Conservation Law Models for Traffic Flow on a Network of Roads

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June 6, 2014

Abstract

The paper develops a model of traffic flow near an intersection, where drivers seeking to enter a congested road wait in a buffer of limited capacity. Initial data comprise the vehicle density on each road, together with the percentage of drivers approaching the intersection who wish to turn into each of the outgoing roads.

If the queue sizes within the buffer are known, then the initial-boundary value problems become decoupled and can be independently solved along each incoming road. Three variational problems are introduced, related to different kind of boundary conditions. From the value functions, one recovers the traffic density along each incoming or outgoing road by a Lax type formula.

Conversely, if these value functions are known, then the queue sizes can be determined by balancing the boundary fluxes of all incoming and outgoing roads. In this way one obtains a contractive transformation, whose fixed point yields the unique solution of the Cauchy problem for traffic flow in an neighborhood of the intersection.

The present model accounts for backward propagation of queues along roads leading to a crowded intersection, it achieves well-posedness for general $\mathbf{L}^\infty$ data, and continuity w.r.t. weak convergence.

1 Introduction

Optimal traffic assignment and dynamic user equilibria on networks have been widely discussed in the engineering literature [10, 11]. For conservation law models of traffic flow on a network of roads, these problems were recently studied in [5]. The basic setting comprises a network with nodes $A_1, \ldots, A_m$, and connecting arcs γ_{ij}. Drivers choose their time of departure and route to destination in order to minimize the sum of a departure cost $\varphi(\tau^d)$ and an arrival cost $\psi(\tau^a)$. The problem is highly nontrivial because the arrival time τ^a depends not only on the departure time τ^d but also on the overall traffic pattern.

On the k-th road of the network, the vehicle density $\rho = \rho_k(t,x)$ is governed by the conservation law

$$\rho_t + [\rho\, v_k(\rho)]_x \;=\; 0\,. \tag{1.1}$$

As in the classical papers [20, 21], we assume that the vehicle speed v_k is a function depending only on the density ρ. These scalar conservation laws must be supplemented by suitable initial conditions and by boundary conditions at road intersections. In [5], the existence of globally optimal traffic assignments, and of Nash equilibrium solutions, was proved for a general network of roads. However, the proof relied on a highly simplified intersection model. Namely, it was assumed that drivers who wish to enter a congested road are placed in a buffer of unlimited capacity, waiting their turn in line. In particular, the model could not account for the backward propagation of queues along roads leading to a crowded intersection.

Aim of the present paper is to develop a new class of models describing traffic flow at intersections, with more realistic features, including the backward propagation of queues. These models lead to Cauchy problems which are well posed within the class of bounded measurable data. As shown in the forthcoming paper [6], they are well suited for the analysis of global optimization and Nash equilibrium problems.

Due to finite propagation speed, to solve the Cauchy problem for traffic flow on an entire network it suffices to construct a local solution in a neighborhood of an intersection. To fix the ideas, consider a junction with m incoming roads, labelled by $i \in \mathcal{I} = \{1, \ldots, m\}$, and n outgoing roads, labelled by $j \in \mathcal{O} = \{m+1, \ldots, m+n\}$. Denote by $\rho_i(t,x)$, $x < 0$ the density of cars on incoming roads, and by $\rho_j(t,x)$, $x > 0$, the density of cars on outgoing roads. At each time t, the boundary conditions will impose suitable restrictions on the $m+n$ boundary values

$$\rho_i(t, 0-), \quad i \in \mathcal{I}, \qquad\qquad \rho_j(t, 0+), \quad j \in \mathcal{O}.$$

In a realistic model, these boundary conditions should depend on

(i) Drivers' turning choices. For every $i \in \mathcal{I}$, $j \in \mathcal{O}$, these are modeled by assigning the fraction θ_{ij} of drivers arriving from the i-th road who wish to turn into the j-th road.

(ii) Relative priority given to incoming roads. For example, if the intersection is regulated by a crosslight, this is modeled by assigning the fraction of time η_i when cars arriving from the i-th road get a green light.

Here $\eta_1, \ldots, \eta_m$ can be taken to be positive constants, with $\sum_i \eta_i = 1$. On the other hand, toward the analysis of optimization problems, the coefficients θ_{ij} cannot be taken as constant but must be determined as part of the solution itself. We illustrate this important point with the aid of Figure 1. Consider two groups of commuters: the first ones drive west-east from road 1 to road 4, while the others drive north-south from road 2 to road 5. All drivers share road 3 as common part of their journey. At the intersection B, the percentage of drivers that turns into road 4 or 5 is not constant, but depends on how many drivers of the two groups are present at the intersection at any given time.

More generally, call $\theta_{ij}(t,x) \in [0,1]$ the fraction of drivers along the i-th incoming road that wish to turn into the j-th outgoing road. These functions θ_{ij} satisfy the obvious relations

$$\sum_{j=1}^{n} \theta_{ij} \;=\; 1\,, \qquad\qquad i = 1, \ldots, m.$$

Calling ρ_i the vehicle density along the i-th road, we have the additional conservation laws

$$(\rho_i \theta_{ij})_t + \Big(\rho_i v_i(\rho_i)\, \theta_{ij}\Big)_x \;=\; 0\,, \qquad i \in \mathcal{I}, \quad j \in \mathcal{O}\,. \tag{1.2}$$

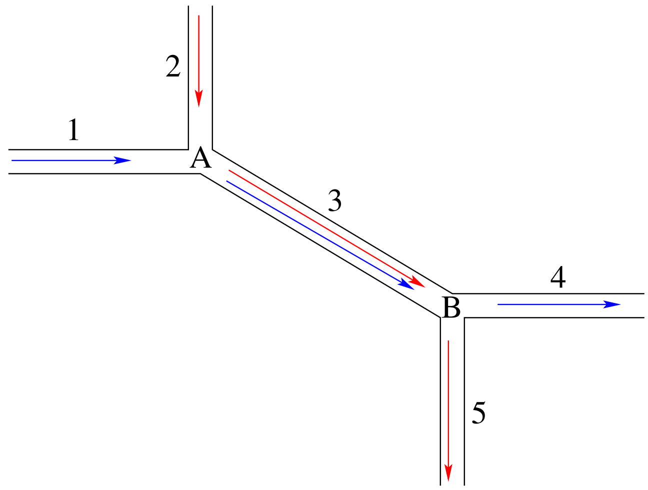

Figure 1: At the intersection B, at any time t the fraction of cars turning left or right is not given a priori but must be computed as part of the solution itself.

By (1.1), these yield the $m \times n$ linear transport equations

$$\theta_{ij,t} + v_i(\rho_i)\,\theta_{ij,x} \;=\; 0 \qquad i \in \mathcal{I}, \quad j \in \mathcal{O}. \tag{1.3}$$

We remark that, to be useful in the analysis of global optimization and Nash equilibrium problems, a model of traffic flow at intersections in terms of the variables ρ_k, θ_{ij} should have two crucial properties:

(I) Well posedness for $\mathbf{L}^\infty$ data.

(II) Continuity w.r.t. weak convergence.

When the flow near an intersection is described in terms of Riemann Solver [8, 13, 14], the counterexamples in [7] show that the total variation of the variables ρ_k, θ_{ij} can become unbounded in finite time, leading to multiple solutions with the same initial data. In addition, even for a simple junction with one incoming and two outgoing roads, Example 5 in [7] shows that the time that drivers need to reach destination does not depend continuously on the variables θ_{ij}, in the topology of weak convergence.

In order to achieve the key properties **(I) - (II)**, at each road intersection our model includes a buffer of limited capacity, as proposed in [12, 15, 16]. We let $q_j(t)$ be the length of the queue in front of the outgoing road $j \in \mathcal{O}$. The rate at which cars enter the intersection is governed by the lengths of these queues. Drivers who are already within the intersection move on to the outgoing roads of their choice, at the maximum rate allowed by the traffic density on these roads.

The main contributions of our analysis can be summarized as follows:

(i) If the queue lengths $q_j(\cdot)$ in front of all outgoing roads are known, then the initial-boundary value problems become decoupled. Indeed, they can be independently solved on each incoming road $i \in I$ and, at a second stage, on each outgoing road $j \in \mathcal{O}$.

Three different optimization problems are introduced, related to different kind of boundary conditions. From the value functions $V_k(t,x)$, $k = 1,\ldots,m+n$, one recovers the traffic densities $\rho_k(t,x) = V_{k,x}(t,x)$ along each road. These densities are explicitly computed by a Lax type formula.

(ii) If the value functions V_k are known, the lengths $q_j(\cdot)$ of the queues can be determined by balancing the boundary fluxes of all incoming and outgoing roads. As shown in Fig. 2, in this way we obtain a contractive transformation $q \mapsto \Lambda(q)$ on a space of Lipschitz continuous functions. The fixed point of this transformation yields the unique solution of the Cauchy problem for traffic flow, in an neighborhood of the intersection.

(iii) Our model of traffic flow at intersections thus achieves well-posedness for general $\mathbf{L}^\infty$ data, and continuity w.r.t. weak convergence. Because of these properties, it is ideally suited to study optimization and Nash equilibrium problems, as shown in the forthcoming paper [6].

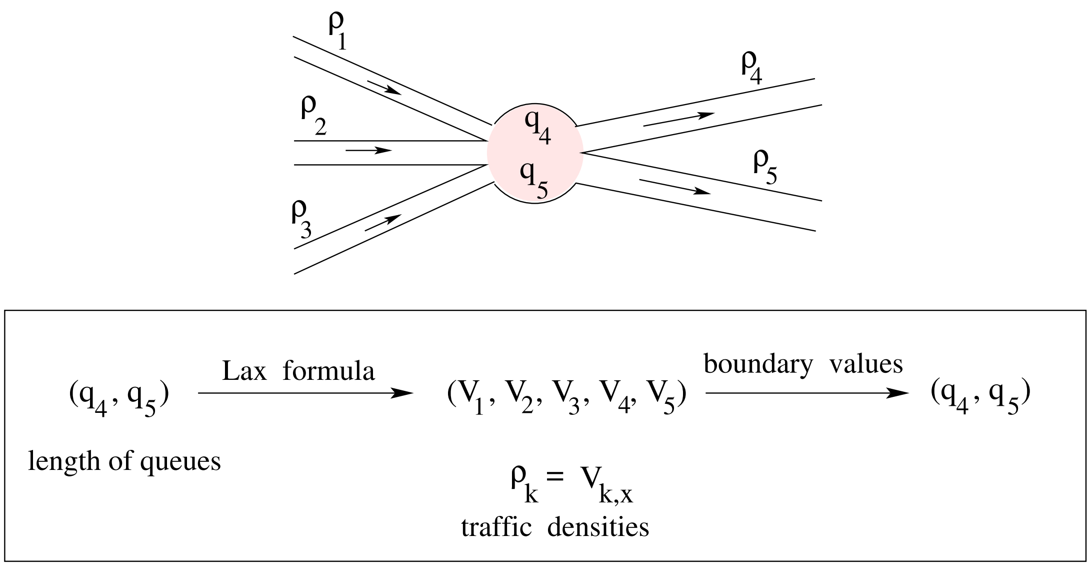

Figure 2: Sketch of the model. Given the length of the queues in front of each outgoing road, by the Lax formula one determines the traffic density $\rho_k = V_{k,x}$, separately on each incoming and each outgoing road. In turn, taking the boundary values of the functions V_k at $x = 0$ one recovers the length of each queue. Ultimately, the solution is achieved as the fixed point of a contractive transformation.

Some relations with earlier work are worth mentioning. Motivated by [2], a natural extension of the Lax formula [18] to the initial-boundary value problem for a scalar conservation law was given in [19]. The boundary conditions are here formulated by assigning values $u_0(t)$ for the conserved quantity, while a variational inequality determines whether these boundary values can be pointwise attained or not. In the present paper, on the other hand, we formulate the boundary conditions by assigning an upper bound on the flux through the boundary, at each time t. In general, this bound depends on the solution itself, through the measurable coefficients θ_{ij}.

A variational approach to the Cauchy problem near a junction of roads was recently introduced in [17]. This is formulated as one single optimization problem, simultaneously for all roads joining at the intersection, and leads to an interesting generalization of the Lax formula on networks. However, the construction is valid only for particular choices of the coefficients θ_{ij}, constant in time.

The remainder of this paper is organized as follows. Section 2 introduces some notation and formulates the main assumptions on the flux functions f_k and on the flow at the intersection, modeled in terms of one or more buffers. In Section 3 we give a definition of admissible solution to the Cauchy problem near a junction, by means of of a generalized Lax formula. Section 4 contains the main result, showing that the Cauchy problem has a globally defined solution, obtained as the unique fixed point of a contractive transformation. In Section 5 we prove that this solution depends continuously on the initial data, in the topology of weak convergence. As remarked earlier, this property is essential toward the analysis of optimization problems.

We observe that, in standard textbooks, one first defines an *admissible solution* to a conservation law by imposing suitable entropy conditions. At a later stage, one checks that the function provided by the Lax formula [9, 18, 22] is indeed an entropy admissible solution. In the present paper we follow a converse approach. Namely, we first give a definition of admissible solution in terms of the Lax formula. Afterwards, we prove that this solution is unique and satisfies the Kruzhkov entropy conditions in the interior of the domain, together with the appropriate initial and boundary conditions required by the model **(SBJ)** or **(MBJ)**.

The second part of this program is achieved in the remaining Sections 6 to 8. Given the initial data and the lengths $q_j(\cdot)$ of the queues at the intersection, three optimization problems are introduced. These correspond to (i) incoming roads for the model **(SBJ)**, (ii) incoming roads for the model **(MBJ)**, and (iii) outgoing roads. In all three cases, we prove that the optimal solutions exist. The value functions V_k are computed by the Lax-type formulas (3.18), (3.28), and (3.22), respectively. From the properties of the value functions V_k, we eventually deduce that the derivatives $\rho_k = V_{k,x}$ provide entropy weak solutions, satisfying the appropriate initial and boundary conditions.

Two lemmas, on the uniqueness of solutions to ODEs with measurable right hand side, are collected in the Appendix.

2 General setting

Consider a family of $n+m$ roads, joining at a node. Indices $i \in \{1,\ldots,m\} = \mathcal{I}$ denote *incoming roads*, while indices $i \in \{m+1,\ldots,m+n\} = \mathcal{O}$ denote *outgoing roads*. On the k-th road, the density of cars $\rho_k(t,x)$ is described by the scalar conservation law

$$\rho_t + f_k(\rho)_x \;=\; 0\,. \tag{2.1}$$

Here $t \geq 0$, while $x \in]-\infty,0]$ for incoming roads and $x \in [0,\infty[$ for outgoing roads. The flux function is $f_k(\rho) = \rho\, v_k(\rho)$, where $v_k(\rho)$ is the speed of cars on the k-th road. We assume that this speed depends only on the density ρ. Moreover, we assume

$$v_k'(\rho) \;\leq\; 0, \qquad f_k \in \mathcal{C}^2, \qquad f_k''(\rho) \;<\; 0, \qquad f_k(0) \;=\; f_k(\rho_k^{jam}) \;=\; 0, \tag{2.2}$$

where ρ_k^{jam} is the maximum possible density of cars on the k-th road. This corresponds to bumper-to-bumper packing, so that the speed of cars is zero. For a given road $k \in \{1,\ldots,m+$

$n\}$, we denote by

$$f_k^{max} \doteq \max_s f_k(s)$$

the maximum flux and

$$\rho_k^{max} \doteq \operatorname{argmax}_s f_k(s) \tag{2.3}$$

the traffic density corresponding to this maximum flux (see Fig. 3).

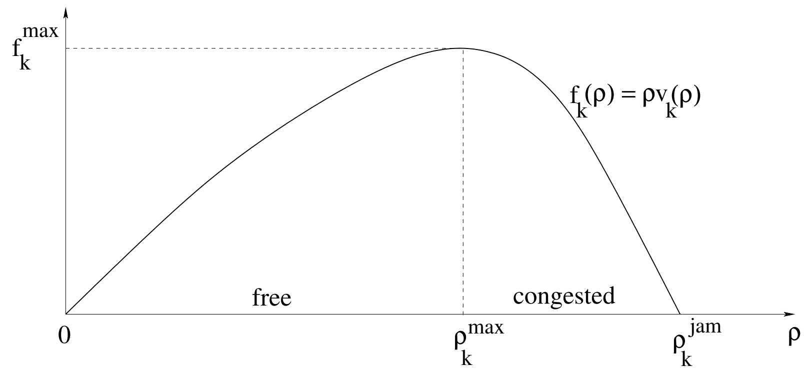

Figure 3: The flux f_k as a function of the density ρ, along the k-th road.

Moreover, we say that

$$\begin{array}{rl} \rho \ \text{ is a } \textbf{free} \text{ state if} & \rho \in \ [0,\, \rho_k^{max}[\,, \\ \rho \ \text{ is a } \textbf{congested} \text{ state if} & \rho \in \ ]\rho_k^{max},\, \rho_k^{jam}]\,. \end{array}$$

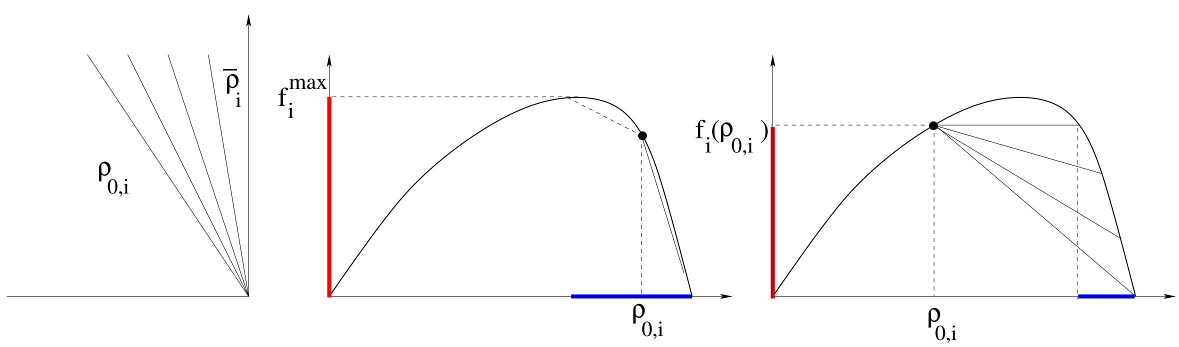

Figure 4: The case of an incoming road $i \in \mathcal{I}$. Given a left state $\rho_{0,i}$, we seek the family of all right states $\bar\rho_k$ which can be connected to $\rho_{0,i}$ by a wave having negative speed. Center: $\rho_{0,i}$ is a congested state, Right: $\rho_{0,i}$ is a free state.

