TRANSPORT-COLLAPSE SCHEME FOR HETEROGENEOUS SCALAR CONSERVATION LAWS

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Abstract. We extend Brenier's transport collapse scheme on heterogeneous scalar conservation laws with initial and boundary data. It is based on averaging out the solution to the corresponding kinetic equation, and it necessarily converges toward the entropy admissible solution of the considered problem. We also provide numerical examples.

1. Introduction

Let $\Omega \subset \mathbf{R}^d$ be a bounded smooth domain. We consider the following Cauchy problem

$$\partial_t u + \mathrm{div}_{\mathbf{x}} f(t,\mathbf{x},u) = 0, \quad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega, \tag{1}$$

$$u|_{t=0} = u_0(\mathbf{x}), \tag{2}$$

$$u|_{\mathbf{R}^+ \times \partial\Omega} = u_B(t,\mathbf{x}), \tag{3}$$

where the function $f = (f_1, \dots, f_d) \in C^2(\mathbf{R}^{d+1}_+)$, $\mathbf{R}^{d+1}_+ = \mathbf{R}^+ \times \mathbf{R}^{d+1}$. We additionally assume that for some constants $a, b \in \mathbf{R}$, it holds

$$f(t,\mathbf{x},a) = f(t,\mathbf{x},b) = 0 \ \text{ and } \ a \leq u_0, u_B \leq b.$$

Latter conditions provide the maximum principle for the entropy admissible solution to (1), (2), (3) (see e.g. [15]).

A typical problem described by (1), (2), (3) arises e.g. in traffic flow models. Namely, if we aim to describe a flow on a finite highway (required to model on and off ramps) we need to use boundary conditions [18]. For instance, optimization of travel time and cost between two points can be obtained by controlling incoming and outgoing car densities [2].

Never the less, it is clear that the boundary conditions cannot be prescribed unless characteristics corresponding to equation (1) leave the boundary (those are characteristics originating from $x = R$ on Figure 1). This means that one needs to introduce a new concept defining what conditions should satisfy the unknown function u in order to be a solution to (1), (2), (3). This was firstly done in [3] via the following definition which we introduce in the form convenient for us (we also include the notion of entropy solution for (1), (2)).

Definition 1. A bounded function u is called an entropy admissible solution to

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a) (1), (2) if for every convex function $V \in C^2(\mathbf{R})$, every $\lambda \in \mathbf{R}$ and every $\varphi \in C^1_c(\mathbf{R}^+ \times \mathbf{R}^d)$, it holds

$$\iint\limits_{\mathbf{R}^+\times\mathbf{R}^d} \big[V(u)\partial_t\varphi + \int_a^u f'_\lambda(t,\mathbf{x},v)\,V'(v)dv\cdot\nabla\varphi + \int_a^u \mathrm{div}_{\mathbf{x}} f(t,\mathbf{x},v)V''(v)dv\,\varphi\big]d\mathbf{x}dt$$
$$+\int_{\mathbf{R}^d} V(u_0(\mathbf{x}))\varphi(0,\mathbf{x})d\mathbf{x} \le 0; \tag{4}$$

b) (1), (2), (3) if for every convex function $V \in C^2(\mathbf{R})$, every $\lambda \in \mathbf{R}$ and every $\varphi \in C^1_c(\mathbf{R}^+ \times \Omega)$, it holds

$$\iint\limits_{\mathbf{R}^+\times\Omega} \big[V(u)\partial_t\varphi + \int_a^u f'_\lambda(t,\mathbf{x},v)\,V'(v)dv\cdot\nabla\varphi + \int_a^u \mathrm{div}_{\mathbf{x}} f(t,\mathbf{x},v)V''(v)dv\,\varphi\big]d\mathbf{x}dt$$
$$+\int_{\mathbf{R}^+}\int_{\partial\Omega}(\varphi\int_a^{u_B} f'_\lambda(t,\mathbf{x},v)\,V'(v)dv)\cdot\nu dsdt + \int_\Omega V(u_0(\mathbf{x}))\varphi(0,\mathbf{x})d\mathbf{x} \le 0, \tag{5}$$

where ν is the unit normal on $\partial\Omega$.

Equivalent and more usual definition of admissible solution is given by the Kruzhkov entropies $V(u) = |u-\lambda|$, $\lambda \in \mathbf{R}$, and it states that a bounded function u is called an entropy admissible solution to (1), (2), (3) if for every $\lambda \in \mathbf{R}$ it holds

$$\partial_t|u-\lambda| + \mathrm{div}_{\mathbf{x}}[\mathrm{sgn}(u-\lambda)(f(t,\mathbf{x},u) - f(t,\mathbf{x},\lambda))] + \mathrm{sgn}(u-\lambda)\mathrm{div}_{\mathbf{x}} f(t,\mathbf{x},\lambda) \le 0 \tag{6}$$

in the sense of distributions on $\mathcal{D}'(\mathbf{R}^+ \times \Omega)$, and

i) it holds $esslim_{t\to 0}\int_\Omega |u(t,\mathbf{x}) - u_0(\mathbf{x})|d\mathbf{x} = 0$;

ii) for every $\lambda \in \mathbf{R}$, it holds
$$(\mathrm{sgn}(u-\lambda) - \mathrm{sgn}(u_B-\lambda))\,\langle f(t,\mathbf{x},u) - f(t,\mathbf{x},u_B), \vec{\nu}\rangle \ge 0$$
on $\partial\Omega$, where $\vec{\nu}$ is normal on $\partial\Omega$.

The expressions in (i) and (ii) are well defined at least when the flux is genuinely nonlinear since then, the strong traces of $\langle \mathrm{sgn}(u-\lambda)(f(t,\mathbf{x},u) - f(t,\mathbf{x},\lambda)), \vec{\nu}\rangle$ and $f(t,\mathbf{x},u), \vec{\nu}\rangle$ exist at $\partial\Omega$ [1, 17]).

Work in the field of numerical methods for conservation laws is rather intensive. Most of the papers deal with Cauchy problems for conservation laws (scalar conservation laws or systems; see e.g. classical books [9, 13] and references therein). As for (1), (2), (3) , there are much less results since the interest for this kind of problem has arisen relatively recently. We mention [5] where stability and convergence results for monotone (first-order) numerical schemes approximating (homogeneous) scalar conservation laws in several space dimensions were obtained. For results in the case of systems, one can consult [16] where one can also find thorough overview of state of the art for the problem.

