ON THE CAUCHY PROBLEM FOR SCALAR CONSERVATION LAWS IN THE CLASS OF BESICOVITCH ALMOST PERIODIC FUNCTIONS[*]

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Abstract. We study the Cauchy problem for a multidimensional scalar conservation law with merely continuous flux vector in the class of Besicovitch almost periodic functions. The existence and uniqueness of entropy solutions are established. We propose also the necessary and sufficient condition for the decay of almost periodic entropy solutions as time $t \to +\infty$.

1. Introduction. In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$, we consider the Cauchy problem for a first order multidimensional conservation law

$$u_t + \operatorname{div}_x \varphi(u) = 0 \tag{1.1}$$

with initial data

$$u(0, x) = u_0(x). \tag{1.2}$$

The flux vector $\varphi(u)$ is supposed to be only continuous:

$$\varphi(u) = (\varphi_1(u), \ldots, \varphi_n(u)) \in C(\mathbb{R}, \mathbb{R}^n).$$

Assume that $u_0(x) \in L^\infty(\mathbb{R}^n)$. Then the notion of entropy solution of (1.1), (1.2) in the sense of S.N. Kruzhkov [6] is well-defined.

DEFINITION 1.1. *A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1.1), (1.2) if for all $k \in \mathbb{R}$*

$$|u - k|_t + \operatorname{div}_x[\operatorname{sign}(u - k)(\varphi(u) - \varphi(k))] \le 0 \tag{1.3}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$);

$$\operatorname*{ess\,lim}_{t \to 0} u(t, \cdot) = u_0 \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

Condition (1.3) means that for all non-negative test functions $f = f(t, x) \in C_0^1(\Pi)$

$$\int_\Pi [|u - k|f_t + \operatorname{sign}(u - k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] dt dx \ge 0$$

(here $\cdot$ denotes the inner product in $\mathbb{R}^n$).

It is known that e.s. is always exists (see [7, 14]) but, in the case under consideration when the flux functions are merely continuous, this e.s. may be nonunique (see examples in [7, 8]). Nevertheless, if initial function is periodic (at least in $n - 1$ independent directions), the uniqueness holds: an e.s. of (1.1), (1.2) is unique and space-periodic, see the proof in [12, 13, 14].

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It is also well-known (see for instance [14, Proposition 1]), that initial requirement can be included in the single integral entropy inequality: for all nonnegative test functions $f = f(t,x) \in C_0^1(\bar{\Pi})$, where $\bar{\Pi} = [0, +\infty) \times \mathbb{R}^n$, for all $k \in \mathbb{R}$

$$\int_{\mathbb{R}^n} |u_0(x) - k| f(0,x) dx + \int_{\Pi} [|u-k| f_t + \operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] dt dx \geq 0, \quad (1.4)$$

that is, $u = u(t,x)$ is an e.s. of (1.1), (1.2) if and only if u satisfies relation (1.4).

We will essentially rely on the following mean L^1-contraction property. Denote by C_R the cube

$$\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x|_\infty = \max_{i=1,\ldots,n} |x_i| \leq R/2 \}, \quad R > 0$$

and let

$$N_p(u) = \limsup_{R \to +\infty} \left(R^{-n} \int_{C_R} |u(x)|^p dx \right)^{1/p}, \quad p \geq 1,$$

be the mean L^p-norm of a function $u(x) \in L^\infty(\mathbb{R}^n)$.

PROPOSITION 1.2 (mean L^1-contraction property). *Let $u(t,x), v(t,x) \in L^\infty(\Pi)$ be e.s. of (1.1), (1.2) with initial functions $u_0(x), v_0(x)$, respectively. Then for almost every (a.e.) $t > 0$*

$$N_1(u(t,\cdot) - v(t,\cdot)) \leq N_1(u_0 - v_0). \quad (1.5)$$

Proof. Applying Kruzhkov doubling of variables method, we obtain the relation (see [6, 14])

$$|u-v|_t + \operatorname{div}_x[\operatorname{sign}(u-v)(\varphi(u) - \varphi(v))] \leq 0 \quad \text{in } \mathcal{D}'(\Pi). \quad (1.6)$$

Let $\rho(s) \in C_0^\infty(\mathbb{R})$ be a function such that $\operatorname{supp} \rho(s) \subset [0,1]$, $\rho(s) \geq 0$, $\int_{-\infty}^{+\infty} \rho(s) ds = 1$. We set for $\nu \in \mathbb{N}$ $\delta_\nu(s) = \nu \rho(\nu s)$, $\theta_\nu(t) = \int_0^t \delta_\nu(s) ds = \int_0^{\nu t} \rho(s) ds$. Obviously, the sequence $\delta_\nu(s)$ converges as $\nu \to \infty$ to the Dirac δ-measure weakly in $\mathcal{D}'(\mathbb{R})$ while the sequence $\theta_\nu(t)$ pointwise converges to the Heaviside function. If $t_1 > t_0 > 0$ then the function $\chi_\nu(t) = \theta_\nu(t - t_0) - \theta_\nu(t - t_1) \in C_0^\infty(\mathbb{R}^+)$, $0 \leq \chi_\nu(t) \leq 1$, and the sequence $\chi_\nu(t)$ pointwise converges as $\nu \to \infty$ to the indicator function $\chi(t)$ of the interval $(t_0, t_1]$. We choose the function $g(y) \in C_0^\infty(\mathbb{R}^n)$ with the properties: $0 \leq g(y) \leq 1$, $g(y) \equiv 1$ in the cube C_1, $g(y) \equiv 0$ in the complement of the cube C_k, $k > 1$ (so that $\operatorname{supp} g(x) \subset C_k$). Applying relation (1.6) to the test function $f = R^{-n} \chi_\nu(t) g(x/R)$, where $R > 0$, we arrive at

$$\int_0^\infty \left(R^{-n} \int_{\mathbb{R}^n} |u(t,x) - v(t,x)| g(x/R) dx \right) (\delta_\nu(t - t_0) - \delta_\nu(t - t_1)) dt +$$

$$R^{-n-1} \int_\Pi \operatorname{sign}(u-v)(\varphi(u) - \varphi(v)) \cdot \nabla_y g(x/R) \chi_\nu(t) dt dx \geq 0. \quad (1.7)$$

Define the set

$$F = \{ t > 0 \mid (t,x) \text{ is a Lebesgue point of } |u(t,x) - v(t,x)| \text{ for a.e. } x \in \mathbb{R}^n \}.$$