Given initial data on each road

$$\rho_k(0,x) = \rho_k^{\diamondsuit}(x) \qquad\qquad k = 1,\ldots,m+n, \tag{2.4}$$

in order to determine a unique solution to the Cauchy problem we must supplement the conservation laws (2.1) with a suitable set of boundary conditions. These provide additional constraints on the limiting values of the vehicle densities

$$\bar\rho_k(t) \doteq \lim_{x\to 0} \rho_k(t,x) \qquad\qquad k = 1,\ldots,m+n \tag{2.5}$$

near the intersection. In a realistic model, these boundary conditions should depend on:

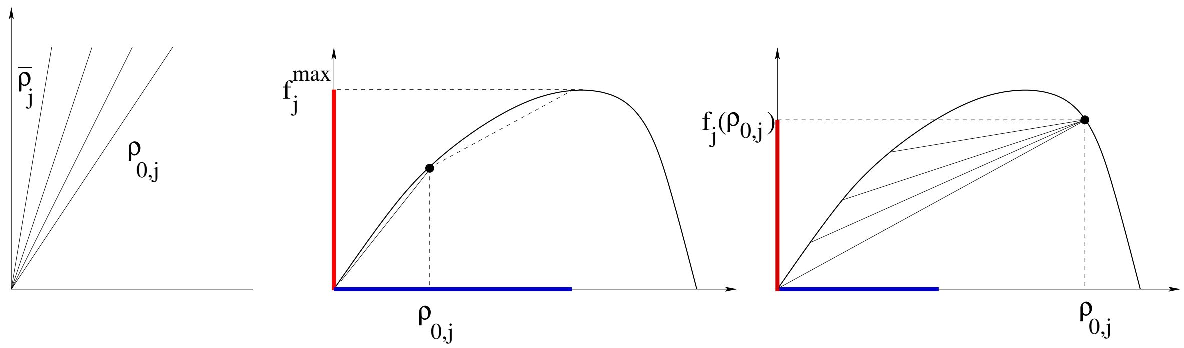

Figure 5: The case of an outgoing road $j \in \mathcal{O}$. Given a right state $\rho_{0,j}$, we seek the family of all left states $\bar{\rho}_j$ which can be connected to $\rho_{0,j}$ by a wave having positive speed. Center: $\rho_{0,j}$ is a free state, Right: $\rho_{0,j}$ is a congested state.

(i) Relative priority given to incoming roads. For example, if the intersection is regulated by a crosslight, the flow will depend on the fraction $\eta_i \in]0,1[$ of time when cars arriving from the i-th road get a green light.

(ii) Drivers' choices. For every $i \in \mathcal{I}$, $j \in \mathcal{O}$, these are modeled by assigning the fraction $\theta_{ij} \in [0,1]$ of drivers arriving from the i-th road who choose to turn into the j-th road. Obvious modeling considerations imply

$$\theta_{ij} \in [0,1], \qquad \sum_{j\in\mathcal{O}} \theta_{ij} = 1 \quad \text{for each } i \in \mathcal{I}. \tag{2.6}$$

In general, the coefficients $\theta_{ij} = \theta_{ij}(t,x)$ need not be constant. Throughout the following, we assume that drivers on the i-th road know in advance their itinerary and do not change their mind. This yields the conservation law

$$(\theta_{ij}\rho_i)_t + [\theta_{ij} f_i(\rho_i)]_x = 0.$$

We can thus regard each θ_{ij} as a passive scalar, transported along the flux:

$$(\theta_{ij})_t + v_i(\rho_i)(\theta_{ij})_x = 0. \tag{2.7}$$

In view of several counterexamples [7], it appears that there is no hope to develop an existence-uniqueness theory for conservation laws on networks based on the Garavello-Piccoli approach, relying on Riemann Solvers. We propose here an alternative approach, modifying the intersection model used in [5]. According to this earlier model, if the flux of cars that want to enter road j is larger than f_j^{max} (the maximum flux allowed on that road), cars are placed in a queue, first-in-first-out. It is assumed that the queue can become arbitrarily large, occupying a buffer of unlimited capacity. As a consequence, there is no backward propagation of queues along the incoming roads.

Here we consider a more realistic model, similar to [12, 15, 16], where at each intersection there is a buffer of limited capacity. The incoming fluxes of cars toward the intersection are constrained by the current degree of occupancy of the buffer. More precisely, consider an intersection with m incoming and n outgoing roads. The state of the buffer at the intersection is described by an n-vector

$$\mathbf{q} = (q_j)_{j\in\mathcal{O}}.$$

Here $q_j(t)$ is the number of cars at the intersection waiting to enter road $j \in \mathcal{O}$ (in other words, the length of the queue in front of road j). Boundary values at the junction will be denoted by

$$\begin{cases} \bar{\theta}_{ij}(t) & \doteq & \lim_{x\to 0-} \theta_{ij}(t,x), \qquad i \in \mathcal{I},\ j \in \mathcal{O}\,, \\ \bar{\rho}_i(t) & \doteq & \lim_{x\to 0-} \rho_i(t,x), \qquad i \in \mathcal{I}\,, \\ \bar{\rho}_j(t) & \doteq & \lim_{x\to 0+} \rho_j(t,x), \qquad j \in \mathcal{O}\,, \\ \bar{f}_i(t) & \doteq & f_i(\bar{\rho}_i(t)) \ = \ \lim_{x\to 0-} f_i(\rho_i(t,x)), \qquad i \in \mathcal{I}\,, \\ \bar{f}_j(t) & \doteq & f_j(\bar{\rho}_j(t)) \ = \ \lim_{x\to 0+} f_j(\rho_j(t,x)), \qquad j \in \mathcal{O}\,. \end{cases} \tag{2.8}$$

Conservation of the total number of cars implies

$$\dot{q}_j \ = \ \sum_{i\in\mathcal{I}} \bar{f}_i \bar{\theta}_{ij} - \bar{f}_j \qquad \text{for all } j \in \mathcal{O}\,, \tag{2.9}$$

at a.e. time $t \geq 0$. Here and in the sequel, the upper dot denotes a derivative w.r.t. time. Following [14], we also define

$$\omega_i \ = \ \omega_i(\bar{\rho}_i) \ \doteq \ \begin{cases} f_i(\bar{\rho}_i) & \text{if } \bar{\rho}_i \text{ is a free state,} \\ f_i^{max} & \text{if } \bar{\rho}_i \text{ is a congested state,} \end{cases} \qquad i \in \mathcal{I}\,, \tag{2.10}$$

the maximum possible flux at the end of an incoming road. Notice that this is the largest flux $f_j(\rho)$ among all states ρ that can be connected to $\bar{\rho}_i$ with a wave of negative speed (Fig. 4).

Similarly, we define

$$\omega_j \ = \ \omega_j(\bar{\rho}_j) \ \doteq \ \begin{cases} f_j(\bar{\rho}_j) & \text{if } \bar{\rho}_j \text{ is a congested state,} \\ f_j^{max} & \text{if } \bar{\rho}_j \text{ is a free state,} \end{cases} \qquad j \in \mathcal{O}\,, \tag{2.11}$$

the maximum possible flux at the beginning of an outgoing road. This is the largest flux $f_j(\rho)$ among all states ρ that can be connected to $\bar{\rho}_j$ with a wave of positive speed (Fig. 5).

We are now ready to introduce two different sets of equations relating the incoming and outgoing fluxes $\bar{f}_i$ and $\bar{f}_j$, depending on the drivers' choices $\bar{\theta}_{ij}$ and on the lengths q_j of the queues in the buffer. We will prove later that both models lead to well posed Cauchy problems.

In the first model, the junction contains one single buffer of size M. Incoming cars are admitted at a rate depending of the amount of free space left in the buffer, regardless of their destination. Once they are within the intersection, cars flow out at the maximum rate allowed by the outgoing road of their choice.

Single Buffer Junction (SBJ). *Consider a constant $M > 0$, describing the maximum number of cars that can occupy the intersection at any given time, and constants $c_i > 0$, $i \in \mathcal{I}$, accounting for priorities given to different incoming roads.*

We then require that the incoming fluxes $\bar{f}_i$ satisfy

$$\bar{f}_i \ = \ \min \left\{ \omega_i\,, \ \ c_i\Big(M - \sum_{j\in\mathcal{O}} q_j\Big) \right\}, \qquad i \in \mathcal{I}\,. \tag{2.12}$$

In addition, the outgoing fluxes $\bar f_j$ should satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar f_j = \omega_j, \\ \\ \text{if } q_j = 0, \text{ then } \bar f_j = \min\left\{\omega_j, \ \sum_{i\in\mathcal{I}} \bar f_i \bar\theta_{ij}\right\}, \end{cases} \qquad j \in \mathcal{O}. \tag{2.13}$$

In our second model, there are n buffers, one for each outgoing road. Incoming drivers are admitted at a rate depending on the length of the queue at the entrance of the road of their choice.

Multiple Buffer Junction (MBJ) *Consider constants M_j, $j \in \mathcal{O}$, describing the size of the buffer at the entrance of the j-th outgoing road, and constants $c_i > 0$, $i \in \mathcal{I}$, accounting for priorities given to different incoming roads.*

We then require that the incoming fluxes $\bar f_i$ satisfy

$$\bar f_i \;=\; \min\left\{\omega_i\,,\ \frac{c_i(M_j - q_j)}{\theta_{ij}},\ \ j \in \mathcal{O}\right\}, \qquad i \in \mathcal{I}. \tag{2.14}$$

As before, the outgoing fluxes $\bar f_j$, should satisfy (2.13).

Remark 1. The difference $M_j - q_j$ in (2.14) describes how much space is left in the buffer at the entrance of the j-th road. When this space shrinks, cars are admitted to the intersection at a slower rate. This difference can decrease exponentially in time, but never becomes zero. Indeed, by (2.9) and (2.14),

$$\frac{d}{dt}(M_j - q_j(t)) \;=\; -\dot q_j(t) \;\geq\; -\left(\sum_{i\in\mathcal{I}} c_i\right)(M_j - q_j(t)). \tag{2.15}$$

The choice $M_j = +\infty$ would correspond to a buffer of unlimited capacity, and leads to the same model considered in [5].

By the same argument, the difference $M - \sum_{j\in\mathcal{O}} q_j$ in (2.12) can decrease exponentially but is never zero.

3 The Cauchy problem

In this section we study the Cauchy problem for the system of equations

$$(\rho_k)_t + f_k(\rho_k)_x \;=\; 0\,, \qquad k \in \mathcal{I}\cup\mathcal{O}, \tag{3.1}$$

$$(\theta_{ij})_t + v_i(\rho_i)(\theta_{ij})_x \;=\; 0\,, \qquad i \in \mathcal{I},\ \ j \in \mathcal{O}\,, \tag{3.2}$$

supplemented by the ODEs

$$\dot q_j \;=\; \sum_{i\in\mathcal{I}} \bar f_i \bar\theta_{ij} - \bar f_j \qquad \text{for all } j \in \mathcal{O}\,, \tag{3.3}$$

and by the boundary conditions (2.12)-(2.13) or (2.14)-(2.13). We consider initial data of the form

$$\begin{cases} \rho_k(0,x) &= \rho_k^\diamondsuit(x) \qquad\qquad k \in \mathcal{I}\cup\mathcal{O}\,, \\ \theta_{ij}(0,x) &= \theta_{ij}^\diamondsuit(x) \qquad\qquad i\in\mathcal{I},\quad j\in\mathcal{O}\,, \\ q_j(0) &= q_j^\diamondsuit \qquad\qquad\qquad j\in\mathcal{O}\,. \end{cases} \tag{3.4}$$

By an admissible solution of the above system we mean a family of functions $(\rho_k, \theta_{ij}, q_j)$, with

$$\rho_k \;\in\; [0, \rho_k^{jam}[\,, \qquad \theta_{ij}\in[0,1], \qquad \sum_{j\in\mathcal{O}} \theta_{ij} = 1\,, \tag{3.5}$$

$$q_j \geq 0, \qquad \begin{cases} \sum_{j\in\mathcal{O}} q_j < M, & \text{in case of } \textbf{(SBJ)}\,, \\ q_j < M_j \quad \text{for every } j\in\mathcal{O}, & \text{in case of } \textbf{(MBJ)}\,, \end{cases} \tag{3.6}$$

and with the following properties.

(P1) The functions ρ_k provide entropy-weak solutions to the conservation laws in (3.1).

(P2) The functions θ_{ij} provide solutions to the linear transport equations in (3.2).

(P3) The functions q_j are Lipschitz continuous and satisfy the ODEs (3.3).

(P4) The initial values of ρ_k, θ_{ij} and q_j satisfy (3.4).

(P5) The boundary values $\bar\rho_k(t)$, $\bar f_k(t)$, $\bar\theta_{ij}(t)$ in (2.8) are well defined in the sense of traces, and satisfy the boundary conditions (2.12)-(2.13) or (2.14)-(2.13) for a.e. $t \geq 0$.

It will be convenient to reformulate the above conditions in terms of the Lax formula, using a set of integrated variables V_k such that

$$V_{k,x}(t,x) \;=\; \rho_k(t,x). \tag{3.7}$$

For each $k\in\mathcal{I}\cup\mathcal{O}$, consider the concave function (see Fig. 6)

$$g_k(v) \;\doteq\; \inf_{u\in[0,\,\rho_k^{jam}]} \{uv - f_k(u)\}\,. \tag{3.8}$$

Notice that g_k is the Legendre transform of the flux function f_k. Indeed

$$g_k(v) \;=\; u^*(v)\,v - f_k(u^*(v)), \tag{3.9}$$

where the map $v\mapsto u^*(v)$ is implicitly defined by

$$f_k'(u^*(v)) \;=\; v\,. \tag{3.10}$$

In particular,

$$g_k'(v) \;=\; u \qquad\Longleftrightarrow\qquad f_k'(u) \;=\; v\,. \tag{3.11}$$

Remark 2. Consider a characteristic $t \mapsto x(t)$ for the conservation law (3.1), with speed $\dot{x} = v$. By (3.9)-(3.10), the Legendre transform can be interpreted as

$$g_k(v) \;=\; -\,[\text{flux of cars from left to right, across the characteristic}]. \tag{3.12}$$

For $v \in \left]f'(\rho_k^{jam}),\, f_k'(0)\right[$, differentiating w.r.t. v, one obtains

$$g_k''(v) \;=\; \frac{\partial}{\partial v} g_k'(u^*(v)) \;=\; \frac{1}{f_k''(u^*(v))} \;<\; 0\,, \tag{3.13}$$

showing that g_k is strictly concave down on this open interval. As shown in Fig. 6, we also have the implications

$$\begin{cases} v \;\leq\; f_k'(\rho_k^{jam}) & \Longrightarrow \qquad g_k(v) = \rho_k^{jam}\, v, \\[2mm] \quad v \;\geq\; f_k'(0) & \Longrightarrow \qquad g_k(v) = 0. \end{cases} \tag{3.14}$$

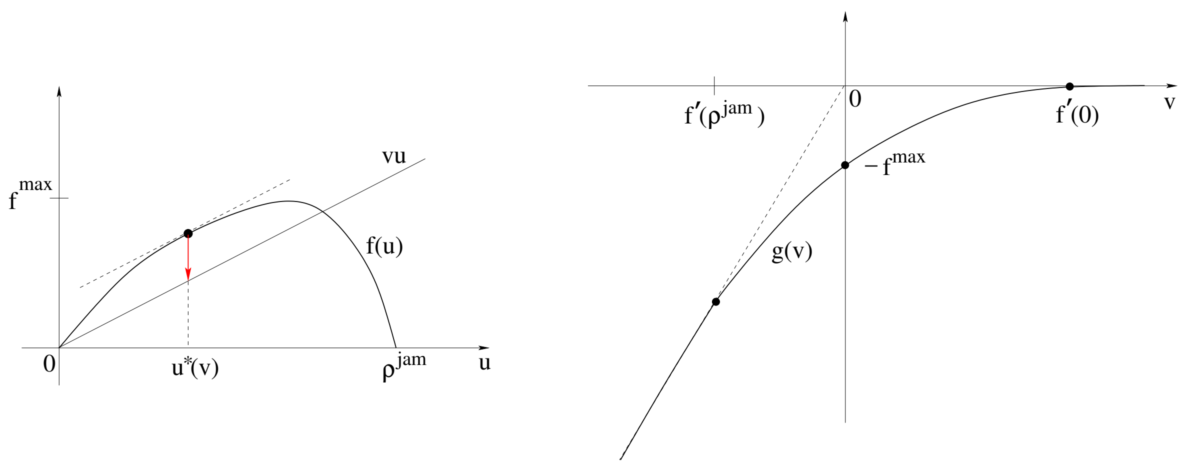

Figure 6: The flux function f and its Legendre transform g defined at (3.8).

In connection with the boundary conditions **(SBJ)**, for $i \in \mathcal{I}$ we also consider the functions

$$h_i(\mathbf{q}) \;\doteq\; \min\; \left\{ f_i^{max}\,,\;\; c_i \cdot \Big(M - \sum_{j \in \mathcal{O}} q_j \Big) \right\}. \tag{3.15}$$

For the junction conditions **(MBJ)** with multiple buffers, these will be replaced by

$$h_i(\mathbf{q}, \theta) \;\doteq\; \min\; \left\{ f_i^{max}\,,\;\; c_i \cdot \frac{M_j - q_j}{\theta_{ij}}\,;\;\; j \in \mathcal{O} \right\}. \tag{3.16}$$

Assume now that the initial data $\rho_k^{\diamondsuit}$, $\theta_{ij}^{\diamondsuit}$, $q_j^{\diamondsuit}$ are given, satisfying the same pointwise estimates as in (3.5)-(3.6). To obtain a solution to the Cauchy problem (3.1)-(3.4), satisfying all conditions (i)–(v), we consider a family of Lipschitz continuous functions $q_j = q_j(t)$ and $V_k = V_k(t, x)$ having the properties **(I)–(III)** below.

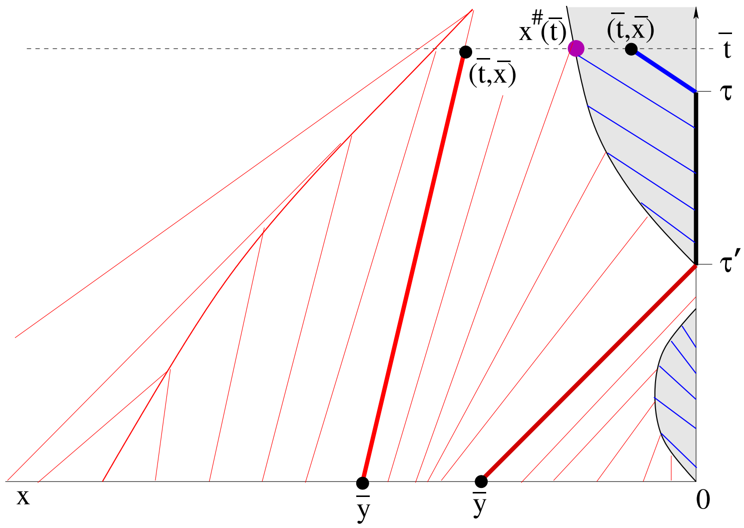

Figure 7: The trajectories leading to the maximum value in (3.18). For each time $t \geq 0$ there exists a unique point $x^\sharp(t) \leq 0$ with the following property. For terminal points $\bar{x} < x^\sharp(t)$, optimal trajectories are affine, while for terminal points $x > x^\sharp(t)$ optimal trajectories are piecewise affine, also taking the value $x = 0$ on some time interval.

