bar\rho_j)$ ranges over the interval

$$\Omega_j = [0,\omega_j], \qquad \omega_j = \begin{cases} f_j^{max} & \text{if} \quad \rho_{0,j} \le \rho_j^{max} \quad (\rho_{0,j} \text{ is a free state}), \\ f(\rho_{0,j}) & \text{if} \quad \rho_{0,j} > \rho_j^{max} \quad (\rho_{0,j} \text{ is a congested state}). \end{cases} \tag{2.16}$$

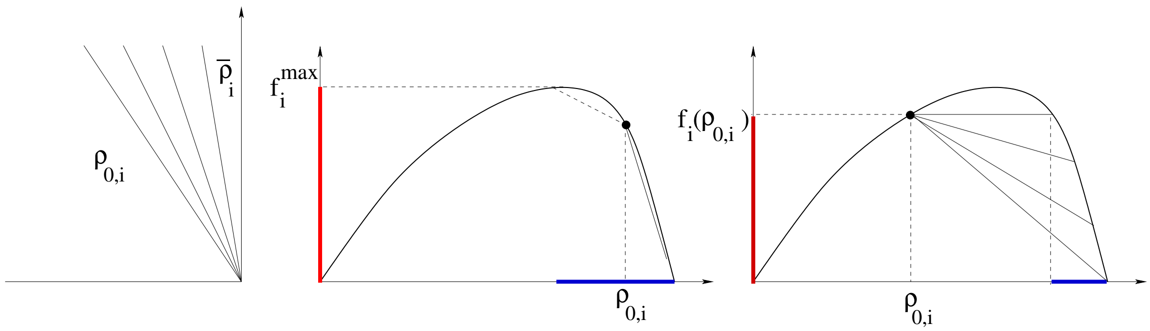

Figure 4: The case of an incoming road $i \in \mathcal{I}$. Given a left state $\rho_{0,i}$, we seek the family of all right states $\bar\rho_i$ which can be connected to $\rho_{i,0}$ by a wave having negative speed. Center: $\rho_{i,0}$ is a congested state, Right: $\rho_{i,0}$ is a free state.

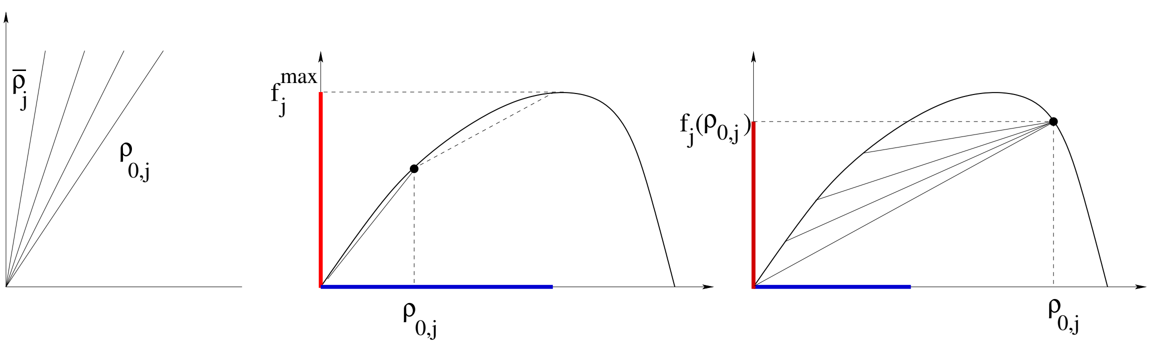

Figure 5: The case of an outgoing road $j \in \mathcal{O}$. Given a right state $\rho_{0,j}$, we seek the family of all left states $\bar\rho_j$ which can be connected to $\rho_{j,0}$ by a wave having positive speed. Center: $\rho_{j,0}$ is a free state, Right: $\rho_{j,0}$ is a congested state.

For $i \in \mathcal{I}$, consider the set of incoming fluxes $a_i = f_i(\bar\rho_i) \in \Omega_i$ determined by the Riemann solver. The identities (2.10) together with $f(\bar\rho_j) \in \Omega_j$ for all $j \in \mathcal{O}$ yield the family of constraints

$$0 \le \ a_i \ \le \omega_i\,, \qquad\qquad i \in \mathcal{I}\,, \tag{2.17}$$

$$\sum_{i\in\mathcal{I}} a_i\theta_{ij} \ \le \ \omega_j\,, \qquad\qquad j \in \mathcal{O}\,. \tag{2.18}$$

Together, the inequalities (2.17)-(2.18) determine a nonempty compact convex polytope $Q = Q(\rho_0, \theta)$. Any admissible Riemann solver will select an m-tuple of fluxes at the end of the m incoming roads

$$(a_1, \ldots, a_m) \ = \ (f_1(\bar\rho_1), \ldots, f_m(\bar\rho_m)) \ \in \ Q. \tag{2.19}$$

In turn, these uniquely determine the fluxes

$$b_j \ = \ f_j(\bar\rho_j) \ = \ \sum_{i\in\mathcal{I}} a_i\theta_{ij}\,, \qquad\qquad j \in \mathcal{O} \tag{2.20}$$

at the beginning of the n outgoing roads.

We recall a specific Riemann solver considered in [7, 13].

Riemann Solver $\mathcal{RS}1$: Let an m-tuple of positive numbers $(\eta_1, \ldots, \eta_m)$ be given, with

$$\eta_i \;>\; 0, \qquad\qquad \sum_{i=1}^m \eta_i \;=\; 1.$$

We can think of η_i as the relative priority given to drivers from the i-th road. For example, if the intersection is regulated by a crosslight, η_i can be the average fraction of time when drivers arriving from the i-th road get green light.

Let $Q \subset I\!\!R^n$ be the polytope determined by the constraints (2.17)-(2.18). Then the vector of incoming fluxes $(a_1, \ldots, a_m) \in Q$ in (2.19) is determined by following two rules.

(i) The total flux through the node is maximized:

$$M \;\doteq\; \sum_{i=1}^m a_i \;=\; \max_{(s_1,\ldots,s_m)\in Q} \sum_{i=1}^m s_i\,. \tag{2.21}$$

(ii) Subject to (2.21), the vector $(a_1, \ldots, a_m)$ is as close as possible to the vector $(M\eta_1, \ldots, M\eta_m)$. Namely

$$\sum_{i=1}^m |a_i - M\eta_i|^2 \;=\; \min\Big\{\sum_{i=1}^m |s_i - M\eta_i|^2\,; \quad s \in Q\,, \quad \sum_i s_i = M\Big\}. \tag{2.22}$$

In turn, the vector of outgoing fluxes $(b_j)_{j\in\mathcal{O}}$ is determined by

$$b_j \;=\; \sum_{i\in\mathcal{I}} a_i\theta_{ij}\,.$$

Finally, the densities $\bar\rho_k$ are uniquely determined by the identities $f_k(\bar\rho_k) = a_k$ together with the admissibility conditions (I).

Example 1. Consider an intersection with two incoming and two outgoing roads. Assume that the maximum flux along the outgoing roads is $f_3^{max} = f_4^{max} = 1$ and let $(\eta_1, \eta_2) = (\frac{2}{3}, \frac{1}{3})$. Moreover, let the initial data satisfy

$$f_1(\rho_{1,0}) \;=\; f_2(\rho_{2,0}) \;=\; 1, \qquad\qquad \rho_{3,0} \;=\; \rho_{4,0} = 0.$$

As for the drivers' turning preferences, we consider two cases.