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It is clear that $F \subset \mathbb{R}_+$ is a set of full Lebesgue measure and each $t \in F$ is a Lebesgue point of the functions $I_R(t) = R^{-n} \int_{\mathbb{R}^n} |u(t,x) - v(t,x)| g(x/R) dx$ for all $R > 0$ and all $g(y) \in C_0(\mathbb{R})$. Now we suppose that $t_0, t_1 \in F$ and pass in (1.7) to the limit as $\nu \to \infty$. As a result, we obtain the inequality

$$I_R(t_1) \le I_R(t_0) + R^{-n-1} \int_\Pi \mathrm{sign}(u-v)(\varphi(u) - \varphi(v)) \cdot \nabla_y g(x/R) \chi(t) dt dx.$$

From this inequality and the initial conditions it follows in the limit as $F \ni t_0 \to 0$ that for all $t = t_1 \in F$

$$I_R(t) \le I_R(0) + R^{-n-1} \int_{(0,t) \times \mathbb{R}^n} \mathrm{sign}(u-v)(\varphi(u) - \varphi(v)) \cdot \nabla_y g(x/R) dt dx, \quad (1.8)$$

where $I_R(0) = R^{-n} \int_{\mathbb{R}^n} |u(0,x) - v(0,x)| g(x/R) dx$. Making the change $y = x/R$ in the last integral in (1.8), we obtain the estimate

$$R^{-n-1} \left| \int_{(0,t) \times \mathbb{R}^n} \mathrm{sign}(u-v)(\varphi(u) - \varphi(v)) \cdot \nabla_y g(x/R) dt dx \right| \le$$

$$R^{-1} \|\varphi(u) - \varphi(v)\|_\infty \int_{(0,t) \times \mathbb{R}^n} |\nabla_y g|(y) dt dy \underset{R \to +\infty}{\to} 0. \quad (1.9)$$

Here we denote by $|v|$ the Euclidean norm of a finite-dimensional vector v. Further, by the properties of $g(y)$, for all $t \ge 0$

$$R^{-n} \int_{C_R} |u(t,x) - v(t,x)| dx \le I_R(t) \le$$

$$R^{-n} \int_{C_{kR}} |u(t,x) - v(t,x)| dx = k^n (kR)^{-n} \int_{C_{kR}} |u(t,x) - v(t,x)| dx,$$

which implies that

$$N_1(u(t,\cdot) - v(t,\cdot)) \le \limsup_{R \to +\infty} I_R(t) \le k^n N_1(u(t,\cdot) - v(t,\cdot)) \quad (1.10)$$

With the help of relations (1.9), (1.10), we derive from (1.8) in the limit as $R \to +\infty$ that $N_1(u(t,\cdot) - v(t,\cdot)) \le k^n N_1(u_0 - v_0)$ for all $t \in F$. To complete the proof it only remains to notice that $k > 1$ is arbitrary. $\square$

REMARK 1.3. *As was established in [15, Corollary 7.1], after possible correction on a set of null Lebesgue measure, any e.s. $u(t,x)$ is continuous on $[0, +\infty)$ as a map $t \to u(t,\cdot) \in L^1_{loc}(\mathbb{R}^n)$. Hence, we may suppose that every e.s. satisfies the property $u(t,\cdot) \in C([0, +\infty), L^1_{loc}(\mathbb{R}^n))$. Then the statement of Proposition 1.2 holds for all $t > 0$. The continuity assumption also allows to replace the essential limit in the initial requirement of Definition 1.1 by the usual limit.*

Denote by $L_0^\infty(\mathbb{R}^n)$ the kernel of the seminorm N_1, that is, $L_0^\infty(\mathbb{R}^n)$ consists of bounded measurable functions $u = u(x)$ such that $N_1(u) = 0$. From Proposition 1.2 it readily follows the weak uniqueness property.

COROLLARY 1.4. *If $u(t,x)$, $v(t,x)$ are e.s. of problem (1.1), (1.2) with initial data $u_0(x), v_0(x)$, respectively, and $v_0 - u_0 \in L_0^\infty(\mathbb{R}^n)$, then $v(t,\cdot) - u(t,\cdot) \in L_0^\infty(\mathbb{R}^n)$ for a.e. $t > 0$.*

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We may consider problem (1.1), (1.2) as the Cauchy problem in the quotient space $\mathcal{L}^\infty(\mathbb{R}^n) \doteq L^\infty(\mathbb{R}^n)/L_0^\infty(\mathbb{R}^n)$. In view of Corollary 1.4, an e.s. $u(t, \cdot) \in \mathcal{L}^\infty(\mathbb{R}^n)$ is unique.

The aim of the present paper is investigation of the well-posedness of problem (1.1), (1.2) in the class of Besicovitch almost periodic functions. Recall, (see [2, 17]) that Besicovitch space is the closure of trigonometric polynomials, i.e., finite sums $\sum a_\lambda e^{2\pi i \lambda \cdot x}$, with $i^2 = -1$, $\lambda \in \mathbb{R}^n$, in the quotient space $\mathcal{B}^p(\mathbb{R}^n) = B^p(\mathbb{R}^n)/B_0^p(\mathbb{R}^n)$, where

$$B^p(\mathbb{R}^n) = \{u \in L_{loc}^p(\mathbb{R}^n) \mid N_p(u) < +\infty\}, \quad B_0^p(\mathbb{R}^n) = \{u \in L_{loc}^p(\mathbb{R}^n) \mid N_p(u) = 0\}.$$

The space $\mathcal{B}^p(\mathbb{R}^n)$ is equipped with the norm $\|u\|_p = N_p(u)$ (we identify classes in the quotient space and their representatives). The space $\mathcal{B}^p(\mathbb{R}^n)$ is a Banach space, it is isomorphic to the completeness of the space $AP(\mathbb{R}^n)$ of Bohr almost periodic functions with respect to the norm N_p.

It is known [2] that for each $u \in \mathcal{B}^p(\mathbb{R}^n)$ there exist the mean value

$$\fint_{\mathbb{R}^n} u(x)dx \doteq \lim_{R \to +\infty} \int_{C_R} u(x)dx$$

and the Bohr-Fourier coefficients

$$a_\lambda = \fint_{\mathbb{R}^n} u(x)e^{-2\pi i \lambda \cdot x}dx, \quad \lambda \in \mathbb{R}^n.$$

The set

$$Sp(u) = \{\ \lambda \in \mathbb{R}^n \mid a_\lambda \neq 0\ \}$$

is called the spectrum of an almost periodic function u. It is known [2] that the spectrum $Sp(u)$ is at most countable. Denote by $M(u)$ the smallest additive subgroup of $\mathbb{R}^n$ containing $Sp(u)$ (notice that $M(u)$ is countable whenever it is different from the zero subgroup).