The aim of the present paper is to extend the transport-collapse scheme [4] for the initial and initial-boundary value problem for heterogeneous scalar conservation laws. Originally, the transport collapse scheme was introduced as a mean for solving the Cauchy problem (1), (2) in the case when the flux is independent of $(t,\mathbf{x}) \in$

$\mathbf{R}^+ \times \mathbf{R}^d$. Although [14] appeared ten years after [4], the transport-collapse scheme is actually based on the kinetic formulation [14] in the frame of which, using the Kruzhkov entropy conditions [11], one reduces the nonlinear equation (1) on the linear (so called kinetic) equation (see Theorem 2 below). However, derivative of a measure figures in the equation (see the right-hand side of (7)) and it has one more variable (so called kinetic or velocity variable). Due to the former reason, the kinetic equation is not convenient for numerical implementation. Never the less, if we neglect the derivative of the measure, and then average out the solution to the obtained linear equation with respect to the kinetic variable, we can obtain entropy solution to the considered problem. Details are provided in the next sections.

In conclusion, the power of the method to be presented is in its ability to transform nonlinear problem into linear. Linear scalar conservation laws are easy to solve numerically since there are a lot of robust numerical schemes available. The cost of that "transformation" in practical computing is adding one more dimension (see (7)).

The paper is organized as follows. In Section 2, we shall prove convergence of the transport-collapse scheme for initial value problems corresponding to (1). In Section 3, we shall introduce a transport-collapse type operator for (1), (2), (3), and the proof of its convergence toward the entropy solution.

2. Initial value problem

In order to extend the transport-collapse in heterogeneous situation, we need appropriate kinetic formulation. It is given in [7] through the following theorem.

Theorem 2. [7] *The function $u \in C([0,\infty); L^1(\mathbf{R}^d)) \cap L^\infty_{loc}((0,\infty); L^\infty(\mathbf{R}^d))$ is the entropy admissible solution to (1), (2) if and only if there exists a non-negative Radon measure $m(t,\mathbf{x},\lambda)$ such that $m((0,T)\times\mathbf{R}^{d+1}) < \infty$ for all $T>0$ and such that the function $\chi(\lambda,u) = \begin{cases} 1, & 0 \le \lambda \le u \\ -1, & u \le \lambda \le 0 \\ 0, & else \end{cases}$, represents the distributional solution to*

$$\partial_t \chi + \operatorname{div}_{(\mathbf{x},\lambda)}[F(t,\mathbf{x},\lambda)\chi] = \partial_\lambda m(t,\mathbf{x},\lambda), \quad (t,\mathbf{x}) \in \mathbf{R}^+ \times \mathbf{R}^d, \tag{7}$$

$$\chi(\lambda, u(t=0,\mathbf{x})) = \chi(\lambda, u_0(\mathbf{x})), \tag{8}$$

where $F = (f'_\lambda, -\sum_{j=1}^d \partial_{x_j} f_j)$.

Let us state properties of the function χ.

Proposition 3. [4, page 1018] *It holds*

a) $\forall u, v \in L^1(\mathbf{R}^d)$ *such that* $u \ge v \implies \chi(\lambda,u) \ge \chi(\lambda,v)$;

b) $\forall u \in L^1(\mathbf{R}^d)$, $\forall g \in L^\infty(\mathbf{R}^d \times \mathbf{R})$, *it holds* $\iint \chi(\lambda,u) g(\mathbf{x},\lambda) d\mathbf{x} d\lambda = \int \left(\int_a^u g(\mathbf{x},\lambda)d\lambda\right) d\mathbf{x}$;
In particular, if $g = G'_\lambda$ *and* $G(a)=0$, *then* $\iint \chi(\lambda,u) g(\mathbf{x},\lambda) d\mathbf{x} d\lambda = \int G(\mathbf{x},u) d\mathbf{x}$

c) $TV(u) = \int TV(\chi(\lambda,\cdot)) d\lambda$.

The idea of the transport collapse scheme for the initial value problem (1), (2) is to solve problem (7), (8) when we omit the right-hand side in (7):

$$\partial_t h + \operatorname{div}_{\mathbf{x},\lambda}[F(t,\mathbf{x},\lambda)h] = 0, \quad h|_{t=0} = \chi(\lambda, u_0(\mathbf{x})). \tag{9}$$

The solution of this equation is obtained via the method of characteristics. They are given by

$$\begin{cases} \dot{\mathbf{x}} = f'_\lambda, \quad \mathbf{x}|_{t=0} = \mathbf{x}_0, \\ \dot{\lambda} = -\sum\limits_{j=1}^{d} \partial_{x_j} f_j(t,\mathbf{x},\lambda), \quad \lambda|_{t=0} = \lambda_0. \end{cases} \tag{10}$$

For later purpose, we rewrite this system in the integral form

$$\begin{cases} \mathbf{x} &= \mathbf{x}_0 + \int_0^t f'_\lambda(t',\mathbf{x},\lambda)dt' \\ \lambda &= \lambda_0 - \int_0^t \sum\limits_{j=1}^{d} \partial_{x_j} f_j(t',\mathbf{x},\lambda)dt'. \end{cases} \tag{11}$$

The solution to (9) has the form

$$h(t,\mathbf{x},\lambda) = \chi(\lambda_0(t,\mathbf{x},\lambda), u_0(\mathbf{x}_0(t,\mathbf{x},\lambda))). \tag{12}$$

To avoid proliferation of symbols, denote

$$\|\nabla_{\mathbf{x}} f\|_\infty = \sup_{\Delta\mathbf{x}>0} \frac{|f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x})|}{\|\Delta\mathbf{x}\|}. \tag{13}$$

We have the following properties of the characteristics.

Proposition 4. *The characteristics* $\mathbf{x}_0 = \mathbf{x}_0(t,\mathbf{x},\lambda)$ *and* $\lambda_0 = \lambda_0(t,\mathbf{x},\lambda)$ *satisfy the following continuity properties:*

$$\begin{aligned} |\mathcal{R}_{\mathbf{x}}| &:= |\mathbf{x}_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda) - \mathbf{x}_0(t,\mathbf{x},\lambda)| \qquad (14)\\ &\le \|\Delta\mathbf{x}\| \left(1 + \int_0^t \max_\lambda \|\nabla_{\mathbf{x}} f'_\lambda(t',\cdot,\lambda))dt'\|_\infty\right). \\ |\mathcal{R}_{\lambda}| &:= |\lambda_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda) - \lambda_0(t,\mathbf{x},\lambda)| \qquad (15)\\ &\le \|\Delta\mathbf{x}\| \int_0^t \max_\lambda \|\nabla_{\mathbf{x}} \mathrm{div}_{\mathbf{x}} f(t',\cdot,\lambda))dt'\|_\infty, \end{aligned}$$

where the norms are given by (13)*.*

Proof: From (11), we have

$$\mathbf{x} = \mathbf{x}_0(t,\mathbf{x},\lambda) + \int_0^t f'_\lambda(t',\mathbf{x},\lambda)dt'$$

$$\mathbf{x}+\Delta\mathbf{x} = \mathbf{x}_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda) + \int_0^t f'_\lambda(t',\mathbf{x}+\Delta\mathbf{x},\lambda)dt'.$$

By subtracting those equations, we obtain

$$\begin{aligned} &|\mathbf{x}_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda) - \mathbf{x}_0(t,\mathbf{x},\lambda)| \qquad (16)\\ &\le \Delta\mathbf{x} + \int_0^t \max_\lambda \|f'_\lambda(t',\mathbf{x}+\Delta\mathbf{x},\lambda) - f'_\lambda(t',\mathbf{x},\lambda)\|_\infty dt' \\ &\le \Delta\mathbf{x} + \|\Delta\mathbf{x}\| \int_0^t \max_\lambda \|\nabla_{\mathbf{x}} f'_\lambda(t',\cdot,\lambda)\|_\infty dt'. \end{aligned}$$

This proves (14).