(I) For $i \in \mathcal{I}$ and $x < 0$, define

$$V_i^\diamondsuit(x) \doteq \int_{-\infty}^x \rho_i^\diamondsuit(y)\,dy\,. \tag{3.17}$$

In the case of boundary conditions **(SBJ)**, recalling (3.15) we require (see Fig. 7)

$$V_i(t,x) \doteq \max\left\{\max_{y\leq 0}\left[V_i^\diamondsuit(y) + t\,g_i\Big(\frac{x-y}{t}\Big)\right],\right.$$
$$\left.\max_{0\leq\tau'\leq\tau\leq t,\ y\leq 0}\left[V_i^\diamondsuit(y) + \tau'\,g_i\Big(\frac{-y}{\tau'}\Big) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s))\,ds + (t-\tau)\,g_i\Big(\frac{x}{t-\tau}\Big)\right]\right\}. \tag{3.18}$$

Here one can think of $V_i(t,x)$ as the total amount of cars which at time t are still inside the half line $]-\infty,x]$. The total amount of cars which have exited from road i during the time interval $[0,t]$ is thus measured by

$$V_i^\diamondsuit(0) - V_i(t,0)\,.$$

To determine how many of these cars wanted to enter road $j \in \mathcal{O}$, we proceed as follows. Let $\xi_i(t)$ be implicitly defined by

$$\xi_i(t) = \max\left\{z \in]-\infty,0]\,;\quad \int_z^0 \rho_i^\diamondsuit(y)\,dy = V_i^\diamondsuit(0) - V_i(t,0)\right\}. \tag{3.19}$$

In other words, $\xi_i(t)$ is the initial position of that particular car on road i which reaches the intersection at time t. The total number of cars that have reached the intersection before time t and wish to turn into road j is thus

$$F_j(t) = q_j^\diamondsuit + \sum_{i\in\mathcal{I}} \int_{\xi_i(t)}^0 \rho_i^\diamondsuit(y)\,\theta_{ij}^\diamondsuit(y)\,dy\,. \tag{3.20}$$

(II) For $j \in \mathcal{O}$ and $x > 0$, defining

$$V_j^\diamondsuit(x) \doteq \int_0^x \rho_j^\diamondsuit(y)\,dy\,, \tag{3.21}$$

we require

$$V_j(t,x) \doteq \max\left\{ \max_{y\geq 0}\left[V_j^\diamondsuit(y) + t\,g_j\Big(\frac{x-y}{t}\Big)\right]\,,\quad \max_{0\leq\tau\leq t}\left[-F_j(\tau) + (t-\tau)\,g_j\Big(\frac{x}{t-\tau}\Big)\right]\right\}, \tag{3.22}$$

where F_j was defined at (3.20).

We observe that

$$V_j^\diamondsuit(x) - V_j(t,x)$$

is the number of cars that have crossed the point x during the time interval $[0,t]$. In particular, $-V_j(t,0)$ is the number of cars that have entered road j (possibly after waiting in a queue) during the time interval $[0,t]$. By (3.20), conservation of the total number of cars implies:

(III) At time t, the length of the queue at the entrance of road j is computed by

$$q_j(t) = F_j(t) + V_j(t,0). \tag{3.23}$$

When dealing with the boundary conditions **(MBJ)**, the formula (3.18) must be modified as follows. For $i \in \mathcal{I}$, $j \in \mathcal{O}$, and $\beta > 0$, we define the point $x_i(\beta)$ implicitly by setting

$$x_i(\beta) = \sup\left\{y \in]-\infty, 0]\,;\quad \int_y^0 \rho_i^\diamondsuit(x)\,dx = \beta\right\}. \tag{3.24}$$

Observe that the function $\beta \mapsto x_i(\beta)$ is decreasing, hence it is differentiable almost everywhere in its domain. Given the initial data $\theta_{ij}^\diamondsuit$, we define the measurable function

$$\theta_{ij}(\beta) \doteq \theta_{ij}^\diamondsuit(x_i(\beta))\,.$$

Finally, given $y \leq 0$ and $0 \leq \tau' \leq \tau$, we define

$$G_i(\tau,\tau';\,y) \doteq \beta(\tau), \tag{3.25}$$

where $s \mapsto \beta(s)$ denotes the solution to the Cauchy problem

$$\frac{d}{ds}\beta(s) = -\,h_i\big(\mathbf{q}(s), \theta_{ij}(\beta(s))\big) \qquad\qquad s \in [\tau',\,\tau]\,, \tag{3.26}$$

$$\beta(\tau') = V_i^\diamondsuit(y) + \tau'\,g_i\Big(\frac{-y}{\tau'}\Big)\,. \tag{3.27}$$

Lemma A1 in the Appendix shows that $\beta(\cdot)$ is well defined, because this Cauchy problem with measurable coefficients admits a unique solution.

In the case **(MBJ)** of a junction with multiple buffers, the formula (3.18) is replaced by

$$V_i(t,x) \doteq \max\Bigg\{ \max_{y\le 0}\left[V_i^{\diamondsuit}(y) + t\,g_i\Big(\frac{x-y}{t}\Big)\right], \qquad \max_{0\le\tau'\le\tau\le t,\ y\le 0}\left[G_i(\tau,\tau';\,y) + (t-\tau)\,g_i\Big(\frac{x}{t-\tau}\Big)\right]\Bigg\}. \tag{3.28}$$

We shall rely on the Lax formulas (3.18), (3.22), and (3.28) to identify a class of admissible solutions to the traffic flow problem, nicely depending on the initial data.

Definition 1. *We say that the functions $\rho_k = \rho_k(t,x)$ and $q_j = q_j(t)$ (with $k \in \mathcal{I}\cup\mathcal{O},\ j\in\mathcal{O}$) provide an admissible solution to the Cauchy problem (3.1)–(3.4) with junction conditions* **(SBJ)** *if there exist Lipschitz continuous functions $V_k = V_k(t,x)$ such that (3.7) holds, together with the following conditions:*

(i) For $i\in\mathcal{I}$, the functions V_i satisfy (3.18).

(ii) For $j\in\mathcal{O}$, the functions V_j satisfy (3.22).

(iii) For $j\in\mathcal{O}$, the functions q_j satisfy (3.23).

In case of the junction conditions **(MBJ)** *, instead of (3.18) the functions V_i are required to satisfy (3.28).*

To justify the above definition, in Sections 6–8 we will show that, if the functions V_k and q_j satisfy the above conditions (i)–(iii), then the derivatives $\rho_k = V_{k,x}$ provide a solution to our traffic flow problem near the intersection, satisfying all the properties (P1)–(P5). As a motivation, one should keep in mind that, for $i\in\mathcal{I}$, the values $\rho_i(t,x)$ are implicitly determined by the identities

$$f'(\rho_i(t,x)) = \frac{x-y}{t} \qquad \text{or} \qquad f'(\rho_i(t,x)) = \frac{x}{t-\tau}\,.$$

These are valid, respectively, if the maximum in (3.18) is achieved by a function whose graph is a single line connecting $(0,y)$ with (t,x), or a polygonal where the last segment connects $(\tau,0)$ with (t,x) (see Fig. 7). Similar representations hold in case of (3.22) and (3.28).

4 Well posedness of the Cauchy problem

This section contains our main result, proving the global well-posedness of the Cauchy problem for traffic flow near an intersection.

Theorem 1. *Let the flux functions f_k satisfy (2.2) and consider initial data as in (3.4), satisfying (3.5)-(3.6). Then, in both cases* **(SBJ)** *and* **(MBJ)** *the Cauchy problem (3.1)–(3.4) has a unique admissible solution in the sense of Definition 1, globally defined for all $t\ge 0$.*

Proof. 1. We claim that, on a sufficiently small time interval $[0, T]$, the solution of the system of equations (3.17)–(3.23) can be obtained as the unique fixed point of a contractive transformation.

The proof will first be given for the single buffer junction **(SBJ)**. Let $t \mapsto q_j(t)$, $j \in \mathcal{O}$, be Lipschitz continuous functions with Lipschitz constant

$$L_q \doteq \sum_{k=1}^{m+n} f_k^{max}, \tag{4.1}$$

and satisfying

$$\sum_{j\in\mathcal{O}} q_j(t) \leq M \qquad\qquad \text{for all } t \geq 0. \tag{4.2}$$

Consider the following sequence of maps:

$$\mathbf{q} = (q_j)_{j\in\mathcal{O}} \mapsto (V_i)_{i\in\mathcal{I}} \mapsto (F_j)_{j\in\mathcal{O}} \mapsto (V_j)_{j\in\mathcal{O}} \mapsto (\Lambda_j(\mathbf{q}))_{j\in\mathcal{O}}. \tag{4.3}$$

Here the functions V_i are defined by (3.18), the functions F_j are defined by (3.19)-(3.20), while the functions V_j are defined by (3.22). Finally, motivated by (3.23), we set

$$\Lambda_j(\mathbf{q})(t) \doteq F_j(t) + V_j(t, 0). \tag{4.4}$$

2. To prove that the map Λ is contractive, consider two Lipschitz continuous functions, say $\mathbf{q} = (q_j)_{j\in\mathcal{O}}$ and $\tilde{\mathbf{q}} = (\tilde{q}_j)_{j\in\mathcal{O}}$. Assume

$$\delta \doteq \sup_{j\in\mathcal{O},\ t\in[0,T]} |q_j(t) - \tilde{q}_j(t)|. \tag{4.5}$$

By (3.18), since the functions h_i are Lipschitz continuous with Lipschitz constant $C_{\mathcal{I}} \doteq \max_{i\in\mathcal{I}} c_i$, one has

$$\sup_{i\in\mathcal{I},\ t\in[0,T],\ x\leq 0} |V_i(t, x) - \widetilde{V}_i(t, x)| \leq C_{\mathcal{I}} \cdot nT\delta. \tag{4.6}$$

In particular,

$$\sup_{i\in\mathcal{I},\ t\in[0,T]} |V_i(t, 0) - \widetilde{V}_i(t, 0)| \leq C_{\mathcal{I}} \cdot nT\,\delta. \tag{4.7}$$

Recalling (3.19) and (3.20), for all $j \in \mathcal{O}$ and $t \in [0, T]$ we now have

$$|F_j(t) - \widetilde{F}_j(t)| \leq \sum_{i\in\mathcal{I}} |V_i(t, 0) - \widetilde{V}_i(t, 0)| \leq C_{\mathcal{I}} \cdot mnT\delta. \tag{4.8}$$

Next, by (3.22) it follows

$$\sup_{j\in\mathcal{I},\ t\in[0,T],x>0} |V_j(t, 0) - \widetilde{V}_j(t, 0)| \leq \sup_{j\in\mathcal{O},t\in[0,T]} |F_j(t) - \widetilde{F}_j(t)| \leq C_{\mathcal{I}} \cdot mnT\,\delta. \tag{4.9}$$

Finally, by (4.4) it follows

$$|\Lambda_j(\mathbf{q})(t) - \Lambda_j(\tilde{\mathbf{q}})(t)| \leq |V_j(t, 0) - \widetilde{V}_j(t, 0)| + |F_j(t) - \widetilde{F}_j(t)| \leq 2C_{\mathcal{I}} \cdot mnT\,\delta. \tag{4.10}$$

By choosing $T \doteq \big(4C_{\mathcal{I}} \cdot mn\big)^{-1}$, we thus have

$$\sup_{t\in[0,T]} |\Lambda_j(\mathbf{q})(t) - \Lambda_j(\tilde{\mathbf{q}})(t)| \;\leq\; \frac{1}{2} \cdot \sup_{j\in\mathcal{O},\; t\in[0,T]} |q_j(t) - \tilde{q}_j(t)|, \tag{4.11}$$

showing that Λ is a strict contraction.

3. We now check that each map $t \mapsto \Lambda_j(\mathbf{q})(t)$ is Lipschitz continuous. Toward this goal, consider any $i \in \mathcal{I}$, $x < 0$, and $0 < t_1 \leq t_2$. If $V_i(t_1, x) = V_i^{\diamondsuit}(y) + t_1\, g_i(\frac{x-y}{t_1})$ for some $y \leq 0$, then the concavity of g_i implies

$$V_i(t_2, x) \;\geq\; V_i^{\diamondsuit}(y) + t_2\, g_i\Big(\frac{x-y}{t_2}\Big) \;\geq\; V_i^{\diamondsuit}(y) + t_1\, g_i\Big(\frac{x-y}{t_1}\Big) \;\geq\; V_i(t_1, x) + (t_2 - t_1) g_i(0).$$

Hence

$$0 \;\leq\; V_i(t_1, x) - V_i(t_2, x) \;\leq\; (t_2 - t_1)\, f_i^{max}. \tag{4.12}$$

Similarly, if

$$V_i(t_1, x) \;=\; V_i^{\diamondsuit}(y) + \tau'\, g_i\Big(\frac{-y}{\tau'}\Big) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s))\, ds + (t_1 - \tau)\, g_i\Big(\frac{x}{t_1 - \tau}\Big)$$

for some $0 \leq \tau' \leq \tau < t_1$ and some $y \leq 0$, then

$$V_i(t_2, x) \;\geq\; V_i^{\diamondsuit}(y) + \tau'\, g_i\Big(\frac{-y}{\tau'}\Big) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s))\, ds + (t_2 - \tau)\, g_i\Big(\frac{x}{t_2 - \tau}\Big).$$

The concavity of g_i implies

$$(t_2 - \tau)\, g_i\Big(\frac{x}{t_2 - \tau}\Big) \;\geq\; (t_1 - \tau)\, g_i\Big(\frac{x}{t_1 - \tau}\Big) + (t_2 - t_1) g_i(0).$$

Therefore, (4.12) again holds. Letting $x \to 0$ and recalling that $h_i(\mathbf{q}) \in [0, f_i^{max}]$, we conclude that the map $t \mapsto V_i(t, 0)$ is Lipschitz continuous with constant f_i^{max}. Of course, this accounts for the fact that the flux of cars exiting from road i at time t is $-V_{i,t}(t, 0) \in [0, f_i^{max}]$.

For $j \in \mathcal{O}$, an entirely similar argument shows that the function V_j in (3.22) satisfies

$$0 \;\leq\; V_j(t_1, x) - V_j(t_2, x) \;\leq\; (t_2 - t_1)\, f_j^{max}. \tag{4.13}$$

for all $0 \leq t_1 < t_2$. Letting $x \to 0$ we conclude that the map $t \mapsto V_j(t, 0)$ is Lipschitz continuous with constant f_j^{max}. This accounts for the fact that the flux of cars entering road j at any time t is $-V_{j,t}(t, 0) \in [0, f_j^{max}]$.

Using (3.20), (3.19), and then (4.12), for any $0 \leq t_1 < t_2$ we now obtain

$$|F_j(t_2) - F_j(t_1)| \;\leq\; \sum_{i\in\mathcal{I}} |V_i(t_2, 0) - V_i(t_1, 0)| \;\leq\; \sum_{i\in\mathcal{I}} (t_2 - t_1)\, f_i^{max}\,. \tag{4.14}$$

Together with (4.13), this implies that the function

$$t \;\mapsto\; \Lambda_j(\mathbf{q})(t) \;=\; V_j(t, 0) + F_j(t)$$

is Lipschitz continuous with Lipschitz constant $f_j^{\max} + \sum_{i\in\mathcal{I}} f_i^{max} \;\leq\; L_q$, as defined at (4.1).

4. Consider the set $S \subset \mathcal{C}([0,T];\ I\!R^n)$ of all Lipschitz continuous maps $\mathbf{q}$, with Lipschitz constant L_q, and such that $q_j(0) = q_j^\diamondsuit$ for all $j \in \mathcal{O}$. Given the initial data $V_i^\diamondsuit$, $V_j^\diamondsuit$, and $q_j^\diamondsuit$, by the previous arguments the map $\mathbf{q} \mapsto \Lambda(\mathbf{q})$ is a strict contraction of S into itself. Therefore it has a unique fixed point. By definition, this provides the unique admissible solution to our Cauchy problem on the time interval $[0,T]$.

5. We now describe the modifications needed in the case of a multiple buffer junction **(MBJ)**.

The Lipschitz continuity of the maps $t \mapsto \Lambda_j(\mathbf{q})(t)$ is proved as in step **3**, with the same Lipschitz constant L_q in (4.1).

Given initial data $\rho_i^\diamondsuit, \theta_{ij}^\diamondsuit$, and $q_j^\diamondsuit$ such that $q_j^\diamondsuit < M_j$ for all $j \in \mathcal{O}$, we again claim that the system of equations (3.17)–(3.23) admits a unique solution, on a suitably small interval $[0,T]$. Indeed, consider two maps $\mathbf{q}(\cdot)$ and $\tilde{\mathbf{q}}(\cdot)$, both satisfying the initial conditions $q_j(0) = q_j^\diamondsuit$, $j \in \mathcal{O}$, and both with Lipschitz constant L_q.