Case 1: $\theta_{13} = \theta_{24} = 1$, $\theta_{14} = \theta_{23} = 0$. In this case, the set of admissible fluxes is the square

$$Q \;=\; \{(s_1, s_2)\,; \quad s_1 \in [0,1]\,, \quad s_2 \in [0,1]\}.$$

The total flux through the intersection is maximized by the pair $(a_1, a_2) = (1, 1)$.

Case 2: $\theta_{13} = \theta_{23} = 1$, $\theta_{14} = \theta_{24} = 0$. In this case, the set of admissible fluxes is the triangle

$$Q \;=\; \{(s_1, s_2)\,; \quad s_1 \geq 0, \quad s_2 \geq 0, \quad s_1 + s_2 \leq 1\}.$$

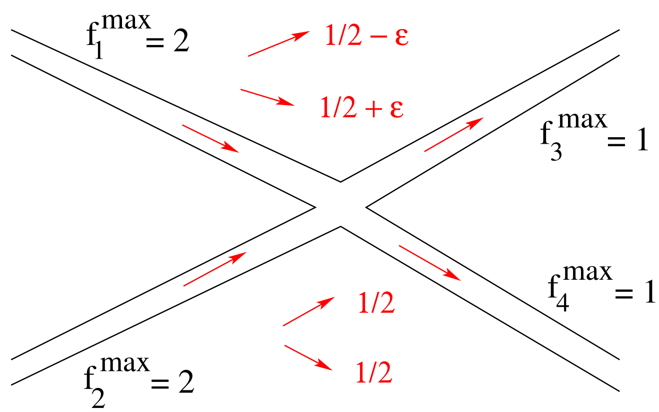

Figure 6: In Example 2, the solution of the Riemann problem, maximizing the total flux across the intersection, changes in a discontinuous way when $\varepsilon = 0$.

All pairs (s_1, s_2) such that $s_1 + s_2 = 1$ maximize the total flux through the intersection. In this case, the additional requirement (2.22) uniquely selects the pair of fluxes $(a_1, a_2) = (\frac{2}{3}, \frac{1}{3})$.

Example 2. Consider an intersection with two incoming and two outgoing roads (Fig. 6). Let the initial densities satisfy

$$\begin{cases} f_1(\rho_{1,0}) &= f_1^{max} &= 2, \\ f_2(\rho_{2,0}) &= f_2^{max} &= 2, \end{cases} \qquad \begin{cases} f_3(\rho_{3,0}) &= f_3^{max} &= 1, \\ f_4(\rho_{4,0}) &= f_4^{max} &= 1. \end{cases}$$

Assume that incoming drivers turn left or right with the following ratios:

$$\begin{cases} \theta_{13} &= \frac{1}{2} + \varepsilon\,, \\ \theta_{14} &= \frac{1}{2} - \varepsilon\,, \end{cases} \qquad \begin{cases} \theta_{23} &= \frac{1}{2}\,, \\ \theta_{24} &= \frac{1}{2}\,. \end{cases}$$

Moreover, let the priority vector be $(\eta_1, \eta_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$. As long as $\varepsilon > 0$, the maximization problem (2.21) has the unique solution

$$f_1(\bar\rho_1) = 0\,, \qquad f_2(\bar\rho_2) = 2\,.$$

However, when $\varepsilon = 0$, there are many ways to maximize the total flux through the intersection. With the above choice of (η_1, η_2), the Riemann solver selects the pair of fluxes

$$f_1(\bar\rho_1) = \frac{4}{3}\,, \qquad f_2(\bar\rho_2) = \frac{2}{3}\,.$$

Observe that this Riemann solver does not satisfy the continuity property **(IV)** at $\varepsilon = 0$. It does not even satisfy the no-jam property **(V)**. Indeed, when $\varepsilon > 0$ the flux coming from road 1 is $f_1(\bar\rho_1) = 0$, hence $\bar\rho_1 = \rho_1^{jam}$.

3 A continuous Riemann Solver

To construct a specific Riemann solver satisfying all conditions **(I)**–**(V)**, we need a rule to select a specific point $a = (a_1, \ldots, a_m) \in Q$, depending continuously on the data $(\rho_{0,1}, \ldots, \rho_{0,m+n})$

and on the coefficients θ_{ij}. For this purpose, we choose m smooth scalar functions $\psi_1, \ldots, \psi_m$, satisfying

$$\psi_i(0) \;=\; 0, \qquad \psi_i'(\xi) \;>\; 0, \qquad \psi_i''(\xi) \;\leq\; 0 \quad \text{for all } \xi \geq 0\,. \tag{3.1}$$

For example, one can choose $\psi_i(\xi) = \frac{c_i \xi}{1+\xi}$, with $c_i > 0$. Different choices of c_i reflect different priorities given to incoming roads.

Riemann Solver $\mathcal{RS}^\sharp$. *Let $(\rho_{0,1}, \ldots, \rho_{0,m+n})$ be given, and let $(\omega_1, \ldots, \omega_{n+m})$ be the corresponding vector of maximum fluxes, defined at (2.15)-(2.16). Assume $\omega_j > 0$ for all $j \in \mathcal{O}$ and consider the convex polytope of admissible fluxes*

$$Q \;\doteq\; \left\{ (s_1, \ldots, s_m)\,; \quad s_i \in [0, \omega_i]\,, \qquad \sum_{i=1}^m s_i \theta_{ij} \;\leq\; \omega_j \quad \textit{for all } j \in \mathcal{O} \right\}. \tag{3.2}$$

The solution $(\bar\rho_1, \ldots, \bar\rho_{m+n})$ is defined in four steps.

1. *First of all, we determine the vector of incoming fluxes $(a_1, \ldots, a_m)$. If $\omega_i = \rho_{0,i} = 0$, then necessarily $a_i = 0$. The remaining incoming fluxes a_i are then found by solving the optimization problem*

$$\textit{maximize:} \;\; \Psi(s) \;\doteq \prod_{i \in \mathcal{I},\ \omega_i > 0} \psi_i(s_i)\,, \qquad \textit{subject to} \;\; (s_1, \ldots, s_m) \in Q\,. \tag{3.3}$$

2. *The outgoing fluxes are then determined by setting* $b_j \doteq \sum_{i=1}^m a_i \theta_{ij}\,, \quad j \in \mathcal{O}$.

3. *For each $i \in \mathcal{I}$, the state $\bar\rho_i$, is uniquely determined by the conditions $f_i(\bar\rho_i) = a_i$ and the requirement that the two states $(\rho_{0,i}, \bar\rho_i)$ should be joined by a backward moving wave. More precisely, if $\bar\rho_i \neq \rho_{0,i}$, then $\bar\rho_i > \rho_i^{max}$.*

4. *For each $j \in \mathcal{O}$, the state $\bar\rho_j$, is uniquely determined by the conditions $f_j(\bar\rho_j) = b_j$ and the requirement that the two states $(\rho_{0,j}, \bar\rho_j)$ should be joined by a forward moving wave. More precisely, if $\bar\rho_j \neq \rho_{0,j}$, then $\bar\rho_j < \rho_j^{max}$.*

As shown in [7], the Riemann solution $(\bar\rho_1, \ldots, \bar\rho_{m+n})$ is completely determined as soon as the incoming fluxes $(a_1, \ldots, a_n)$ are assigned. The only difference between the Riemann solvers $\mathcal{RS}^\sharp$ and $\mathcal{RS}1$ is that in (3.3) we are maximizing a weighted product of incoming fluxes, rather than their sum. This makes a qualitative difference, providing the continuous dependence w.r.t. initial data. Indeed, the following analysis will show that the Riemann solver $\mathcal{RS}^\sharp$ satisfies all properties **(I)–(V)**.