Inequality (15) is proved analogously. It holds

$$\lambda = \lambda_0(t, \mathbf{x} + \Delta\mathbf{x}, \lambda) - \int_0^t \sum_{j=1}^d \partial_{x_j} f_j(t', \mathbf{x} + \Delta\mathbf{x}, \lambda)dt',$$

$$\lambda = \lambda_0(t, \mathbf{x}, \lambda) - \int_0^t \sum_{j=1}^d \partial_{x_j} f_j(t', \mathbf{x}, \lambda)dt'$$

and it is enough to subtract the last two equalities, and to follow the procedure from (16). ❑

Let us now define the transport-collapse operator T.

Definition 5. The transport collapse operator $T(t)$ is defined for every $u \in L^1(\mathbf{R}^d)$ by

$$T(t)u(\mathbf{x}) = \int \chi(\lambda_0(t, \mathbf{x}, \lambda), u(\mathbf{x}_0(t, \mathbf{x}, \lambda)))d\lambda. \tag{17}$$

It satisfies the following properties which are the same as the ones from [4, Proposition 1].

Proposition 6. *It holds for every $u, v \in L^1(\mathbf{R}^d)$*

a) *$u \leq v$ a.e. implies $T(t)u \leq T(t)v$ a.e;*
b) *$\int T(t)u(\mathbf{x})d\mathbf{x} = \int u(\mathbf{x})d\mathbf{x}$;*
c) *the operator $T(t)$ is non-expansive*

$$\|T(t)u - T(t)v\|_{L^1(\mathbf{R}^d)} \leq \|u - v\|_{L^1(\mathbf{R}^d)},$$

and, in particular, $\|T(t)u\|_{L^1(\mathbf{R}^d)} \leq \|u\|_{L^1(\mathbf{R}^d)}$;
d) *$TV(T(t)u) \leq (1 + C_1 t)TV(u) + tC_2$, where TV is the total variation and C_1 and C_2 are appropriate constants depending on the C^2-bounds of the flux f;*
e) *$\|T(t)u - u\|_{L^1(\mathbf{R}^d)} \leq C_2 TV(u)t + tC_1$ for the constants C_1 and C_2 from the previous item;*

Proof: Item a) directly follows from the definition of the transport collapse operator $T(t)$.

As for the item b), denote by $Z = (\mathbf{x}, \lambda)$ characteristics from (10). Notice that, since $\operatorname{div}_{(\mathbf{x},\lambda)} F = 0$, it holds

$$\Big|\det \frac{\partial Z(t, \mathbf{x}_0, \lambda_0)}{\partial(\mathbf{x}_0, \lambda_0)}\Big| = 1. \tag{18}$$

Therefore, according to Proposition 3,

$$\int T(t)u(\mathbf{x})d\mathbf{x} = \int_{\mathbf{R}^{d+1}} \chi(\lambda_0(t, \mathbf{x}, \lambda), u(\mathbf{x}_0(t, \mathbf{x}, \lambda))d\mathbf{x}d\lambda$$
$$= \begin{pmatrix} \mathbf{x}_0(t, \mathbf{x}, \lambda) = \mathbf{y} \\ \lambda_0(t, \mathbf{x}, \lambda) = \eta \end{pmatrix} = \int_{\mathbf{R}^{d+1}} \chi(\eta, u(\mathbf{y}))\Big|\det \frac{\partial Z(t, \mathbf{x}_0, \lambda_0)}{\partial(\mathbf{x}_0, \lambda_0)}\Big|d\mathbf{y}d\eta = \int u(\mathbf{y})d\mathbf{y}.$$

Item c) now follows from a) and b) according to the Crandall-Tartar lemma about non-expansive order preserving mappings [6, Proposition 3.1].

Let us now prove item d). We have

$$\int\limits_{\mathbf{R}^d} |T(t)u(\mathbf{x}+\Delta\mathbf{x}) - T(t)(\mathbf{x})|d\mathbf{x}$$

$$= \int\limits_{\mathbf{R}^d} |\int\limits_{\mathbf{R}} \chi(\lambda_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda))) - \chi(\lambda_0(\mathbf{x}_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))d\lambda|d\mathbf{x}$$

$$\leq \int\limits_{\mathbf{R}^{d+1}} |\chi(\lambda_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda))) - \chi(\lambda_0(\mathbf{x}_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))|d\mathbf{x}d\lambda$$

We next write $\mathbf{x}_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda) = \mathbf{x}_0(t,\mathbf{x},\lambda) + \mathcal{R}_{\mathbf{x}}(t,\mathbf{x},\lambda)$ and $\lambda_0(t,\mathbf{x}+\Delta\mathbf{x},\lambda) = \lambda_0(t,\mathbf{x},\lambda) + \mathcal{R}_\lambda(t,\mathbf{x},\lambda)$, where $\mathcal{R}_{\mathbf{x}}$ and $\mathcal{R}_\lambda$ are estimated in (14), and introduce the change of variables $\mathbf{x}_0(t,\mathbf{x},\lambda) = \mathbf{y}$, $\lambda_0(t,\mathbf{x},\lambda) = \eta$ (keep in mind (18)). We obtain

$$\begin{aligned}
&\int_{\mathbf{R}^d} |T(t)u(\mathbf{x}+\Delta\mathbf{x}) - T(t)u(\mathbf{x})|d\mathbf{x} \\
&\leq \int_{\mathbf{R}^{d+1}} |\chi(\eta+\mathcal{R}_\lambda, u(\mathbf{y}+\mathcal{R}_{\mathbf{x}})) - \chi(\eta, u(\mathbf{y}))|d\mathbf{y}d\eta \\
&\leq \int_{\mathbf{R}^{d+1}} |\chi(\eta+\mathcal{R}_\lambda, u(\mathbf{y}+\mathcal{R}_{\mathbf{x}})) - \chi(\eta, u(\mathbf{y}+\mathcal{R}_{\mathbf{x}}))|d\mathbf{y}d\eta \\
&\quad + \int_{\mathbf{R}^{d+1}} |\chi(\eta, u(\mathbf{y}+\mathcal{R}_{\mathbf{x}})) - \chi(\eta, u(\mathbf{y}))|d\mathbf{y}d\eta \\
&\leq \|\mathcal{R}_\lambda\|_\infty TV(\chi) + \|\mathcal{R}_{\mathbf{x}}\|_\infty \int_{\mathbf{R}} TV(\chi(\eta, u(\cdot)))d\eta = 4\|\mathcal{R}_\lambda\|_\infty + \|\mathcal{R}_{\mathbf{x}}\|_\infty TV(u),
\end{aligned}$$

since the characteristics are of C^1-class, $TV(\chi) = 4$, and since (3), item c) holds. Having in mind Proposition 4, we conclude the proof of d). We remark that

$$C_1 = 4\max_{t,\lambda} \|\nabla_{\mathbf{x}} f(t,\cdot,\lambda)\|_\infty, \quad C_2 = \max_{t,\lambda} \|\nabla_{\mathbf{x}} \mathrm{div}_{\mathbf{x}} f(t,\cdot,\lambda)\|_\infty.$$