Our first result is the following

THEOREM 1.5. *Let $u_0(x) \in \mathcal{B}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ be a bounded Besicovitch almost periodic function, and $u(t,x)$ be an e.s. of problem (1.1), (1.2). Then, after possible correction on a set of t of null measure, $u(t, \cdot) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n)) \cap L^\infty(\Pi)$ and for all $t > 0$ $M(u(t, \cdot)) \subset M(u_0)$.*

The next our result concerns the decay property of e.s. as the time $t \to +\infty$.

THEOREM 1.6. *Assume that*

$$\forall \xi \in M_0 \doteq M(u_0), \xi \neq 0 \quad \textit{the functions } u \to \xi \cdot \varphi(u)$$
$$\textit{are not affine on non-empty intervals} \tag{1.11}$$

(the linear non-degeneracy condition). Then

$$\lim_{t \to +\infty} \fint_{\mathbb{R}^n} |u(t,x) - C|dx = 0, \quad \textit{where} \quad C = \fint_{\mathbb{R}^n} u_0(x)dx. \tag{1.12}$$

Moreover, condition (1.11) is exact: if it fails, then there exists an initial function $u_0 \in \mathcal{B}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ such that $Sp(u_0) \subset M_0$ and the e.s. of (1.1), (1.2) does not satisfy the decay property (1.12).

Notice that for periodic initial data $u_0(x)$ with the lattice of periods $L \subset \mathbb{R}^n$ the group M_0 coincides with the dual lattice $L' = \{\, \xi \in \mathbb{R}^n \mid \xi \cdot x \in \mathbb{Z} \; \forall x \in L \,\}$, and the result of Theorem 1.6 reduces to the decay property for periodic e.s. from [16]: the linear non-degeneracy condition

$$\forall \xi \in L', \xi \neq 0 \;\; \text{the function } u \to \xi \cdot \varphi(u)$$
$$\text{is not affine on non-empty intervals} \tag{1.13}$$

is necessary and sufficient for the decay of every space-periodic (with the lattice of periods L) e.s. $u(t,x)$:

$$\lim_{t \to +\infty} \int_{\mathbb{T}^n} |u(t,x) - C| dx = 0, \quad C = \int_{\mathbb{T}^n} u_0(x) dx. \tag{1.14}$$

Here $\mathbb{T}^n = \mathbb{R}^n / L$ is a torus (which can be identified with the fundamental parallelepiped

$$P = \{\, x = \sum_{i=1}^{n} \alpha_i e_i \mid \alpha_i \in [0,1), \; i = 1, \ldots, n \,\},$$

where e_i, $i = 1, \ldots, n$ is a basis in L), dx is the normalized Lebesgue measure on $\mathbb{T}^n$. Notice that for a periodic function $w(x) \in L^1(\mathbb{T}^n)$

$$\fint_{\mathbb{R}^n} w(x) dx = \frac{1}{|P|} \int_P w(x) dx = \int_{\mathbb{T}^n} w(x) dx,$$

where $|P|$ denote the Lebesgue measure of P. If the lattice of periods is not fixed and may depend on a solution, the decay property holds under the stronger assumption

$$\forall \xi \in \mathbb{R}^n, \xi \neq 0, \;\; \text{the function } u \to \xi \cdot \varphi(u)$$
$$\text{is not affine on non-empty intervals.} \tag{1.15}$$

This result generalizes the decay property established by G.-Q. Chen and H. Frid [3] under the conditions $\varphi(u) \in C^2(\mathbb{R}, \mathbb{R}^n)$ and

$$\forall (\tau, \xi) \in \mathbb{R}^{n+1}, (\tau, \xi) \neq 0, \quad \text{meas}\{\, u \in \mathbb{R} \mid \tau + \varphi'(u) \cdot \xi = 0 \,\} = 0 \tag{1.16}$$

(by meas A we denote the Lebesgue measure of a measurable set A). Obviously, condition (1.15) is strictly weaker than (1.16) even in the case of smooth flux $\varphi(u)$. For completeness sake we confirm it by the following simple example.

Let $n = 1$, $A \subset [0,1]$ be a closed nowhere dense set with positive Lebesgue measure (a so-called fat Cantor set). There exists a smooth function $\varphi(u) \in C^\infty(\mathbb{R})$ such that $A = \varphi^{-1}(0)$ and $\varphi(u)$ is not affine on non-empty intervals. For instance, we may define $\varphi(u) = e^{-\frac{1}{(u-a)(b-u)}}$ on each component interval (a,b) of the complement $\mathbb{R} \setminus A$ (in the cases when $a = -\infty$ or when $b = +\infty$ we set $\varphi(u) = e^{-\frac{1}{(b-u)}}$, $\varphi(u) = e^{-\frac{1}{(u-a)}}$, respectively), and set $\varphi(u) = 0$ for $u \in A$. Obviously, the function $\varphi(u)$ satisfies condition (1.15) (notice that A does not contain any interval) but, since $A \subset \{\, u \in \mathbb{R} \mid \varphi'(u) = 0 \,\}$, condition (1.16) fails.

REMARK 1.7. *In the case when the flux vector $\varphi(u)$ satisfies the Lipschitz condition on any segment in $\mathbb{R}$, any e.s. $u(t,x)$ of (1.1), (1.2) exhibits the property of finite speed of propagation, which implies in particular that for all $p, q \in \mathbb{R}^n$ and $t > 0$*

$$\int_{C_R} |u(t,x+p) - u(t,x+q)| dx \leq \int_{C_{R+Lt}} |u_0(x+p) - u_0(x+q)| dx, \tag{1.17}$$

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where $R > 0$, and L is the Lipschitz constant of $\varphi(u)$ on the segment $[-M, M]$, $M = \|u\|_\infty$.

Suppose that the initial data $u_0(x)$ is a Stepanov almost periodic function (see the definition in [2]). It readily follows from (1.17) that $u(t, \cdot)$ is a Stepanov almost periodic function as well for all $t > 0$. The decay property (1.12) for such solutions was established in [4] under non-degeneracy condition (1.16) (with smooth flux) and rather restrictive assumptions on the dependence of the length $l(\varepsilon)$ of inclusion intervals for ε-almost periods of u_0 on the parameter ε.