Lemma 1. *Let $\psi_1, \ldots, \psi_\nu$ be functions that satisfy (3.1), and let $c_0, R > 0$ be given. Then for every $c \geq c_0$ the sup-level set*

$$\Lambda_c \;\doteq\; \left\{ (s_1, \ldots, s_\nu)\,; \quad \Psi(s) \doteq \prod_{i=1}^\nu \psi_i(s_i) \;\geq\; c, \qquad s_i > 0\,, \qquad \sum_i s_i^2 \leq R^2 \right\}$$

is uniformly convex.

Proof. Let $c \geq c_0 > 0$ be given and consider two ν-tuples $s = (s_1, \ldots, s_\nu)$ and $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_\nu)$, with $s, \tilde{s} \in \Lambda_c$ and $\Psi(s) = \Psi(\tilde{s}) = c$. We need to show that

$$\left\langle \frac{\nabla\Psi(s)}{|\nabla\Psi(s)|}, \, \tilde{s} - s \right\rangle \;\geq\; \delta_0 \, |\tilde{s} - s|^2 \,, \tag{3.4}$$

for some constant $\delta_0 > 0$ depending only on c_0, R. As long as $\Psi(s)$ remains uniformly positive, say $\Psi(s) \geq c_0 > 0$, we can equivalently work with the logarithm $g(s) = \ln \Psi(s)$. It thus suffices to prove that

$$\langle \nabla g(s), \;\tilde{s} - s\rangle \;\geq\; \delta \, |\tilde{s} - s|^2 \tag{3.5}$$

for some constant $\delta > 0$.

Using the fact that all functions ψ_i are concave down, and that the logarithm function is uniformly concave down on bounded sets, for any $\lambda \in [0, 1]$ we obtain

$$\begin{aligned}
\ln\Big(\Psi(\lambda\tilde{s} + (1-\lambda)s)\Big) \;&=\; \sum_{i=1}^{\nu} \ln\Big(\psi_i(\lambda\tilde{s}_i + (1-\lambda)s_i)\Big) \;\geq\; \sum_{i=1}^{\nu} \ln\Big(\lambda\psi_i(\tilde{s}_i) + (1-\lambda)\psi_i(s_i)\Big) \\
&\geq\; \sum_{i=1}^{\nu} \Big(\lambda \ln(\psi_i(\tilde{s}_i)) + (1-\lambda)\ln(\psi_i(s_i)) + \delta\,\lambda(1-\lambda)|\tilde{s}_i - s_i|^2\Big) \\
&=\; \lambda \ln(\Psi(\tilde{s})) + (1-\lambda)\ln(\Psi(s)) + \delta\,\lambda(1-\lambda)|\tilde{s} - s|^2 \,,
\end{aligned}$$

for some constant $\delta > 0$ depending on the lower bound on the quantities $\psi_i(s_i)$, $\psi_i(\tilde{s}_i)$, $i = 1, \ldots, \nu$, hence on c_0 and R.

If $\Psi(s) = \Psi(\tilde{s}) = c \geq c_0$, we now obtain

$$\begin{aligned}
\langle \nabla g(s), \;\tilde{s} - s\rangle \;&=\; \lim_{\lambda \to 0+} \frac{g(\lambda\tilde{s} + (1-\lambda)s) - g(s)}{\lambda} \\
&=\; \lim_{\lambda \to 0+} \frac{g(\lambda\tilde{s} + (1-\lambda)s) - \lambda g(s) - (1-\lambda)g(s)}{\lambda} \\
&\geq\; \lim_{\lambda \to 0} \frac{\delta\,\lambda(1-\lambda)|\tilde{s} - s|^2}{\lambda} \;=\; \delta\,|\tilde{s} - s|^2 \,.
\end{aligned}$$

This establishes (3.5), completing the proof. □

Lemma 2. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. If the maximum outgoing fluxes in (3.2) satisfy $\omega_j \geq \varepsilon$ for all $j \in \mathcal{O}$, then the optimal incoming fluxes a_i in (3.3) satisfy*

$$a_i \;\geq\; \min\{\omega_i, \,\delta\} \qquad \textit{for all } i \in \mathcal{I}. \tag{3.6}$$

Proof. 1. Assume that $\omega_j \geq \varepsilon$ for all $j \in \mathcal{O}$. If $a_i < \varepsilon/m$ for every i, then

$$\sum_{i=1}^{m} a_i\theta_{ij} \;<\; m \cdot \frac{\varepsilon}{m} \;\leq\; \omega_j \qquad \text{for all } j \in \mathcal{O}.$$

Hence none of the above constraints is satisfied as an equality. Since all functions ψ_i are strictly increasing, the necessary conditions for optimality in (3.3) imply

$$a_i = \omega_i \qquad \text{for all } i \in \mathcal{I}.$$

2. Next, we consider the case where the set of indices

$$\mathcal{B} \doteq \left\{ i \in \mathcal{I};\ \ a_i \geq \frac{\varepsilon}{2m} \right\}$$

is nonempty. For every $j \in \mathcal{O}$ we have

$$\sum_{i\in\mathcal{I}} a_i \theta_{ij} \leq \sum_{i\in\mathcal{B}} a_i\theta_{ij} + \sum_{i\notin\mathcal{B}} a_i < \sum_{i\in\mathcal{B}} a_i\theta_{ij} + \frac{\varepsilon}{2}.$$

Hence

$$\text{either} \quad \sum_{i\in\mathcal{B}} a_i\theta_{ij} \geq \frac{\varepsilon}{2} \qquad \text{or else} \quad \sum_{i\in\mathcal{I}\setminus\mathcal{B}} a_i\theta_{ij} < \omega_j. \tag{3.7}$$

3. Consider an index $\ell \in \mathcal{I}\setminus\mathcal{B}$ such that $0 \leq a_\ell < \omega_\ell$. We claim that, if ω_ℓ is sufficiently small, then the vector $(a_1, \ldots, a_m)$ cannot be optimal. Indeed, consider a perturbed vector $(a_1(\xi), \ldots, a_m(\xi))$, where

$$\begin{cases} a_\ell(\xi) = a_\ell + \xi, \\ a_i(\xi) = a_i & i \in \mathcal{I}\setminus\mathcal{B}, \\ a_i(\xi) = a_i - \frac{2}{\varepsilon}\xi a_i & i \in \mathcal{B}. \end{cases}$$

We claim that, for $\xi > 0$ small, this perturbation still satisfies all constraints. Indeed, if the j-th constraint is satisfied as an equality, by (3.7) it follows that $\sum_{i\in\mathcal{B}} a_i\theta_{ij} \geq \varepsilon/2$. Hence

$$\frac{d}{d\xi}\left[\sum_{i=1}^m a_i(\xi)\theta_{ij}\right] = \theta_{\ell j} - \sum_{i\in\mathcal{B}} \frac{2}{\varepsilon}\, a_i\theta_{ij} \leq 1 - \frac{2}{\varepsilon}\cdot\frac{\varepsilon}{2} \leq 0.$$