It remains to prove item e). Using (11), as in to the proof of item d), we have

$$\begin{aligned}
&\|T(t)u - u\|_{L^1(\mathbf{R}^d)} \leq \int_{\mathbf{R}^{d+1}} |\chi(\lambda_0(t,\mathbf{x},\lambda), u(x_0(t,\mathbf{x},\lambda))) - \chi(\lambda, u(\mathbf{x}))|d\mathbf{x}d\lambda \\
&= \int_{\mathbf{R}^{d+1}} |\chi(\lambda+\mathcal{R}_\lambda, u(\mathbf{x}+\mathcal{R}_{\mathbf{x}})) - \chi(\lambda, u(\mathbf{x}))|d\lambda d\mathbf{x} \\
&\leq C_1\, t\, TV(u) + C_2\, t,
\end{aligned}$$

which immediately gives e). ❑

We also need the following result.

Proposition 7. *For any smooth positive test function φ, and Lipschitz function $V : \mathbf{R} \to \mathbf{R}$, we have*

$$\begin{aligned}
\int (V(T(t)u) - V(u))(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \leq \int_0^t \int B_V(t',\mathbf{x},u(\mathbf{x}))\nabla\varphi d\mathbf{x}dt' \qquad (19)\\
+ \int_0^t \int \int_a^u \mathrm{div}_{\mathbf{x}} f(t,\mathbf{x},\lambda)V''(\lambda)d\lambda dt' + o(t), \quad t \to 0
\end{aligned}$$

where $B_V(t,\mathbf{x},u) = \int_a^u f'_\lambda(t,\mathbf{x},\lambda)V'(\lambda)d\lambda$.

Proof: Remark first that for any fixed $(t, \mathbf{x})$, from the definition of the function χ, it follows for any C^1-function G

$$\int G'(\lambda)\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))d\lambda = \sum_{k=0}^{2p}(-1)^k G'(\omega_k) - G(0), \tag{20}$$

where the increasing sequence (ω_k), $k = 0, \dots, 2p$, belongs to the set $\{\lambda \in [a,b] : \lambda_0(t,\mathbf{x},\lambda) = u(\mathbf{x}_0(t,\mathbf{x},\lambda))\}$ (since the entropy solution to (1), (2) takes values in the interval (a,b)). Remark that the set has odd cardinality since the multivalued solution is obtained by continuous transformation from the graph of initial value [4, page 1016]. Moreover, due to the mean value theorem, the following relation holds for any convex function V (see e.g. [8, p. 40]):

$$V(\sum_{k=0}^{2p}(-1)^k\omega_k) \le \sum_{k=0}^{2p}(-1)^k V(\omega_k). \tag{21}$$

From (20) and (21), it follows

$$V(T(t)u(\mathbf{x})) = V(\int \chi(\lambda_0(t,\mathbf{x},\lambda), u(t,\mathbf{x}_0(t,\mathbf{x},\lambda)))d\lambda) = V(\sum_{k=0}^{2p}(-1)^k\omega_k)$$

$$\le \sum_{k=0}^{2p}(-1)^k V(\omega_k) = \int V'(\lambda)\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))d\lambda + V(0).$$

We have from here

$$\int \left(V(T(t)u(\mathbf{x})) - V(u(\mathbf{x}))\right)\varphi(\mathbf{x})d\mathbf{x} \tag{22}$$

$$\le \iint \left(V'(\lambda)\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda))) - V'(\lambda)\chi(\lambda, u(\mathbf{x}))\right)\varphi(\mathbf{x})d\mathbf{x}d\lambda$$

$$= \iint V'(\lambda_0(t,\mathbf{x},\lambda))\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))(\varphi(\mathbf{x}) - \varphi(\mathbf{x}_0(t,\mathbf{x},\lambda)))d\mathbf{x}d\lambda \tag{23}$$

$$+ \iint (V'(\lambda) - V'(\lambda_0(t,\mathbf{x},\lambda)))\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))\varphi(\mathbf{x})d\mathbf{x}d\lambda \tag{24}$$

$$+ \Big(\iint V'(\lambda_0)\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))\,\varphi(\mathbf{x}_0(t,\mathbf{x},\lambda))d\mathbf{x}d\lambda$$

$$- \iint V'(\lambda)\chi(\lambda, u(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}d\lambda\Big). \tag{25}$$

The two terms from (25) cancel according to (18). Let us consider the term from (23). Using the Taylor formula

$$\iint V'(\lambda_0(t,\mathbf{x},\lambda))\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))(\varphi(\mathbf{x}_0(t,\mathbf{x},\lambda)) - \varphi(\mathbf{x}))d\mathbf{x}d\lambda \tag{26}$$

$$= \iint V'(\lambda_0(t,\mathbf{x},\lambda))\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))(\mathbf{x}_0(t,\mathbf{x},\lambda) - \mathbf{x}) \cdot \nabla\varphi(\mathbf{x}_0(t,\mathbf{x},\lambda))d\mathbf{x}d\lambda$$

$$+ \iint V'(\lambda_0(t,\mathbf{x},\lambda))\chi(\lambda_0(t,\mathbf{x},\lambda), u(\mathbf{x}_0(t,\mathbf{x},\lambda)))D^2\varphi(\tilde{\mathbf{x}})\,(\mathbf{x}_0(t,\mathbf{x},\lambda) - \mathbf{x})^2 d\mathbf{x}d\lambda.$$

