On one criterion of the uniqueness of generalized solutions for linear transport equations with discontinuous coefficients

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Abstract

We study generalized solutions of multidimensional transport equation with bounded measurable solenoidal field of coefficients $a(x)$. It is shown that any generalized solution satisfies the renormalization property if and only if the operator $a \cdot \nabla u$, $u \in C_0^1(\mathbb{R}^n)$ in the Hilbert space $L^2(\mathbb{R}^n)$ is an essentially skew-adjoint operator, and this is equivalent to the uniqueness of generalized solutions. We also establish existence of a contractive semigroup, which provides generalized solutions, and give a criterion of its uniqueness.

1 Introduction

We study the following evolutionary linear transport equation

$$u_t + \sum_{i=1}^{n} a_i(x) u_{x_i} = 0, \tag{1.1}$$

where $u = u(t,x)$, $(t,x) \in \Pi = (0,+\infty) \times \mathbb{R}^n$.

In the case when the field of coefficients $a = (a_1(x), \dots, a_n(x)) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ the theory of solutions (both classical and generalized) to the Cauchy problem for equation (1.1) is well-known and it is covered by the method of characteristics. The case when the coefficients are generally discontinuous is more interesting and more complicated. The well-posedness of Cauchy problem for such equations is established under some additional restrictions on coefficients. Some results in this direction could be found in papers [8, 2]. The equations like (1.1) with general solenoidal vector of coefficients naturally arise in the study of some important nonlinear conservation laws (see for instance, [3]). The solenoidality condition

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div$a(x) = 0$ (in distributional sense) allows to rewrite the equation in divergence form

$$u_t + \mathrm{div}_x(a(x)u) = 0$$

and introduce generalized solutions (g.s.) of the corresponding Cauchy problem with initial data

$$u(0,x) = u_0(x). \tag{1.2}$$

The coefficients a_i, $i = 1,\dots,n$ are supposed to be bounded: $a(x) \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. We denote $\bar\Pi = [0,+\infty) \times \mathbb{R}^n$.

Definition 1.1. A function $u = u(t,x) \in L^1_{loc}(\bar\Pi)$ is called a g.s. of the problem (1.1), (1.2) if for all $f = f(t,x) \in C_0^\infty(\bar\Pi)$

$$\int_\Pi [uf_t + au\cdot\nabla_x f]dtdx + \int_{\mathbb{R}^n} u_0(x)f(0,x)dx = 0. \tag{1.3}$$

Here and below we use the notation $\cdot$ for the scalar multiplication on $\mathbb{R}^n$.

Taking in (1.3) test functions $f \in C_0^\infty(\Pi)$, we derive that

$$u_t + \mathrm{div}_x(a(x)u) = 0 \tag{1.4}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$). Besides, (1.3) readily implies that

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \ \text{ in } \mathcal{D}'(\mathbb{R}^n). \tag{1.5}$$

Actually, (1.3) is equivalent to (1.4), (1.5). For the details see [10, Proposition 2].

For classical solutions $u(t,x) \in C^1(\bar\Pi)$ of transport equations (1.1), it is clear that compositions $g(u)$ remain to be solutions for every $g(u) \in C^1(\mathbb{R})$. This fact, called the renormalization property, is readily follows from the chain rule. For generalized solutions the renormalization property may fail (cf. [1, 7]). This induce us to introduce the specific notion of *a renormalized solution.*

Definition 1.2. A function $u = u(t,x) \in L^1_{loc}(\bar\Pi)$ is called a renormalized solution of the problem (1.1), (1.2) if for any $g(u) \in C(\mathbb{R})$ such that $g(u_0(x)) \in L^1_{loc}(\mathbb{R}^n)$, $g(u(t,x)) \in L^1_{loc}(\bar\Pi)$ the function $g(u(t,x))$ is a g.s. of problem (1.1), (1.2) with initial data $g(u_0(x))$.

We need the following simple a-priory estimate for nonnegative g.s. (below we denote by $|x|$ the Euclidean norm of a finite-dimensional vector x).

Proposition 1.1. *Let $u = u(t,x) \geq 0$ be a g.s. of the problem* (1.1), (1.2) *. Then for a.e. $t > 0$ for each $R > 0$*

$$\int_{|x|<R} u(t,x)dx \leq \int_{|x|<R+Nt} u_0(x)dx, \tag{1.6}$$

$$\int_{|x|>R+Nt} u(t,x)dx \leq \int_{|x|>R} u_0(x)dx, \tag{1.7}$$

where $N = \|a\|_\infty$.

Proof. Choose a function $\beta(s) \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\beta(s) \subset [0,1]$, $\beta(s) \geq 0$, and $\displaystyle\int \beta(s)ds = 1$ and set for $\nu \in \mathbb{N}$ $\beta_\nu(s) = \nu\beta(\nu s)$, $\theta_\nu(t) = \displaystyle\int_{-\infty}^t \beta_\nu(s)ds$. It is clear that $\beta_\nu(s) \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\beta_\nu(s) \subset [0,1/\nu]$, $\beta_\nu(s) \geq 0$, $\displaystyle\int \beta_\nu(s)ds = 1$. Therefore, the sequence $\beta_\nu(s)$ converges to Dirac δ-function in $\mathcal{D}'(\mathbb{R})$ as $\nu \to \infty$, and the sequence $\theta_\nu(t)$ is bounded ($0 \leq \theta_\nu(t) \leq 1$) and converges pointwise to the Heaviside function $\theta(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$ Let $p = p(s) \in C^\infty(\mathbb{R})$, $p'(s) \geq 0$, $p(s) = 0$ for $s \leq -1$, $p(s) = 1$ for $s \geq 0$. Set for $t_0 > 0$, $r > Nt_0$, $\nu \in \mathbb{N}$ $f = f(t,x) = p(r - Nt - |x|)\theta_\nu(t_0 - t)$. Then $f \in C_0^\infty(\bar{\Pi})$ and by identity (1.3)

$$\theta_\nu(t_0)\int_{\mathbb{R}^n} u_0(x)p(r-|x|)dx - \int_0^{+\infty}\int_{\mathbb{R}^n} u(t,x)p(r-Nt-|x|)dx\delta_\nu(t_0-t)dt$$
$$- \int_\Pi [N+a(x)\cdot x/|x|]p'(r-Nt-|x|)u(t,x)\theta_\nu(t_0-t)dtdx = 0. \quad (1.8)$$

Since $|a(x)\cdot x/|x|| \leq |a(x)| \leq N$ and $p'(s) \geq 0$, the last integral in (1.8) is nonnegative. Therefore, (1.8) implies the inequality

$$\int_0^{+\infty}\int_{\mathbb{R}^n} u(t,x)p(r-Nt-|x|)dx\delta_\nu(t_0-t)dt \leq \theta_\nu(t_0)\int_{\mathbb{R}^n} u_0(x)p(r-|x|)dx. \quad (1.9)$$

Let $\mathcal{E} \subset \mathbb{R}_+$ be the set of full measure consisting of values $t > 0$ such that $u(t,x) \in L^1_{loc}(\mathbb{R}^n)$ and t is a Lebesgue point of functions $F_r(t) = \displaystyle\int_{\mathbb{R}^n} u(t,x)p(r-Nt-|x|)dx$ for all rational r. Since $F_r(t)$ depends continuously on the parameter r then $t \in \mathcal{E}$ is a Lebesgue point of $F_r(t)$ for all real r. Let $t_0 \in \mathcal{E}$. Passing to the limit in (1.9) as $\nu \to \infty$, we obtain that

$$F_r(t_0) \leq F_r(0) = \int_{\mathbb{R}^n} u_0(x)p(r-|x|)dx.$$

Thus $\forall t = t_0 \in \mathcal{E}$, $r > Nt$

$$\int_{\mathbb{R}^n} u(t,x)p(r-Nt-|x|)dx \leq \int_{\mathbb{R}^n} u_0(x)p(r-|x|)dx. \quad (1.10)$$

Obviously, the set $\mathcal{E}$ of full measure could be chosen common for a countable family of functions $p = p_k(s)$, approximating the Heaviside function. Taking $p = p_k$ in (1.10) and passing to the limit as $k \to \infty$, we conclude that $\forall t \in \mathcal{E}$, $r > Nt$

$$\int_{|x|<r-Nt} u(t,x)dx \leq \int_{|x|<r} u_0(x)dx$$

and to complete the proof of (1.6) it only remains to substitute $r = R + Nt$ in the obtained inequality.