We now compute

$$\begin{aligned} \frac{d}{d\xi} \prod_{\omega_i>0} \psi_i(a_i(\xi))\bigg|_{\xi=0} &= \psi'_\ell(a_\ell) \prod_{\omega_i>0, i\neq\ell} \psi_i(a_i) - \psi_\ell(a_\ell) \sum_{k\in\mathcal{B}} \left(\frac{2a_k}{\varepsilon} \psi'_k(a_k) \cdot \prod_{\omega_i>0, i\neq\ell,k} \psi_i(a_i) \right) \\ &= \left(\prod_{\omega_i>0, i\neq\ell} \psi_i(a_i) \right) \cdot \left(\psi'_\ell(a_\ell) - \psi_\ell(a_\ell) \cdot \sum_{k\in\mathcal{B}} \frac{2}{\varepsilon} \frac{a_k}{\psi_k(a_k)} \cdot \psi'_k(a_k) \right) \\ &\leq \left(\prod_{\omega_i>0, i\neq\ell} \psi_i(a_i) \right) \cdot \left(\psi'_\ell(a_\ell) - \psi_\ell(a_\ell) \cdot \frac{\kappa}{\varepsilon^2} \right), \end{aligned}$$

where the constant κ is defined as

$$\kappa \doteq \sum_{k\in\mathcal{I}} 2f_k^{max} \left(\min_{s\in[0,f_k^{max}]} |\psi'_k(s)| \right)^{-1} \cdot \max_{s\in[0,f_k^{max}]} |\psi'_k(s)|.$$

Given $\varepsilon > 0$, we now choose $\delta > 0$ so that, for every $\ell \in \mathcal{I}$,

$$\psi_\ell'(s) - \psi_\ell(s) \cdot \frac{\kappa}{\varepsilon^2} \; > \; 0 \qquad \text{whenever} \quad 0 < s \leq \delta\,.$$

This is certainly possible because $\psi_\ell(s) \to 0$ as $s \to 0$. By the previous analysis, with this choice of δ the property (3.6) holds. □

Consider the set of drivers' preferences

$$\Theta \;\doteq\; \left\{ \theta = (\theta_{ij})\,; \quad \sum_{j\in\mathcal{O}} \theta_{ij} \;=\; 1 \quad \text{for all } i \in \mathcal{I} \right\}. \tag{3.8}$$

Given an $(m+n)$-tuple of maximum fluxes $\omega = (\omega_1, \ldots, \omega_{m+n})$, let Q be as in (3.2) and let $a = (a_1, \ldots, a_m)$ be determined by

$$\begin{cases} \qquad\qquad a_i \;=\; 0 \qquad \text{for all } i \in \mathcal{I} \text{ such that } \omega_i = 0\,, \\[2ex] (a_1, \ldots, a_m) \;=\; \operatorname{argmax} \left\{ \displaystyle\prod_{i\in\mathcal{I},\ \omega_i>0} \psi_i(s_i)\,; \qquad (s_1, \ldots, s_m) \in Q \right\}. \end{cases} \tag{3.9}$$

We recall that the Hausdorff distance between two compact sets Q, Q' is

$$d_H(Q, Q') \;\doteq\; \max \left\{ \max_{x\in Q}\; d(x, Q')\;,\;\; \max_{x'\in Q'}\; d(x', Q) \right\},$$

where $d(x, Q') \;=\; \min_{x'\in Q'} |x - x'|$.

Lemma 3. *Let $\Omega \subset I\!\!R^m$ be a compact, convex domain, and let $\Psi : \Omega \mapsto I\!\!R$ be a continuous function whose level sets are $\mathcal{C}^1$ and satisfy the convexity condition (3.4). For every compact convex set Q, let*

$$a \;\doteq\; \operatorname*{argmax}_{s\in Q} \Psi(s)$$

be the point where Ψ attains its maximum. Then the map $Q \mapsto a$ is Hölder continuous with exponent $1/2$, w.r.t. the Hausdorff distance in the space of compact convex sets.

Proof. Consider two compact convex sets Q, Q', say with Hausdorff distance $d_H(Q, Q') = \delta$. To fix the ideas, assume

$$\Psi(a) \;=\; \max_{s\in Q} \Psi(s) \;=\; M \;\leq\; M' \;=\; \max_{s\in Q'} \Psi(s) \;=\; \Psi(a')\,.$$

Consider the unit vector

$$\mathbf{e} \;=\; \frac{\nabla\Psi(a)}{|\nabla\Psi(a)|}\,.$$

Then

$$Q \;\subseteq\; \{s \in I\!\!R^m\,;\;\; \langle \mathbf{e}, s\rangle \;\leq\; \langle \mathbf{e}, a\rangle\},$$

$$a' \;\in\; Q' \;\subseteq\; \{s \in I\!\!R^m\,;\;\; \langle \mathbf{e}, s\rangle \;\leq\; \langle \mathbf{e}, a\rangle + \delta\}.$$

With reference to Fig. 7, recalling (3.4) one obtains

$$\delta_0|a'-a|^2 \;\leq\; \langle \mathbf{e}\,,\; a'-a\rangle \;\leq\; \delta\,.$$

We thus conclude

$$|a'-a| \;\leq\; \sqrt{\frac{\delta}{\delta_0}} \;=\; \frac{1}{\sqrt{\delta_0}}\cdot\sqrt{d_H(Q,Q')}\,.$$

proving the Hölder continuity with exponent 1/2. □

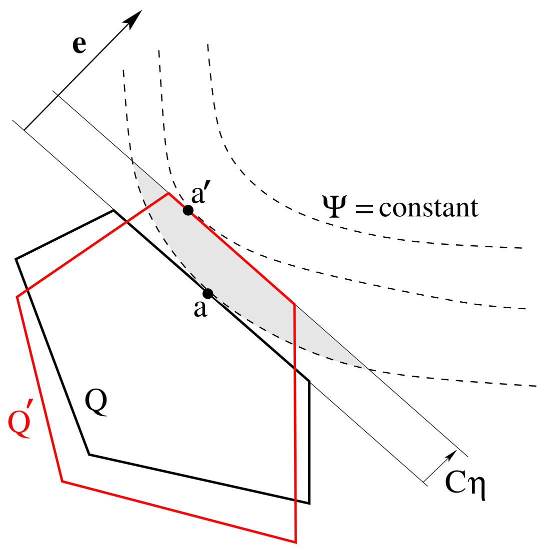

Figure 7: Proving Proposition 1. Here a, a' are the points which maximize the function Ψ over Q and Q' respectively. Since the Hausdorff distance satisfies $d_H(Q,Q') \leq C\eta$ and the sup-level sets of the function Ψ are uniformly convex, this implies $|a'-a| \leq C_0\eta^{1/2}$.

Our first main result shows that, for the Riemann Solver $\mathcal{RS}^\sharp$, the boundary fluxes depend Hölder continuously on all the data.