Introduce the constants

$$M^\diamondsuit \ \doteq\ \min_{j\in\mathcal{O}} \,(M - q_j^\diamondsuit)\ >\ 0, \qquad\qquad T_1 \ \doteq\ \frac{1}{2}\cdot\frac{M^\diamondsuit}{L_q}\,.$$

Notice that

$$q_j(t)\ \leq\ F_j(t)\ \leq\ q_j^\diamondsuit + t\cdot\sum_{i\in\mathcal{I}} f_i^{\max}\,,$$

and the same is true for $\tilde{q}_j$. Therefore,

$$\min\ \bigl\{M_j - q_j(t)\,,\ M_j - \tilde{q}_j(t)\bigr\}\ \geq\ \frac{1}{2}\cdot M^\diamondsuit \qquad\qquad \text{for all } t\in[0,T_1]\,.$$

By Lemma A2 in the Appendix, there exists $0 < T < T_1$ such that, for all $0 \leq \tau' \leq \tau \leq T$, one has

$$|G_i(\tau',\tau;\,y) - \widetilde{G}_i(\tau',\tau;\,y)|\ \leq\ nC_1|\tau'-\tau|\,\delta\ \leq\ nC_1 T\,\delta,$$

for some constant C_1. Here δ is the distance defined at (4.5). Recalling (3.28) we thus obtain

$$|V_i(t,x) - \widetilde{V}_i(t,x)|\ \leq\ nC_1^\diamondsuit T\delta$$

for some constant $C_1^\diamondsuit$ and all $i \in \mathcal{I}$, $t \in [0,T]$, and $x < 0$. This inequality replaces (4.6). The remainder of the proof, showing that for $T > 0$ the transformation $\mathbf{q} \mapsto \Lambda(\mathbf{q})$ is a strict contraction in $\mathcal{C}([0,T]; I\!R^n)$, is the same as in step **4.**

6. To complete the proof, we now show that the above construction can be iterated on a sequence of time intervals $[0,T_1]$, $[T_1,T_2]$, …, with

$$\lim_{\nu\to\infty} T_\nu\ =\ +\infty\,. \tag{4.15}$$

In case of a single buffer junction **(SBJ)**, the definition (3.15) yields the a priori bound on the growth of the queue

$$\frac{d}{dt}\Bigl(M - \sum_{j\in\mathcal{O}} q_j(t)\Bigr)\ \leq\ \sum_{i\in\mathcal{I}} c_i\Bigl(M - \sum_{j\in\mathcal{O}} q_j(t)\Bigr).$$

In turn, this implies

$$\Big(M - \sum_{j\in\mathcal{O}} q_j(t)\Big) \;\geq\; \exp\left\{-t\cdot\sum_{i\in I} c_i\right\}\cdot\Big(M - \sum_{j\in O} q_j^{\diamondsuit}\Big). \tag{4.16}$$

According to the analysis in step **2**, the contraction property (4.11) can be achieved by choosing the length of these time intervals to be $T_\nu - T_{\nu-1} = (4C_{\mathcal{I}}\cdot mn)^{-1}$. Of course, this yields (4.15).

In case of a multiple buffer junction **(MBJ)**, the definition (3.16) yields the a priori bound

$$\frac{d}{dt}\Big(M_j - \sum_{j\in\mathcal{O}} q_j(t)\Big) \;\leq\; \sum_{i\in\mathcal{I}} c_i\Big(M_j - \sum_{j\in\mathcal{O}} q_j(t)\Big).$$

In turn, this implies

$$M_j - q_j(t) \;\geq\; \exp\left\{-t\cdot\sum_{i\in I} c_i\right\}\cdot(M_j - q_j^{\diamondsuit}). \tag{4.17}$$

According to the analysis in steps **3** and **5**, the contraction property (4.11) can be achieved by choosing these time intervals $[T_\nu, T_{\nu-1}]$ sufficiently small. By Lemma A2 in the Appendix, the size $T_\nu - T_{\nu-1}$ needs to satisfy a constraint depending only on the Lipschitz constant L_q of the functions $\mathbf{q}$ and on the lower bound on $M_j - q_j(t)$. By (4.17) these quantities remain uniformly positive on any bounded time interval $[0, \overline{T}]$. Hence, as long as $T_\nu < \overline{T}$, the lengths $T_\nu - T_{\nu-1}$ of these intervals can be taken uniformly positive. This yields (4.15). □

5 Continuity w.r.t. weak convergence

In this section we prove that the solution constructed in Theorem 1 depends continuously on the initial data, in the topology of weak convergence.

Theorem 2. *Consider a sequence of initial data $(\hat\rho_k^\nu, \hat\theta_{ij}^\nu, \hat q_j^\nu)_{\nu\geq 1}$ in (3.4) such that, as $\nu\to\infty$, one has*

$$\hat q_j^\nu \;\to\; q_j^{\diamondsuit} \qquad j\in\mathcal{O}, \tag{5.1}$$

together with the weak convergence

$$\begin{cases} \hat\rho_i^\nu \hat\theta_{ij}^\nu \;\rightharpoonup\; \rho_i^{\diamondsuit}\theta_{ij}^{\diamondsuit}, & i\in\mathcal{I},\quad j\in\mathcal{O},\\[2ex] \hat\rho_j^\nu \;\rightharpoonup\; \rho_j^{\diamondsuit}, & j\in\mathcal{O}\,. \end{cases} \tag{5.2}$$

Calling $\rho_k^\nu = V_{k,x}^\nu$ and q_j^ν the corresponding solutions, for every $t>0$ one has the convergence $q_j^\nu(t)\to q_j(t)$ uniformly for t on bounded sets, and the strong convergence in $\mathbf{L}^1_{loc}$

$$\rho_k^\nu(t,\cdot) \;\to\; \rho_k(t,\cdot) \qquad k\in\mathcal{I}\cup\mathcal{O}\,. \tag{5.3}$$

Here $\rho_k = V_{k,x}$ and q_j are the components of the unique solution corresponding to initial data $(\rho_k^{\diamondsuit}, \theta_{ij}^{\diamondsuit}, q_j^{\diamondsuit})$. The result holds both in the case **(SBJ)** *of a single buffer and in the case* **(MBJ)** *of multiple buffers.*

Proof. 1. We first prove theorem in the case **(SBJ)** of a junction with a single buffer. For every $\nu \geq 1$, let $V_i^\nu, V_j^\nu, q_j^\nu$ be the components of the solution constructed in Theorem 1, replacing the initial data $(\rho_k^\diamondsuit, \theta_{ij}^\diamondsuit, q_j^\diamondsuit)$ with $(\hat\rho_k^\nu, \hat\theta_{ij}^\nu, \hat q_j^\nu)$.

For any $i \in \mathcal{I}$, $t \in]0,T[$, and $x \leq 0$, by (3.17) and (3.18) we have the bound

$$|V_i^\nu(t,x) - V_i(t,x)| \;\leq\; \|\widehat{V}_i^\nu - V_i^\diamondsuit\|_{\mathbf{L}^\infty(]-\infty,0])} + nC_{\mathcal{I}}T \cdot \|\hat{\mathbf{q}}^\nu - \mathbf{q}^\diamondsuit\|_{\mathbf{L}^\infty([0,T])}. \tag{5.4}$$

On the other hand, let F_j^ν be defined as in (3.20). By (3.20) and (3.19) it follows

$$\begin{aligned}
|F_j^\nu(t) - F_j(t)| \;&\leq\; \sum_{i\in\mathcal{I}} \left| \int_{\xi_i(t)}^0 \Big(\hat\rho_i^\nu(y)\hat\theta_{ij}^\nu(y) - \rho_i^\diamondsuit(y)\theta_{ij}^\diamondsuit(y)\Big)\,dy \right| + \sum_{i\in\mathcal{I}} \left| \int_{\xi_i^\nu(t)}^{\xi_i(t)} \hat\rho_i^\nu(y)\hat\theta_{ij}^\nu(y)\,dy \right| \\
&\leq\; \sum_{i\in\mathcal{I}} \left| \int_{\xi_i(t)}^0 \Big(\hat\rho_i^\nu(y)\hat\theta_{ij}^\nu(y) - \rho_i^\diamondsuit(y)\theta_{ij}^\diamondsuit(y)\Big)dy \right| + \sum_{i\in\mathcal{I}} \left| \int_{\xi_i^\nu(t)}^{\xi_i(t)} \hat\rho_i^\nu(y)\,dy \right|,
\end{aligned}$$

$$\left| \int_{\xi_i^\nu(t)}^{\xi_i(t)} \hat\rho_i^\nu(y)dy \right| \;\leq\; |V_i^\nu(t,0) - V_i(t,0)| + \left| \int_{-\infty}^{\xi_i(t)} \Big(\hat\rho_i^\nu(y) - \rho_i^\diamondsuit(y)\Big)\,dy \right|.$$

Therefore

$$\begin{aligned}
|F_j^\nu(t) - F_j(t)| \;\leq\; \sum_{i\in\mathcal{I}} \Bigg\{ &|V_i^\nu(t,0) - V_i(t,0)| + \|\widehat{V}_i^\nu - V_i^\diamondsuit\|_{\mathbf{L}^\infty(]-\infty,0])} \\
&+ \left| \int_{\xi_i(t)}^0 \Big(\hat\rho_i^\nu(y)\hat\theta_{ij}^\nu(y) - \rho_i^\diamondsuit(y)\theta_{ij}^\diamondsuit(y)\Big)\,dy \right| \Bigg\}.
\end{aligned} \tag{5.5}$$

Moreover, for every $j \in \mathcal{O}$, $t \geq 0$, and $x \geq 0$, by (3.22) one has

$$|V_j^\nu(t,x) - V_j(t,x)| \;\leq\; \sup_{\tau\in[0,t]} |F_j^\nu(\tau) - F_j(\tau)|. \tag{5.6}$$

Combining (5.5) with (5.6) we obtain

$$\begin{aligned}
\left|q_j^\nu(t) - q_j(t)\right| \;\leq\; 2 \cdot \sup_{\tau\in[0,t]} \sum_{i\in\mathcal{I}} \Bigg\{ &|V_i^\nu(t,0) - V_i(t,0)| + \|\widehat{V}_i^\nu - V_i^\diamondsuit\|_{\mathbf{L}^\infty(]-\infty,0])} \\
&+ \left| \int_{\xi_i(t)}^0 \Big(\hat\rho_i^\nu(y)\hat\theta_{ij}^\nu(y) - \rho_i^\diamondsuit(y)\theta_{ij}^\diamondsuit(y)\Big)\,dy \right| \Bigg\}.
\end{aligned} \tag{5.7}$$

This proves the convergence of the queue sizes $q_j^\nu \to q_j$.

2. Using (5.4) and (5.7), we obtain

$$\begin{aligned}
\sum_{i\in\mathcal{I}} \|V_i^\nu - V_i^\diamondsuit\|_{\mathbf{L}^\infty([0,T]\times]-\infty,0])} \;&\leq\; (2n^2 mC_{\mathcal{I}}T + m) \cdot \sum_{i\in\mathcal{I}} \|\widehat{V}_i^\nu - V_i^\diamondsuit\|_{\mathbf{L}^\infty(]-\infty,0])} \\
+\, 2n^2 mC_{\mathcal{I}}T \cdot \Bigg\{ \sup_{\tau\in[0,t]} &\left| \int_{\xi_i^\diamondsuit(\tau)}^0 \hat\rho_i^\nu(y)\hat\theta_{ij}^\nu(y) - \rho_i^\diamondsuit(y)\theta_{ij}^\diamondsuit(y)\,dy \right| + \sum_{i\in\mathcal{I}} \|V_i^\nu - V_i\|_{\mathbf{L}^\infty([0,T]\times]-\infty,0])} \Bigg\}.
\end{aligned} \tag{5.8}$$

Therefore, by choosing $T = \big(4n^2 m C_{\mathcal{I}}\big)^{-1}$ we obtain

$$\sum_{i\in\mathcal{I}} \|V_i^\nu - V_i\|_{\mathbf{L}^\infty([0,T]\times]-\infty,0])} \;\leq\; (2m+1)\cdot\sum_{i\in\mathcal{I}} \|\widehat{V}_i^\nu - V_i^\diamondsuit\|_{\mathbf{L}^\infty(]-\infty,0])} + \sup_{\tau\in[0,T]}\Big|\int_{\xi_i(\tau)}^0 \hat\rho_i^\nu(y)\hat\theta_{ij}^\nu(y) - \rho_i^\diamondsuit(y)\theta_{ij}^\diamondsuit(y)\,dy\Big|. \quad (5.9)$$

This implies

$$\lim_{\nu\to\infty}\sum_{i\in\mathcal{I}} \|V_i^\nu - V_i\|_{\mathbf{L}^\infty([0,T]\times]-\infty,0])} \;=\; 0.$$

From the uniform convergence of the Lipschitz functions $V_i^\nu \to V_i$ in $\mathcal{C}\big([0,T]\times]0,\infty]\big)$, it now follows the weak convergence $\rho_i^\nu(t,\cdot) \rightharpoonup \rho_i(t,\cdot)$, for every $i\in\mathcal{I}$, $t\in[0,T]$. In a similar way, recalling (5.5) and (5.6), we obtain the convergence $V_j^\nu(t,\cdot)\to V_j(t,\cdot)$ in $\mathcal{C}\big([0,\infty]\big)$, for every $t\in[0,T]$ and $j\in\mathcal{O}$. Again, this implies the weak convergence $\rho_j^\nu(t,\cdot) \rightharpoonup \rho_j(t,\cdot)$.

By Oleinik's estimates, the solutions ρ_k^ν satisfy uniform BV bounds, on any compact domain D bounded away from the x-axis and from the t-axis. Hence the weak convergence implies strong convergence in $\mathbf{L}^1_{loc}$.

3. As in step **6** of the proof of Theorem 1, we can repeat these same estimates on a sequence of time intervals $[0,T_1]$, $[T_1,T_2]$, etc.... By induction, the convergence still holds for every $t>0$.

4. In the case of a multiple buffer junction **(MBJ)**, the proof is entirely similar. It suffices to show that (5.4) holds with another constant $C_{\mathcal{I}}$. Indeed, from lemma A2 and (3.22), we obtain that

$$|V_i^\nu(t,x) - V_i(t,x)| \;\leq\; \|\widehat{V}_i^\nu - V_i^\diamondsuit\|_{\mathbf{L}^\infty(]-\infty,0])} + nC_0 e^{C_0 T} T\cdot\|\mathbf{q}^\nu - \mathbf{q}^\diamondsuit\|_{\mathbf{L}^\infty([0,T])}\,,$$

for a suitable constant C_0. □

6 Variational formulation of (SBJ)

In this and in the following two sections we introduce three optimization problems. In each case, we show that the optimal solution is piecewise affine, and the value function V_k admits the explicit representation (3.18), (3.22), or (3.28), respectively. In turn, this variational representation allows us to prove that the derivative $\rho_k = V_{k,x}$ yields an entropy weak solution to the conservation law (1.1), satisfying the appropriate initial and boundary conditions.

The junction conditions **(SBJ)** lead to:

Optimization Problem 1. For any $i\in\mathcal{I}$, given the function $V_i^\diamondsuit$ in (3.17) and the length of the queues q_j, $j\in\mathcal{O}$, consider the following variational problem.

$$\text{maximize:} \qquad J_i(x(\cdot)) \;\doteq\; V_i^\diamondsuit(x(0)) + \int_0^{\bar t} L_i(x(t),\dot x(t))\,dt\,. \qquad (6.10)$$

Recalling (3.8) and (3.15), the payoff function is here defined as

$$L_i(x(t),\dot x(t)) \;=\; \begin{cases} g_i(\dot x(t)) & \text{if} \quad x(t)<0, \\[2mm] -h_i(\mathbf{q}(t)) & \text{if} \quad x(t)=0. \end{cases} \tag{6.11}$$

The maximum is sought among all absolutely continuous functions $x:[0,\bar t]\mapsto I\!R$ such that

$$x(\bar t) \;=\; \bar x, \qquad\qquad x(t)\leq 0 \quad \text{for all } t\in[0,\bar t]. \tag{6.12}$$

The following lemma shows that, for any optimal solution $x(\cdot)$, the set of times where $x(t)=0$ must be an interval.

Lemma 1. *Consider an absolutely continuous map $x:[0,\bar t]\mapsto\,]-\infty,0]$ satisfying (6.12). Define the times*

$$a \;\doteq\; \min\bigl\{t\in[0,\bar t]\,;\quad x(t)=0\bigr\}, \qquad\qquad b \;\doteq\; \max\bigl\{t\in[0,\bar t]\,;\quad x(t)=0\bigr\} \tag{6.13}$$

and the function

$$x^\sharp(t) \;\doteq\; \begin{cases} 0 & \textit{if} \;\; x\in[a,b]\,, \\ x(t) & \textit{if} \;\; x\notin[a,b]\,. \end{cases} \tag{6.14}$$

Then, $x^\sharp$ satisfies (6.12) and achieves a larger payoff, namely

$$J_i(x(\cdot)) \;\leq\; J_i(x^\sharp(\cdot)). \tag{6.15}$$

Proof. 1. Consider any subinterval $[a',b']\subseteq[a,b]$ such that $x(a')=x(b')=0$ and $x(t)<0$ for all $t\in]a',b'[$. Define the function $x^\flat(\cdot)$ by setting

$$x^\flat(t) \;\doteq\; \begin{cases} 0 & \text{if} \;\; x\in[a',b']\,, \\ x(t) & \text{if} \;\; x\notin[a',b']\,. \end{cases} \tag{6.16}$$

We claim that

$$\overline V_i(x(0))+\int_0^{\bar t} L_i(x(t),\dot x(t))\,dt \;\leq\; \overline V_i(x^\flat(0))+\int_0^{\bar t} L_i(x^\flat(t),\dot x^\flat(t))\,dt\,. \tag{6.17}$$

Indeed, recalling the definition of L_i at (6.11), the above inequality is equivalent to

$$\int_{a'}^{b'} g_i(\dot x(t))\,dt \;\leq\; \int_{a'}^{b'} -h_i(\mathbf{q}(t))\,dt\,. \tag{6.18}$$

To prove (6.18), observe that by (3.15)

$$-h_i(\mathbf{q}(t)) \;\geq\; -\,f_i^{max} \;=\; g_i(0)$$

(see Fig. 6). Applying Jensen's inequality to the concave function g_i we thus obtain

$$\int_{a'}^{b'} -h_i(\mathbf{q}(t))dt \;\geq\; \int_{a'}^{b'} g_i(0)dt \;\geq\; \int_{a'}^{b'} g_i(\dot x(t))\,dt\,. \tag{6.19}$$

2. Let $[a'_\nu, b'_\nu]$, $\nu \geq 1$ be the family of all subintervals of $[a,b]$ with $x(a'_\nu) = x(b'_\nu) = 0$, $x(t) > 0$ for $t \in]a'_\nu, b'_\nu[$. For each $N \geq 1$, call

$$x^N(t) \doteq \begin{cases} 0 & \text{if } x \in \cup_{\nu=1}^N [a'_\nu, b'_\nu]\,, \\ x(t) & \text{otherwise}\,. \end{cases} \tag{6.20}$$

By the previous step, the sequence of payoffs $J_i\big(x^N(\cdot)\big)$ is monotone increasing. Since $x^N \to x^\sharp$ as $N \to \infty$, we have

$$J(x^\sharp(\cdot)) = \lim_{N\to\infty} J_i(x^N(\cdot)) \geq J_i(x(\cdot)).$$

□

Proposition 1. *Let a continuous function $t \mapsto \mathbf{q}(t) = (q_j(t))_{j\in\mathcal{O}}$ be given, together with initial data $\rho_i^\diamondsuit(x)$, $\theta_{ij}^\diamondsuit(x)$ for $x < 0$, satisfying the conditions (3.5)-(3.6). For $i \in \mathcal{I}$, define $V_i^\diamondsuit$ as in (3.17) and consider the variational problem (6.10)–(6.12). Then the following holds.*

(i) For every given $\bar t > 0$ and $\bar x < 0$, an optimal solution $x^*(\cdot)$ exists. This solution is piecewise affine and satisfies $\dot x^*(t) \in [f_i'(\rho_i^{jam}), f_i'(0)]$ for a.e. $t \in [0, \bar t]$.