Similarly, to establish (1.7) we choose the test function $f = f(t,x) = \chi(t,x)\theta_\nu(t_0 - t) \in C^\infty(\bar{\Pi})$, where $\chi(t,x) = (p(R - Nt - |x|) - p(r + Nt - |x|))$, $R > r > 0$, $R > Nt_0$. By (1.3) we obtain

$$\theta_\nu(t_0)\int_{\mathbb{R}^n} u_0(x)\chi(0,x)dx - \int_0^{+\infty}\int_{\mathbb{R}^n} u(t,x)\chi(t,x)dx\delta_\nu(t_0-t)dt + \int_\Pi u(t,x)[\chi_t + a(x)\cdot\nabla_x\chi]\theta_\nu(t_0-t)dtdx = 0. \quad (1.11)$$

Since $\chi_t = -N((p'(R - Nt - |x|) + p'(r + Nt - |x|)) \le 0$ while

$$|a(x)\cdot\nabla_x\chi| \le |a(x)||\nabla_x\chi| \le N|p'(R - Nt - |x|) - p'(r + Nt - |x|)| \le N(p'(R - Nt - |x|) + p'(r + Nt - |x|)),$$

we see that the last integral in (1.11) is nonpositive and from (1.8) it follows that

$$\int_0^{+\infty}\int_{\mathbb{R}^n} u(t,x)(p(R - Nt - |x|) - p(r + Nt - |x|))dx\delta_\nu(t_0-t))dt \le \theta_\nu(t_0)\int_{\mathbb{R}^n} u_0(x)(p(R - |x|) - p(r - |x|))dx. \quad (1.12)$$

Obviously, the set $\mathcal{E}_1$ of common Lebesgue points of all functions of the kind

$$F(t) = \int_{\mathbb{R}^n} u(t,x)(p(R - Nt - |x|) - p(r + Nt - |x|))dx$$

has full Lebesgue measure. Assuming that $t_0 \in \mathcal{E}_1$ and passing to the limit as $\nu \to \infty$, we arrive at the inequality

$$\int_{\mathbb{R}^n} u(t_0,x)(p(R - Nt_0 - |x|) - p(r + Nt_0 - |x|))dx \le \int_{\mathbb{R}^n} u_0(x)(p(R - |x|) - p(r - |x|))dx.$$

Taking in this estimate $p = p_k(s)$, $k \in \mathbb{N}$ (recall that this sequence converges to the Heaviside function) and passing to the limit as $k \to \infty$, we obtain that for all $t \in \mathcal{E}_1$

$$\int_{r+Nt<|x|<R+Nt} u(t,x)dx \le \int_{r<|x|<R} u_0(x)dx.$$

To complete the proof, we pass to the limit in this inequality as $R \to \infty$ and replace r by R. $\square$

Let us introduce the linear operator $A_0 = \operatorname{div}(au) = a(x)\cdot\nabla u(x)$ in the real Hilbert space $L^2 = L^2(\mathbb{R}^n)$. This operator is defined on a dense subspace $C_0^1(\mathbb{R}^n) \subset L^2$. For every $u, v \in C_0^1(\mathbb{R}^n)$

$$(Au,v)_2 = \int_{\mathbb{R}^n}(a(x)\cdot\nabla u(x))v(x)dx = -\int_{\mathbb{R}^n} u(x)a(x)\cdot\nabla v(x)dx + \int_{\mathbb{R}^n} a(x)\cdot\nabla(u(x)v(x))dx = -\int_{\mathbb{R}^n} u(x)a(x)\cdot\nabla v(x)dx = -(u,Av)_2,$$

where we use the fact that div$a = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. Here we denote by $(f,g)_2$ the scalar multiplication in L^2: $(f,g)_2 = \int_{\mathbb{R}^n} f(x)g(x)dx$.

The obtained identity means that A_0 is skew-symmetric operator. Therefore, it admits the closure, which we define by A. A is a closed skew-symmetric operator: $-A \subset A^*$. It is easy to see that the conjugate operator is defined as follows $v = A^*u$ if and only if $u, v \in L^2$ and $-\mathrm{div}(au) = v$ in $\mathcal{D}'(\mathbb{R}^n)$.

Our main results are the following criteria.

Theorem 1.1. *(i) The necessary and sufficient condition for any g.s. $u(t,x) \in L^2_{loc}(\bar{\Pi})$ to be a renormalized solution of (1.1), (1.2) (with $u_0 \in L^2_{loc}(\mathbb{R}^n)$) is that the operator A is skew-adjoint; (ii) The same condition is necessary and sufficient for the uniqueness of any g.s. $u(t,x) \in L^2_{loc}(\bar{\Pi})$.*

In Theorem 6.1 below we also give a necessary and sufficient condition of uniqueness of contraction semigroups on $L^2(\mathbb{R}^n)$, which provide g.s.

2 The case of smooth coefficients

In the case when the coefficients $a_i(x) \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $i = 1, \ldots, n$, are smooth the existence and uniqueness of g.s. is well known. In this case a g.s. of the problem (1.1), (1.2) can be found by the method of characteristics, see [10, Proposition 3] for details. The characteristics of equation (1.1) are integral curves $(t, x(t))$ of the system of ordinary differential equations

$$\dot{x} = a(x), \tag{2.1}$$

and they are defined for all $t \in \mathbb{R}$ since the right-hand side of (2.1) is bounded. For $(t_0, x_0) \in \Pi$ we denote by $x(t; t_0, x_0)$ the solution of (2.1) such that $x(t_0) = x_0$, we also denote $y(t_0, x_0) = x(0; t_0, x_0)$ (i.e., the source of characteristic $x(t; t_0, x_0)$). Then any g.s. $u(t,x)$ of the problem (1.1), (1.2) should be constant on characteristics (possibly after correction on a set of null Lebesgue measure), which implies that $u(t,x) = u_0(y(t,x))$. We observe that the map $(t,x) \to (t, y(t,x))$ is a diffeomorphism on Π, which implies that $u(t,x)$ is measurable and the correspondence $u_0 \to u$ keeps the relation of equality almost everythere. Besides, in view of the solenoidality assumption for each $t \in \mathbb{R}$ the map $x \to y(t,x)$ conserves the Lebesgue measure. This readily implies that for all $t \in \mathbb{R}$

$$\int_{\mathbb{R}^n} u(t,x)dx = \int u_0(x)dx \tag{2.2}$$

whenever these integrals exist. The above observations allow to obtain the following properties of g.s.

Proposition 2.1. *Assume that $u(t,x) = u_0(y(t,x))$ be the unique g.s. of problem (1.1), (1.2) (defined for all real times t). Then* *(i) For every continuous function $g(u)$ such that $g(u_0) \in L^1_{loc}(\mathbb{R}^n)$ the composition $g(u(t,x))$ is a g.s. of (1.1), (1.2) with initial function $g(u_0(x))$ (renormalization property);*

(ii) If $u_0 \le v_0$ almost everywhere (a.e.) on $\mathbb{R}^n$, and $u = u(t,x)$, $v = v(t,x)$ are g.e.s. of (1.1), (1.2) with initial functions u_0, v_0, respectively, then $u(t,x) \le v(t,x)$ a.e. on $\mathbb{R}^{n+1}$ (monotonicity);

(iii) Let $T_t u = u(y(t,x))$. Then $T_{t+s}u = T_t(T_s u)$ (group property);

(iv) If $u_0(x) \in L^p(\mathbb{R}^n)$, where $1 \le p \le \infty$, then $u(t,\cdot) \in L^p(\mathbb{R}^n)$ for all $t \in \mathbb{R}$ and $\|u(t,\cdot)\|_p = \|u_0\|_p$. Moreover, if $p < \infty$, then

$$\|u(t+h,\cdot) - u(t,\cdot)\|_p \le \omega_p(h) \doteq \inf_{v \in C_0^1(\mathbb{R}^n)} (2\|u_0 - v\|_p + NC(v)|h|) \underset{h\to 0}{\to} 0, \quad (2.3)$$

where the constant $C(v)$, given below in (2.5), depends only on v. In particular, the map $t \to T_t u_0 = u(t,\cdot) \in L^p(\mathbb{R}^n)$ is uniformly continuous on $\mathbb{R}$.