Proposition 1. *Let the functions ψ_i satisfy (3.1). For any given $\varepsilon > 0$, the map $(\omega,\theta) \mapsto a$ defined by (3.9) is Hölder continuous of exponent $\gamma = 1/2$ restricted to the compact set*

$$\Omega_\varepsilon \;\doteq\; \Big\{(\omega,\theta);\quad \omega_i \in [0, f_i^{max}]\;\; \textit{for all } i \in \mathcal{I},\quad \omega_j \in [\varepsilon, f_j^{max}]\;\; \textit{for all } j \in \mathcal{O},\quad \theta \in \Theta\Big\}.$$

Proof. 1. Given $\varepsilon > 0$, let $\delta > 0$ be the corresponding constant in Lemma 2. Calling $\mathcal{B} \subset \mathcal{I}$ the set of indices such that $\omega_i > 2\delta/3$, we can write

$$a_i \;=\; \omega_i \qquad\qquad i \in \mathcal{I}\setminus\mathcal{B}\,, \tag{3.10}$$

$$(a_i)_{i\in\mathcal{B}} \;=\; \text{argmax}\left\{\prod_{i\in\mathcal{B}}\psi_i(s_i)\,;\quad s\in Q\right\}, \tag{3.11}$$

where

$$Q \;=\; Q(\omega,\theta) \;\doteq\; \left\{(s_i)_{i\in\mathcal{B}}\,;\qquad s_i\in[0,\omega_i],\quad \sum_{i\in\mathcal{B}} s_i\theta_{ij} \;\leq\; \omega_j - \sum_{i\in\mathcal{I}\setminus\mathcal{B}}\omega_i\theta_{ij},\quad j\in\mathcal{O}\right\}. \tag{3.12}$$

2. Now consider a second set of data: $(\omega', \theta') \in \Omega_\varepsilon$ with $|\omega'_i - \omega_i| \le \delta/3$ for all $i \in \mathcal{I}$. Observe that this implies $\omega'_i \le \delta$ for all $i \in \mathcal{I} \setminus \mathcal{B}$. Hence the corresponding solution $a' = (a'_1, \ldots, a'_{m+n})$ still satisfies

$$\begin{aligned} a'_i &= \omega'_i \qquad\qquad i \in \mathcal{I} \setminus \mathcal{B}\,, \\ (a'_i)_{i\in\mathcal{B}} &= \text{argmax}\left\{ \prod_{i\in\mathcal{B}} \psi_i(s_i)\,; \quad s \in Q' \right\}, \end{aligned}$$

with Q' defines as in (3.12), with (ω, θ) replaced by (ω', θ').

3. To fix the ideas, assume

$$\Psi(a) \doteq \prod_{i\in\mathcal{B}} \psi_i(a_i) \le \prod_{i\in\mathcal{B}} \psi_i(a'_i) = \Psi(a').$$

Since the polytope Q is convex and the sup-level sets of Ψ are also convex, by optimality these two domains can be linearly separated:

$$\begin{aligned} s \in Q \qquad &\Longrightarrow \qquad \langle \nabla\Psi(a),\, s - a\rangle \le 0\,, \\ \Psi(s) \ge \Psi(a) \qquad &\Longrightarrow \qquad \langle \nabla\Psi(a),\, s - a\rangle \ge 0\,. \end{aligned}$$

Call

$$\eta \doteq \max_{k\in\mathcal{I}\cup\mathcal{O}} |\omega_k - \omega'_k| + \max_{i\in\mathcal{I},\, j\in\mathcal{O}} |\theta_{ij} - \theta'_{ij}|$$

the distance between the two sets of data (ω, θ) and (ω', θ'). This implies

$$Q' \subseteq \widetilde{Q} \doteq \Bigg\{ (\tilde{s}_i)_{i\in\mathcal{B}}\,; \qquad \tilde{s}_i \in [0, \omega_i + \eta],$$

$$\sum_{i\in\mathcal{B}} \tilde{s}_i(\theta_{ij} - \eta) \le (\omega_j + \eta) - \sum_{i\in\mathcal{I}\setminus\mathcal{B}} (\omega_i - \eta)(\theta_{ij} - \eta) \quad \text{for all} \ \ j \in \mathcal{O} \Bigg\}.$$

We claim that the Hausdorff distance can be estimated by

$$d_H(Q, \widetilde{Q}) \le C\eta \tag{3.13}$$

for a suitable constant $C > 0$, provided that $\eta > 0$ is small enough. Indeed, assume $\tilde{s} \in \widetilde{Q}$ and consider the vector s with components $s_i = \tilde{s}_i - C\eta$. We are here assuming $\eta < \delta/3C$ so that $s_i > 0$.

If we choose $C \ge 1$, then $s_i \in [0, \omega_i]$. Moreover, for every $j \in \mathcal{O}$ we have

$$\sum_{i\in\mathcal{B}} s_i\theta_{ij} = \sum_{i\in\mathcal{B}} (\tilde{s}_i - C\eta)\theta_{ij} \le \omega_j + C'\eta - \sum_{i\in\mathcal{I}\setminus\mathcal{B}} \omega_i\theta_{ij} + C\eta \cdot \sum_{i\in\mathcal{B}} \theta_{ij}\,,$$

for some constant C' depending only on the maximum fluxes f_j^{max}. Given C', for each $j \in \mathcal{O}$ we consider two cases.

Case 1: $\sum_{i\in\mathcal{B}} f_i^{max}\theta_{ij} < \delta/6$. In this case the constraint

$$\sum_{i\in\mathcal{B}} s_i\theta_{ij} \le \frac{\delta}{6} \le \omega_j - \sum_{j\in\mathcal{B}\setminus\mathcal{B}} \omega_i\theta_{ij}$$

is automatically satisfied.

Case 2: $\sum_{i\in\mathcal{B}} f_i^{max}\theta_{ij} \geq \delta/6$. In this case it suffices to choose

$$C \;\geq\; \frac{6C'}{\delta\cdot\max_i f_i^{max}}\,.$$

4. By the previous analysis, the set $Q = Q(\omega,\theta)$ defined in (3.12) depends Lipschitz continuously on the vector $(\omega_i,\theta_{ij})_{i\in\mathcal{B}}$. By Lemma 3, the corresponding point $a = (a_i)_{i\in\mathcal{B}}$, where the maximum in (3.11) is attained, varies in a Hölder continuous way. On the other hand, the remaining components $(a_i)_{i\in\mathcal{I}\setminus\mathcal{B}}$ are given by (3.10). Trivially, these components depend on the vector $\omega = (\omega_1,\ldots,\omega_m)$ in a Lipschitz continuous way. □

We now show that the Riemann Solver defined at (3.2)-(3.3) has a number of good properties.

Proposition 2. *Let the functions ψ_i satisfy (3.1). Then the Riemann solver $\mathcal{RS}^\sharp$ satisfies the properties* **(I)**–**(V)**.

Proof. 1. The mass conservation property **(I)** follows from the definition of the outgoing fluxes b_j, in Step 2. The admissibility condition **(II)** is an immediate consequence of Steps 3-4 in the construction of the Riemann solver.

2. To check the consistency condition **(III)**, assume

$$\mathcal{RS}^\sharp(\rho_{01},\ldots\rho_{0,m+n}) \;=\; (\bar\rho_1,\ldots\bar\rho_{m+n}).$$

For notational simplicity, we discuss here the case where $\rho_{i,0} > 0$ for every $i\in\mathcal{I}$. In (3.3) we are thus maximizing the function

$$\Psi(s) \;\doteq\; \prod_{i=1}^m \psi_i(s_i)\,.$$

With a minor modification, the same arguments can be adapted to the general case.