Notice also that Stepanov almost periodic functions are strictly embedded in $\mathcal{B}^1(\mathbb{R}^n)$.

In the present paper we remove all these unnecessary assumptions and proof the decay property under exact condition (1.11) for general equation (1.1) with merely continuous flux and with arbitrary Besicovitch almost periodic initial data.

2. Initial data with finite spectrum. Reduction to the periodic case.

In this section we study the case when $u_0(x) = \sum\limits_{\lambda \in \Lambda} a_\lambda e^{2\pi i \lambda \cdot x}$ is a trigonometric polynomial. Here $\Lambda \subset \mathbb{R}^n$ is a finite set and it is assumed that $a_\lambda \neq 0$ for $\lambda \in \Lambda$, i.e., $\Lambda = Sp(u_0)$. We assume that $u_0(x)$ is a real function, this means that $-\Lambda = \Lambda$ and $a_{-\lambda} = \overline{a_\lambda}$ (as usual, $\bar{z}$ denotes the complex conjugate to $z \in \mathbb{C}$). The minimal additive subgroup $M_0 \doteq M(u_0)$ containing Λ is a finite generated torsion free abelian group and therefore it is a free abelian group of finite rank (see [9]). Thus, there exists a basis $\lambda_j \in M_0$, $j = 1, \ldots, m$. Every element $\lambda \in M_0$ can be uniquely represented as $\lambda = \lambda(\bar{k}) = \sum\limits_{j=1}^m k_j \lambda_j$, $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$. In particular, vectors λ_j, $j = 1, \ldots, m$ are linearly independent over the field of rational numbers $\mathbb{Q}$. We introduce the finite set $J = \{\ \bar{k} \in \mathbb{Z}^m \mid \lambda(\bar{k}) \in \Lambda\ \}$. Then we can represent the initial function as follows

$$u_0(x) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \sum_{j=1}^m k_j \lambda_j \cdot x}, \quad a_{\bar{k}} \doteq a_{\lambda(\bar{k})}.$$

From this representation we find that $u_0(x) = v_0(y(x))$, where

$$v_0(y) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \bar{k} \cdot y}$$

is a periodic function in $\mathbb{R}^m$ with the standard lattice of periods $\mathbb{Z}^m$, and $y(x)$ is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$ defined as $y_j = \lambda_j \cdot x = \sum\limits_{k=1}^n \lambda_{jk} x_k$, λ_{jk}, $k = 1, \ldots, n$, being coordinates of the vector λ_j, $j = 1, \ldots, m$. We consider the conservation law

$$v_t + \operatorname{div}_y \tilde{\varphi}(v) = 0, \quad v = v(t, y), \ t > 0, \ y \in \mathbb{R}^m, \tag{2.1}$$

with the flux functions

$$\varphi_j(v) = \sum_{k=1}^n \lambda_{jk} \varphi_k(v) \in C(\mathbb{R}), \quad j = 1, \ldots, m.$$

There exists a unique e.s. $v(t, y) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^m)$ of the Cauchy problem for (2.1) with initial data $v_0(y)$, and this e.s. is y-periodic: $v(t, y+e) = v(t, y)$ a.e. in $\mathbb{R}_+ \times \mathbb{R}^m$ for all $e \in \mathbb{Z}^m$. Moreover, in view of [15, Corollary 7.1] $v(t, \cdot) \in C([0, +\infty), L^1(\mathbb{T}^m))$, where

$\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$ is an m-dimensional torus (it may be identified with the fundamental cube $[0, 1)^m$ in $\mathbb{R}^m$).

THEOREM 2.1. *For a.e. $z \in \mathbb{R}^m$ the function $u(t, x) = v(t, z + y(x))$ is an e.s. of (1.1), (1.2) with initial function $v_0(z + y(x))$.*

Proof. Firstly notice that the function $v(t, y)$ is an e.s. of the Cauchy problem for equation (2.1) considered in the half-space $t > 0$, $(y, x) \in \mathbb{R}^{m+n}$, with initial data $v_0(y)$ (we just attach the additional variables x). We make the non-degenerate linear change of variables $(z, x) \to (y, x)$, with $y = z + y(x)$. After this change the function $u(t, z, x) = v(t, z + y(x))$ satisfies for each $k \in \mathbb{R}$ the relation

$$|u - k|_t + \mathrm{div}_x[\mathrm{sign}(u - k)(\varphi(u) - \varphi(k))] =$$

$$|v - k|_t + \sum_{l=1}^{n} \sum_{j=1}^{m} [\mathrm{sign}(v - k)(\varphi_l(v) - \varphi_l(k))]_{y_j} \frac{\partial y_j(x)}{\partial x_l} =$$

$$|v - k|_t + \sum_{j=1}^{m} \sum_{l=1}^{n} [\mathrm{sign}(v - k)(\varphi_l(v) - \varphi_l(k))]_{y_j} \lambda_{jl} =$$

$$|v - k|_t + \sum_{j=1}^{m} [\mathrm{sign}(v - k)(\tilde{\varphi}_j(u) - \tilde{\varphi}_j(k))]_{y_j} \le 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^{m+n}).$$

It is clear that this function satisfies also the initial condition

$$\lim_{t \to 0+} u(t, z, x) = u_0(z, x) \doteq v_0(z + y(x)) \quad \text{in } L^1_{loc}(\mathbb{R}^{m+n}).$$