(ii) The maximum attainable value $V_i(\bar t, \bar x)$ is given by the formula (3.18).

(iii) The corresponding density $\rho_i(t,x) = V_{i,x}(t,x)$ is well defined a.e., and provides an entropy weak solution to the conservation law

$$\rho_t + f_i(\rho)_x = 0, \tag{6.21}$$

with initial data as in (3.4) and boundary fluxes (2.12).

More precisely, the last statement will be proved by showing that the following conditions hold.

(i) On the open set

$$\Omega \doteq \{(t,x)\,;\ t > 0,\ x < 0\} \tag{6.22}$$

the function $\rho_i = V_{i,x}$ provides an entropy weak solution to (6.21).

(ii) For a.e. $t > 0$ the limits

$$\bar\rho_i(t) \doteq \lim_{x\to 0-} \rho_i(t,x), \qquad \overline{V}_i(t) \doteq \lim_{x\to 0-} V_i(t,x), \tag{6.23}$$

are well defined and satisfy

$$f_i(\overline{\rho}_i(t)) = \min\left\{\omega_i(t)\,,\ c_i \cdot \Big(M - \sum_{j\in\mathcal{O}} q_j(t)\Big)\right\}. \tag{6.24}$$

Here

$$\omega_i(t) \doteq \begin{cases} f_i(\bar\rho_i(t)) & \text{if } \bar\rho_i(t) \text{ is a free state,} \\ f_i^{max} & \text{if } \bar\rho_i(t) \text{ is a congested state.} \end{cases} \tag{6.25}$$

(iii) For every test function $\phi \in \mathcal{C}_c^\infty(I\!R^2)$, one has

$$\int_0^\infty \int_{-\infty}^0 \big\{\phi_t\,\rho_i + \phi_x f(\rho_i)\big\}\,dxdt + \int_{-\infty}^0 \phi(0,x)\,V_{i,x}^\diamondsuit(x)\,dx - \int_0^\infty \phi(t,0)\,\overline{V}_{i,t}(t)\,dt \;=\; 0. \tag{6.26}$$

Proof. 1. The existence of an optimal solution will be proved by the direct method of the Calculus of Variations. Let $(x_n)_{n\geq 1}$ be a maximizing sequence of absolutely continuous functions satisfying the admissibility conditions (6.12). This means

$$\lim_{n\to\infty} \left\{ V_i^\diamondsuit(x_n(0)) + \int_0^{\bar t} L_i(x_n(t), \dot x_n(t))\,dt \right\} \;=\; B, \tag{6.27}$$

where B is the supremum among all payoffs achieved by admissible functions $x(\cdot)$. In this first step we prove some a priori estimates. Two cases will be considered.

CASE 1: There exists $N_0 > 0$ such that

$$x_n(t) \;<\; 0, \qquad \text{for all } t\in[0,\bar t]\,, \quad n > N_0.$$

In this case, for all $n > N_0$ we have

$$V_i^\diamondsuit(x_n(0)) + \int_0^{\bar t} L_i(x_n(t),\dot x_n(t))\,dt \;=\; V_i^\diamondsuit(x_n(0)) + \int_0^{\bar t} g_i(\dot x_n(t))\,dt, \quad \text{for all } n > N_0\,. \tag{6.28}$$

Applying Jensen's inequality to the concave function g_i, we obtain

$$\int_0^{\bar t} g_i(\dot x_n(t))\,dt \;\leq\; \bar t\cdot g_i\Big(\frac{\bar x - x_n(0)}{\bar t}\Big).$$

Hence, using (6.27) and (6.28) we conclude

$$\lim_{n\to\infty} \left\{ V_i^\diamondsuit(x_n(0)) + \bar t\cdot g_i\Big(\frac{\bar x - x_n(0)}{\bar t}\Big)\right\} \;=\; B\,. \tag{6.29}$$

The following argument shows that, without loss of generality, we can assume

$$\frac{\bar x - x_n(0)}{\bar t} \;\in\; [f_i'(\rho_i^{jam}), f_i'(0)] \qquad \text{for every } n\geq 1. \tag{6.30}$$

- If $x_n(0) \leq x_n^- \doteq \bar x - \bar t\cdot f_i'(0)$, recalling that $V_{i,x}^\diamondsuit = \rho_i^\diamondsuit \geq 0$, by (3.14) one obtains

$$V_i^\diamondsuit(x_n(0)) + \bar t\cdot g_i\Big(\frac{\bar x - x_n(0)}{\bar t}\Big) \;=\; V_i^\diamondsuit(x_n(0)) \;\leq\; V_i^\diamondsuit(x_n^-) \;=\; V_i^\diamondsuit(x_n^-) + \bar t\cdot g_i\Big(\frac{\bar x - x_n^-}{\bar t}\Big).$$

- If $x_n(0) \geq x_n^+ \doteq \bar x - \bar t\cdot f_i'(\rho_i^{jam})$, recalling that $V_{i,x}^\diamondsuit = \rho_i^\diamondsuit \leq \rho_i^{jam}$, by (3.14) one obtains

$$\begin{aligned} & V_i^\diamondsuit(x_n(0)) + \bar t\cdot g_i\Big(\frac{\bar x - x_n(0)}{\bar t}\Big) \\ &\leq\; \Big[V_i^\diamondsuit(x_n^+) + \rho_i^{jam}(x_n(0) - x_n^+)\Big] + \bar t\cdot \rho_i^{jam}\cdot \frac{\bar x - x_n(0)}{\bar t} \\ &=\; V_i^\diamondsuit(x_n^+) + \rho_i^{jam}(\bar x - x_n^+) \;=\; V_i^\diamondsuit(x_n^+) + \bar t\cdot g_i\Big(\frac{\bar x - x_n^+}{\bar t}\Big). \end{aligned}$$

By possibly replacing $x_n(0)$ with x_n^- or x_n^+, we can thus assume that (6.30) holds. Since the sequence $(x_n(0))_{n\geq 1}$ is bounded, we can now extract a subsequence $\{n_k\}$ and a point $\bar y$ such that $\lim_{k\to\infty} x_{n_k}(0) = \bar y$. This implies

$$V_i^{\diamondsuit}(\bar y) + \bar t \cdot g_i\Big(\frac{\bar x - \bar y}{\bar t}\Big) \;=\; B\,.$$

Therefore, the affine function

$$x(t) \;=\; \bar y + t\cdot \frac{\bar x - \bar y}{\bar t} \tag{6.31}$$

is an optimal solution of the variational problem (6.10)–(6.11). In particular, the representation formula (3.18) is valid.

CASE 2: For infinitely many n, the set of times $\big\{t\in[0,\bar t]\,;\quad x_n(t) = 0\big\}$ is nonempty.

Because of Lemma 1, we can assume that, for each n, the set of times where $x_n(t) = 0$ is a closed interval, say

$$\big\{t\in[0,\bar t]\,;\;\; x_n(t) = 0\big\} \;=\; [a_n, b_n]\,.$$

Using Jensen's inequality, we thus obtain

$$\begin{aligned}\int_0^{\bar t} L_i(x_n(t),\dot x_n(t))\,dt \;&=\; \int_0^{a_n} g_i(\dot x_n(t))\,dt - \int_{a_n}^{b_n} h_i(\mathbf{q}(t))\,dt + \int_{b_n}^{\bar t} g_i(\dot x_n(t))\,dt\\ &\leq\; a_n\cdot g_i\Big(\frac{-x_n(0)}{a_n}\Big) - \int_{a_n}^{b_n} h_i(\mathbf{q}(t))\,dt + (\bar t - b_n)\cdot g_i\Big(\frac{\bar x}{\bar t - b_n}\Big).\end{aligned} \tag{6.32}$$

The following argument shows that, without loss of generality, we can assume

$$x_n(0) \;\geq\; -\,a_n\, f_i'(0)\,,\qquad\quad \frac{\bar x}{\bar t - b_n} \;\geq\; f_i'(\rho_i^{jam})\,,\qquad\qquad \text{for every } n\geq 1. \tag{6.33}$$

- If $x_n(0) < x_n^- \doteq -a_n\, f_i'(0)$, recalling that $V_{i,x}^{\diamondsuit}\geq 0$, by (3.14) one obtains

$$V_i^{\diamondsuit}(x_n(0)) + a_n\cdot g_i\Big(\frac{-x_n(0)}{a_n}\Big) \;=\; V_i^{\diamondsuit}(x_n(0)) \;\leq\; V_i^{\diamondsuit}(x_n^-) \;=\; V_i^{\diamondsuit}(x_n^-) + a_n\cdot g_i\Big(\frac{-x_n^-}{a_n}\Big).$$

- If $b_n > b_n' \doteq \bar t - \frac{\bar x}{f_i'(\rho_i^{jam})}$, we consider two cases.

  Case 1. If $b_n' \geq a_n$, recalling that $h_i(\mathbf{q}(t))\geq 0$, by (3.14) one obtains

$$\begin{aligned}\int_{b_n'}^{b_n} -h_i(\mathbf{q}(t))\,dt + (\bar t - b_n) g_i\Big(\frac{\bar x}{\bar t - b_n}\Big) \;&\leq\; (\bar t - b_n) g_i\Big(\frac{\bar x}{\bar t - b_n}\Big)\\ &=\; \rho_i^{jam}\bar x \;=\; (\bar t - b_n') g_i\Big(\frac{\bar x}{\bar t - b_n'}\Big).\end{aligned}$$

  Case 2. If $b_n' < a_n$, repeating the previous argument with a_n in place of b_n' we obtain

$$\int_{a_n}^{b_n} -h_i(\mathbf{q}(t))\,dt + (\bar t - b_n) g_i\Big(\frac{\bar x}{\bar t - b_n}\Big) \;\leq\; (\bar t - a_n) g_i\Big(\frac{\bar x}{\bar t - a_n}\Big).$$

In this case, calling R the right hand side of (6.32), we have the bound

$$R \;\leq\; a_n \cdot g_i\Big(\frac{-x_n(0)}{a_n}\Big) + (\bar t - a_n)\cdot g_i\Big(\frac{\bar x}{\bar t - a_n}\Big) \;\leq\; \bar t\, g_i\Big(\frac{\bar x - x_n(0)}{\bar t}\Big).$$

By earlier analysis, we already know that the bound (6.30) holds.

2. By the previous step, there exists a maximizing sequence of functions $x_n(\cdot)$, whose derivatives satisfy

$$\dot x_n(t) \in [f_i'(\rho_i^{jam}),\ f_i'(0)] \qquad \text{for a.e.} \quad t\in[0,\bar t], \tag{6.34}$$

and satisfying one of the following properties.

(i) Either x_n is affine. In this case, for some $y_n \leq 0$ we have

$$x_n(t) \;=\; y_n + t\cdot \frac{\bar x - y_n}{\bar t}\,. \tag{6.35}$$

(ii) Or else x_n is piecewise affine. In this case, for some $y_n\leq 0$ and $0<a_n<b_n<\bar t$ we have

$$x_n(t) \;=\; \begin{cases} y_n - t\cdot \dfrac{y_n}{a_n} & \text{if} \quad t\in[0,a_n]\,,\\[2mm] 0 & \text{if} \quad t\in[a_n,b_n]\,,\\[2mm] \bar x + (t-\bar t)\cdot \dfrac{\bar x}{\bar t - b_n}\cdot & \text{if} \quad t\in[b_n,\bar t]\,.\end{cases} \tag{6.36}$$

Thanks to the uniform bounds (6.34) we can extract a uniformly convergent subsequence, say, $x_{n_k}(t)\to x^*(t)$ for all $t\in[0,\bar t]$. By (6.35)-(6.36), this function x^* will have one of the following properties.

(i) Either $x^*(\cdot)$ is affine. In this case, for some $\bar y = \lim_{k\to\infty} y_{n_k}\leq 0$ we have

$$x^*(t) \;=\; \bar y + t\cdot\frac{\bar x - \bar y}{\bar t}\,. \tag{6.37}$$

(ii) Or else $x^*(\cdot)$ is piecewise affine. In this case, assuming

$$y_{n_k}\to\bar y,\qquad a_{n_k}\to a,\qquad b_{n_k}\to b\qquad \text{as}\quad k\to\infty,$$

we have

$$x^*(t) \;=\; \begin{cases} \bar y - t\cdot \dfrac{\bar y}{a} & \text{if} \quad t\in[0,a]\,,\\[2mm] 0 & \text{if} \quad t\in[a,b]\,,\\[2mm] \bar x + (t-\bar t)\cdot \dfrac{\bar x}{\bar t - b}\cdot & \text{if} \quad t\in[b,\bar t]\,.\end{cases} \tag{6.38}$$

By the strong convergence $\dot x_n \to \dot x^*$, it follows

$$V_i^{\diamondsuit}(x^*(0)) + \int_0^{\bar t} L_i\big(x^*(t), \dot x^*(t)\big)\, dt \;=\; B, \tag{6.39}$$

Therefore x^* is an optimal solution of (6.10)–(6.12). This achieves the proof of statement (i). Statement (ii) is an immediate consequence of (6.37)-(6.38).

3. We now work toward a proof of (iii). We observe that the value function $(t, x) \mapsto V_i(t, x)$ in (3.18) is Lipschitz continuous. Indeed, this follows easily from the Lipschitz continuity of the function $V_i^{\diamondsuit}$, the bound $h_i(\mathbf{q}) \in [0, f_i^{max}]$, together with the fact that the maximum in (3.18) is attained when the quantities

$$\frac{x-y}{t}, \qquad \frac{-y}{\tau'}, \qquad \frac{x}{t-\tau},$$

are all contained inside the interval $[f_i'(\rho_i^{jam}),\ f_i'(0)]$.

Fix any time $\tau \geq 0$ and define

$$V^\dagger(x) \;\doteq\; V_i(\tau, x).$$

Moreover, consider the open domain

$$\Omega^\tau \;\doteq\; \Big\{(t, x)\,;\quad t > \tau, \quad x < (t-\tau)\cdot f_i'(\rho_i^{jam})\Big\}.$$

By the dynamic programming principle and by finite propagation speed, restricted to Ω^τ the value function V_i is given by

$$V_i(\bar t, \bar x) \;=\; \max\; \left\{V^\dagger(x(\tau)) + \int_\tau^{\bar t} g_i(\dot x(s))\, ds\,; \qquad x(\bar t) = \bar x, \quad \dot x(s) \in [f_i'(\rho_i^{jam}),\, f_i'(0)]\right\}. \tag{6.40}$$

This is a classical problem in optimal control. In this case, it is well known [1, 9] that V_i provides a viscosity solution to the Hamilton-Jacobi equation

$$V_{i,t} + f_i(V_{i,x}) \;=\; 0 \tag{6.41}$$

restricted to Ω^τ. Moreover, the derivative $\rho_i(t, x) = V_{i,x}(t, x)$ exists a.e. and provides an entropy weak solution to the conservation law (6.21).

We now observe that, as τ varies, the union of the sets Ω^τ covers $\Omega \doteq \{(t, x)\,;\ t > 0,\ x < 0\}$. Therefore, $\rho_i = V_{i,x}$ is an entropy solution of (6.21) on the entire open domain Ω.

Consider any test function $\phi \in \mathcal{C}_c^\infty(I\!R^2)$. Since V_i is Lipschitz and satisfies (6.41) pointwise a.e., integrating twice by parts we obtain

$$\begin{aligned} 0 \;&=\; \int_0^\infty \int_{-\infty}^0 \big\{\phi_x V_{i,t} + \phi_x f(V_{i,x})\big\}\, dxdt \\ &=\; \int_0^\infty \int_{-\infty}^0 \big\{\phi_t V_{i,x} + \phi_x f(V_{i,x})\big\}\, dxdt \\ &\qquad + \int_{-\infty}^0 \phi(0, x) V_{i,x}^{\diamondsuit}(x)\, dx - \int_0^\infty \phi(t, 0)\, \overline{V}_{i,t}(t)\, dt\,. \end{aligned} \tag{6.42}$$

This yields (6.26).

Remark 3. (i) If the optimal trajectory is given by (6.37), then the function

$$x \;\mapsto\; V_i^{\diamondsuit}(\bar y) + \bar t \cdot g_i\left(\frac{x-\bar y}{\bar t}\right)$$

provides a lower bound on the value function $V_i(\bar t, x)$. In particular, $g_i'\left(\frac{\bar x-\bar y}{\bar t}\right)$ is a subdifferential for the map $x \mapsto V_i(\bar t, x)$ at the point $\bar x$. By Lipschitz continuity, $V_{i,x}$ exists for a.e. $\bar x$ and we have

$$\rho_i(\bar t, \bar x) \;=\; V_{i,x}(\bar t, \bar x) \;=\; g_i'\left(\frac{\bar x-\bar y}{\bar t}\right). \tag{6.43}$$

By (3.11), this implies

$$f_i'(\rho_i(\bar t, \bar x)) \;=\; \frac{\bar x-\bar y}{\bar t}\,, \tag{6.44}$$

showing that optimal trajectories are characteristic curves of the conservation law.

(ii) If the optimal trajectory is given by (6.38), then the function

$$x \;\mapsto\; V_i^{\diamondsuit}(\bar y) + a\, g_i\Big(\frac{-\bar y}{a}\Big) - \int_a^b h_i(\mathbf{q}(s))\, ds + (\bar t - b)\cdot g_i\left(\frac{x}{\bar t - b}\right)$$

provides a lower bound on the value function $V_i(\bar t, x)$. In particular, $g_i'\left(\frac{\bar x}{\bar t-b}\right)$ is a subdifferential for the map $x \mapsto V_i(\bar t, x)$ at the point $\bar x$. By Lipschitz continuity, $V_{i,x}$ exists for a.e. $\bar x$ and we have

$$\rho_i(\bar t, \bar x) \;=\; V_{i,x}(\bar t, \bar x) \;=\; g_i'\left(\frac{\bar x}{\bar t - b}\right). \tag{6.45}$$

By (3.11), this implies

$$f_i'(\rho_i(\bar t, \bar x)) \;=\; \frac{\bar x}{\bar t - b}\,, \tag{6.46}$$

showing again that optimal trajectories are characteristic curves for the conservation law.