From (11), we conclude by expanding the function $f'_\lambda(t', \mathbf{x}, \lambda)$ into the Taylor expansion around $\mathbf{x}_0$:

$$\begin{aligned}\mathbf{x}_0(t,\mathbf{x},\lambda)-\mathbf{x}&=\int_0^t f_\lambda'(t',\mathbf{x},\lambda)dt'=\int_0^t f_\lambda'(t',\mathbf{x}_0(t,\mathbf{x},\lambda),\lambda)dt' \\ &+(\mathbf{x}_0(t,\mathbf{x},\lambda)-\mathbf{x})\int_0^t \nabla_x f_\lambda'(t',\tilde{\mathbf{x}},\lambda)dt'=\int_0^t f_\lambda'(t',\mathbf{x}_0(t,\mathbf{x},\lambda),\lambda)dt'+\mathcal{O}(t^2),\end{aligned} \tag{27}$$

since clearly $\mathbf{x}_0(t,\mathbf{x},\lambda)-\mathbf{x}=\int_0^t f_\lambda'(t',\mathbf{x},\lambda)dt'=\mathcal{O}(t)$. Inserting this into (26) and applying the change of variables from (18), we conclude using item b) from (3):

$$\begin{aligned}&\iint V'(\lambda_0)\chi(\lambda,u(\mathbf{x}))(\varphi(\mathbf{x}_0(t,\mathbf{x},\lambda))-\varphi(\mathbf{x}))d\mathbf{x}d\lambda \\ &=\int_0^t B_V(t',\mathbf{x},u(\mathbf{x}))\nabla\varphi d\mathbf{x}dt'+\mathcal{O}(t^2).\end{aligned} \tag{28}$$

To deal with the remaining term from (24), we shall expand the function V' into the Taylor series around λ_0. We have

$$\begin{aligned}&\iint (V'(\lambda)-V'(\lambda_0))\chi(\lambda_0(t,\mathbf{x},\lambda),u(\mathbf{x}_0(t,\mathbf{x},\lambda)))\,\varphi(\mathbf{x})d\mathbf{x}d\lambda \\ &=\iint V''(\lambda_0(t,\mathbf{x},\lambda))(\lambda-\lambda_0((t,\mathbf{x},\lambda)))\chi(\lambda_0(t,\mathbf{x},\lambda),u(\mathbf{x}_0(t,\mathbf{x},\lambda)))\,\varphi(\mathbf{x})d\mathbf{x}d\lambda \\ &+\mathcal{O}(\|\lambda-\lambda_0((t,\mathbf{x},\lambda))\|^2_{L^1(\mathrm{supp}(\varphi)\times(a,b))})\end{aligned} \tag{29}$$

Applying the procedure as in (27), we reach to the estimate

$$\lambda_0(t,\mathbf{x},\lambda)-\lambda=-\int_0^t\sum_{j=1}^d f_j(t',\mathbf{x}_0(t,\mathbf{x},\lambda),\lambda)dt'+\mathcal{O}(t^2). \tag{30}$$

If we notice that $\|\lambda-\lambda_0((t,\mathbf{x},\lambda))\|^2_{L^1(\mathrm{supp}(\varphi)\times(a,b))}=\mathcal{O}(t^2)$, from (29) and (30), we conclude

$$\begin{aligned}&\iint (V'(\lambda)-V'(\lambda_0))\chi(\lambda_0(t,\mathbf{x},\lambda),u(\mathbf{x}_0(t,\mathbf{x},\lambda)))\,\varphi(\mathbf{x}_0(t,\mathbf{x},\lambda))d\mathbf{x}d\lambda \\ &=-\int_0^t\int\int_a^u\sum_{j=1}^d f_j(t',\mathbf{x}_0(t,\mathbf{x},\lambda),\lambda)dt'\,\varphi(\mathbf{x})d\mathbf{x}+\mathcal{O}(t^2)\end{aligned} \tag{31}$$

Combining (22), (28), and (31), we conclude the theorem. ❑

A consequence of Proposition 6 and Proposition 7 is the following theorem:

Theorem 8. *Denote*

$$S_n(t)u=(1-\alpha)T(\frac{t}{n})^k u\,+\,\alpha T(\frac{t}{n})^{k+1}u, \tag{32}$$

where

$$t=\frac{(k+\alpha)}{n},\quad k\in\mathbf{N},\quad \alpha\in[0,1). \tag{33}$$

For each initial value $u_0\in L^1(\mathbf{R}^d)$ such that $a\le u_0\le b$, the unique entropy solution of (1), (2) at time t is given by the formula

$$u(t,\cdot)=L^1-\lim_{n\to\infty}S_n(t)u.$$

Proof: First, fix an arbitrary $t>0$. Consider the sequence of functions $u_n(t,\cdot)=S_n(t)u$. We aim to prove that the sequence $(u_n(t,\cdot))$ is strongly precompact in $L^1(\mathbf{R}^d)$. To this end, we shall use the Kolmogorov criterion stating that a functional sequence bounded in $L^1(\mathbf{R}^d)$ is strongly precompact in $L^1(\mathbf{R}^d)$ if it is uniformly $L^1(\mathbf{R}^d)$ continuous. In other words, we need to prove that

a) $\|u_n(t,\cdot)\|_{L^1(\mathbf{R}^d)}\leq C$ for every $n\in\mathbf{N}$ and some constant C;
b) for any relatively compact $K\subset\subset\mathbf{R}^d$, any $\varepsilon>0$, there exists $\Delta x>0$ such that $\|u_n(t,\mathbf{x}+\Delta\mathbf{x})-u_n(t,\mathbf{x})\|_{L^1(\mathbf{R}^d)}\leq\varepsilon$.

Item a) follows from Proposition 6, item c).

As for the item b), we shall use property d) from Proposition 6. Taking into account definition of the total variation and form of the sequence $(u_n(t,\cdot))$, simple calculations show that (with the notations from Proposition 6)

$$\begin{aligned}\|u_n(t,\cdot+\Delta\mathbf{x})-u_n(t,\cdot)\|_{L^1(\mathbf{R}^d)}&\leq 2\Delta\mathbf{x}\left(\left(1+\frac{tC_1}{n}\right)^n+\frac{C_2t}{n}\sum_{j=1}^n\left(1+\frac{tC_1}{n}\right)^j\right)\\&\leq e^{Ct}\Delta\mathbf{x},\end{aligned}$$

for an appropriate constant C. This clearly implies L^1-equicontinuity of the sequence $(u_n(t,\cdot))$. This means that for every fixed $t>0$, we can choose a strongly converging subsequence (not relabeled) $(u_n(t,\cdot))$ of the sequence $(u_n(t,\cdot))$. By taking a dense countable subset $E\subset\mathbf{R}^+$, we can choose the same converging subsequence $(u_n(t,\cdot))$ for every $t\in E$.

Now, by the continuity property given in item e) from Proposition 6, we conclude that the subsequence $(u_n(t,\cdot))$ strongly converges in $C([0,T];L^1(\mathbf{R}^d)$ for every $T\in\mathbf{R}^+$ toward a function $u\in C([0,T];L^1(\mathbf{R}^d)$.