Thus, $u(t, z, x)$ is an e.s. of the modified problem (1.1), (1.2) in the extended space $\mathbb{R}_+ \times \mathbb{R}^{m+n}$. Define the set $E \subset \mathbb{R}^m$ consisting of $z \in \mathbb{R}^m$ such that (t, z, x) is a Lebesgue point of $u(t, z, x)$ for almost all $(t, x) \in \Pi$. Then the set E has full Lebesgue measure in $\mathbb{R}^m$. We demonstrate that for fixed $z_0 \in E$ the function $u(t, z_0, x)$ is an e.s. of (1.1), (1.2) with initial function $u_0(z_0, x)$. For that, we have to verify integral relation (1.4). Let us take test function $f(t, x) \in C_0^1(\bar{\Pi})$, $f(t, x) \ge 0$, and set $f_\nu(t, z, x) = f(t, x)g_\nu(z - z_0)$, where the sequence $g_\nu(y) = \prod_{j=1}^{m} \delta_\nu(y_i)$, $\nu \in \mathbb{N}$, is an approximate unit in $\mathbb{R}^m$ (so that it weakly converges as $\nu \to \infty$ to the Dirac δ-function in $\mathcal{D}'(\mathbb{R}^m)$), the functions $\delta_\nu(s)$ were defined in the proof of Proposition 1.2 above. Obviously, $f = f_\nu(t, z, x) \in C_0^1([0, +\infty) \times \mathbb{R}^{m+n})$, $f_\nu(t, z, x) \ge 0$. Taking $f = f_\nu(t, z, x)$ in relation (1.4) for the e.s. $u(t, z, x)$, we obtain that for each $k \in \mathbb{R}$

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |u_0(z, x) - k| f(0, x) dx \right) g_\nu(z - z_0) dz + \int_{\mathbb{R}^m} \left(\int_{\Pi} [|u(t, z, x) - k| f_t(t, x)] \right.$$

$$\left. + \mathrm{sign}(u(t, z, x) - k)(\varphi(u(t, z, x)) - \varphi(k)) \cdot \nabla_x f(t, x)] dt dx \right) g_\nu(z - z_0) dz \ge 0. \quad (2.2)$$

It can be easily verified that $z_0 \in E$ is a Lebesgue point of the functions

$$I_0(z) \doteq \int_{\mathbb{R}^n} |u_0(z, x) - k| f(0, x) dx, \quad I(z) \doteq \int_{\Pi} [|u(t, z, x) - k| f_t(t, x) +$$

$$\mathrm{sign}(u(t, z, x) - k)(\varphi(u(t, z, x)) - \varphi(k)) \cdot \nabla_x f(t, x)] dt dx.$$

Therefore, it follows from (2.2) in the limit as $\nu \to \infty$ that $I_0(z_0) + I(z_0) \ge 0$, i.e.,

$$\int_{\mathbb{R}^n} |u_0(z_0, x) - k| f(0, x) dx + \int_{\Pi} [|u(t, z_0, x) - k| f_t(t, x) +$$

$$\mathrm{sign}(u(t, z_0, x) - k)(\varphi(u(t, z_0, x)) - \varphi(k)) \cdot \nabla_x f(t, x)] dt dx \ge 0$$

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for each $k \in \mathbb{R}$ and all nonnegative test functions $f(t,x) \in C_0^1(\bar{\Pi})$. Hence, the function $u = u(t, z_0, x)$ satisfies (1.4) with $u_0 = u_0(z_0, x)$ and therefore it is an e.s. of (1.1), (1.2) with initial function $u_0 = v_0(z_0 + y(x))$ for all z_0 from the set E of full measure. The proof is complete. $\square$

The additive group $\mathbb{R}^n$ acts on the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ by the shift transformations $S_x z = z + y(x)$. Notice that by linearity of $y(x)$, $S_{x_1+x_2} = S_{x_1} S_{x_2}$, so that S_x, $x \in \mathbb{R}^m$ is an m-parametric group of measure preserving transformations of $\mathbb{T}^m$. We demonstrate that this action is ergodic. Let $A \subset \mathbb{T}^m$ be a measurable invariant set, and $\chi_A(y) = \sum_{\bar{k} \in \mathbb{Z}^m} a_{\bar{k}} e^{2\pi i \bar{k} \cdot y}$ be the Fourier series for the indicator function $\chi_A(y)$ of the set A. Since this function is invariant, then $\chi_A(z) = \chi_A(z + y(x))$ for all $x \in \mathbb{R}^n$, and

$$a_{\bar{k}} = \int_{\mathbb{T}^m} \chi_A(z) e^{-2\pi i \bar{k} \cdot z} dz = \int_{\mathbb{T}^m} \chi_A(z + y(x)) e^{-2\pi i \bar{k} \cdot z} dz =$$

$$\int_{\mathbb{T}^m} \chi_A(y) e^{-2\pi i \bar{k} \cdot (y - y(x))} dy = e^{2\pi i \bar{k} \cdot y(x)} \int_{\mathbb{T}^m} \chi_A(y) e^{-2\pi i \bar{k} \cdot y} dy = e^{2\pi i \bar{k} \cdot y(x)} a_{\bar{k}}. \quad (2.3)$$

Assume that $a_{\bar{k}} \neq 0$. Then it follows from (2.3) that $e^{2\pi i \bar{k} \cdot y(x)} = 1$ for all $x \in \mathbb{R}^n$, which readily implies the relation $\sum_{j=1}^m k_j \lambda_j = 0$, where k_j, $j = 1, \ldots, m$, are the coordinates of $\bar{k}$. Since the vectors λ_j, $j = 1, \ldots, m$, form a basis in M_0, we derive that all $k_j = 0$, that is, $\bar{k} = 0$. Thus, only one Fourier coefficient a_0 may be different from zero. This yields $\chi_A(y) \equiv c = $ const. It is clear that $c = 0$ or $c = 1$, which means that $m(A) = 0$ or $m(A) = 1$, where $m = dy$ is the Lebesgue measure on the torus $\mathbb{T}^m$. Thus, the action $S_x y = y + y(x)$ of the group $\mathbb{R}^n$ on the torus $\mathbb{T}^m$ is ergodic. By the variant of Birkhoff individual ergodic theorem [5, Chapter VIII] for each $w(y) \in L^1(\mathbb{T}^m)$ for almost all $z \in \mathbb{T}^m$ there exists the mean value

$$\fint_{\mathbb{R}^n} w(z + y(x)) dx = \int_{\mathbb{T}^m} w(y) dy. \quad (2.4)$$

Moreover, if $w(y) \in C(\mathbb{T}^m)$, then (2.4) holds for all $z \in \mathbb{T}^m$ (this follows from uniform continuity of $w(z + y(x))$ with respect to z) and $w^z(x) \doteq w(z + y(x))$ are Bohr almost periodic functions for all $z \in \mathbb{T}^m$.

Property (2.4) allows to establish the following important statement.