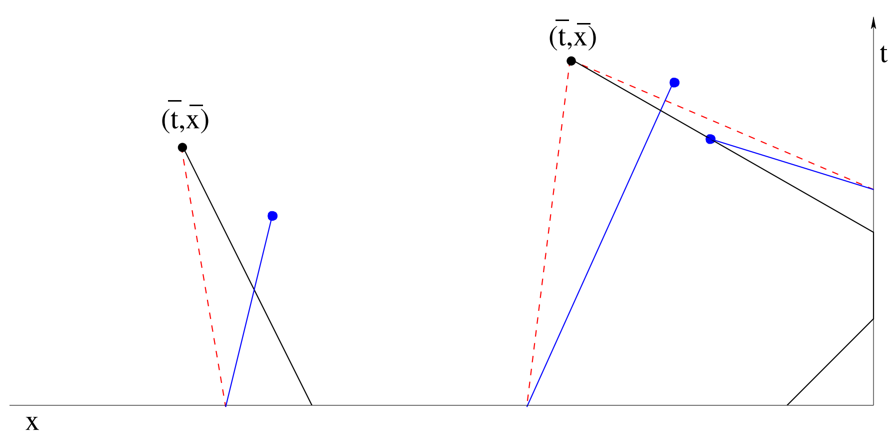

Figure 8: If two optimal trajectories (black and blue) cross each other, then by the strict concavity of the Legendre transform g_i, the dashed red trajectories would yield strictly larger payoffs, leading to a contradiction.

4. It remains to prove that the boundary conditions (6.24) are satisfied. Toward this goal, we recall that optimal trajectories for (6.10)–(6.12) coincide with characteristics of the conservation law (6.21). By the strict concavity of the Legendre transform g_i in (3.8), (3.13), these lines never cross each other(see Fig. 8). Therefore, as shown in Fig. 7, there exists a Lipschitz continuous function $t \mapsto x^\sharp(t)$ such that

- if $\bar x < x^\sharp(\bar t)$, then the optimal trajectory has the form (6.37), for some $\bar y < 0$,
- if $x^\sharp(\bar t) < \bar x < 0$, then the optimal trajectory has the form (6.38), for some $0 \le \tau' < \tau < \bar t$ and $\bar y \le 0$.

Two cases will be considered.

CASE 1: $x^\sharp(t) = 0$. In this case, for each $x \le 0$ there exists a point $y^x \le 0$ such that

$$V_i(t,x) \;=\; V_i^\diamondsuit(y^x) + t\, g_i\left(\frac{x-y^x}{t}\right).$$

Since the map $x \mapsto y^x$ is nondecreasing, there exists the limit $y^x \to y^0$ as $x \to 0-$. By continuity,

$$V_i(t,0) \;=\; V_i^\diamondsuit(y^0) + t\, g_i\left(\frac{-y^0}{t}\right).$$

Therefore

$$f_i'(\rho_i(t,x)) \;=\; \frac{x-y^x}{t} \;\to\; \frac{-y^0}{t} \;\ge\; 0\,. \tag{6.47}$$

The limit

$$\bar\rho_i(t) \;=\; \lim_{x\to 0-} \rho_i(t,x)$$

is thus well defined. By (6.47), since the characteristic speed is nonnegative, it is clear that $\bar\rho_i(t) \le \rho_i^{max}$. Hence the maximum outgoing flux in (6.24) is is $\omega_i(t) = f_i(\bar\rho_i(t))$.

To complete the proof, it remains to show that

$$f_i(\bar\rho_i(t)) \;\le\; h_i(\mathbf{q}(t)) \;\doteq\; \min\left(f_i^{max}\,,\; c_i\cdot\Big(M - \sum_{j\in\mathcal{O}} q_j(t)\Big)\right)\,. \tag{6.48}$$

If (6.48) fails, by the continuity of the maps q_j there exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ small such that

$$f_i(\bar\rho_i(t)) \;>\; h_i(\mathbf{q}(\tau)) + \delta_0 \qquad \text{for all} \quad \tau \in [t-\varepsilon_0,\, t]\,.$$

We claim that in this case the trajectory

$$x^*(s) \;=\; \begin{cases} \left(1 - \dfrac{s}{t-\varepsilon}\right) y^0 & \text{if} \;\; s < t-\varepsilon, \\[2ex] 0 & \text{if} \;\; s \ge t-\varepsilon, \end{cases} \tag{6.49}$$

achieves a strictly larger payoff for $\varepsilon \in]0,\varepsilon_0]$ sufficiently small. Indeed, this payoff is computed by

$$-\int_{t-\varepsilon}^t h_i(\mathbf{q}(\tau))d\tau + (t-\varepsilon)\cdot g_i\left(\frac{-y^0}{t-\varepsilon}\right) \;>\; -\varepsilon\cdot f_i(\bar\rho_i(t)) + (t-\varepsilon)\cdot g_i\left(\frac{-y^0}{t-\varepsilon}\right) + \delta_0\varepsilon. \tag{6.50}$$

On the other hand, from (6.47), one has that

$$f_i'(\bar\rho_i(t)) \;=\; \frac{-y_0}{t}\,.$$

Thus, by (3.9)–(3.11), it holds

$$f_i(\bar\rho_i(t)) \;=\; g_i'\Big(\frac{-y_0}{t}\Big)\cdot\frac{-y_0}{t} - g_i\Big(\frac{-y_0}{t}\Big)\,.$$

Therefore,

$$\begin{aligned}
&-\varepsilon\cdot f_i(\bar\rho_i(t)) + (t-\varepsilon)\cdot g_i\left(\frac{-y^0}{t-\varepsilon}\right) - t\cdot g_i\left(\frac{-y^0}{t}\right)\\
&\quad=\; \varepsilon\cdot g_i'\left(\tfrac{-y^0}{t}\right)\cdot\tfrac{y^0}{t} + (t-\varepsilon)\cdot g\left(\tfrac{-y^0}{t-\varepsilon}\right) - (t-\varepsilon)\cdot g_i\left(\tfrac{-y^0}{t}\right)\\
&\quad=\; \frac{-\varepsilon y^0}{t}\cdot\Big[\int_0^1 g_i'\left(s\cdot\frac{-y^0}{t-\varepsilon} + (1-s)\cdot\frac{-y^0}{t}\right)ds - g_i'\left(\frac{-y^0}{t}\right)\Big]\\
&\quad\geq\; -\,\varepsilon^2\cdot\frac{y_0^2\cdot\|g_i''\|_{\mathbf{L}^\infty}}{2t^2(t-\varepsilon)}\,.
\end{aligned}\tag{6.51}$$

Combine the above inequality and (6.50), we finally obtain that

$$-\int_{t-\varepsilon}^{t} h_i(\mathbf{q}(\tau))d\tau + (t-\varepsilon)\cdot g_i\left(\frac{-y^0}{t-\varepsilon}\right) - t\cdot g_i\left(\frac{-y^0}{t}\right) \;>\; \varepsilon\cdot\left[\delta_0 - \varepsilon\cdot\frac{y_0^2\cdot\|g_i''\|_{\mathbf{L}^\infty}}{2t^2(t-\varepsilon)}\right] \;>\; 0$$

for $\varepsilon\in]0,\varepsilon_0]$ sufficiently small. By contradiction, this proves (2.12).

CASE 2: $x^\sharp(t) < 0$. In this case we can find $\delta > 0$ such that, for every terminal point $(\bar t,\bar x)\in[t-\delta,t]\times[-\delta,0]$, the optimal trajectory has the form (6.38). For $s\in[t-\delta,t+\delta]$, this implies

$$\overline{V}_i(s) \;= V_i(s,0) \;=\; V_i(t-\delta,0) - \int_{t-\delta}^{s} h_i(\mathbf{q}(s))\,ds.$$

If $h_i(\mathbf{q}(s)) = f_i^{max}$, the inequality (6.48) is trivial.

If $h_i(\mathbf{q}(s)) < f_i^{max}$, the necessary conditions for optimality of a trajectory of type (6.38) yield

$$\begin{aligned}
0 \;&=\; \frac{d}{d\tau}\left[\int_{\tau'}^{\tau} -h_i(\mathbf{q}(t))\,dt + (\bar t-\tau)g_i\Big(\frac{\bar x}{\bar t-\tau}\Big)\right]\\
&=\; -\,h_i(\mathbf{q}(\tau)) - g_i\left(\frac{\bar x}{\bar t-\tau}\right) + \frac{\bar x}{\bar t-\tau}\,g_i'\left(\frac{\bar x}{\bar t-\tau}\right)\\
&=\; -\,h_i(\mathbf{q}(\tau)) - \left[\rho\cdot\frac{\bar x}{\bar t-\tau} - f_i(\rho)\right] + \frac{\bar x}{\bar t-\tau}\,\rho\,.
\end{aligned}$$

This implies $f_i(\rho) = h_i(\mathbf{q}(\tau))$. In other words, the density ρ along the characteristic reaching the point $(\bar t,\bar x)$ yields precisely the flux $h_i(\mathbf{q}(\tau))$. Letting $(\bar t,\bar x)\to(t,0)$, we obtain $f_i(\rho_i(\bar t,\bar x))\to h_i(\mathbf{q}(t))$. In this case, (6.48) is satisfied as an equality. □

7 Variational formulation of (MBJ)

Next, we perform a similar analysis in connection with the multi-buffer junction conditions **(MBJ)**. These lead to:

Optimization Problem 2. For any $i \in I$, given the function $V_i^\diamondsuit$ in (3.17) and the length of the queues q_j, $j \in \mathcal{O}$ such that

$$q_j(t) \ < \ M_j, \quad \text{for all } t > 0,$$

consider the following variational problem.

$$\text{maximize:} \qquad J_i(x(\cdot)) \ \doteq \ V_i^\diamondsuit(x(0)) + \int_0^{\bar t} L_i(x(\cdot), \dot x(t))\, dt \tag{7.1}$$

over the set of all absolutely continuous functions such that

$$x(\bar t) \ = \ \bar x, \qquad\qquad x(t) \le 0 \quad \text{for all } t \in [0, \bar t], \tag{7.2}$$

and such that the set $\{t \in [0, \bar t]\,; \ \ x(t) = 0\}$ is the union of at most finitely many intervals. In order to define the payoff function, recalling (3.26) we introduce the Lipschitz continuous function $t \mapsto \beta(t)$, defined as

$$\dot\beta(t) \ = \ \begin{cases} g_i(\dot x(t)) & \text{if} \quad x(t) < 0\,, \\ -h_i\big(\mathbf{q}(t), \theta(\beta(t))\big) & \text{if} \quad x(t) = 0\,, \end{cases} \qquad \text{with} \quad \beta(0) = \overline{V}_i(x(0)). \tag{7.3}$$

Instead of (6.11), we consider the payoff function

$$L_i(x(t), \dot x(t)) \ = \ \begin{cases} g(\dot x(t)) & \text{if} \quad x(t) < 0, \\ -h_i(\mathbf{q}(t), \theta(\beta(t))) & \text{if} \quad x(t) = 0. \end{cases} \tag{7.4}$$

The following lemma, similar to Lemma 1, shows that the requirement about the set of zeroes of the function $x(\cdot)$ is not really a restriction. Indeed, the maximum is always achieved when this set is either empty or one single interval.

Lemma 2. *Consider an absolutely continuous map $x : [0, \bar t] \mapsto]-\infty, 0]$ satisfying (7.2). Define the times a, b as in (6.13) and the function $x^\sharp(\cdot)$ as in (6.14). Then, in connection with the integrand function L_i in (7.4)-(7.3), the inequality (6.15) remains valid.*

Proof. Consider any subinterval $[a', b'] \subseteq [a(x), b(x)]$ such that $x(a') = x(b') = 0$ and $x(t) < 0$ for all $t \in]a', b'[$, and define $x^\flat(\cdot)$ as in (6.16).

The lemma will be proved by showing that (6.17) still holds. Let β and $\beta^\flat$ be the solutions of (7.3) associated with x and $x^\flat$ respectively. Clearly, $\beta(t) = \beta^\flat(t)$ for all $t \in [0, a']$. Moreover, using Jensen's inequality and recalling that $-h_i(\mathbf{q}(t), \theta_{ij}) \ge -f_i^{max} = g_i(0)$, we obtain

$$\int_{a'}^{b'} g_i(\dot x(t))dt \ \le \ \int_{a'}^{b'} g_i(0)dt \ \le \ \int_{a'}^{b'} -h_i(\mathbf{q}(t), \theta_{ij}(\beta^\flat(t)))dt.$$

Thus,

$$\beta(b') \;=\; \beta(a') + \int_{a'}^{b'} g(\dot x(t))dt \;\leq\; \beta^\flat(a') + \int_{a'}^{b'} -h_i\big(\mathbf{q}(t), \theta_{ij}(\beta^\flat(t))\big)\,dt \;=\; \beta^\flat(b')\,.$$

Next, choose the times

$$b' \;=\; b_0 \;\leq\; a_1 \;<\; b_1 \;\leq\; a_2 \;<\; b_2 \;\leq\; \cdots \;\leq\; b_{n-1} \;<\; a_n \;=\; \bar t,$$

so that

$$\begin{cases} x(t) = 0 & \text{if} \quad t\in [b_{\ell-1}, a_\ell]\,,\\ x(t) < 0 & \text{if} \quad t\in \,]a_\ell, b_\ell[\,. \end{cases}$$

Note that this is possible because of the structural assumption we are making on $x(\cdot)$. By a comparison argument for solutions to the ODE (7.3) describing $\beta(\cdot)$, we obtain the implications

$$\begin{array}{rcl} \beta(b_{\ell-1}) \;\leq\; \beta^\flat(b_{\ell-1}) & \Longrightarrow & \beta(a_\ell) \;\leq\; \beta^\flat(a_\ell),\\ \beta(a_\ell) \;\leq\; \beta^\flat(a_\ell) & \Longrightarrow & \beta(b_\ell) \;\leq\; \beta^\flat(b_\ell), \end{array}$$

for every $\ell\geq 1$. By induction, this implies $\beta(t)\leq \beta^\flat(t)$ for all $t\in [b',\bar t]$. Hence (6.17) holds.
□

Proposition 2. *Let a continuous function $t\mapsto \mathbf{q}(t) = (q_j(t))_{j\in\mathcal{O}}$ be given, together with initial data $\rho_i^\diamondsuit(x)$, $\theta_{ij}^\diamondsuit(x)$ for $x<0$, satisfying the conditions (3.5)-(3.6). Define $V_i^\diamondsuit$ as in (3.17) and consider the variational problem (7.1)–(7.2). Then the following holds.*

- *(i) For every given $\bar t>0$ and $\bar x<0$, an optimal solution $x^*(\cdot)$ exists. This solution is piecewise affine and satisfies $\dot x^*(t)\in [f_i'(\rho_i^{jam}), f_i'(0)]$ for a.e. $t\in [0,\bar t]$.*
- *(ii) The maximum attainable value $V_i(\bar t,\bar x)$ is given by the formula (3.28).*
- *(iii) The corresponding density $\rho_i(t,x) = V_{i,x}(t,x)$ is defined a.e., and provides a solution to the conservation law (2.1) with initial data as in (3.4) and boundary conditions (2.14).*

Proof. 1. Given any $\bar t>0$ and $\bar x<0$, call B the supremum among all payoffs in (7.1), and let $(x_n)_{n\geq 1}$ be a minimizing sequence. We thus assume that each x_n satisfies (7.2) and

$$\lim_{n\to\infty} \overline{V}_i(x_n(0)) + \int_0^{\bar t} L_i(x_n(t), \dot x_n(t))\,dt \;=\; B.$$

As in the proof of Proposition 1, without loss of generality we can assume that each $x_n(\cdot)$ is piecewise affine, having the form (6.35) or (6.36). Indeed, two cases must be considered.

CASE 1: There exists $N_0>0$ such that for every $n>N_0$

$$x_n(t) \;<\; 0, \quad \text{for all } t\in [0,\bar t[.$$

This is the same as CASE 1 in the proof of Proposition 1. By the same arguments, we conclude that there exists a point $\bar y\leq 0$ such that the affine function (6.31) yields the maximum payoff. In particular, the representation formula (3.28) holds.

CASE 2: For infinitely many n, the set of times $\{t \in [0,\bar{t}]; \quad x_n(t)=0\}$ is nonempty.

Because of Lemma 2, we can assume that, for each n, the set of times where $x_n(t)=0$ is a closed interval, say

$$\{t \in [0,\bar{t}]; \quad x_n(t)=0\} \;=\; [a_n,b_n].$$

Applying Jensen inequality to g_i, we obtain

$$\int_0^{\bar{t}} L_i(x_n(t),\dot{x}_n(t))\,dt \;\leq\; \int_0^{a_n} g(\dot{x}_n(t))\,dt + \int_{a_n}^{b_n} -h_i(\mathbf{q}(t),\theta_{ij}(\beta_n(t)))\,dt + (\bar{t}-b_n)\cdot g_i\Big(\frac{\bar{x}}{\bar{t}-b_n}\Big). \tag{7.5}$$

where $\beta_n(\cdot)$ is the solution of (7.3) with $\beta_n(a_n) = \overline{V}_i(x_n(0)) + \displaystyle\int_0^{a_n} g(\dot{x}_n(t))dt$.

Moreover, let $\bar{\beta}_n$ be the solution of the second equation in (7.3) in $[a_n,b_n]$ with $\bar{\beta}_n(a_n) = \overline{V}_i(x_n(0)) + a_n\cdot g_i\Big(\frac{-x_n(0)}{a_n}\Big)$. One can see that $\beta_n(a_n)\leq\bar{\beta}_n(a_n)$. Thus, $\beta_n(b_n)\leq\bar{\beta}_n(b_n)$, i.e.,

$$\begin{aligned}\overline{V}_i(x_n(0)) &+ \int_0^{a_n} g(\dot{x}_n(t))dt + \int_{a_n}^{b_n} -h_i(\mathbf{q}(t),\theta_{ij}(\beta_n(t)))\,dt\\ &\leq \overline{V}_i(x_n(0)) + a_n\cdot g_i\Big(\frac{-x_n(0)}{a_n}\Big) + \int_{a_n}^{b_n} -h_i(\mathbf{q}(t),\theta_{ij}(\beta_n(t)))\,dt\end{aligned}$$

Combining with (7.5), we obtain that

$$\int_0^{\bar{t}} L_i(x_n(t),\dot{x}_n(t))\,dt \;\leq\; a_n\cdot g_i\Big(\frac{-x_n(0)}{a_n}\Big) + \int_{a_n}^{b_n} -h_i(\mathbf{q}(t),\theta_{ij}(\bar{\beta}_n(t)))\,dt + (\bar{t}-b_n)\cdot g_i\Big(\frac{\bar{x}}{\bar{t}-b_n}\Big). \tag{7.6}$$

Thus, we can assume that $\dot{x}_n(t) = \frac{-x_n(0)}{a_n}$ for all $t\in]0,a_n[$.