Call $\mathcal{S}\subseteq\mathcal{I}\cup\mathcal{O} = \{1,\ldots,m+n\}$ the set of saturated indices for the optimization problem (3.9). More precisely, define

$$\mathcal{S} \;\doteq\; \{i\in\mathcal{I}\,;\;\; a_i=\omega_i\}\cup\Big\{j\in\mathcal{O}\,;\;\;\sum_{i\in\mathcal{I}} a_i\theta_{ij} \;=\; \omega_j\Big\}\,,$$

$$Q' \;\doteq\; \left\{(s_1,\ldots,s_m)\,;\quad s_i\in[0,\omega_i]\;\text{ for } i\in\mathcal{I}\cap\mathcal{S},\qquad \sum_{i=1}^m s_i\theta_{ij} \;\leq\; \omega_j\quad\text{for } j\in\mathcal{O}\cap\mathcal{S}\right\}.$$

By the strict convexity of the set $\{s\,;\;\;\Psi(s)\geq\Psi(a)\}$, we have

$$a \;=\; \text{argmax}\left\{\prod_{i=1}^m \psi_i(s_i)\,;\qquad (s_1,\ldots,s_m)\in Q'\right\}. \tag{3.14}$$

Call $(\omega_1, \ldots \omega_{m+n})$ the vector of maximum possible fluxes given the states $(\rho_{0,k})$, and let $(\bar\omega_1, \ldots \bar\omega_{m+n})$ be the corresponding vector for the states $(\bar\rho_k)$. Call $\overline{Q}$ the polytope defined at (3.2), with ω replaced by $\bar\omega$. Because of the admissibility condition **(II)** we now have

$$\bar\omega_k \;=\; \omega_k \qquad\qquad \text{for all} \quad k\in\mathcal{S}.$$

Observing that $a\in\overline{Q}\subseteq Q'$, by (3.14) we conclude

$$a \;=\; \mathrm{argmax}\left\{\prod_{i=1}^m \psi_i(s_i)\,;\qquad (s_1,\ldots,s_m)\in\overline{Q}\right\}.$$

3. Since the maps $\rho_{0,k}\mapsto\omega_k$ are Lipschitz continuous, the Hölder continuity of the map $(\rho_0,\theta)\mapsto(a_1,\ldots,a_m)$ is an immediate consequence of Proposition 1. In turn, the outgoing fluxes are determined by

$$f(\bar\rho_j) \;=\; \sum_{i\in\mathcal{I}} a_i\theta_{ij}\,.$$

This achieves the continuity property **(IV)**.

4. Finally, we check that the "no-jam" condition **(V)** is satisfied. It is not restrictive to assume that $\omega_i>0$ for some $i\in\mathcal{I}$, otherwise the conclusion is trivial. By the properties of the functions ψ_i in (3.1) we have

$$\prod_{\omega_i>0}\psi_i(a_i) \;=\; \max_{(s_1,\ldots,s_m)\in Q}\prod_{\omega_i>0}\psi_i(s_i) \;>\; 0.$$

The product on the left hand side is nonzero if and only if $a_i>0$ for every $i\in\mathcal{I}$ such that $\omega_i>0$. Hence, either $\omega_i=\rho_{i,0}=\bar\rho_i=0$, or else $f_i(\bar\rho_i)=a_i>0$. In both cases, the i-th incoming road is not jammed. □

4 Non-uniqueness for data having unbounded variation

We show by a counterexample that, if the functions θ_{ij} have unbounded variation, then the solution need not be unique.

Example 3. We consider a junction with two incoming roads ($i=1,2$) and two outgoing roads ($j=3,4$). The flux functions are all the same:

$$f_1(\rho) \;=\; f_2(\rho) \;=\; f_3(\rho) \;=\; f_4(\rho) \;=\; 2\rho-\rho^2\,. \tag{4.1}$$

This implies that the maximum outgoing flux is

$$f_3^{max} \;=\; f_4^{max} \;=\; 1\,.$$

Moreover, as prioritizing function Ψ in (3.3) we choose a smooth function whose sup-level sets are strictly convex and which satisfies (see Fig. 8)

$$\mathrm{argmax}\Big\{\Psi(s_1,s_2)\,;\quad s_1\leq 1\,,\quad s_2\leq 1\Big\} \;=\; (1,1)\,, \tag{4.2}$$

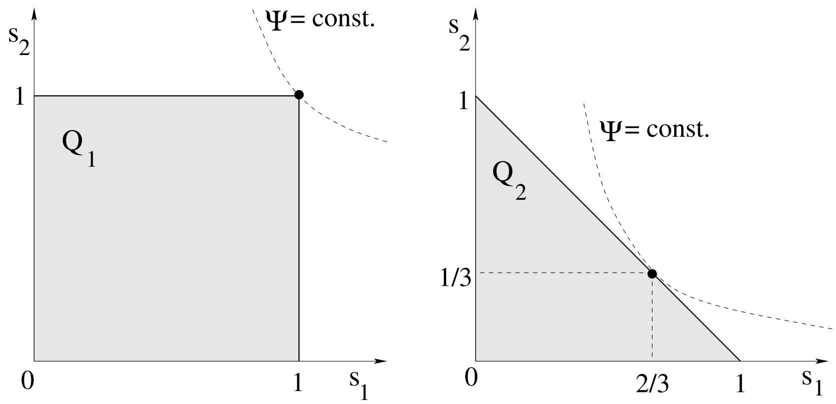

Figure 8: Left: the set Q_1 of possible incoming fluxes in the solution (4.6). Right: the set Q_2 of possible incoming fluxes in the solution described at (4.7)-(4.8). In this example it suffices to choose any function Ψ which attains its maximum over Q_1 at the point $(1,1)$, and its maximum over Q_2 at the point $(\frac{2}{3}, \frac{1}{3})$.

$$\operatorname{argmax} \Big\{ \Psi(s_1, s_2)\,; \quad s_1 + s_2 \le 1 \Big\} \;=\; \left(\frac{2}{3}, \frac{1}{3}\right). \tag{4.3}$$

As initial data, we choose the constant densities

$$\rho_1(0,x) \;=\; \rho_2(0,x) \;=\; \rho_3(0,x) \;=\; \rho_4(0,x) \;=\; 1. \tag{4.4}$$

Finally, as initial values for the drivers' preferences we choose

$$\bar\theta_{13}(x) \;=\; \bar\theta_{24}(x) \;=\; \begin{cases} 1 & \text{if } \; -2^{-n} < x < -2^{-n-1}, \qquad n \text{ even}, \\[2ex] 0 & \text{if } \; -2^{-n} < x < -2^{-n-1}, \qquad n \text{ odd}. \end{cases} \tag{4.5}$$

Of course, this implies

$$\bar\theta_{14}(x) \;=\; \bar\theta_{23}(x) \;=\; \begin{cases} 0 & \text{if } \; -2^{-n} < x < -2^{-n-1}, \qquad n \text{ even}, \\[2ex] 1 & \text{if } \; -2^{-n} < x < -2^{-n-1}, \qquad n \text{ odd}. \end{cases}$$

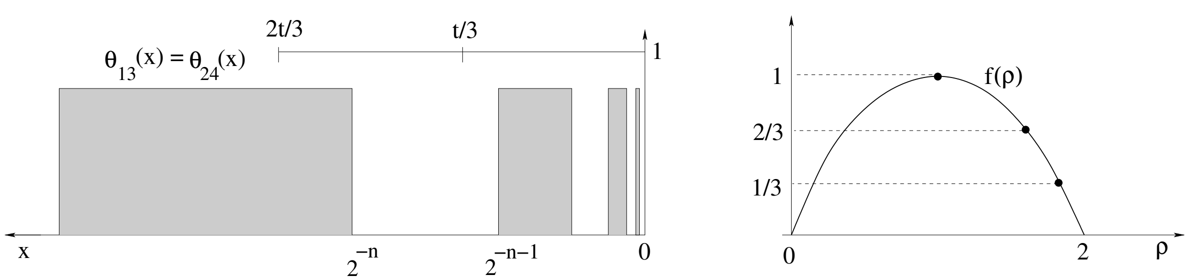

Figure 9: Left: the functions $\theta_{13} = \theta_{24}$ in (4.5) have unbounded variation. Right: the flux functions in (4.1).