Proof. Properties (i), (ii) readily follows from the representations $u = u_0(y(t,x))$, $v = v_0(y(t,x))$. To prove (iii), notice that $y(t+s,x) = x(0;t+s,x) = x(0;s,x(s;t+s,x)) = x(0;s,x(0;t,x)) = y(s,y(t,x))$, where we used that $x(t+h;t_0+h,x_0) \equiv x(t;t_0,x_0)$ $\forall h \in \mathbb{R}$ because characteristic system (2.1) is autonomous. This readily implies the group property

$$T_{t+s}u(x) = u(y(t+s,x)) = u(y(s,y(t,x))) = (T_s u)(y(t,x)) = T_t(T_s u)(x).$$

If $u_0 \in L^\infty(\mathbb{R}^n)$, the representation $u(t,x) = u_0(y(t,x))$ yields $u(t,\cdot) \in L^\infty(\mathbb{R}^n)$, $\|u(t,\cdot)\|_\infty = \|u_0\|_\infty$. If $p < \infty$, then by assertion (i) with $g(u) = |u|^p$ and identity (2.2) we find

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx = \int_{\mathbb{R}^n} |u_0(x)|^p dx \ \forall t \in \mathbb{R},$$

that is, $u(t,\cdot) \in L^p(\mathbb{R}^n)$, $\|u(t,\cdot)\|_p = \|u_0\|_p$. Finally, let $u_0 \in L^p(\mathbb{R}^n)$, $v = u(t,\cdot) = T_t u_0$. Then by group property (iii) we find

$$\|u(t+h,\cdot) - u(t,\cdot)\|_p = \|T_t(T_h u_0 - u_0)\|_p = \|T_h u_0 - u_0\|_p = \|u_0(y(h,x)) - u_0(x)\|_p.$$

We observe that $y(h,x) - x = x(0) - x(h)$, where $x(t) = x(t;h,x)$, and since $\dot{x}(t) = a(x(t))$, then

$$|y(h,x) - x| = \left|\int_0^h a(x(t))dt\right| \le \left|\int_0^h |a(x(t))|dt\right| \le N|h|,$$

$N = \|a\|_\infty$. If $v(x) \in C_0^1(\mathbb{R}^n)$, then

$$\|T_h v - v\|_p = \|v(y(h,x)) - v(x)\|_p = \left(\int_{A_v \cup A_v^h} |v(y(h,x)) - v(x)|^p dx\right)^{1/p}$$
$$\le \|\nabla v\|_\infty \left(\int_{A_v \cup A_v^h} |y(h,x) - x|^p dx\right)^{1/p} \le$$
$$\|\nabla v\|_\infty (m(A_v) + m(A_v^h))^{1/p} N|h|, \quad (2.4)$$

where A_v, A_v^h are subsets of $\mathbb{R}^n$, determined by the relations $v(x) \neq 0$, $v(y(h,x)) \neq 0$, respectively, and by $m(A)$ we denote the Lebesgue measure of a measurable set A. Since the map $y(h,\cdot)$ keeps the Lebesgue measure, $m(A_v^h) = m(y(h,\cdot)^{-1}(A_v)) = m(A_v)$ and, in view of (2.4),

$$\|T_h v - v\|_p \leq NC(v)|h|,$$

where

$$C(v) = C(v)\|\nabla v\|_\infty (2m(A_v))^{1/p} \tag{2.5}$$

(notice that, in view of assumption $v \in C_0^1(\mathbb{R}^n)$, the set A_v is bounded and, therefore, $m(A_v) < \infty$). Therefore, for all $v \in C_0^1(\mathbb{R}^n)$

$$\|T_h u_0 - u_0\|_p \leq \|T_h u_0 - T_h v\|_p + \|T_h v - v\|_p + \|v - u_0\|_p = \\ \|T_h v - v\|_p + 2\|u_0 - v\|_p \leq 2\|u_0 - v\|_p + NC(v)|h|,$$

and (2.3) follows. Let us show that $\omega_p(h) \to 0$ as $h \to 0$. For arbitrary $\varepsilon > 0$ we can find $v \in C_0^1(\mathbb{R}^n)$ such that $\|u_0 - v\|_p \leq \varepsilon/2$. Then

$$\omega_p(h) \leq 2\|u_0 - v\|_p + NC(v)|h| \leq \varepsilon + NC(v)|h|.$$

Hence,

$$\limsup_{h\to 0} \omega_p(h) \leq \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, we derive that $\lim\limits_{h\to 0} \omega_p(h) = 0$. This completes the proof. □

As follows from assertions (iii), (iv) of Proposition 2.1, the linear operators $T_t u_0 = u(t,\cdot) = u_0(y(t,x))$ generate the C_0-group of linear isomorphisms on $L^p(\mathbb{R}^n)$. In the particular case $p = 2$ the operators T_t, $t \in \mathbb{R}$ is a group of unitary operators in the Hilbert space $L^2(\mathbb{R}^n)$. Let $Bu = \lim\limits_{t\to 0} \frac{T(t)u - u}{t}$ be the infinitesimal generator of this group. This operator is defined in the domain $D(B)$ consisting on such $u \in L^2(\mathbb{R}^n)$ that $\lim\limits_{t\to 0} \frac{T(t)u - u}{t}$ exists in L^2. It is known that $D(B)$ is a dense subspace and B is a closed, possibly unbounded, operator. Since $T(t)$ is an unitary group, then by Stone's theorem B is a skew-adjoint operator. If $u(t,x) = T_t u(x)$, then $u_t = -\mathrm{div} au$ in $\mathcal{D}'(\mathbb{R}^{n+1})$. Hence, it is natural to expect that $B = -A$, where the operator A was defined above, in the end of Introduction.

Theorem 2.1. *The operator B coincides with $-A$. In particular, the operator $A = -B$ is skew-adjoint.*

Proof. First, we remark that $-A_0 \subset B$. Indeed, if $u(x) \in C_0^1(\mathbb{R}^n) = D(A_0)$, then $u(t,x) = T_t u(x) \in C^1(\mathbb{R}^{n+1})$ is a classic solution of (1.1). Therefore,

$$\lim_{t\to 0} \frac{T_t u(x) - u(x)}{t} = u_t(0,x) = -a(x)\cdot\nabla u(x) = -A_0 u(x).$$

Obviously, this limit is uniform with respect to $x \in \mathbb{R}^n$, which implies that

$$\lim_{t\to 0} \frac{T_t u - u}{t} = -A_0 u \ \text{ in } L^2.$$

Hence, $u \in D(B)$ and $Bu = -A_0 u$. Since B is closed, then also $-A \subset B$ (recall that A is the closure of operator A_0). In particular, $B = -B^* \subset A^*$. We will show that actually $B = A^*$. Let $u \in D(A^*)$. Then $f = u + A^* u \in L^2$. Since B is skew-adjoint, the operator $E + B$ is invertible and $(E+B)^{-1}$ is a bounded operator on L^2. Let $v = (E+B)^{-1} f \in D(B)$. Then $v + Bv = v + A^* v = f = u + A^* u$, and the function $w = u - v$ satisfies the relation $w - \mathrm{div} aw = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. As follows from DiPerna-Lions renormalization lemma [8, Lemma II.1], $2w^2 = 2w\mathrm{div} aw = \mathrm{div} aw^2$ in $\mathcal{D}'(\mathbb{R}^n)$. Applying this relation to the test function $\rho(\varepsilon x)$, where $\rho(y) \in C_0^1(\mathbb{R}^n)$, $\rho(y) \ge 0$, $\rho(0) = 1$, and $\varepsilon > 0$, we arrive at the equality

$$2\int_{\mathbb{R}^n} w^2 \rho(\varepsilon x) dx = -\varepsilon \int_{\mathbb{R}^n} w^2 a(x) \cdot \nabla_y \rho(\varepsilon x) dx.$$

Passing in this equality to the limit as $\varepsilon \to 0$, we deduce that $\|w\|_2 = 0$. Hence, $u = v \in D(B)$. We have proven that $D(A^*) = D(B)$. This means that $B = A^*$. This, in turn, implies $B = -B^* = -A^{**} = -A$. The proof is complete. $\square$

3 Main result: the necessity

Now we consider the case of general solenoidal field of coefficients $a = a(x) \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Let

$$\gamma_\nu(\xi) = \nu^n \prod_{i=1}^n \beta(\nu \xi_i)$$

be a sequence of averaging kernels (approximate unity), where $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{N}$, and the function $\beta(s)$ was defined above in the proof of Proposition 1.1. Introduce sequences of averaged coefficients, setting for $x \in \mathbb{R}^n$

$$a_\nu(x) = (a_{1\nu}(x), \dots, a_{n\nu}(x)) = a * \gamma_\nu(x) = \int_{\mathbb{R}^n} a(x - \xi)\gamma_\nu(\xi) d\xi.$$

By the known property of averaging functions, $a_\nu \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $\|a_\nu\|_\infty \le \|a\|_\infty \doteq N$, and $\mathrm{div} a_\nu(x) = (\mathrm{div} a) * \gamma_\nu(x) = 0$. As was demonstrated in the previous section, there exists a unique g.s. $u = u_\nu(t, x)$ of the Cauchy problem for the regularized equation

$$u_t + a_\nu(x) \cdot \nabla_x u = u_t + \mathrm{div}(a_\nu u) = 0 \qquad (3.1)$$

with initial condition (1.2), which may be considered for all time $t \in \mathbb{R}$. By the renormalization property (i) for any $r \ge 0$ the function $(|u_\nu(t, x)| - r)^+ =$