The following argument shows that, without loss of generality, we can also assume

$$x_n(0) \;\geq\; -\,a_n\,f_i'(0), \qquad \frac{\bar{x}}{\bar{t}-b_n} \;\geq\; f_i'(\rho_i^{jam}), \qquad \text{for every } n\geq 1. \tag{7.7}$$

- If $x_n(0) < x_n^- \doteq -a_n\,f_i'(0)$, one has

  $$V_i^{\diamondsuit}(x_n(0)) + a_n\cdot g_i\Big(\frac{-x_n(0)}{a_n}\Big) \;=\; V_i^{\diamondsuit}(x_n(0)) \;\leq\; V_i^{\diamondsuit}(x_n^-) \;=\; V_i^{\diamondsuit}(x_n^-) + a_n\cdot g_i\Big(\frac{-x_n^-}{a_n}\Big).$$

  As in the above argument, let $\tilde{\beta}_n$ be the solution of the second equation in (7.3) in $[a_n,b_n]$ with $\tilde{\beta}_n(a_n)\leq\; V_i^{\diamondsuit}(x_n^-)$. We have

  $$V_i^{\diamondsuit}(x_n(0)) + \int_0^{\bar{t}} L_i(x_n(t),\dot{x}_n(t))\,dt \;\leq\; \tilde{\beta}_n(b_n) + (\bar{t}-b_n)\cdot g_i\Big(\frac{\bar{x}}{\bar{t}-b_n}\Big).$$

- the proof of the second inequality in (7.7) is similar to the proof of the second inequality in (6.33).

2. By the previous step, there exists a maximizing sequence of piecewise affine functions $x_n(\cdot)$, whose derivatives satisfy

$$\dot{x}_n(t)\in[f_i'(\rho_i^{jam}),\, f_i'(0)] \qquad \text{for a.e.}\quad t\in[0,\bar{t}], \tag{7.8}$$

and satisfying (6.35) or (6.36). By taking a subsequence, we can assume the uniform convergence $x_n(\cdot)\to x^*(\cdot)$ on $[0,\bar{t}]$. The function x^* satisfies (6.37) or (6.38).

- If x^* satisfies (6.37), then by the convergence $y_n \to \bar{y}$ and the strong convergence $\dot{x}_n \to \dot{x}^*$ in $\mathbf{L}^1$, it follows

$$V_i^{\diamondsuit}(x^*(0)) + \int_0^{\bar{t}} L_i\big(x^*(t), \dot{x}^*(t)\big)\, dt \;=\; B,$$

- If x^* satisfies (6.38), then by the convergence $y_n \to \bar{y}$, $a_n \to a$, $b_n \to b$, and the strong convergence $\dot{x}_n \to \dot{x}^*$, it follows and

$$\lim_{n_k \to \infty} \beta_n(a_n) \;=\; \beta^*(a).$$

From Lemma A1, we have

$$\lim_{n_k \to \infty} \beta_{n_k}(b_{n_k}) \;=\; \beta^*(b).$$

This implies

$$\beta^*(b) + (\bar{t} - b) \cdot g_i\Big(\frac{\bar{x}}{\bar{t} - b}\Big) \;=\; \lim_{n \to \infty} \beta_{n_k}(b_n) + (\bar{t} - b_n) \cdot g_i\Big(\frac{\bar{x}}{\bar{t} - b_n}\Big) \;=\; B.$$

Statement (ii) is an immediate consequence of (6.37)-(6.38).

3. As in the proof of Proposition 1, we conclude that the value function V_i is a viscosity solution of the Hamilton-Jacobi equation (6.41) on the open set $\Omega = \{(t,x)\,;\ T > 0, x < 0\}$. Hence $\rho_i = V_{i,x}$ is an entropy weak solution to the conservation law (6.21) on Ω, with the prescribed initial data (3.4).

To prove that the boundary conditions (2.14) are also satisfied, we proceed as follows. Let $t \mapsto x^\sharp(t)$ be a Lipschitz continuous function such that

- if $\bar{x} < x^\sharp(\bar{t})$, then the optimal trajectory has the form (6.37), for some $\bar{y} < 0$,
- if $x^\sharp(\bar{t}) < \bar{x} < 0$, then the optimal trajectory has the form (6.38), for some $0 \le \tau' < \tau < \bar{t}$ and $\bar{y} \le 0$.

Two cases will be considered.

CASE 1: $x^\sharp(t) = 0$. This case is treated as in the proof of Proposition 1, with one modification. To prove that the inequality

$$f_i(\bar{\rho}_i(t)) \;\le\; h_i(\mathbf{q}(t), \theta(t)) \;\doteq\; \min\ \left\{ f_i^{max}\,,\quad c_i \cdot \frac{M_j - q_j(t)}{\theta_{ij}(t)}\,;\quad j \in \mathcal{O} \right\} \tag{7.9}$$

is a.e. satisfied, let t be a Lebesgue point for the maps $t \mapsto h_i(\mathbf{q}(t), \theta(t))$ and $t \mapsto \bar{\rho}_i(t)$. Assume that, on the contrary,

$$f_i(\bar{\rho}_i(t)) \;>\; h_i(\mathbf{q}(t), \theta(t)) + 2\delta_0 \tag{7.10}$$

for some constant $\delta_0 > 0$.

Since t is a Lebesgue point of the map $t \mapsto h_i(\mathbf{q}(t), \theta(t))$, there exists $\varepsilon_0 > 0$ such that

$$\int_{t-\varepsilon}^{t} \big|h_i(\mathbf{q}(t), \theta(t)) - h_i(\mathbf{q}(\tau), \theta(\tau))\big|\, d\tau \;\le\; \delta_0 \varepsilon, \quad \text{for all } \varepsilon \in]0, \varepsilon_0[\,.$$

Recalling (7.10), we obtain that

$$-\int_{t-\varepsilon}^{t} h_i(\mathbf{q}(\tau),\theta(\tau))d\tau \;>\; -\,\varepsilon\, f_i(\bar\rho_i(t))+\delta_0\varepsilon, \quad \text{for all } \varepsilon\in]0,\varepsilon_0[\,.$$

As in the proof of Proposition 1, for $\varepsilon\in[0,\varepsilon_0[$ sufficiently small the modified function x^*_ε defined at (6.49) yields a strictly larger payoff. Indeed, this follows from

$$-\int_{t-\varepsilon}^{t} h_i(\mathbf{q}(\tau),\theta(\tau))d\tau + (t-\varepsilon)\cdot g_i\left(\frac{-y^0}{t-\varepsilon}\right) \;>\; -\,\varepsilon\cdot f_i(\bar\rho_i(t)) + (t-\varepsilon)\cdot g_i\left(\frac{-y^0}{t-\varepsilon}\right)+\delta_0\varepsilon.$$

and (6.51). By contradiction, this proves (2.14).

CASE 2: $x^\sharp(t)>0$. By continuity, there exists $\delta>0$ such that $x^\sharp(s)>0$ for $s\in[t-\delta,\,t+\delta]$. Since optimal trajectories do not cross, this implies that the optimal trajectory $x^*(\cdot)$ through the terminal point $(t,0)$ satisfies $x^*(s)=0$ for $s\in[t-\delta,t]$. By the definition (7.4), this implies

$$V_{i,t}(s,0) \;=\; -\,h_i(\mathbf{q}(s),\theta(\beta(s))) \qquad\qquad \text{for a.e. } \; s\in[t-\delta,\,t].$$

Since $-V_{i,t}(s,0)=f_i(\bar\rho_i(s))$ measures the outgoing flux through the boundary, this shows that in this case the relation (7.9) is satisfied as an equality. □

8 Variational formulation for the flow on outgoing roads

In this section we introduce one more optimization problem, whose solution describes the traffic density along each outgoing road. In the case where $V_j^\diamondsuit\equiv 0$, a very similar variational problem was considered in [4].

Optimization Problem 3. For any $j\in\mathcal{O}$ and any terminal point $(\bar t,\bar x)$ with $\bar t>0$, $\bar x>0$, given the functions $V_j^\diamondsuit$ and F_j in (3.20), consider the problem of maximizing the functional

$$J(x(\cdot)) \;\doteq\; \max\left\{ V_j^\diamondsuit(x(0))+\int_0^{\bar t} g_j(\dot x(t))\,dt\,,\qquad \max_{\tau\ge 0,\ x(\tau)=0}\left(-F_j(\tau)+\int_\tau^{\bar t} g_j(\dot x(t))\,dt\right)\right\}. \tag{8.11}$$

The maximum is sought among all absolutely continuous functions $x:[0,\bar t]\mapsto I\!R$ such that

$$x(\bar t) \;=\; \bar x, \qquad\qquad x(t)\ge 0 \quad \text{for all } t\in[0,\bar t]. \tag{8.12}$$

Notice that, if $x(t)>0$ for all $t\in[0,\bar t]$, then $J(x(\cdot))$ is defined by the first term within brackets in (8.11). However, if $x(\tau)=0$ for some $0<\tau<\bar t$, then the maximum can be attained by the second term.

Proposition 3. *For $j\in\mathcal{O}$, let a continuous function $t\mapsto F_j(t)\ge 0$ be given, together with initial data $\rho_j^\diamondsuit(x)\in[0,\rho_j^{jam}]$, for $x>0$. Define $V_j^\diamondsuit$ as in (3.21) and consider the above variational problem (8.11)–(8.12). Then the following holds.*

(i) For every given $\bar t>0$ and $\bar x>0$, an optimal solution $x^*(\cdot)$ exists. This solution is affine, with constant derivative satisfying $\dot x^*(t)\in[f_j'(\rho_j^{jam}),\,f_j'(0)]$.

(ii) The maximum attainable value $V_j(\bar t,\bar x)$ is given by the formula (3.22).

(iii) The corresponding density $\rho_j(t,x) = V_{j,x}(t,x)$ is defined a.e., and provides a solution to the conservation law

$$\rho_t + f_j(\rho)_x \;=\; 0 \tag{8.13}$$

with initial data as in (3.4) and boundary conditions (2.13).

Proof. 1. Given $\bar t > 0$ and $\bar x > 0$, let B be the supremum of all possible payoffs in (8.11). Consider a maximizing sequence $(x_n)_{n\geq 1}$, such that

$$J(x_n) \;\to\; B.$$

Two cases must be considered.

CASE 1: For infinitely many indices n, one has

$$J(x_n) \;=\; V_j^{\diamondsuit}(x_n(0)) + \int_0^{\bar t} g_j(\dot x_n(t))\,dt\,.$$

In this case, since the function g_j is concave down, we obtain

$$\int_\tau^{\bar t} g_j(\dot x_n(t))\,dt \;\leq\; \bar t\cdot g_j\left(\frac{\bar x}{\bar t}\right).$$

We can thus replace x_n with the affine function

$$t \;\mapsto\; x_n(0) + t\,\frac{\bar x - x_n(0)}{\bar t}$$

without lowering the payoff.

CASE 2: For infinitely many indices n, one has

$$J(x_n) \;=\; -F_j(\tau_n) + \int_{\tau_n}^{\bar t} g_j(\dot x(t))\,dt,$$

for some $\tau_n \in [0,\bar t]$ with $x_n(\tau_n) = 0$. In this case the concavity of g_j implies

$$\int_{\tau_n}^{\bar t} g_j(\dot x_n(t))\,dt \;\leq\; (\bar t - \tau_n)\cdot g_j\left(\frac{\bar x}{\bar t - \tau_n}\right).$$

We can thus replace x_n with a piecewise affine function $\tilde x_n$ such that

$$\tilde x_n(t) \;=\; \frac{t-\tau_n}{\bar t - \tau_n}\,\bar x$$

without lowering the payoff.

As in the proof of Proposition 1, one can show that the derivatives $\dot x_n$ can be taken uniformly bounded. More precisely,

$$\dot x_n(t) \;\in\; [f_j'(\rho_j^{jam})\,,\; f_j'(0)]\,.$$

Indeed, in CASE 1 this can be proved as in Proposition 1.

Let us now consider CASE 2. Observe first that $\dot x_n(t) \geq 0 > f_j'(\rho_j^{jam})$ for a.e. $t \in [0, \bar t]$. To show that $\dot x_n(t) \leq f_j'(0)$, assume that, on the contrary,

$$\frac{\bar x}{\bar t - \tau_n} \;>\; f_j'(0)\,.$$

This implies $g_j'(\frac{\bar x}{\bar t - \tau_n}) = 0$ and thus the payoff is $J(x_n) = -F(\tau_n) < 0$. We consider two subcases:

- If $\bar x \geq \bar t \cdot f_j'(0)$ then $J(x_n) < J(x_n^+) = 0$ where x_n^+ is the linear function defined as $x_n^+(t) \doteq t\bar x/\bar t$. The conclusion thus follows from the analysis of CASE 1.
- If $\bar x < \bar t \cdot f_j'(0)$, we then set $\tau_n^+ \doteq \bar t - \frac{\bar x}{f_j'(0)}$ and define the function

$$x_n^+(t) \;=\; \begin{cases} \dfrac{t - \tau_n^+}{\bar t - \tau_n}\,\bar x & \text{if} \quad t \in [\tau_n^+, \bar t]\,, \\[2ex] 0 & \text{if} \quad t \in [0, \tau_n^+]\,. \end{cases}$$

  Observing that $g_j'(\frac{\bar x}{\bar t - \tau_n^+}) = 0$ and $\tau_n^+ < \tau_n$, since F_j is nondecreasing function, we conclude

$$J(x_n^+) \;=\; -F(\tau_n^+) \;\geq\; -F(\tau_n) \;=\; J(x_n)\,.$$

  We can thus replace x_n by x_n^+ without decreasing the payoff.

To complete the proof of (i) and (ii), in CASE 1 we choose a subsequence such that $x_n(0) \to \bar y$ and obtain an affine function

$$x^*(t) \;=\; \bar y + t\,\frac{\bar x - \bar y}{\bar t} \tag{8.14}$$

which achieves the maximum payoff. In CASE 2, choosing a subsequence such that $\tau_n \to \tau$, we obtain a piecewise affine function such that

$$x^*(t) \;=\; \frac{t - \tau}{\bar t - \tau}\,\bar x \tag{8.15}$$

achieving the maximum payoff. This proves the existence of an optimal solution, together with the representation formula (3.22) for the value function.

2. The Lipschitz continuity of the value function $V_j(t,x)$ is an immediate consequence of the Lipschitz continuity of the boundary data $V_j^\diamond$ and F_j.

Next, for a given $\tau \geq 0$, consider the open domain

$$\Omega^\tau \;\doteq\; \{(t,x)\,;\;\; t > \tau, \;\; x > f'(0)(t - \tau)\}$$

and define $V^\dagger(x) \;\doteq\; V_j(\tau, x)$. By the dynamic programming principle and by finite propagation speed, restricted to Ω^τ the value function V_j is given by

$$V_j(\bar t, \bar x) \;=\; \max\;\left\{ V^\dagger(x(\tau)) + \int_\tau^{\bar t} g_j(\dot x(s))\,ds\,; \quad x(\bar t) = \bar x, \;\; \dot x(s) \in [f_j'(\rho_j^{jam}),\, f_j'(0)] \right\}. \tag{8.16}$$

Hence V_j provides a viscosity solution to the Hamilton-Jacobi equation

$$V_{j,t} + f_j(V_{j,x}) \;=\; 0 \tag{8.17}$$

restricted to Ω^τ. Moreover, the derivative $\rho_j(t,x) = V_{j,x}(t,x)$ exists a.e. and provides an entropy weak solution to the conservation law (8.13).

We now observe that, as τ varies, the union of the sets Ω^τ covers $\Omega \doteq \{(t,x)\,;\; t>0,\; x>0\}$. Therefore, $\rho_j = V_{j,x}$ is an entropy solution of (8.13) on the entire open domain Ω. Moreover, the initial data $V_j(0,x) = V^\diamondsuit(x)$ are clearly satisfied.

3. To show that the boundary conditions (2.13) are also satisfied, as in the previous proofs we consider a Lipschitz continuous function $t \mapsto x^\sharp(t)$ such that

- if $\bar x > x^\sharp(\bar t)$, then the optimal trajectory has the form (8.14), for some $\bar y > 0$,
- if $0 < \bar x < x^\sharp(\bar t)$, then the optimal trajectory has the form (8.15), for some $0 \le \tau < \bar t$.

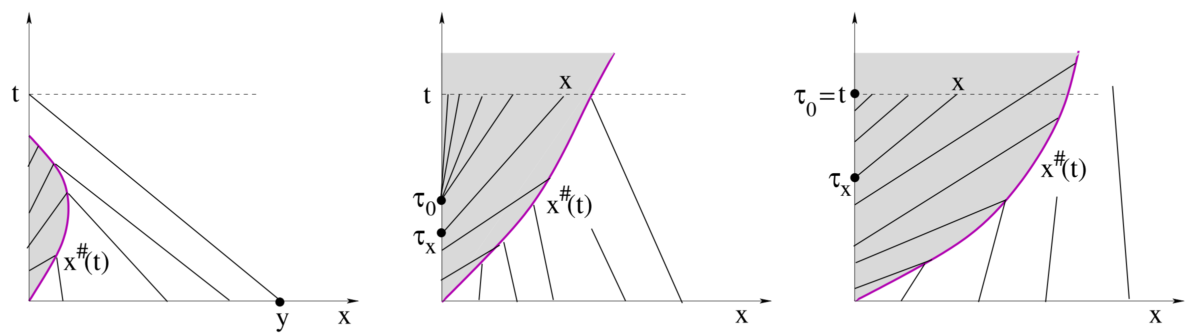

Figure 9: Various cases considered in the proof of Proposition 3. Here $t \mapsto x^\sharp(t)$ is the Lipschitz curve separating characteristics which originate from the x-axis and from the t-axis.

For a fixed $t > 0$, two cases will be considered.

CASE 1. If $x^\sharp(t) = 0$, then

$$V_j(t,0) \;=\; V_j^\diamondsuit(y) + t\, g_j\left(\frac{-y}{t}\right)$$

for some $y \ge 0$ (see Fig. 9, left). This implies that the vector

$$(\partial_t V_j,\ \partial_x V_j) \;=\; \left(g_j\Big(\frac{-y}{t}\Big) + \frac{y}{t} g_j'\Big(\frac{-y}{t}\Big)\ ,\ g_j'\Big(\frac{-y}{t}\Big)\right) \tag{8.18}$$

lies in the subdifferential of V at the point $(t,0)$.