A first, explicit solution of this problem is easily found:

$$\rho_1(t,x) \;=\; \rho_2(t,x) \;=\; \rho_3(t,x) \;=\; \rho_4(t,x) \;=\; 1. \tag{4.6}$$

Indeed, in this case the incoming fluxes at the intersections are

$$f_1(t,0-) \;=\; f_2(t,0-) \;=\; 1.$$

Therefore, the drivers' turning preferences are

$$\theta_{ij}(t,0-) \;=\; \bar\theta_{ij}(-t) \qquad\qquad i=1,2, \quad j=3,4, \quad t>0\,.$$

At any time t, drivers arriving to the intersection from road 1 and from road 2 always wish to turn into different outgoing roads. Therefore, no queue is ever created.

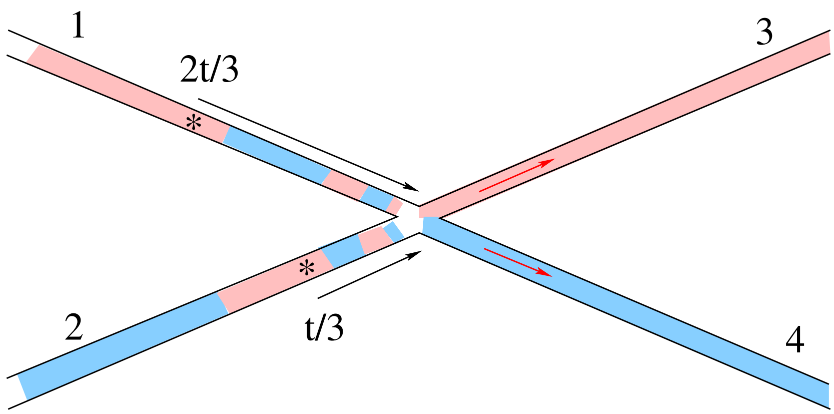

Figure 10: The second solution described in Example 3. When the intersection is congested, the flow of cars from road 1 is twice as large as the flow from road 2. For every $t>0$, drivers from road 1 and 2 arriving at the intersection at time t wish to turn into the same outgoing road. Hence the intersection remains congested.

We claim that there exists a second solution, where the incoming fluxes at the intersection are

$$f_1(t,0-) \;=\; \frac{2}{3}\,, \qquad\qquad f_2(t,0-) \;=\; \frac{1}{3}\,. \tag{4.7}$$

In this second solution, the traffic densities on the two incoming roads are piecewise constant, with a backward moving shock. Namely:

$$\rho_1(t,x) \;=\; \begin{cases} \quad 1 & \text{if} \quad x<\lambda_1 t\,,\\[2mm] 1+\sqrt{1/3} & \text{if} \quad \lambda_1 t<x<0\,,\end{cases}$$

$$\rho_2(t,x) \;=\; \begin{cases} \quad 1 & \text{if} \quad x<\lambda_2 t\,,\\[2mm] 1+\sqrt{2/3} & \text{if} \quad \lambda_2 t<x<0\,,\end{cases}$$

where the shock speeds are

$$\lambda_1 \;=\; -\sqrt{1/3}, \qquad\qquad \lambda_2 \;=\; -\sqrt{2/3}\,.$$

We claim that the junction conditions at the intersection are satisfied, for every time $t>0$. Indeed, since the flux from road 1 is twice as large as the flux from road 2, we have

$$\begin{cases} \theta_{1j}(t,0-) \;=\; \bar\theta_{1j}(-2t/3),\\[2mm] \theta_{2j}(t,0-) \;=\; \bar\theta_{2j}(-t/3),\end{cases} \qquad j=3,4, \quad t>0. \tag{4.8}$$

By (4.5), this implies

$$\theta_{1j}(t,0-) \;=\; \theta_{2j}(t,0-) \qquad j=3,4, \quad t>0\,.$$

At any given time $t>0$, drivers arriving to the intersection from road 1 and from road 2 always wish to turn into the same outgoing road. Hence the total flux through the intersection is $f_1(t,0-)+f_2(t,0-) \;=\; 1$. Since the prioritizing function Ψ satisfies (4.3), we conclude that this second solution satisfies the junction conditions as well.

It is worth mentioning that a very similar initial data with unbounded variation was considered in [6]. This provided an example of a Cauchy problem for a strictly hyperbolic system, admitting multiple solutions.

5 Multiple solutions for small BV data on a network

We now give another counterexample, showing that if the initial density $\hat{\rho}_i$ on some roads is allowed to be zero, even if the total variation is initially small the solution can be non-unique.

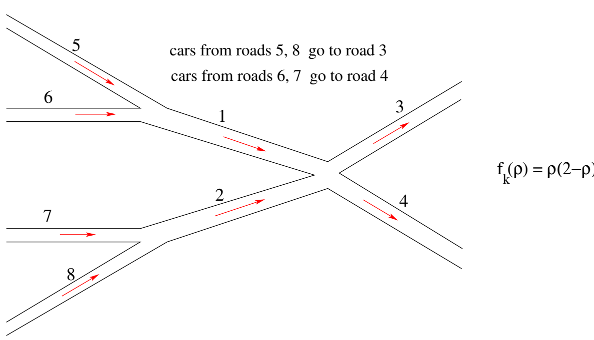

Figure 11: The network considered in Example 4.

Example 4. Consider a network consisting of eight roads, as in Fig. 11. We assume that roads 1 and 2 have unit length, while the other roads have infinite length. We shall construct a BV initial data such that, at time $t=1$, the solution along roads 1,2,3,4 will be the same as the initial data for Example 3.

We consider the flux functions:

$$f_1(\rho) \;=\; \cdots \;=\; f_8(\rho) \;=\; 2\rho-\rho^2\,.$$

Concerning drivers' preferences, assume that

- All drivers which are initially on roads 1, 5 and 7 eventually turn into road 3,
- All drivers which are initially on roads 2, 6 and 8 eventually turn into road 4.