PROPOSITION 2.2. *Let $p \geq 1$, $v(y) \in L^p(\mathbb{T}^m)$, and $a_{\bar{k}} = \int_{\mathbb{T}^m} v(y) e^{-2\pi i \bar{k} \cdot y} dy$, $\bar{k} \in \mathbb{Z}^m$ be the Fourier coefficients of $v(y)$. Then for a.e. $z \in \mathbb{T}^m$ the function $u(x) \doteq v(z + y(x)) \in \mathcal{B}^p(\mathbb{R}^n)$. Moreover, $Sp(u) \subset M_0$, and the Fourier-Bohr coefficients of the function $u(x)$ are*

$$a_\lambda = a_{\bar{k}} e^{2\pi i \bar{k} \cdot z}, \quad \lambda = \lambda(\bar{k}). \quad (2.5)$$

Proof. There exists a sequence $v_r = \sum_{\bar{k} \in I_r} a_{r\bar{k}} e^{2\pi i \bar{k} \cdot y}$ of trigonometric polynomials, where $I_r \subset \mathbb{Z}^m$ being finite sets, such that $v_r \to v$ in $L^p(\mathbb{T}^m)$ as $r \to \infty$ (for instance we may choose the Fejér approximations). According to (2.4), we may find a set

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$E \subset \mathbb{T}^m$ of full Lebesgue measure such that for each $z \in E$ and all $r \in \mathbb{N}$

$$(N_p(u - u_r))^p = \fint_{R^n} |v(z + y(x)) - v_r(z + y(x))|^p dx =$$

$$\int_{\mathbb{T}^m} |v(y) - v_r(y)|^p dy \underset{r \to \infty}{\to} 0. \qquad (2.6)$$

Here $u_r(x) = v_r(z + y(x)) = \sum_{\bar{k} \in I_r} a_{r\bar{k}} e^{2\pi i \bar{k} \cdot z} e^{2\pi i \lambda(\bar{k}) \cdot x}$, where $\lambda(\bar{k}) = \sum_{j=1}^{m} k_j \lambda_j \in M_0$ for $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$. We see that $u_r(x)$ is a trigonometric polynomial and $Sp(u_r) \subset M_0$ for all $r \in \mathbb{N}$. In view of (2.6) we see that $u(x) \in \mathcal{B}^p(\mathbb{R}^n)$ and $Sp(u) \subset M_0$. Moreover, in view of (2.6) again, the Fourier-Bohr coefficients

$$a_\lambda = \lim_{r \to \infty} a_{r\bar{k}} e^{2\pi i \bar{k} \cdot z} = a_{\bar{k}} e^{2\pi i \bar{k} \cdot z}, \quad \lambda = \lambda(\bar{k}),$$

and (2.5) follows. The proof is complete. $\square$

Now we are ready to prove our Theorems 1.5, 1.6 in the case when $u_0(x)$ has a finite spectrum. Recall that M_0 is the minimal additive subgroup of $\mathbb{R}^n$ containing $Sp(u_0)$.

THEOREM 2.3. *Suppose that $u(t, x)$ be an e.s. of (1.1), (1.2) with initial function $u_0(x)$ being a trigonometric polynomial. Then $u(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n)) \cap L^\infty(\Pi)$ and $Sp(u(t, \cdot)) \subset M_0$ for all $t > 0$. Moreover, if for all $\xi \in M_0$ the functions $u \to \xi \cdot \varphi(u)$ are not affine on non-empty intervals, then the decay property (1.12) holds.*

Proof. As above, we introduce the periodic function $v_0(y)$ on $\mathbb{R}^m$, where m is the rank of M_0, and an e.s. $v(t, y) \in C([0, +\infty), L^1(\mathbb{T}^m)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^m)$ of the Cauchy problem for (2.1) with initial data $v_0(y)$. By Theorem 2.1 there exists a set $E_1 \subset \mathbb{R}^m$ of full Lebesgue measure such that for all $z \in E_1$ the function $u^z(t, x) = v(t, z + y(x))$ is an e.s. of (1.1), (1.2) with initial function $u_0^z(x) = v_0(z + y(x))$. Here the linear map $y(x)$ was defined before the formulation of Theorem 2.1 above. Let

$$v_r(t, y) = \int_{\mathbb{T}^m} v(t, z) \Phi_r(y - z) dz$$

be the Fejér approximations of $v(t, y)$, where

$$\Phi_r(z) = \sum_{\bar{k} \in \mathbb{Z}^m, |\bar{k}|_\infty \leq r} \prod_{j=1}^{m} \left(1 - \frac{|k_j|}{r}\right) e^{2\pi i \bar{k} \cdot z} = r^{-m} \prod_{j=1}^{m} \frac{\sin^2 \frac{r z_j}{2}}{\sin^2 \frac{z_j}{2}}$$

are the Fejér kernels. Then

$$v_r(t, y) = \sum_{\bar{k} \in \mathbb{Z}^m, |\bar{k}|_\infty \leq r} a_{r\bar{k}}(t) e^{2\pi i \bar{k} \cdot y}$$

are trigonometric polynomials with respect to the variables y, and $v_r(t, \cdot) \to v(t, \cdot)$ as $r \to \infty$ in $L^1(\mathbb{T}^m)$ for all $t \geq 0$. In view of Proposition 2.2, for all $t > 0$ there exists a set $E(t) \subset \mathbb{R}^m$ of full Lebesgue measure consisting of $z \in \mathbb{R}^m$ such that relation (2.6) holds with $u = u^z(t, x) = v(t, z + y(x))$, $u_r = u_r^z = v_r(t, z + y(x))$, and $p = 1$:

$$N_1(u^z(t, \cdot) - u_r^z(t, \cdot)) = \fint_{\mathbb{R}^n} |v(t, z + y(x)) - v_r(t, z + y(x))| dx =$$

$$\int_{\mathbb{T}^m} |v(t, y) - v_r(t, y)| dy \underset{r \to \infty}{\to} 0. \qquad (2.7)$$

It is clear that the set G of pairs $(t, z) \in [0, +\infty) \times \mathbb{R}^m$, which satisfy for all $r \in \mathbb{N}$ the equality

$$\fint_{\mathbb{R}^n} |v(t, z + y(x)) - v_r(t, z + y(x))| dx =$$

$$\lim_{N \to \infty} N^{-n} \int_{C_N} |v(t, z + y(x)) - v_r(t, z + y(x))| dx =$$

$$\int_{\mathbb{T}^m} |v(t, y) - v_r(t, y)| dy, \qquad (2.8)$$

is measurable. We use here the fact that, due to the relation

$$\left(\frac{N}{N+1} \right)^n N^{-n} \int_{C_N} w(x) dx \leq R^{-n} \int_{C_R} w(x) dx \leq$$

$$\left(\frac{N+1}{N} \right)^n (N+1)^{-n} \int_{C_{N+1}} w(x) dx, \quad \forall N \in \mathbb{N}, \ N \leq R < N+1,$$

which holds for every nonnegative function $w(x) \in L^1_{loc}(\mathbb{R}^n)$, existence of the mean value $\fint_{\mathbb{R}^n} w(x) dx$ is equivalent to existence of the limit

$$\lim_{N \to \infty} N^{-n} \int_{C_N} w(x) dx = \fint_{\mathbb{R}^n} w(x) dx.$$