By Legendre duality (3.11), one has

$$f_j'(\rho) \;=\; \frac{-y}{t} \qquad \Longleftrightarrow \qquad g_j'\Big(\frac{-y}{t}\Big) \;=\; \rho. \tag{8.19}$$

Choosing ρ so that (8.19) holds, we thus have

$$g_j\Big(\frac{-y}{t}\Big) + \frac{y}{t} g_j'\Big(\frac{-y}{t}\Big) \;=\; \left[\rho\cdot\Big(\frac{-y}{t}\Big) - f_j(\rho)\right] + \frac{y}{t}\,\rho \;=\; -f_j(\rho).$$

By Lipschitz continuity, the partial derivative $V_{j,t}(t,0)$ is well defined and must coincide with the first component of the vector in (8.18) for a.e. time t. Since $\rho = \rho(t,x)$ has locally bounded variation restricted to the set $\{(t,x)\,;\;\; t>0,\; x\geq x^\sharp(t)\}$, for a.e. t such that $x^\sharp(t)=0$ one has

$$V_{j,t}(t,0) \;=\; -\,\bar f_j(t) \;=\; -\,f_j(\bar\rho_j(t))\,, \qquad\qquad \bar\rho_j(t) \;\doteq\; \lim_{x\to 0+}\rho_j(t,x).$$

Observing that $\bar\rho_j(t)\geq \rho_j^{max}$, by (2.11) we have $f_j(\bar\rho_j(t)) \;=\; \omega_j(\bar\rho_j(t))$. Therefore, in this case we only need to show that, if $q_j(t)=0$, then

$$\bar f_j(t) \;=\; \min\,\Big\{\omega_j(\bar\rho_j(t))\;,\;\sum_{i\in I}\bar f_i(t)\bar\theta_{ij}(t)\Big\}\,. \tag{8.20}$$

Let t be a time where the maps $\tau\mapsto V_j(\tau,0)$ and $\tau\mapsto F_j(\tau)$ are both differentiable, and assume that $q_j(t)=0$. Then $V_j(t,0) \;=\; -\,F_j(t)$. Therefore,

$$0 \;=\; \lim_{h\to 0+}\frac{V_j(t+h,0)-V_j(t,0)}{h} - V_{j,t}(t,0) \;=\; \lim_{h\to 0+}\frac{V_j(t+h,0)+F_j(t)}{h} + \omega_j(\bar\rho_j(t)).$$

Observing that $V_j(t+h,0)\geq -F_j(t+h)$, we obtain

$$\begin{aligned} 0 \;&=\; \lim_{h\to 0+}\frac{V_j(t+h,0)+F_j(t)}{h} + \omega_j(\bar\rho_j(t)) \\ &\geq\; \lim_{h\to 0+}\frac{-F_j(t+h)+F_j(t)}{h} + \omega_j(\bar\rho_j(t)) \;=\; -\,F_j'(t)+\omega_j(\bar\rho_j(t)). \end{aligned}$$

This implies

$$\omega_j(\bar\rho_j(t)) \;\leq\; F_j'(t)\,. \tag{8.21}$$

On the other hand, from (3.17), (3.19) and (3.20), for a.e. $t>0$,

$$F_j'(t) \;=\; -\sum_{i\in\mathcal I}\dot\xi_i(t)\cdot\rho_i^\diamondsuit(\xi_i(t))\cdot\theta_{ij}^\diamondsuit(\xi_i(t)) = -\sum_{i\in\mathcal I}V_{i,t}(t,0)\cdot\theta_{ij}^\diamondsuit(\xi_i(t)) \;=\; \sum_{i\in\mathcal I}\bar f_i(t)\theta_{ij}^\diamondsuit(\xi_i(t)).$$

For every i,j, the linear transport equation (3.2) and the boundary conditions in (3.4) yield the identity

$$\bar\theta_{ij}(t) \;=\; \theta_{ij}^\diamondsuit(\xi_i(t)),$$

for a.e. $t>0$. Therefore

$$F_j'(t) \;=\; \sum_{i\in\mathcal I}\bar f_i(t)\bar\theta_{ij}(t)\,.$$

Together with (8.21), this implies $\omega_j(\bar\rho_j(t))\leq\sum_{i\in\mathcal I}\bar f_i(t)\bar\theta_{ij}(t)$, proving (8.20).

CASE 2. If $x^\sharp(t)>0$ then for every $x\in\,]0,x^\sharp(t)[$ the optimal solution starting from $(\bar t,x)$ connects to a point $(\tau_x,0)$ for some $\tau_x\in[0,t[$. That means

$$V_j(t,x) \;=\; -\,F_j(\tau_x)+(t-\tau_x)\cdot g_j\Big(\frac{x}{t-\tau_x}\Big)\,. \tag{8.22}$$

Moreover, for a.e. $x\in[0,x^\sharp(t)]$,

$$\rho_j(t,x) \;=\; g_j'\Big(\frac{x}{t-\tau_x}\Big) \qquad\text{and}\qquad f_j(\rho_j(t,x)) \;=\; \frac{x}{t-\tau_x}\cdot g_j'\Big(\frac{x}{t-\tau_x}\Big)-g_j\Big(\frac{x}{t-\tau_x}\Big).$$

For $x\in[0,x^\sharp(t)]$, the map $x\mapsto\tau_x$ is nonincreasing. The limit $\tau_0\doteq\lim_{x\to 0+}\tau_x$ is thus well defined. Two sub-cases will be considered.

(a) If $q_j(t) > 0$, then $\tau_0 < t$ (see Fig. 9, center). Indeed, assume by a contradiction that $\lim_{x\to 0+} \tau_x = t$. We then have $\lim_{x\to 0+} F_j(\tau_x) \;=\; F_j(t)$ and

$$\lim_{x\to 0+} \left|(t-\tau_x)\cdot g_j\Big(\frac{x}{t-\tau_x}\Big)\right| \;=\; \lim_{x\to 0+} (t-\tau_x)\cdot \left|g_j\Big(\frac{x}{t-\tau_x}\Big) - g_j(0)\right| \;\leq\; \lim_{x\to 0+} \rho_j^{jam}\, x \;=\; 0\,.$$

Recalling (8.22), we thus obtain

$$V_j(t,0) \;=\; \lim_{x\to 0+} -F_j(\tau_x) + (t-\tau_x)\cdot g_j\Big(\frac{x}{t-\tau_x}\Big) \;=\; -\,F_j(t)\,,$$

and hence $q_j(t) = V_j(t,0) + F_j(t) = 0$. This yields a contradiction.

In the case where $q_j(t) > 0$ we thus have

$$\bar\rho_j(t) \;=\; \lim_{x\to 0} \rho_j(t,x) = g_j'(0) \quad \text{and} \quad \bar f_j(t) \;=\; f_j(\bar\rho_j(t)) \;=\; -\,g_j(0) \;=\; f_j^{\max}\,.$$

Therefore, $\bar f_j(t) = \omega_j(t)$ and (2.13) holds.

(b) If $q_j(t) \;=\; 0$ then $V_j(t,0) = -F_j(t)$. Assume that the Lipschitz continuous functions $\tau \mapsto F_j(\tau)$ and $\tau \mapsto V_j(\tau,0)$ are both differentiable at t. Two possibilities must be considered.

If $\tau_0 = t$ (as in Fig. 9, right), then from (8.22) we obtain

$$F_j'(t) \;=\; \lim_{x\to 0} \frac{F(t) - F(\tau_x)}{t-\tau_x} \;=\; -\lim_{x\to 0+} g_j\Big(\frac{x}{t-\tau_x}\Big) \;=\; \bar f_j(t) \;=\; \omega_j(t)\,. \tag{8.23}$$

If $\tau_0 < t$ (Fig. 9, center), then as in the previous case one has

$$\bar f_j(t) \;=\; \lim_{x\to 0+} f_j(\rho_j(t,x)) \;=\; -\,g_j(0) \;=\; f_j^{max}\,. \tag{8.24}$$

Moreover,

$$V_j(\tau,0) \;=\; -\,F_j(\tau_0) + (\tau-\tau_0)g_j(0) \qquad\qquad \text{for all } \tau\in[\tau_0,t],$$

hence

$$V_{j,t}(t,0) \;=\; g_j(0)\,. \tag{8.25}$$

Recalling that $V_j(\tau,0) \geq -F_j(\tau)$ for every τ, while $V_j(t,0) = -F_j(t)$, by (8.24) and (8.25) we obtain

$$F_j'(t) \;\geq\; -\,V_{j,t}(t,0) \;=\; -\,g_j(0) \;=\; f_j^{max} \;=\; \bar f_j(t) \;=\; \omega_j(t)\,. \tag{8.26}$$

On the other hand, for a.e. $t > 0$ one has

$$F_j'(t) \;=\; \sum_{i\in\mathcal{I}} \bar f_i(t)\bar\theta_{ij}(t). \tag{8.27}$$

Together, (8.26)-(8.27) yield (2.13), for a.e. $t > 0$.

9 Appendix

In Section 4, the function $\beta(\cdot)$ was defined in (3.26) as the solution to a Cauchy problem for an ODE with discontinuous right hand side. Since the existence and uniqueness of such a solution does not follow from standard ODE theory, we supply here a proof. We recall that $q_j(t)$ is the length of queue on road j at time t, while $\mathbf{q} \doteq (q_j)_{j\in\mathcal{O}}$.

Lemma A1. *Let $\theta = (\theta_{ij})_{i\in\mathcal{I},j\in\mathcal{O}}$ be measurable functions satisfying (2.6), and let $t \mapsto q_j(t) \geq 0$ be Lipschitz continuous functions such that*

$$m_0 \ \doteq \inf_{j\in\mathcal{O},\,\tau\in[0,T]} \big(M_j - q_j(t)\big) \ > \ 0\,. \tag{9.28}$$

Consider the ODE

$$\frac{d}{ds}\beta(s) \ = \ -\,h_i(\mathbf{q}(s), \theta_{ij}(\beta(s))) \quad \text{for a.e.} \quad s \in [\tau,\, T]\,, \tag{9.29}$$

where h_i is the function defined at (3.16). Then the following holds.

(i) Given any $\beta_0 \in I\!R$, (9.29) admits a unique solution $\beta(\cdot)$ with $\beta(\tau) = \beta_0$. Moreover,

$$|\beta(t) - \beta(s)| \ \leq \ f_i^{\max} \cdot |t - s| \qquad \text{for all } s, t \in [\tau, T]. \tag{9.30}$$

(ii) Let $\beta_1(\cdot)$ and $\beta_2(\cdot)$ be the solutions of (9.29) with $\beta_1(\tau) = \bar\beta_1$ and $\beta_2(\tau) = \bar\beta_2$, respectively. Then,

$$|\beta_2(t) - \beta_2(t)| \ \leq \ C\,|\bar\beta_2 - \bar\beta_1| \quad \text{for all } t \in [\tau, T], \tag{9.31}$$

where the constant C depends only on τ, T, m_0 and the Lipschitz constant of $\mathbf{q}$.

Proof. 1. To prove (i), observe that h_i is strictly positive. If β is a solution of (9.29) then the map $t \mapsto \beta(t)$ is strictly decreasing. Hence, the inverse function $\beta \mapsto S(\beta)$ provides a solution to the Cauchy problem

$$\frac{d}{d\beta}\,S(\beta) \ = \ G_i(S, \beta)\,, \qquad S(\beta_0) \ = \ \tau\,, \tag{9.32}$$

where

$$G_i(S, \beta) \ \doteq \ -\,\frac{1}{h_i(\mathbf{q}(S), \theta(\beta))}\,. \tag{9.33}$$

We claim that (9.32) has a unique, strictly decreasing solution. Indeed, this follows from Carathéodory's theorem, because G_i is Lipschitz continuous w.r.t. S and measurable w.r.t β. Finally, from (3.16) it follows

$$\big|h_i(\mathbf{q}(s), \theta_{ij}(\beta(s)))\big| \ \leq \ f_i^{\max} \qquad \text{for all } s \in [\tau, T],$$

which yields (9.30).

2. To prove (ii), consider the inverse functions $S_1 \doteq \beta_1^{-1}$ and $S_2 \doteq \beta_2^{-1}$. Then S_1 and S_2 are solutions of (9.32) with $S_1(\bar\beta_1) = S_2(\bar\beta_2) = \tau$. Observing that (9.28) yields a lower bound on the flux h_i, it follows

$$|S_1(\bar\beta_2) - S_1(\bar\beta_1)| \ \leq \ C_1 \cdot |\bar\beta_1 - \bar\beta_2|\,,$$

where $C_1 > 0$ depends on the lower bound m_0 in (9.28). Using Gronwall's inequality one obtains

$$|S_1(\beta_2(t)) - S_2(\beta_2(t))| \leq C_2(\tau, T) \cdot |\bar{\beta}_2 - \bar{\beta}_1| \quad \text{for all } t \in [\tau, T].$$

Observing that $S_2(\beta_2(t)) = S_1(\beta_1(t)) = t$, we obtain

$$|S_1(\beta_2(t)) - S_1(\beta_1(t))| \leq C_2(\tau, T) \cdot |\bar{\beta}_2 - \bar{\beta}_1| \quad \text{for all } t \in [\tau, T].$$

The proof of (9.31) is now achieved by observing that

$$|S_1(\beta_2(t)) - S_1(\beta_1(t))| \geq \frac{1}{f_i^{\max}} \cdot |\beta_2(t) - \beta_1(t)| \quad \text{for all } t \in [\tau, T].$$

□

The next lemma provides the continuous dependence of the solution of (9.29) on the function $\mathbf{q} = (q_j)_{j\in\mathcal{O}}$.

Lemma A2. *Let $\theta = (\theta_{ij})_{i\in\mathcal{I}, j\in\mathcal{O}}$ be measurable functions satisfying (2.6), and let $\mathbf{q} = (q_j)$, $\tilde{\mathbf{q}} = (\tilde{q}_j)$ be Lipschitz continuous functions, with Lipschitz constant L_q, and such that*

$$\min\Big\{q_i(t),\ \tilde{q}_j(t)\Big\} \geq 0, \qquad \min\Big\{M_j - q_j(t),\ M_j - \tilde{q}_j(t)\Big\} \geq m_0, \tag{9.34}$$

for some $m_0 > 0$ and all $t \in [\tau, T]$, $j \in \mathcal{O}$. Let $\beta, \tilde{\beta}$ be the corresponding solutions of (9.29) with the same initial data

$$\beta(\tau) = \tilde{\beta}(\tau) = \beta_0.$$

Then,

$$\|\beta - \tilde{\beta}\|_{\mathbf{L}^\infty([\tau,T])} \leq C_0 e^{C_0(T-\tau)}(T-\tau) \cdot \|\mathbf{q} - \tilde{\mathbf{q}}\|_{\mathbf{L}^\infty([\tau,T])} \tag{9.35}$$

for some constant $C_0 > 0$ depending only on m_0, L_q.

Proof. Let $\widetilde{S}$ and S be the solutions of (9.32) with respect to $\tilde{\mathbf{q}}$ and $\mathbf{q}$. By (9.34) and (3.16), we have

$$|h_i(\tilde{\mathbf{q}}, \theta) - h_i(\mathbf{q}, \theta)| \leq C_1 \cdot |\tilde{\mathbf{q}} - \mathbf{q}|,$$

where $C_1 > 0$ depends only on the lower bound m_0. This implies

$$\begin{aligned}
\frac{d}{d\beta}|\tilde{S}(\beta) - S(\beta)| &\leq \left|\frac{1}{h_i(\tilde{\mathbf{q}}(\tilde{S}(\beta)), \theta(\beta))} - \frac{1}{h_i(\mathbf{q}(S(\beta)), \theta(\beta))}\right| \leq \frac{C_1}{m_0^2} \cdot \left|\tilde{\mathbf{q}}(\tilde{S}(\beta)) - \mathbf{q}(S(\beta))\right| \\
&\leq \frac{C_1}{m_0^2} \cdot \Big[\left|\tilde{\mathbf{q}}(\tilde{S}(\beta)) - \tilde{\mathbf{q}}(S(\beta))\right| + \left|\tilde{\mathbf{q}}(S(\beta)) - \mathbf{q}(S(\beta))\right|\Big] \\
&= \frac{C_1 L_q}{m_0^2} \cdot \left|\tilde{S}(\beta) - S(\beta)\right| + \frac{C_1}{m_0^2} \cdot \|\tilde{\mathbf{q}} - \mathbf{q}\|_{\mathbf{L}^\infty([\tau,T])},
\end{aligned}$$

for all $\beta \in [\beta_0, \min\{\beta(T), \tilde{\beta}(T)\}]$. Therefore, by Gronwall's inequality,

$$|\widetilde{S}(\beta) - S(\beta)| \leq \frac{C_1}{m_0^2}(\beta - \beta_0) \cdot e^{\frac{C_1 L_q}{m_0^2}\cdot(\beta-\beta_0)} \cdot \|\tilde{\mathbf{q}} - \mathbf{q}\|_{\mathbf{L}^\infty([\tau,T])},$$

for all $\beta \in [\beta_0, \min\{\beta(T), \tilde{\beta}(T)\}]$. Moreover, recalling (9.30) that

$$|\beta(s) - \beta_0| = |\beta(s) - \beta(\tau)| \leq f_i^{\max} \cdot |T - \tau| \quad \text{for all } s \in [\tau, T],$$

we obtain

$$|\widetilde{S}(\tilde{\beta}(s)) - S(\tilde{\beta}(s))| \ \leq \ C_2(T-\tau)e^{C_2(T-\tau)}\|\tilde{\mathbf{q}} - \mathbf{q}\|_{\mathbf{L}^\infty([\tau,T])} \quad \text{for all } s \in [\tau, T]\,, \tag{9.36}$$

with $C_2 \doteq \max\left\{\frac{C_1 f_i^{\max}}{m_0^2}, \frac{C_1 L_q f_i^{\max}}{m_0^2}\right\}$. Since $\widetilde{S}(\tilde{\beta}(s)) = S(\beta(s)) = s$, by (9.36) one obtains

$$|S(\beta(s)) - S(\tilde{\beta}(s))| \ \leq \ C_2(T-\tau)e^{C_2(T-\tau)}\|\tilde{\mathbf{q}} - \mathbf{q}\|_{\mathbf{L}^\infty([\tau,T])} \quad \text{for all } s \in [\tau, T]\,.$$

On the other hand, (9.30) implies

$$|S(\beta') - S(\beta)| \ \geq \ \frac{1}{f_i^{\max}} \cdot |\beta' - \beta|. \tag{9.37}$$

Combining (9.36) and (9.37), we conclude

$$|\tilde{\beta}(s) - \beta(s)| \ \leq \ f_i^{max} C_2(T-\tau)e^{C_2(T-\tau)}\|\tilde{\mathbf{q}} - \mathbf{q}\|_{\mathbf{L}^\infty([\tau,T])} \quad \text{for all } s \in [\tau, T]\,.$$

This yields (9.35), with the constant $C_0 \doteq C_2 \cdot (1 + f_i^{max})$. □

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