Let the initial density be

$$\hat{\rho}_i(x) \;=\; 1 \qquad i=1,2,3,4.$$

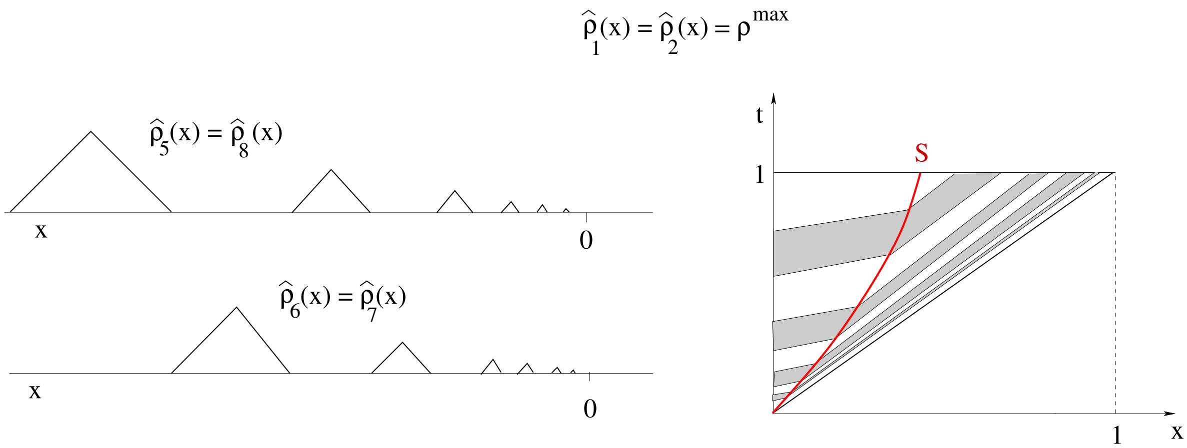

Figure 12: Left: the initial densities $\hat{\rho}_i$ on the incoming roads 5,6,7,8. Right: the car trajectories along road 1. Here $S(t)$ denotes the position of a shock. For $x > S(t)$ the car density is $\rho_1(t,x) = 1$. At any time t, the shaded regions denote the positions of drivers coming from road 5, and hence eventually turning into road 1.

To define the remaining initial densities, we first construct the points

$$y_n \doteq \sum_{k \geq n} 2^{-k/2}, \qquad\qquad z_n \doteq \frac{y_n + y_{n+1}}{2}.$$

Observe that the function

$$\phi_n(x) \doteq \max\{2^{-n/2} - |x - z_n|,\ 0\}$$

has a triangular shape and vanishes for $x \leq y_n$ and for $x \geq y_{n+1}$. Moreover,

$$\int \phi_n(x)\, dx = 2^{-n}.$$

For $x < 0$, we now define the initial densities on the incoming roads $5, 6, 7, 8$, by setting

$$\begin{cases} \hat{\rho}_i(x) \doteq \displaystyle\sum_{n=1}^{\infty} \phi_{2n}(-x), & i = 5, 8, \\[2em] \hat{\rho}_i(x) \doteq \displaystyle\sum_{n=1}^{\infty} \phi_{2n+1}(-x), & i = 6, 7. \end{cases}$$

All these densities have bounded variation. Indeed

$$\text{Tot.Var.}\{\hat{\rho}_5\} = \sum_{n=1}^{\infty} 2 \cdot 2^{-n} = 2, \qquad\qquad \text{Tot.Var.}\{\hat{\rho}_6\} = \sum_{n=1}^{\infty} 2 \cdot 2^{-(2n+1)/2} = \sqrt{2}.$$

With these initial data, the flux at the entrance of road 1 consists of

- cars coming from road 5, hence eventually turning into road 3, if $t \in [y_{n+1}, y_n]$, n odd,
- cars coming from road 6, hence eventually turning into road 4, if $t \in [y_{n+1}, y_n]$, n even.

Similarly, the flux at the entrance of road 2 consists of

- cars coming from road 8, hence eventually turning into road 4, if $t \in [y_{n+1}, y_n]$, n odd,
- cars coming from road 7, hence eventually turning into road 3, if $t \in [y_{n+1}, y_n]$, n even.

At time $t = 1$, the first cars coming from roads 5 or 6 start reaching the end of road 1 and entering either road 3 or 4. At exactly the same time, the first cars coming from roads 7 or 8 start reaching the end of road 2 and entering either road 3 or 4.

In a neighborhood of this last intersection, at time $t = 1$ we have thus reached exactly the same configuration as the initial data in Example 3. Hence, for $t > 1$, we obtain two distinct solutions.

6 Discontinuous dependence w.r.t. weak convergence

In view of the previous examples, it seems that the only type of intersections where the Cauchy problem is well posed are T-junctions. However, even for T-junctions with 1 incoming and n outgoing roads, the solution does not depend continuously on the data, w.r.t. the topology of weak convergence.

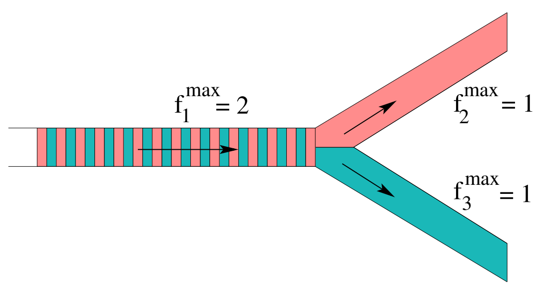

Figure 13: As shown in Example 5, the solution of the Cauchy problem cannot depend continuously on the drivers' choices θ_{ij}, in the topology of weak convergence.

Example 5. We show that the solution of the Cauchy problem cannot depend continuously on the θ_{ij} w.r.t. the topology of weak convergence. Consider a node with one incoming road and two outgoing roads. Assume that the maximum flux on each of the outgoing roads is $f_2^{max} = f_3^{max} = 1$. Let the initial densities be constant:

$$\rho_1(0,x) = \bar{\rho}_1, \qquad f_1(\bar{\rho}_1) = 2, \qquad \rho_2(0,x) = \rho_3(0,x) = 0. \tag{6.1}$$

Consider a sequence of highly oscillating drivers' preferences

$$(\theta_{12}^\nu, \theta_{13}^\nu)(x) = \begin{cases} (1,0) & \text{if } x \in \,](2k-1)\,2^{-\nu},\ 2k\,2^{-\nu}]\,, \\ (0,1) & \text{if } x \in \,]2k\,2^{-\nu},\ (2k+1)\,2^{-\nu}]\,. \end{cases}$$

For each $\nu \geq 1$, the Riemann solver will allow the maximum possible flux out of the road 1, namely $f_1(\rho_1^\nu(t, 0-)) = 1$.

On the other hand, as $\nu \to \infty$, the weak limit is

$$(\theta_{12}^\nu, \theta_{13}^\nu) \rightharpoonup \left(\frac{1}{2}, \frac{1}{2}\right).$$

Corresponding to this weak limit, the solution of the Cauchy problem with initial data (6.1) allows for an outgoing flux

$$f_1(\rho_1(t, 0-)) = 2 \neq 1 = \lim_{\nu\to\infty} f_1(\rho_1^\nu(t, 0-)).$$

7 Concluding remarks

Models of traffic flow at intersections based on Riemann Solvers, as developed in [7, 12, 13], work very well as long as the drivers' turning preferences θ_{ij} are assumed to be constant. In this case, for initial densities having bounded variation, solutions are known to be unique and depend continuously on the initial data. Unfortunately, in many situations the assumption that the θ_{ij} are constant is not realistic. In particular, this setting does not allow the analysis of optimization problems and of Nash equilibria [3].

Assuming that the initial data $\rho_k(0, x)$ and $\theta_{ij}(0, x)$ have bounded variation, in a neighborhood of a single intersection one still expects to find unique solutions. However, as shown by our Examples 3 and 4, on a network with several nodes the total variation of the θ_{ij} can blow up in finite time. Afterwards, uniqueness of solutions can be lost.

Because of these difficulties, one is led to consider alternative models, where the intersection is not just a point but occupies some region in physical space, such as a traffic circle [4, 15, 11]. By inserting one or more buffers in front of each outgoing road, in [4] the authors were able to construct intersection models where (i) the Cauchy problem is well posed within the general class of $\mathbf{L}^\infty$ initial data, and (ii) solutions depend continuously w.r.t. the topology of weak convergence. These are the two key properties needed to study global optima and Nash equilibria on a network of roads [5].

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