Since all sections

$$G_t = \{ \ z \in \mathbb{R}^m \mid (t, z) \in G \ \} = E(t)$$

of the set G has full measure, it has full Lebesgue measure as well, by Fubini's theorem. By Fubini's theorem again, there exists a set $E_2 \subset \mathbb{R}^m$ of full measure such that the sections

$$F(z) = G^z = \{ \ t \geq 0 \mid (t, z) \in G \ \}$$

are sets of full Lebesgue measure in $[0, +\infty)$ for each $z \in E_2$.

Now we choose a sequence $z_l \in E_1 \cap E_2$ converging to zero as $l \to \infty$. By Proposition 1.2 for a.e. $t > 0$

$$I_1(u^{z_l}(t, x) - u(t, x)) \leq I_1(u_0^{z_l}(x) - u_0(x)) \leq \sum_{\bar{k} \in J} |a_{\bar{k}}| |e^{2\pi i \bar{k} \cdot z_l} - 1| \underset{l \to \infty}{\to} 0. \qquad (2.9)$$

Here $J \subset \mathbb{Z}^m$ is a finite set (the spectrum of $v_0(y)$). Notice that the e.s. $u^{z_l}(t, \cdot)$ are not generally continuous in $L^1_{loc}(\mathbb{R}^m)$ with respect to time t. Hence, the relation (2.9) may fails for a set of t of null measure. We define by $F_{1l} \subset [0, +\infty)$ the set of t, which satisfy (2.9). Let $F_{2l} = F(z_l) \subset [0, +\infty)$ be the sets of full measures, consisting of $t \geq 0$ such that equality (2.8) holds with $z = z_l$ for all $r \in \mathbb{N}$. Observe that in view of the relation $v_r(t, \cdot) \underset{r \to \infty}{\to} v(t, y)$ in $L^1(\mathbb{T}^m)$, it follows from (2.8) that $u^{z_l}(t, \cdot) \in \mathcal{B}^1(\mathbb{R}^n)$ for $t \in F_{2l}$. We set $F = \bigcap_{l \in \mathbb{N}} (F_{1l} \cap F_{2l})$. Then $F \subset [0, +\infty)$ is a set of full measure, and for each $t \in F$ $u^{z_l}(t, x) \in \mathcal{B}^1(\mathbb{R}^n)$ for all $l \in \mathbb{N}$, and $I_1(u^{z_l}(t, x) - u(t, x)) \to 0$ as $l \to \infty$. This implies that the limit function $u(t, \cdot)$ belongs to the Besicovitch space

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$\mathcal{B}^1(\mathbb{R}^n)$. Let us demonstrate continuity of the map $t \to u(t, \cdot) \in \mathcal{B}^1(\mathbb{R}^n)$. Suppose that $t, t' \in F$. Obviously, for each $r \in \mathbb{N}$

$$\fint_{R^n} |v_r(t', z + y(x)) - v_r(t, z + y(x))| dx = \int_{\mathbb{T}^m} |v_r(t', y) - v_r(t, y)| dy,$$

This relation in the limit as $r \to \infty$ implies, with the help of (2.8), that for all $l \in \mathbb{N}$

$$N_1(|u^{z_l}(t', \cdot) - u^{z_l}(t, \cdot)|) = \fint_{\mathbb{R}^n} |v(t', z_l + y(x)) - v(t, z_l + y(x))| dx =$$

$$\int_{\mathbb{T}^m} |v(t', y) - v(t, y)| dy. \quad (2.10)$$

Passing to the limit in (2.10) as $z = z_l \to 0$ and taking into account (2.9), we arrive at the relation

$$N_1(|u(t', \cdot) - u(t, \cdot)|) = \int_{\mathbb{T}^m} |v(t', y) - v(t, y)| dy \underset{t' \to t}{\to} 0.$$

This relation show that the map $t \to u(t, \cdot)$ is uniformly continuous in $\mathcal{B}^1(\mathbb{R}^n)$ on a $F \cap [0, T]$ for arbitrary $T > 0$. Therefore, it can be uniquely extended as a continuous map $t \to u(t, \cdot) \in \mathcal{B}^1(\mathbb{R}^n)$ on the whole half-line $[0, +\infty)$. We see that, after possible correction of $u(t, \cdot)$ on a set of null measure, this map is continuous in $\mathcal{B}^1(\mathbb{R}^n)$. Notice also that $u(0, x) = u_0(x)$ (this readily follows from the fact that $0 \in F$). Since $u(t, x)$ is bounded, we conclude that $u(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n)) \cap L^\infty(\Pi)$. The first statement is proved.

To prove the decay property, we notice that for every $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$

$$\bar{k} \cdot \tilde{\varphi}(u) = \sum_{j=1}^{m} \sum_{k=1}^{n} k_j \lambda_{jk} \varphi_k(u) = \lambda(\bar{k}) \cdot \varphi(u),$$

where $\lambda(\bar{k}) = \sum_{j=1}^{m} k_j \lambda_j \in M_0$. It now follows from our assumption (1.11) that the functions $u \to \bar{k} \cdot \tilde{\varphi}(u)$ are not affine on non-empty intervals. This means that non-degeneracy condition (1.13) is satisfied (with $L = L' = \mathbb{Z}^m$). As follows from the result of [16], the periodic e.s. $v(t, y)$ satisfies the decay relation:

$$\lim_{t \to +\infty} \int_{\mathbb{T}^m} |v(t, y) - C| dy = 0, \text{ where} \quad (2.11)$$

$$C = \int_{\mathbb{T}^m} v_0(y) dy = \fint_{\mathbb{R}^n} u_0(x) dx.$$

Now we choose a vanishing sequence $z_l \in E_1 \cap E_2$ and the set $F \subset [0, +\infty)$ of full measure as in the first part of our proof. By (2.4) with $w(y) = |v_r(t, y) - C| \in C(\mathbb{T}^m)$

$$\fint_{\mathbb{R}^n} |u_r^{z_l}(t, x) - C| dx = \fint_{\mathbb{R}^n} |v_r(t, z_l + y(x)) - C| dx = \int_{\mathbb{T}^m} |v_r(t, y) - C| dy.$$