$\max(|u_\nu(t,x)|-r,0)$ is a g.s. of (3.1), (1.2) with initial function $(|u_0(x)|-r)^+$. By Proposition 1.1 we have the estimate:

$$\int_{|x|<R}(|u_\nu(t,x)|-r)^+dx \le \int_{|x|<R+Nt}(|u_0(x)|-r)^+dx \underset{r\to+\infty}{\to} 0.$$

By Danford-Pettis criterion, this estimate implies weak compactness of the sequence $u_\nu(t,x)$ in $L^1_{loc}(\bar\Pi)$. Therefore, there exists a subsequence $u_k = u_{\nu_k}(t,x)$, $k\in\mathbb{N}$, with $\nu_k\to\infty$ as $k\to\infty$ such that $u_k \underset{k\to\infty}{\rightharpoonup} u = u(t,x)$ weakly in $L^1_{loc}(\bar\Pi)$. Since the sequence $a_k(x) \doteq a_{\nu_k}(x) \to a(x)$ as $k\to\infty$ strongly in $L^1_{loc}(\mathbb{R}^n,\mathbb{R}^n)$ and this sequence is uniformly bounded, then $u_k(t,x)a_k(x) \underset{k\to\infty}{\rightharpoonup} u(t,x)a(x)$ weakly in $L^1_{loc}(\bar\Pi,\mathbb{R}^n)$. This allows to pass to the limit as $k\to\infty$ in relation (1.3) corresponding to problem (3.1), (1.2):

$$\int_\Pi [u_k f_t + u_k a_k\cdot\nabla_x f]dtdx + \int_{\mathbb{R}^n} u_0(x)f(0,x)dx = 0 \quad \forall f=f(t,x)\in C_0^1(\bar\Pi)$$

and obtain that

$$\int_\Pi [u f_t + u a\cdot\nabla_x f]dtdx + \int_{\mathbb{R}^n} u_0(x)f(0,x)dx = 0 \quad \forall f=f(t,x)\in C_0^1(\bar\Pi).$$

By Definition 1.1, this means that u is a g.s. of original problem (1.1), (1.2). We established the existence of a g.s. to (1.1), (1.2) for arbitrary initial function $u_0\in L^1_{loc}(\mathbb{R}^n)$ (in the case $u_0\in L^\infty(\mathbb{R}^n)$ this follows from [10, Theorem 1]). Concerning the uniqueness, generally it fails, see examples in [4, 7, 10]. It is clear, that the uniqueness follows from the renormalization property. Indeed, let $u(t,x)\in L^1_{loc}(\bar\Pi)$ be a g.s. of (1.1), (1.2) with zero initial data. Then $|u(t,x)|$ be a nonnegative g.s. of the same problem. By Proposition 1.1 we see that for a.e. $t>0$

$$\int_{|x|<R}|u(t,x)|dx \le \int_{|x|<R+Nt}|u_0(x)|dx = 0 \quad \forall R>0,$$

which implies that $u=0$ a.e. on Π. By the linearity the uniqueness follows.

Suppose that the following requirement is fulfilled.

(R) Any g.s. $u(t,x)$ of (1.1), (1.2) such that $u_0, u(t,\cdot)\in L^2$, $\|u(t,\cdot)\|_2\le \text{const}$, satisfies the renormalization property.

As we will demonstrate below in this case g.s. of (1.1), (1.2) form the C_0-semigroup $T_t=e^{-At}$ governed by a skew-adjoint generator $A=-A^*$. First, we prove that trajectories T_tu_0 of such semigroups are necessary g.s. of (1.1), (1.2). More precisely, the following criterion holds.

Lemma 3.1. *Let B be an infinitesimal generator of C_0-semigroup T_t in L^2. Then the function $u(t,x)=T_tu_0(x)$ is a g.s. of problem (1.1), (1.2) for every $u_0\in L^2$ if and only if $B\subset A^*$.*

Proof. First, we assume that $B \subset A^*$ and $u_0 \in D(B)$. Then $u(t,\cdot) = T_t u_0(x)$ is a C^1-function with values in L^2: $\dot{u} = BT_t u_0 = Bu(t,\cdot)$. This implies that for arbitrary $g = g(x) \in C_0^1(\mathbb{R}^n)$

$$\frac{d}{dt}(u(t,\cdot),g)_2 = (Bu(t,\cdot),g)_2 = (A^*u(t,\cdot),g)_2 = (u(t,\cdot),Ag)_2,$$

where $Ag = \mathrm{div} ag = a \cdot \nabla g$, that is,

$$\frac{d}{dt}\int_{R^n} u(t,x)g(x)dx - \int_{\mathbb{R}^n} u(t,x)a(x)\cdot\nabla_x g(x)dx = 0.$$

Multiplying this relation by a function $h(t) \in C_0^1([0,+\infty))$ and integrating over t, we obtain with the help of integration by part formula that

$$\int_{\mathbb{R}^n} u_0(x)f(0,x)dx + \int_\Pi u[f_t + a\cdot\nabla_x f]dtdx = 0, \tag{3.2}$$

where $f = g(x)h(t)$. Since the linear span of such functions f is dense in $C_0^1(\bar{\Pi})$, we see that (3.2) holds for every $f = f(t,x) \in C_0^1(\bar{\Pi})$. Hence, $u(t,x)$ is a g.s. of (1.1), (1.2). If $u_0(x) \in L^2$ is an arbitrary function, then we can find a sequence $u_{0k} \in D(B)$ converging to u_0 as $k \to \infty$ in L^2 (notice that by the Hille-Yosida theorem $D(B)$ is dense in L^2). Then $u_k(t,x) = T_t u_{0k}(x)$ are g.s. of (1.1), (1.2) with initial data u_{0k}, $k \in \mathbb{N}$, and

$$\|u_k(t,\cdot) - u(t,\cdot)\|_2 \le \|T_t\|\|u_{0k} - u_0\|_2 \underset{k\to\infty}{\longrightarrow} 0$$

uniformly in t on any segment $[0,T]$. In particular $u_k \to u$ as $k \to \infty$ in $L^1_{loc}(\bar{\Pi})$. Passing to the limit as $k \to \infty$ in the relation

$$\int_{\mathbb{R}^n} u_{0k}(x)f(0,x)dx + \int_\Pi u_k[f_t + a\cdot\nabla_x f]dtdx = 0, \quad f = f(t,x) \in C_0^1(\bar{\Pi}),$$

we arrive at the identity (3.2). Therefore, $u(t,x)$ is a g.s. of (1.1), (1.2), as was to be proved.

Conversely, assume that all the functions $u(t,x) = T_t u_0$, $u_0 \in L^2$, are g.s. of (1.1), (1.2). If $u_0 \in D(B)$, then $u(t,\cdot) = T_t u_0 \in C^1([0,+\infty), L^2)$, and $u'(0) = Bu_0$. This implies that for each function $g(x) \in C_0^1(\mathbb{R}^n)$ the scalar function

$$I(t) = \int_{\mathbb{R}^n} u(t,x)g(x)dx = (u(t,\cdot),g)_2 \in C^1([0,+\infty)),\ I'(0) = (g,Bu_0)_2. \tag{3.3}$$

On the other hand for all $h(t) \in C_0^1([0,+\infty))$

$$\int_0^{+\infty} I(t)h'(t)dt = \int_\Pi u(t,x)g(x)h'(t)dtdx =$$
$$-h(0)\int_{\mathbb{R}^n} u_0(x)g(x)dx - \int_\Pi u(t,x)a(x)\cdot\nabla g(x)h(t)dxdt,$$

by virtue of (1.3) with $f = h(t)g(x)$. Taking in this relation $h(t) = \theta_\nu(t_0 - t)$ and passing to the limit as $\nu \to \infty$ we obtain the equality

$$I(t_0) - I(0) = \int_0^{t_0} \int_{\mathbb{R}^n} u(t,x)a(x) \cdot \nabla g(x) dx dt = \int_0^{t_0} (Ag, u(t,\cdot))_2 dt,$$

which implies the relation $I'(0) = (Ag, u_0)_2$. In view of (3.3) we find $(Ag, u_0)_2 = (g, Bu_0)_2$ for all $g \in C_0^1(\mathbb{R}^n)$. Therefore, $u_0 \in D(A^*)$ and $A^* u_0 = Bu_0$. Hence $B \subset A^*$. The proof is complete. □

Now we are ready to prove the following statement analogous to Theorem 2.1 (that is, the necessity statement in Theorem 1.1).