Now, we need to check that u satisfies the entropy admissibility conditions. First, notice that for every t, as $n\to\infty$, it holds that $\alpha\to0$. Thus, it is enough to notice that the main part of the transport-collapse operator given by $T(\frac{t}{n})^ku\to u$ as $n\to\infty$ along the previously chosen subsequence and to consider

$$\begin{aligned}\int_{\mathbf{R}^d}(V(T(\frac{t}{n})^ku)-V(u))\varphi(\mathbf{x})d\mathbf{x}&=\sum_{j=0}^{k-1}\int_{\mathbf{R}^d}(V(T(\frac{t}{n})^{j+1}u)-V(T(\frac{1}{n})^ju))\varphi(\mathbf{x})d\mathbf{x}\\&\overset{(19)}{\leq}\sum_{j=0}^{k-1}\int_{jt/n}^{(j+1)t/n}\int_{\mathbf{R}^d}B_V(t',\mathbf{x},T(\frac{t}{n})^ju(\mathbf{x}))\nabla\varphi d\mathbf{x}dt'+\mathcal{O}(t/n).\end{aligned}$$

Now, we simply let $n\to\infty$ and keep in mind arbitrariness of t to infer that the function u satisfies the entropy admissibility conditions from Definition 1, a).

Remark also that this implies convergence of the entire sequence given by (32) due to uniqueness of entropy solutions to (1), (2). ❏

3. Boundary value problem

First, notice that the kinetic formulation from Theorem 2 still holds in the interior of $\mathbf{R}^+\times\Omega$. This means that in order to adapt the transport collapse scheme for the problem (1), (2), (3) we can apply the same method as in the previous section. We cannot use the method of characteristics directly since the characteristics entering the boundary determine the value at the boundary. However, since we are re-iterating the procedure after a short period of time (see (32) and (38)), we can adjust the (small part of) initial data so that the method of characteristics is well defined. We provide the details below.

Let us also remark that a kinetic formulation which includes boundary conditions is derived in [10] but we were not able to use it here.

Accordingly, in order to generalize the transport collapse scheme to the mixed problem corresponding to (1), let us consider for a short period of time the kinetic formulation to (1) augmented with the initial and boundary conditions as follows.

$$\partial_t h_\varepsilon + \mathrm{div}_{\mathbf{x},\lambda}[F(t,\mathbf{x},\lambda)h_\varepsilon] = 0, \tag{34}$$

$$h_\varepsilon|_{t=0} = \chi(\lambda, u_0^\varepsilon(\mathbf{x})), \quad h_\varepsilon|_{\mathbf{R}^+\times\partial\Omega} = \chi(\lambda, u_B(t,\mathbf{x})). \tag{35}$$

Above, the approximation u_0^ε satisfy the compatibility conditions with u_B in the following sense. We are keeping fixed u_B and we adapt u_0 so that it coincides with u_B at the edges of $\partial\Omega\times\mathbf{R}^+$. More precisely, denote by $\partial\Omega_\varepsilon$ common part of Ω and the ε-neighborhood of $\partial\Omega$. Assume that the characteristics issuing from $(0,x_0)\in\{t=0\}\times\partial\Omega_\varepsilon$ hits the boundary at (t_0,y_0). Then, we replace the value $u_0(x_0)$ of the initial function u_0 by $u_B(t_0,y_0)$ (see Figure 1).

Notice that from (10), it follows that $t_0\in(0,C\varepsilon)$ for any $x_0\in\Omega_\varepsilon$ where $C=\max_{t,\mathbf{x},\lambda}|F(t,\mathbf{x},\lambda)|$. In the sequel, we shall assume that $C=1$

Under such assumptions, for a short period of time, we can solve (34), (35) using the method of characteristics where the characteristics will emanate not only from $t=0$, but also from the boundaries. Remark that if a characteristic originates from $\{t=0\}$, we simply use the system of characteristics (10). If a characteristic originates from the boundary, we then write:

$$\begin{cases} \dot{t} = 1, \quad t(0)=t_0 \\ \dot{\mathbf{x}} = f'_\lambda, \quad \mathbf{x}|_{t=t_0} = \mathbf{x}_0 \\ \dot{\lambda} = -\sum\limits_{j=1}^{d}\partial_{x_j} f_j(t,\mathbf{x},\lambda), \quad \lambda|_{t=t_0}=\lambda_0 \\ \dot{h_\varepsilon} = 0, \quad h_\varepsilon|_{t=0} = \chi(\lambda, u_B(t_0,\mathbf{x}_0)) \end{cases} \tag{36}$$

where $(t_0,\mathbf{x}_0)$ is the point from the boundary.

Now, the solution has the same form as for the initial value problem. The value of the unknown function h_ε at a point $(t,\mathbf{x},\lambda)$ is obtained by drawing a characteristic through it. By going back along the characteristic, we shall either hit the boundary or the line $t=0$. Thus, the boundary value or the initial value will determine the value of h_ε at $(t,\mathbf{x},\lambda)$.

To be more precise, denote by $W_B\subset\mathbf{R}^+_t\times\Omega_{\mathbf{x}}\times\mathbf{R}_\lambda$ set of all points through which characteristics issuing from the boundary pass, and by $W_I\subset\mathbf{R}^+_t\times\Omega_{\mathbf{x}}\times\mathbf{R}_\lambda$ set of all points through which characteristics issuing from the initial plane $t=0$ pass. We can rewrite the solution in the form

$$h_\varepsilon(t,\mathbf{x},\lambda) = \chi(\lambda_0(t,\mathbf{x},\lambda), u_0^\varepsilon(\mathbf{x}_0(t,\mathbf{x},\lambda))\kappa_{W_I} + u_B(\mathbf{x}_0(t,\mathbf{x},\lambda))\kappa_{W_B}), \tag{37}$$

at least in the set $[0,\varepsilon)\times\partial\Omega_\varepsilon$. Now, we can generalize the transport collapse scheme to the initial boundary problem for heterogeneous scalar conservation laws.

Theorem 9. *Denote*

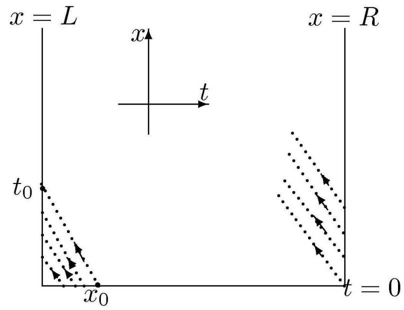

FIGURE 1. The characteristics are denoted by dotted lines. We choose the correction u_0^ε of u_0 so that $u_0^\varepsilon(x_0) = u_B(t_0, L)$.

$$T_n(t)(u_0, u_B)(\mathbf{x}) = \int h_{1/n}(t, \mathbf{x}, \lambda) d\lambda.$$

The entropy admissible solution to the initial boundary value problem (1), (2), (3) *is given by the formula*

$$u(t, \cdot) = L^1 - \lim_{n\to\infty} T_n(\frac{t}{n})^n (u_0^{1/n}, u_B). \tag{38}$$

Remark 10. Notice that after $t = 1/n$ we stop the time and then re-iterate the procedure. This means that the sequence of functions $(u_n) = (T_n(\frac{t}{n})^n(u_0^{1/n}, u_B))$ is well defined.