Passing in this relation to the limit as $r \to \infty$ and after as $l \to \infty$ and taking into account relations (2.7), (2.9), we obtain that for all $t \in F$

$$\fint_{\mathbb{R}^n} |u(t, x) - C| dx = \int_{\mathbb{T}^m} |v(t, y) - C| dy.$$

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Since both parts of this equality are continuous with respect to $t \in [0, +\infty)$, it remains valid for all $t > 0$ In view of (2.11), this implies the desired decay property (1.12):

$$\lim_{t \to +\infty} \fint_{\mathbb{R}^n} |u(t, x) - C| dx = 0.$$

The proof is complete. $\square$

3. The general case.   Proof of Theorems 1.5, 1.6. Assume that $u_0 \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is an arbitrary bounded Besicovitch almost periodic function, and $u(t, x) \in L^\infty(\Pi)$ is an e.s. of (1.1), (1.2).

Denote by $M(u_0)$ the minimal additive subgroup of $\mathbb{R}^n$ containing $Sp(u_0)$. Let u_{0l} be the sequence of Bochner-Fejér trigonometric polynomials for u_0. Then $u_{0l} \to u_0$ as $l \to \infty$ in $\mathcal{B}^1(\mathbb{R}^n)$, and $Sp(u_{0l})$ is finite and contained in $M(u_0)$. We denote by $u_l(t, x)$ an e.s. of problem (1.1), (1.2) with initial function $u_{0l}(x)$. By Proposition 1.2, there exists a set $F \subset \mathbb{R}_+$ of full Lebesgue measure such that for all $t \in F$

$$N_1(u_l(t, \cdot) - u(t, \cdot)) \leq N_1(u_{0l} - u_0) \underset{l \to \infty}{\to} 0. \tag{3.1}$$

Since the function $u_{0l}(x)$ has finite spectrum, then by Theorem 2.3 we see that $u_l(t, x) \in C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ and $Sp(u_l(t, \cdot)) \subset M(u_0)$ for all $l \in \mathbb{N}$. In view of uniform limit relation (3.1) we conclude, in the limit as $l \to \infty$, that $u(t, \cdot)$ is uniformly continuous map on $F \cap [0, T]$ into $\mathcal{B}^1(\mathbb{R}^n)$ for every $T > 0$, and $Sp(u(t, \cdot) \subset M(u_0)$ for all $t \in F$. This allows to extend $u(t, x)$ as a function $C([0, +\infty), \mathcal{B}^1(\mathbb{R}^n))$ on the whole half-line $[0, +\infty)$. By continuity, we conclude that $Sp(u(t, \cdot)) \subset M(u_0)$ for $t > 0$. The proof of Theorem 1.5 is complete.

Denote by M_l the minimal additive subgroup of $\mathbb{R}^n$ contained $Sp(u_{0l})$. Then $M_l \subset M(u_0)$ Assume that the linear non-degeneracy condition is satisfied. Then for each $l \in \mathbb{N}$ and for all $\xi \in M_l$, $\xi \neq 0$, the function $u \to \xi \cdot \varphi(u)$ is not affine on non-empty intervals. By Theorem 2.3 again

$$\lim_{t \to +\infty} \fint_{\mathbb{R}^n} |u_l(t, x) - C_l| dx = 0, \tag{3.2}$$

where

$$C_l = \fint_{\mathbb{R}^n} u_{0l}(x) dx \underset{l \to \infty}{\to} C = \fint_{\mathbb{R}^n} u_0(x) dx. \tag{3.3}$$

In view of (3.1), (3.3), it readily follows from (3.2) in the limit as $l \to \infty$ that

$$\lim_{t \to +\infty} \fint_{\mathbb{R}^n} |u(t, x) - C| dx = 0,$$

and the decay property holds.

Conversely, assume that non-degeneracy condition (1.11) fails. Then there exists a nonzero vector $\xi \in M_0$ such that the function $\xi \cdot \varphi(u) = \tau u + c$ on some segment $[a, b]$, where $\tau, c \in \mathbb{R}$. Then, as is easy to verify, the function

$$u(t, x) = \frac{a + b}{2} + \frac{b - a}{2} \sin(2\pi(\xi \cdot x - \tau t))$$

is an e.s. of (1.1), (1.2) with the periodic initial data $u(0, x) = \frac{a+b}{2} + \frac{b-a}{2} \sin(2\pi(\xi \cdot x))$ such that $Sp(u_0) = \{-\xi, \xi\} \subset M_0$. Obviously, the e.s. $u(t, x)$ does not satisfy the decay property.

REMARK 3.1. *The results of this paper can be easily extended to the case of unbounded almost periodic solutions $u(t,x) \in C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$. Indeed, if $u_0(x) \in \mathcal{B}_1(\mathbb{R}^n)$ then there exists a sequence $u_l(x) \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (for instance, the sequence of Bochner-Fejér approximations for u_0) such that $u_l(x) \to u(x)$ as $l \to \infty$ and that $Sp(u_l) \subset M(u_0)$. By Proposition 1.2 for all $t > 0$, $k, l \in \mathbb{N}$*

$$N_1(u_k(t,\cdot) - u_l(t,\cdot)) \le N_1(u_k - u_l) \underset{k,l \to \infty}{\to} 0.$$

This implies that $u_l(t,x)$, $l \in \mathbb{N}$, is a Cauchy sequence in $C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$. Since this space is complete, we claim that $u_l(t,x)$ converges as $l \to \infty$ to a function $u(t,x)$ in $C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$. Obviously, this limit function does not depend on the choice of a sequence $u_l(x) \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. It is natural to call the function $u(t,x) \in C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$ a renormalized solution to problem (1.1), (1.2), cf. the notion of renormalized solution $u(t,x) \in C([0,+\infty), L^1(\mathbb{R}^n))$ defined in [1], see also further results in [10, 11]. It readily follows from the decay property for bounded e.s. u_l in the limit as $u_l \to u$ that any renormalized solution also satisfies the decay property (1.12) under non-degeneracy condition (1.11).

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