Theorem 3.1. *Suppose that assumption (R) is satisfied. Then the operator A (recall that it is the closure of operator* $\operatorname{div} au$, $u \in C_0^1(\mathbb{R}^n)$ *) is skew-adjoint.*

Proof. Let A_ν be the closure of operator $\operatorname{div}(a_\nu u)$, where $a_\nu(x) = a * \gamma_\nu(x)$, $\nu \in \mathbb{N}$, is the above defined sequence of averaged coefficients. If $u_0(x) \in L^2(\mathbb{R}^n)$ and $u_\nu = u_\nu(t,x)$ is a unique g.s. of the approximate problem (3.1), (1.2), then $(u_\nu)^2$ is a g.s. of (3.1), (1.2) with initial data $(u_0)^2 \in L^1(\mathbb{R}^n)$ in view of Proposition 2.1(i). We know that there exists a subsequence (not relabeled) such that $u_\nu \rightharpoonup u$, $(u_\nu)^2 \rightharpoonup v$ as $\nu \to \infty$ weakly in $L^1_{loc}(\bar{\Pi})$, where u, v are g.s. of original problem (1.1), (1.2) with initial data u_0, $(u_0)^2$, respectively. Observe that since a g.s. of problem (1.1), (1.2) is unique, then the above limit relations remain valid for the original sequences, without extraction of subsequences. By the renormalization property we have $v = u^2$, which implies the strong convergence $u_\nu \underset{\nu\to\infty}{\to} u$ in $L^2_{loc}(\bar{\Pi})$. Indeed, in view of Proposition 2.1(iv) $\int_0^T |u_\nu(t,x)|^2 dt dx = T\|u_0\|_2$, therefore the sequence u_ν is bounded in $L^2_{loc}(\bar{\Pi})$. This readily implies that this sequence converges to u weakly in $L^2_{loc}(\bar{\Pi})$. Hence, for each nonnegative $\rho(t,x) \in C_0(\bar{\Pi})$

$$\int_\Pi (u_\nu - u)^2 \rho dt dx = \int_\Pi ((u_\nu)^2 - u^2)\rho dt dx - 2\int_\Pi (u_\nu - u) u \rho dt dx \underset{\nu\to\infty}{\to} 0.$$

Thus, $u_\nu \underset{\nu\to\infty}{\to} u$ in $L^2_{loc}(\bar{\Pi})$. Extracting a subsequence (not relabeled) we can assume that for almost all $t > 0$ $u_\nu(t,\cdot) \to u(t,\cdot)$ as $\nu \to \infty$ in $L^2_{loc}(\mathbb{R}^n)$. By estimate (1.7) we can find sufficiently large $R > NT$ such that for a.e. $t \in (0,T)$

$$\int_{|x|>R} (u_\nu(t,x))^2 dx \le \int_{|x|>R-Nt} (u_0(x))^2 dx < \varepsilon/4,$$
$$\int_{|x|>R} (u(t,x))^2 dx \le \int_{|x|>R-Nt} (u_0(x))^2 dx < \varepsilon/4,$$

where ε is an arbitrary positive number. This implies that

$$\int_{\mathbb{R}^n} (u_\nu(t,x) - u(t,x))^2 dx \le \int_{|x|<R} (u_\nu(t,x) - u(t,x))^2 dx +$$
$$\int_{|x|>R} (u_\nu(t,x) - u(t,x))^2 dx \le \int_{|x|<R} (u_\nu(t,x) - u(t,x))^2 dx +$$
$$2\int_{|x|>R} (u_\nu(t,x))^2 dx + 2\int_{|x|>R} (u(t,x))^2 dx \le$$
$$\int_{|x|<R} (u_\nu(t,x) - u(t,x))^2 dx + \varepsilon.$$

Since $u_\nu(t,\cdot) \underset{\nu\to\infty}{\to} u(t,\cdot)$ in $L^2_{loc}(\mathbb{R}^n)$, we obtain the relation

$$\limsup_{\nu\to\infty} \int_{\mathbb{R}^n} (u_\nu(t,x) - u(t,x))^2 dx \le \varepsilon$$

for all $\varepsilon > 0$. Therefore,

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^n} (u_\nu(t,x) - u(t,x))^2 dx = 0,$$

that is, $u_\nu(t,\cdot) \to u(t,\cdot)$ as $\nu \to \infty$ in L^2 for a.e. $t > 0$.

Let us show that actually this convergence is uniform with respect to t on any segment $[0,T]$. For that we use estimate (2.3) with $p = 2$. By this estimate for all $\nu \in \mathbb{N}$

$$\|u_\nu(t+h,\cdot) - u_\nu(t,\cdot)\|_2 \le \omega_2(h) = \inf_{v\in C_0^1(\mathbb{R}^n)} (2\|u_0 - v\|_2 + NC(v)|h|)\,. \quad (3.4)$$

Since the above estimate is uniform in ν and $u_\nu(t,\cdot) \underset{\nu\to\infty}{\to} u(t,\cdot)$ in L^2 for a.e. $t > 0$, we conclude that this convergence holds for all $t > 0$ and it is unform on any segment $[0,T]$. From (3.4) it follows in the limit as $\nu \to \infty$ that

$$\|u(t+h,\cdot) - u(t,\cdot)\|_2 \le \omega_2(h) \quad \forall t, t+h \ge 0.$$

Thus, the operators $T_t u_0 = u(t,\cdot)$ form a C_0-semigroup of linear operators on L^2, and the sequence of the unitary groups $T_t^\nu u_0 = u_\nu(t,\cdot)$ converges to T_t uniformly on any segment $[0,T]$. It is clear that $\|T_t u_0\|_2 = \lim\limits_{\nu\to\infty} \|T_t^\nu u_0\|_2 = \|u_0\|_2$. Observe that by the same reasons as above we can establish that for each $\tau > 0$ the sequence $\tilde{u}_\nu(t,\cdot) = T_{t-\tau}^\nu u_0$ converges uniformly on $[0,\tau]$ to a g.s. $\tilde{u}(t,x)$ of problem (1.1), (1.2) with some initial function $\tilde{u}_0(x)$. By the construction $T_\tau \tilde{u}_0 = \tilde{u}(\tau,\cdot) = u_0$. We see that the operator T_τ is invertible, $\tilde{u}_0 = (T_\tau)^{-1} u_0$. Hence T_t are unitary operators and they form the unitary group $T(t)$ (for negative t we set $T(t) = (T(-t))^{-1} = (T(-t))^*$). By Stone' theorem the infinitesimal generator B of this group is a skew-adjoint operator on L^2. By the Trotter–Kato theorem, the convergence $T_t^\nu \to T_t$ of semigroups, which we have established above, implies the convergence of the resolvents $(E+A_\nu)^{-1}u \to (E-B)^{-1}u$ in L^2 as $\nu \to \infty$. Recall that A_ν is the closure of operator $\operatorname{div}(a_\nu u)$, $u \in C_0^1(\mathbb{R}^n)$. By Theorem 2.1 this operator is skew-adjoint and $-A_\nu$ is the generator of semigroup (group) T_t^ν. Denote $v_\nu = (E+A_\nu)^{-1}u$, $v = (E-B)^{-1}u$. Then $v_\nu \to v$ as $\nu \to \infty$ in L^2 and $v_\nu + A_\nu v_\nu = v - Bv = u$. Therefore, $A_\nu v_\nu \to -Bv$ as $\nu \to \infty$ in L^2. Since $A_\nu = -(A_\nu)^*$, we claim that in $\mathcal{D}'(\mathbb{R}^n)$ $A_\nu v_\nu = \operatorname{div}(a_\nu(x)v_\nu(x)) \underset{\nu\to\infty}{\to} -Bv$. Passing to the limit as $\nu \to \infty$, we obtain $Bv = -\operatorname{div}(av)$, that is, $v \in D(A^*)$, $Bv = A^*v$. Hence, $B \subset A^*$ and $A = A^{**} \subset B^* = -B$, so that B is a skew-adjoint extension of the skew-symmetric operator $-A$. If $B \neq -A$ then this extension cannot be unique (because the deficiency indices of the symmetric operator $-iA$ are identical and nonzero). If $\tilde{B}$ is another skew-adjoint extension of $-A$ then $\tilde{B} = -\tilde{B}^* \subset A^*$. The operator $\tilde{B}$ generates the unitary group $\tilde{T}_t = e^{\tilde{B}t}$ different of T_t (since $\tilde{B} \neq B$). Therefore, we can find $u_0 \in L^2$ such that $\tilde{u}(t,x) = \tilde{T}(t)u_0(x) \not\equiv u(t,x) = T_t u_0(x)$. However, in view of Lemma 3.1 both functions $\tilde{u}(t,x)$, $u(t,x)$ are g.s. of the same Cauchy problem (1.1), (1.2). By the uniqueness we see that $\tilde{u} \equiv u$. The obtained contradiction shows that $B = -A$. Hence, the operator $A = -B$ is skew-adjoint, as was to be proved. □

4 The group solutions

We are going to establish the inverse statement to Theorem 3.1 claiming that if the operator A is skew-adjoint, then any g.s. $u(t,x) \in L^2_{loc}(\bar{\Pi})$ of problem (1.1), (1.2) satisfies the renormalization property.