Proof: The form of the transport-collapse operator (38) is almost the same as from (17). Therefore, the proof that the TV bound of the sequence (u_n) is finite is the same. We provide the details below.

$$\begin{aligned}
&TV(T_n(t)(u_0, u_B)) = TV(\int h_{1/n}(t, \mathbf{x}, \lambda) d\lambda)\\
&\overset{(37)}{\leq} TV(\int [\chi(\lambda_0(t, \mathbf{x}, \lambda), u_0^{1/n}(\mathbf{x}_0(t, \mathbf{x}, \lambda))\kappa_{W_I} + u_B(\mathbf{x}_0(t, \mathbf{x}, \lambda))\kappa_{W_B}) d\lambda]\\
&\overset{Prop.6}{\leq} (1 + C_1 t) \max\{TV(u_0), TV(u_B)\} + tC_2.
\end{aligned}$$

Repeating the procedure from the proof of Theorem 8, we conclude that the sequence $(u_n) = (T_n(\frac{t}{n})^n(u_0, u_B))$ satisfies

$$TV(u_n) \leq e^{C_3 t} \mathbf{\Delta x} \leq C_4 \mathbf{\Delta x}$$

for some constants C_3 and C_4.

Thus, we conclude that the total variation of the sequence (u_n) remains uniformly bounded from where, according to the Kolmogorov criterion, it follows that the sequence (u_n) is strongly precompact in $L^1_{loc}(\mathbf{R}^+ \times \Omega)$ toward a function u. The limit of the sequence satisfies the entropy admissibility conditions from Definition 1 which makes it a unique entropy solution to (1)-(3).

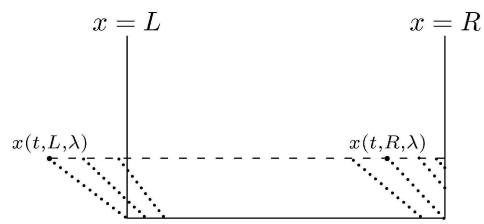

FIGURE 2. The characteristics are denoted by dotted lines. They transform the interval (L,R) into the interval $(x(t,L,\lambda),x(t,R,\lambda))$.

The proof of the latter fact is similar to the proof of Proposition 7 and Theorem 8. From there, we see that it is enough to prove that

$$\int (V(T(t)u)-V(u))(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \leq \int_0^t \int B_V(t',\mathbf{x},u(\mathbf{x}))\nabla\varphi d\mathbf{x}dt' \tag{39}$$
$$+\int_0^t\int\int_a^u \mathrm{div}_{\mathbf{x}} f(t,\mathbf{x},\lambda)V''(\lambda)d\lambda dt' + \int\int_0^t f'_\lambda(t,L,\lambda)dt'V'(\lambda)\chi(\lambda,u_B^l)\,\varphi(L)d\lambda$$
$$+\int\int_0^t f'_\lambda(t,R,\lambda)dt'V'(\lambda)\chi(\lambda,u_B^r)\,\varphi(R)d\lambda + o(t), \quad t\to 0.$$

The only difference between the proof of Proposition 7 and the situation that we have here is in the term (25) (see relation (22) in Proposition 7). Namely, after applying the change of variables (18) the domain of integration is changed for the first term from (25) and therefore, the two terms from (25) will not subtract (see Figure 2). In order to explain technical details more concisely, we shall assume that $\mathbf{x}\in\mathbf{R}$ (i.e. that we are in the one-dimensional situation) and that the boundary function u_B is continuously differentiable. This means that we have the following boundary conditions for some real numbers $L<R$

$$u|_{x=L}=u_B^l(t), \quad u|_{x=R}=u_B^r(t).$$

Remark that the change of variables (18) maps the interval (L,R) into the interval $(\mathbf{x}(t,L,\lambda),\mathbf{x}(t,R,\lambda))$, $(t,\lambda)\in\mathbf{R}^+\times\mathbf{R}$. We have after the change of variables (18) (in the first integral below):

$$\int_L^R\int_{\mathbf{R}} V'(\lambda_0)\chi(\lambda_0(t,\mathbf{x},\lambda),u(\mathbf{x}_0(t,\mathbf{x},\lambda)))\,\varphi(\mathbf{x}_0(t,\mathbf{x},\lambda))d\mathbf{x}d\lambda \tag{40}$$
$$-\int_L^R\int_{\mathbf{R}} V'(\lambda)\chi(\lambda,u(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}d\lambda$$
$$=\int_{\mathbf{R}}\int_{x(t,L,\lambda)}^{x(t,R,\lambda)} V'(\lambda)\chi(\lambda,u(\mathbf{x}))\,\varphi(\mathbf{x})d\mathbf{x}d\lambda - \int_L^R\int_{\mathbf{R}} V'(\lambda)\chi(\lambda,u(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}d\lambda$$
$$=\int\left(\int_{x(t,L,\lambda)}^{L} + \int_R^{x(t,R,\lambda)}\right)\int_{\mathbf{R}} V'(\lambda)\chi(\lambda,u(\mathbf{x}))\,\varphi(\mathbf{x})d\mathbf{x}d\lambda.$$

Since the boundary data are continuously differentiable and compatible with the initial data,

$$
\begin{aligned}
&\left(\int_{x(t,L,\lambda)}^{L}+\int_{R}^{x(t,R,\lambda)}\right)\int_{\mathbf{R}} V'(\lambda)\chi(\lambda,u(\mathbf{x}))\,\varphi(\mathbf{x})d\mathbf{x}d\lambda\\
&=\int (x(t,L,\lambda)-L)V'(\lambda)\chi(\lambda,u_B^l(t))\,\varphi(L)d\lambda\\
&\quad+\int (R-x(t,R,\lambda))V'(\lambda)\chi(\lambda,u_B^r(t))\,\varphi(R)d\lambda+t\,o(1),\quad t\to 0.
\end{aligned}
$$

Finally, taking into account (11):

$$
\begin{aligned}
\mathbf{x}(t,L,\lambda)-L&=\int_0^t f_\lambda'(t',L,\lambda)dt'=\mathcal{O}(t),\\
\mathbf{x}(t,R,\lambda)-R&=\int_0^t f_\lambda'(t',R,\lambda)dt'=\mathcal{O}(t),
\end{aligned}
$$

we conclude from here and (40):

$$
\begin{aligned}
&\int_L^R\int_{\mathbf{R}} V'(\lambda_0)\chi(\lambda_0(t,\mathbf{x},\lambda),u(\mathbf{x}_0(t,\mathbf{x},\lambda)))\,\varphi(\mathbf{x}_0(t,\mathbf{x},\lambda))d\mathbf{x}d\lambda\\
&\qquad-\int_L^R\int_{\mathbf{R}} V'(\lambda)\chi(\lambda,u(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}d\lambda\\
&=\int\int_0^t f_\lambda'(t,L,\lambda)dt'V'(\lambda)\chi(\lambda,u_B^l(t))\,\varphi(L)d\lambda\\
&\quad+\int\int_0^t f_\lambda'(t,R,\lambda)dt'V'(\lambda)\chi(\lambda,u_B^r(t))\,\varphi(R)d\lambda+o(1)+\mathcal{O}(t)
\end{aligned}
$$