Observe that in this case $T_t = e^{-At}$ is an unitary C_0-group on L^2 governed by the skew-adjoint operator $-A$. We call a function $u(t,x) = T_t u_0(x)$ *a group solution* of problem (1.1), (1.2). By Lemma 3.1 the group solution is a g.s. of (1.1), (1.2). First, we establish that the approximate sequence $u_\nu = T_t^\nu u_0(x)$ converges strongly as $\nu \to \infty$ to the group solution.

Proposition 4.1. *Let $T_t^\nu = e^{-A_\nu t}$ be the group with generator $-A_\nu$ (being the closure of operator $-\operatorname{div}(a_\nu u)$), so that $T_t^\nu u_0 = u_\nu(t,x)$ is the unique g.s. of approximate problem (3.1), (1.2). Then $u_\nu(t,\cdot) \to u(t,\cdot) = T_t u_0$ as $\nu \to \infty$ in L^2 uniformly on any segment $|t| \le T$.*

Proof. Assume that $f \in L^2$, $h \neq 0$. We set $v_\nu = (E+hA_\nu)^{-1}f \in D(A_\nu)$, $\nu \in \mathbb{N}$; $v = (E+hA)^{-1}f \in D(A)$. Then $v_\nu + hA_\nu v_\nu = f$, $v + hAv = f$. Since $A_\nu = -(A_\nu)^*$, $A = -A^*$, these equalities mean that

$$v_\nu(x) + h\operatorname{div}(a_\nu(x)v_\nu(x)) = v(x) + h\operatorname{div}(a(x)v(x)) = f(x) \text{ in } \mathcal{D}'(\mathbb{R}^n). \quad (4.1)$$

Since A_ν, A are skew-symmetric,

$$\|v_\nu\|_2^2 = (f,v_\nu)_2 + h(A_\nu v_\nu, v_\nu)_2 = (f,v_\nu)_2,\ \|v\|_2^2 = (f,v)_2 + h(Av,v)_2 = (f,v)_2. \quad (4.2)$$

From (4.2) it follows that $\|v_\nu\|_2 \le \|f\|_2$ for all $\nu \in \mathbb{N}$. Therefore, possibly after extraction of a subsequence (not relabeled), we can assume that $v_\nu \rightharpoonup w$ as $\nu \to \infty$ weakly in L^2, $w = w(x) \in L^2$. Passing to the limit as $\nu \to \infty$ in (4.1) and taking into account that the sequence $a_\nu(x) \underset{\nu\to\infty}{\to} a(x)$ in $L^1_{loc}(\mathbb{R}^n)$ and uniformly bounded, we find $w(x) + h\mathrm{div}(a(x)w(x)) = f(x)$ in $\mathcal{D}'(\mathbb{R}^n)$, which means $w + hAw = f$. Hence $v - w + hA(v-w) = 0$ and we conclude that $w = v$ because the operator $E + hA$ is invertible. Thus, $v_\nu \rightharpoonup v$ as $\nu \to \infty$ weakly in L^2. Then $(f, v_\nu)_2 \underset{\nu\to\infty}{\to} (f, v)_2$ and from (4.2) it follows that $\|v_\nu\|_2 \underset{\nu\to\infty}{\to} \|v\|_2$. It is well-known that this implies the strong convergence $v_\nu \underset{\nu\to\infty}{\to} v$ in L^2. Notice that the limit function v does not depend on the choice of weakly convergent subsequence. Therefore, the original sequence converges to the same limit strongly in L^2. We have established the strong convergence of resolvents $(E + hA_\nu)^{-1} \to (E + hA)^{-1}$. By the Trotter–Kato theorem the sequence of groups T_t^ν converges to the group T_t in the sense indicated in the formulation of our theorem. The proof is complete. □

Corollary 4.1. *Let $u_0 \in L^2(\mathbb{R}^n)$. Then $u(t,x) = T_t u_0(x)$ is a renormalized solution of (1.1), (1.2).*

Proof. Let $g(u)$ be a bounded continuous function, $u_\nu(t,x) = T_t^\nu u_0(x)$, $\nu \in \mathbb{N}$ be g.s. of approximate problem (3.1), (1.2). By Proposition 2.1(i) $u_\nu(t,x)$ is a renormalized solution of (3.1), (1.2). Therefore, $g(u_\nu(t,x))$ is a g.s. of (3.1), (1.2) with initial data $g(u_0(x))$, that is, $\forall f = f(t,x) \in C_0^1(\bar\Pi)$

$$\int_{\mathbb{R}^n} g(u_0(x)) f(0,x)dx + \int_\Pi g(u_\nu(t,x))[f_t(t,x) + a_\nu(x)\cdot\nabla_x f(t,x)]dtdx = 0. \quad (4.3)$$

By Proposition 4.1 the sequence $g(u_\nu(t,x)) \to g(u(t,x))$ as $\nu \to \infty$ in $L^1_{loc}(\bar\Pi)$, which allows to pass to the limit as $\nu \to \infty$ in (4.3) and obtain the relation: $\forall f = f(t,x) \in C_0^1(\bar\Pi)$

$$\int_{\mathbb{R}^n} g(u_0(x)) f(0,x)dx + \int_\Pi g(u(t,x))[f_t(t,x) + a(x)\cdot\nabla_x f(t,x)]dtdx = 0, \quad (4.4)$$

showing that $g(u)$ is a g.s. of (1.1), (1.2). Consider now the general case $g(u) \in C(\mathbb{R})$, $g(u_0(x)) \in L^1_{loc}(\mathbb{R}^n)$, $g(u(t,x)) \in L^1_{loc}(\bar\Pi)$. Let $g_k(u) = \max(-k, \min(g(u), k))$, $k \in \mathbb{N}$, be cut-off functions. Then $g_k(u) \in C(\mathbb{R})$, $|g_k(u)| \le k$, $g_k(u) \underset{k\to\infty}{\to} g(u)$ $\forall u \in \mathbb{R}$, $|g_k(u)| = \min(|g(u)|, k) \le |g(u)|$. The latter implies the estimates $|g_k(u_0(x))| \le |g(u_0(x))|$, $|g_k(u(t,x))| \le |g(u(t,x))|$. As we already proved, $g_k(u(t,x)$ are g.s. of (1.1), (1.2) with initial functions $g_k(u_0(x))$. Therefore, identity (4.4) holds with $g = g_k$. Passing to the limit in this relation as $k \to \infty$, with the help of Lebesgue dominated convergence theorem, we arrive at the same identity (4.4) with the limit function g. We conclude that $g(u)$ is a g.s. of (1.1), (1.2) with initial data $g(u_0)$. Thus, u is a renormalized solution of (1.1), (1.2). □

Corollary 4.2. . *Assume that the operator A is skew-adjoint. Then for every $u_0(x) \in L^2_{loc}(\mathbb{R}^n)$ there exists a renormalized solution $u(t,x) \in L^2_{loc}(\bar{\Pi})$ of the problem (1.1), (1.2).*

Proof. Let $u_r = u_r(t,x) \in C(\mathbb{R}, L^2(\mathbb{R}^n))$ be a group solution of (1.1), (1.2) with initial function $u_{0r} = u_0(x)\theta(r - |x|) \in L^2(\mathbb{R}^n)$ (recall that $\theta(s)$ is the Heaviside function). By Corollary 4.1 $u_r(t,x)$ is a renormalized solution of (1.1), (1.2) for each $r \in \mathbb{N}$. Since the difference $u_l - u_r$ is a group solution and, therefore, also a renormalized solution of problem (1.1), (1.2) with initial data $u_{0l} - u_{0r}$, $l, r \in \mathbb{N}$, then $|u_l - u_r|$ is a nonnegative g.s. of this problem with initial function $|u_{0l} - u_{0r}|$. By Proposition 1.1, we find that for all $t > 0$

$$\int_{|x|<r-Nt} |u_l(t,x) - u_r(t,x)|dx \le \int_{|x|<r} |u_{0l}(x) - u_{0r}(x)|dx = 0, \ \forall l > r,$$

and $u_l(t,x) = u_r(t,x)$ almost everywhere in the cone $C_r = \{ \ (t,x) \in \Pi \mid |x| < r - Nt \ \}$. This implies that the sequence u_r converges as $r \to \infty$ to a function $u = u(t,x)$, where $u = u_r(t,x)$ whenever $(t,x) \in C_r$ for some $r \in \mathbb{N}$. It is clear that $u(t,x) \in L^2_{loc}(\bar{\Pi})$. Let us demonstrate that u is the desired renormalized solution. Let a function $g(u) \in C(\mathbb{R})$ be such that $g(u_0(x)) \in L^1_{loc}(\mathbb{R}^n)$, $g(u(t,x)) \in L^1_{loc}(\bar{\Pi})$, and $f = f(t,x) \in C^1_0(\bar{\Pi})$. Then one can choose a sufficiently large $r \in \mathbb{N}$ such that $\operatorname{supp} f \subset C_r$. Since $u = u_r$ in C_r while u_r is a renormalized solution, we conclude that

$$\int_{\mathbb{R}^n} g(u_0(x))f(0,x)dx + \int_{\Pi} g(u(t,x))[f_t + a(x)\cdot\nabla_x f]dtdx =$$
$$\int_{\mathbb{R}^n} g(u_{0r}(x))f(0,x)dx + \int_{\Pi} g(u_r(t,x))[f_t + a(x)\cdot\nabla_x f]dtdx = 0.$$

Hence, u is a renormalized solution of (1.1), (1.2). □

Theorem 4.1. *Assume that A is a skew-adjoint operator, and* $\operatorname{div}(a(x)u(x)) = 0$ *in $\mathcal{D}'(\mathbb{R}^n)$, where $u(x) \in L^2_{loc}(\mathbb{R}^n)$. Then* $\operatorname{div}(a(x)g(u(x))) = 0$ *in $\mathcal{D}'(\mathbb{R}^n)$ for any $g(u) \in C(\mathbb{R})$ such that $g(u(x)) \in L^1_{loc}(\mathbb{R}^n)$.*

Proof. Let $p(y) \in C^1_0(\mathbb{R}^n)$ be a function equaled 1 in the unit ball $|y|^2 \le 1$. We set $u_r(x) = u(x)p(x/r) \in L^2(\mathbb{R}^n)$. By our assumption the operator A is skew-adjoint and, in view of equality $A = -(A)^*$, this operator may be considered in distributional sense. Obviously, for all $r > 0$

$$Au_r(s,x) = v_r(x) \doteq u(x)Ap(x/r) = \frac{1}{r}u(x)a(x)\cdot(\nabla_y p)(x/r) \ \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Since $v_r(x) \in L^2(\mathbb{R}^n)$, then $u_r(x) \in D(A)$. Now let $U_r(t,x) = e^{-At}u_r(x)$ be the group solution of (1.1), (1.2) with initial data $u_r(x)$. As we demonstrated above, $u_r(x) \in D(A)$. Therefore, $U_r(t,\cdot) \in C^1(\mathbb{R}, L^2(\mathbb{R}^n))$, and

$$V_r \doteq \frac{d}{dt}U_r(t,\cdot) = -e^{-At}Au_r = -e^{-At}v_r.$$

We see that $V_r(t,x)$ is a renormalized solution to the Cauchy problem (1.1), (1.2) with initial data $-v_r(x)$. By Corollary 4.1 $|V_r(t,x)|$ is a g.s. of this problem with initial function $|v_r(x)|$. Let $T,R>0$, $r>R+NT$. Then by Proposition 1.1 for all $t\in[0,T]$

$$\int_{|x|<R}|V_r(t,x)|dx\le\int_{|x|<r}|v_r(x)|dx=0$$

(since $(\nabla_y p)(x/r)=0$ for $|x|<r$).

We find that $V_r=\frac{d}{dt}U_r\equiv 0$ in the cylinder $C_{R,T}=\{\ (t,x)\mid |x|<R,\ t\in(0,T)\ \}$. This implies that $U_r\equiv u_r=u$ in this cylinder. Now, let $g(u)$ be a bounded continuous function. By Corollary 4.1 the function $g(U_r)$ is a g.s. of (1.1), (1.2). Therefore this function satisfies (1.1) in $\mathcal{D}'(C_{R,T})$. Since $g(U_r)\equiv g(u)$ in $C_{R,T}$, we obtain that $\operatorname{div}_x(ag(u))=0$ in $\mathcal{D}'(V_R)$, where V_R denotes the open ball $|x|<R$. In view of arbitrariness of R we conclude that $\operatorname{div}_x(ag(u))=0$ in $\mathcal{D}'(\mathbb{R}^n)$. In the general case when $g(u)\in C(\mathbb{R})$, $g(u(t,x))\in L^1_{loc}(\mathbb{R}^n)$, we construct the sequence of cut-off functions $g_k(u)=\max(-k,\min(g(u),k))$. Then $\operatorname{div}_x(ag_k(u))=0$ in $\mathcal{D}'(\mathbb{R}^n)$ for all $k\in\mathbb{N}$. Since $g_k(u(x))\to g(u(x))$ as $k\to\infty$ in $L^1_{loc}(\mathbb{R}^n)$ (cf. the proof of Corollary 4.1), we can pass to the limit as $k\to\infty$ in the relation $\operatorname{div}_x(ag_k(u))=0$ and conclude that $\operatorname{div}_x(ag(u))=0$ in $\mathcal{D}'(\mathbb{R}^n)$. □

5 Main result: the sufficiency

We are going to establish the much stronger result than the statement of Corollary 4.1, claiming that any generalized solution $u(t,x)$ of (1.1), (1.2) is a renormalized solution, that is, the sufficiency statement of our main Theorem 1.1.

We define the operator $\tilde{A}_0=\frac{\partial}{\partial s}+A_0$ acting on $C_0^1(\mathbb{R}^{n+1})$, so that $\tilde{A}_0u(s,x)=\frac{\partial u(s,x)}{\partial s}+a(x)\nabla_x u(s,x)$. Let $\tilde{A}$ be a closure of $\tilde{A}_0$ in $L^2(\mathbb{R}^{n+1})$. We will prove that $\tilde{A}$ is a skew-adjoint operator whenever A is a skew-adjoint operator on $L^2(\mathbb{R}^n)$. First, we observe that, at least formally, operator $-\tilde{A}$ should coincide with the infinitesimal generator of the unitary group $G_tu(s,\cdot)=T_tu(s-t,\cdot)$, $u(s,x)\in L^2(\mathbb{R}^{n+1})$. By Stone's theorem $G_t=e^{-Bt}$, where B is a skew-adjoint operator on $L^2(\mathbb{R}^{n+1})$. The following statement justifies this formal observation.

Lemma 5.1. *The equality $\tilde{A}=B$ holds. In particular, the operator $\tilde{A}$ is skew-adjoint.*

Proof. We denote by X the space $L^2(\mathbb{R}^n)$ and by X_0 the space $D(A)$ equipped with the graph norm $\|x\|_2+\|Ax\|_2$. Since the operator A is closed, X_0 is a Banach space. Let F be a subspace of $L^2(\mathbb{R}^{n+1})=L^2(\mathbb{R},X)$ consisting of functions $u(s,\cdot)\in L^2(\mathbb{R},X_0)$, such that $\frac{d}{ds}u(s,\cdot)\in L^2(\mathbb{R},X)$. We show that $F\subset D(B)\cap D(\tilde{A})$ and $Bu=\tilde{A}u$ on F. Thus, assume that $u(s,x)\in F$. Then,

$$\frac{G_tu-u}{t}=T_t\frac{u(s-t,\cdot)-u(s,\cdot)}{t}+\frac{T_tu(s,\cdot)-u(s,\cdot)}{t}$$

Since

$$\lim_{t\to 0}\frac{u(s-t,\cdot)-u(s,\cdot)}{t}=-\frac{d}{ds}u(s,\cdot),$$

$$\lim_{t\to 0}\frac{T_t u(s,\cdot)-u(s,\cdot)}{t}=-Au(s,\cdot)\ \text{ in } L^2(\mathbb{R},X),$$

we find that there exists

$$-Bu=\lim_{t\to 0}\frac{G_t u-u}{t}=-\frac{d}{ds}u(s,\cdot)-Au(s,\cdot)\ \text{ in } L^2(\mathbb{R},X),$$

that is, $u\in D(-B)=D(B)$ and

$$Bu=\frac{d}{ds}u(s,\cdot)+Au(s,\cdot). \tag{5.1}$$

Let us show that the same holds for the operator $\tilde{A}$. Assume firstly that

$$u(s,x)=\sum_{j=1}^{N}\alpha_j(s)v_j(x),\quad \alpha_j(s)\in C_0^1(\mathbb{R}),\ v_j\in X_0,\ j=1,\dots,N. \tag{5.2}$$

If $v_j(x)\in C_0^1(\mathbb{R}^n)$, then $u(s,x)\in C_0^1(\mathbb{R}^{n+1})=D(\tilde{A}_0)$ and then

$$\tilde{A}u(s,x)=\tilde{A}_0u(s,x)=\frac{\partial}{\partial s}u(s,x)+a(x)\cdot\nabla_x u(s,x)=\frac{d}{ds}u(s,\cdot)+Au(s,\cdot).$$