Combining this with (22), we conclude that (39) holds. This concludes the theorem. ❑

Corresponding numerical examples are given below. It is one-dimensional scalar conservation law defined on $[0,0.5]\times[-1,1]$ with the flux $f(x,u)=H_\varepsilon(x)(1-u)(u+1)+4H_\varepsilon(-x)(1-u)(u+1)$, where H_ε is a standard regularization of the Heaviside function with $\varepsilon=10^{-4}$. In the first simulation boundary conditions are $u|_{x=-1}=0$, $u|_{x=1}=1$ and the initial condition is $u|_{t=0}=H_\varepsilon(x)$. In the second simulation boundary conditions are $u|_{x=-1}=1$, $u|_{x=1}=0$ and the initial condition is $u|_{t=0}=H_\varepsilon(-x)$.

References

[1] J. Aleksić, D. Mitrović, *Strong traces for averaged solutions of heterogeneous ultra-parabolic transport equations*, J. of Hyperbolic Differential Equations **10** (2014), 659–676.

[2] F. Ancona, A. Marson *On the attainable set for scalar nonlinear conservation laws with boundary control*, SIAM J. Control Optim. **36** (1998), 290172

[3] C. Bardos, A. Y. Le Roux, J.-C. Nédélec, *First order quasi-linear equation with boundary conditions*, Comm. Partial Diff. Equations **9** (1979), 1017–1034.

[4] Y. Brenier, *Averaged multivalued solutions for scalar conservation laws*, SIAM J. on Numerical Analysis **21** (1984), 1013–1037.

[5] B Cockburn, F Coquel, PG LeFloch, *Convergence of the finite volume method for multidimensional conservation laws*, SIAM Journal on Numerical Analysis **32** (1995), 687–705.

[6] M. G. Crandall, A. Majda, *Monotone difference approximations for scalar conservation laws*, Math. Comput. **34** (1981), 1–21.

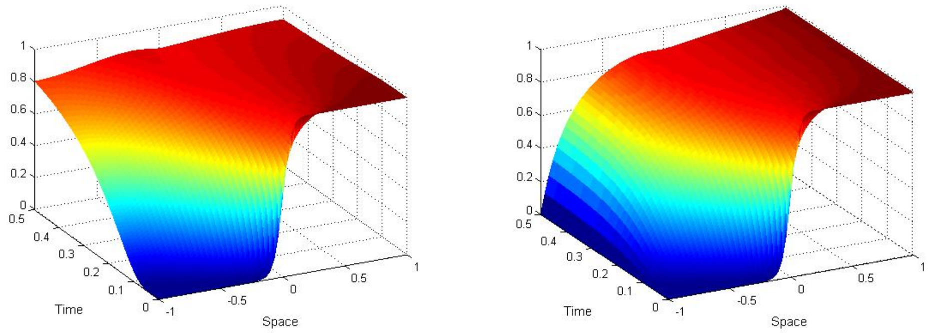

FIGURE 3. Cauchy problem (left) and boundary problem (right) with the initial condition $u_0(x) = H_\varepsilon(x)$.

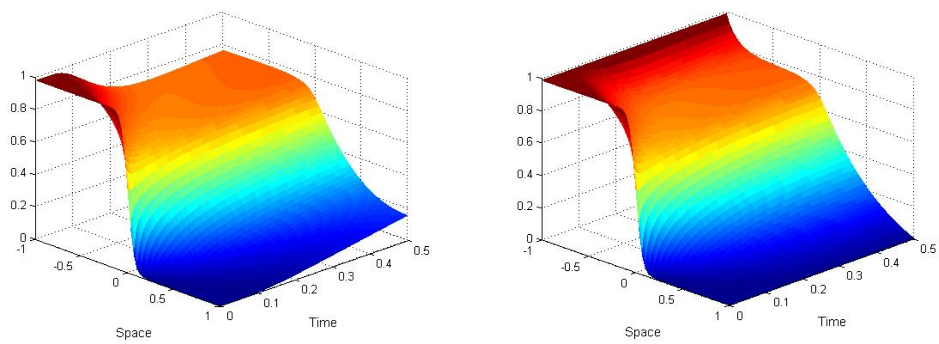

FIGURE 4. Cauchy problem (left) and boundary problem (right) with the initial condition $u_0(x) = H_\varepsilon(-x)$.

[7] A. L. Dalibard, *Kinetic formulation for heterogeneous scalar conservation laws*, **23** (2006), 475-500

[8] J. Dieudonne, *Calcul infinitemsimal*, Hermann, Paris, 1968.

[9] H. Holden, N. H. Risebro, *Front tracking for hyperbolic conservation laws*, Applied Mathematics Sciences 152, Springer, 2011.

[10] C. Imbert, J. Vovelle, *Kinetic formulation for multidimensional scalar conservation laws with boundary conditions and applications*, SIAM J. Math. Anal. **36** (2004), 214-172.

[11] S. N. Kruzhkov, *First order quasilinear equations in several independent variables*, Mat. Sb., **81** (1970), 21717243.

[12] O. A. Ladyenskaja, V. A. Solonnikov, N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type.* (Russian) Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1968 xi+648 pp.

[13] R. LeVeque, *Numerical Methods for Conservation Laws*, Lectures in Mathematics, ETH-Zurich Birkhauser-Verlag, Basel, 1990.

[14] P. L. LIONS, B. PERTHAME, E. TADMOR, *A kinetic formulation of multidimensional scalar conservation law and related equations*, J. Amer. Math. Soc. **7** (1994), 169–191.

[15] S. Martin, *First order quasilinear equations with boundary conditions in the L^∞- framework*, J. of Differential Equations **236** (2007), 375-406.

[16] S. Mishra, M. Svard, , Z. Anfew. Math. Phys.
[17] E. Yu. Panov, *Existence of strong traces for quasi-solutions of multi-dimensional conservation laws*, J. of Hyperbolic Differential Equations **4** (2007), 729–770.
[18] I. S. Strub, A. M. Bayen, *Mixed Initial-Boundary Value Problems for Scalar Conservation Laws: Application to the Modeling of Transportation Networks*, Hybrid Systems: Computation and Control Lecture Notes in Computer Science **3927** (2006), 552–567

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