In the case of arbitrary $v_j\in X_0$ we can find sequences $v_{jr}\in C_0^1(\mathbb{R}^n)$, $r\in\mathbb{N}$, converging to v_j as $r\to\infty$ in X_0 (because A is the closure of A_0). Then the sequences

$$u_r(s,x)=\sum_{j=1}^{N}\alpha_j(s)v_{jr}(x)\underset{r\to\infty}{\to}u(s,x),\ \tilde{A}u_r=\sum_{j=1}^{N}\alpha_j'(s)v_{jr}(x)+$$

$$\sum_{j=1}^{N}\alpha_j(s)Av_{jr}(x)\underset{r\to\infty}{\to}\sum_{j=1}^{N}\alpha_j'(s)v_j(x)+\sum_{j=1}^{N}\alpha_j(s)Av_j(x)=\frac{d}{ds}u(s,\cdot)+Au(s,\cdot)$$

in $L^2(\mathbb{R},X)$. Since the operator $\tilde{A}$ is closed, we conclude that $u(s,x)\in D(\tilde{A})$ and $\tilde{A}u(s,\cdot)=\dfrac{d}{ds}u(s,\cdot)+Au(s,\cdot)$. Now we consider the general case $u(s,x)\in F$. Then, as is easy to verify, there exists a sequence $u_m(s,x)$, $m\in\mathbb{N}$, of functions having form (5.2) such that $u_m(s,\cdot)\underset{m\to\infty}{\to}u(s,\cdot)$ in $L^2(\mathbb{R},X_0)$, $\dfrac{d}{ds}u_m(s,\cdot)\underset{m\to\infty}{\to}\dfrac{d}{ds}u(s,\cdot)$ in $L^2(\mathbb{R},X)$. Then $u_m(s,\cdot)\underset{m\to\infty}{\to}u(s,\cdot)$, $\tilde{A}u_m(s,\cdot)\underset{m\to\infty}{\to}\dfrac{d}{ds}u(s,\cdot)+Au(s,\cdot)$ in $L^2(\mathbb{R},X)$, which implies that $u\in D(\tilde{A})$, $\tilde{A}u(s,\cdot)=\frac{d}{ds}u(s,\cdot)+Au(s,\cdot)$ again due to the closedness of $\tilde{A}$.

In view of (5.1) we conclude that $F \subset D(B) \cap D(\tilde{A})$, and $B = \tilde{A}$ on F. By the known representation of the resolvent $(E+B)^{-1}$, we find

$$u(s,\cdot) = (E+B)^{-1}f(s,\cdot) = \int_0^{+\infty} e^{-t}G_t f dt = \int_0^{+\infty} e^{-t}T_t f(s-t,\cdot)dt = \int_{-\infty}^{s} e^{t-s}T_{s-t}f(t,\cdot)dt.$$

Notice that X_0 is an invariant space for a group T_t and since $\|T_t u\|_2 = \|u\|_2$, $\|AT_t u\|_2 = \|T_t Au\|_2 = \|Au\|_2$, then $\|T_t u\|_{X_0} = \|u\|_{X_0}$. Therefore, taking $f(s,x) \in L^2(\mathbb{R}, X_0)$, we find

$$U(s) \doteq \|u(s,\cdot)\|_{X_0} \le \int_{-\infty}^{s} e^{t-s}\|T_{s-t}f(t,\cdot)\|_{X_0}dt = \int_{-\infty}^{s} e^{t-s}\|f(t,\cdot)\|_{X_0}dt = (\gamma * F)(s),$$

where $F(t) = \|f(t,\cdot)\|_{X_0}$, $\gamma(t) = \theta(t)e^{-t}$ (recall that $\theta(t)$ is the Heaviside function). It is clear that $\|\gamma\|_1 = 1$ and by the known property of convolutions $\|U\|_2 \le \|F\|_2$, that is, $u(s,\cdot) \in L^2(\mathbb{R}, X_0)$, $\|u(s,\cdot)\|_{L^2(\mathbb{R},X_0)} \le \|f(t,\cdot)\|_{L^2(\mathbb{R},X_0)}$. Further, there exists the derivative

$$\frac{d}{ds}u(s,\cdot) = \frac{d}{ds}\int_{-\infty}^{s} e^{t-s}T_{s-t}f(t,\cdot)dt = f(s,\cdot) - \int_{-\infty}^{s} e^{t-s}T_{s-t}f(t,\cdot)dt - \int_{-\infty}^{s} e^{t-s}AT_{s-t}f(t,\cdot)dt = f(s,\cdot) - u(s,\cdot) - Au(s,\cdot) \in L^2(\mathbb{R}, X).$$

We see that $u(s,\cdot) \in F$. Assume that $u(s,\cdot) \in D(B)$. Then, there exists a unique $f(s,\cdot) \in L^2(\mathbb{R}, X)$ such that $u(s,\cdot) = (E+B)^{-1}f(s,\cdot)$. Evidently, $L^2(\mathbb{R}, X_0)$ is dense in $L^2(\mathbb{R}, X)$, which implies existence of a sequence $f_k(s,\cdot) \in L^2(\mathbb{R}, X_0)$, $k \in \mathbb{N}$, such that $f_k \to f$ as $k \to \infty$ in $L^2(\mathbb{R}, X)$. We define the corresponding sequence $u_k = u_k(s,\cdot) = (E+B)^{-1}f_k$. Then $u_k \to u$, $\tilde{A}u_k = Bu_k \to Bu$ as $k \to \infty$ in $L^2(\mathbb{R}, X)$. Since $\tilde{A}$ is a closed operator, we derive that $u \in D(\tilde{A})$ and $\tilde{A}u = Bu$. Hence, $B \subset \tilde{A}$. Conversely, $\tilde{A}_0 \subset B$ (since, evidently, $D(\tilde{A}_0) \subset F$), which implies $\tilde{A} \subset B$ as the closure of $\tilde{A}_0$. We conclude that $\tilde{A} = B$, as required. □

Now, we are ready to prove the renormalization property.

Theorem 5.1. *Assume that operator A is skew-adjoint and $u_0 \in L^2_{loc}(\mathbb{R}^n)$. Then any g.s. $u(t,x) \in L^2_{loc}(\bar{\Pi})$ of the problem (1.1), (1.2) is a renormalized solution of this problem and, therefore, is unique.*

Proof. We may extend $u(t,x)$ to a g.s. of (1.1), (1.2) on the whole space $\mathbb{R}^{n+1}$, setting $u(-t,x) = v(t,x)$, where $v(t,x) \in L^2_{loc}(\bar{\Pi})$ is a renormalized solution of the problem $v_t - \operatorname{div}(a(x)v) = 0$, $v(0,x) = u_0(x)$. Since the operator $-A$

is skew-adjoint, this renormalized solution exists due to Corollary 4.2. Then $u_t + \operatorname{div}(a(x)u) = 0$ in $D'(\mathbb{R}^{n+1})$. By Lemma 5.1 the operator $\frac{\partial}{\partial t} + \operatorname{div}(au)$ is skew-adjoint on $L^2(\mathbb{R}^{n+1})$. Then, by Theorem 4.1 $g(u)_t + \operatorname{div}(a(x)g(u)) = 0$ in $D'(\mathbb{R}^{n+1})$ whenever $g(u) \in L^1_{loc}(\mathbb{R}^{n+1})$. This easily implies that $u(t,x)$ is a renormalized solution of (1.1), (1.2). □

Remark 5.1. In the case of more general transport equation

$$u_t + a(t,x) \cdot \nabla_x u = u_t + \operatorname{div}_x(a(t,x)u) = 0 \tag{5.3}$$

with $a(t,x) = (a_1(t,x), \dots, a_n(t,x)) \in L^\infty(\Pi, \mathbb{R}^n)$, $\operatorname{div}_x a(t,x) = 0$, we may extend the field $a(t,x)$ on the whole space $(t,x) \in \mathbb{R}^{n+1}$, setting $a(t,x) = -a(-t,x)$ for $t < 0$. It is clear that the vector field $\tilde{a}(t,x) = \frac{\partial}{\partial t} + a(t,x)$ is bounded and solenoidal on $\mathbb{R}^{n+1}$, and for any g.s. $u(t,x) \in L^1_{loc}(\Pi)$ of (5.3) the function $\tilde{u}(t,x) = u(|t|,x)$ is a g.s. of (5.3) in the whole space $\mathbb{R}^{n+1}$.

For equation (5.3) the following analogue of Theorem 2.1 holds.

Theorem 5.2. *Any g.s. of the Cauchy problem (5.3), (1.2) is a renormalized solution if and only if the operator $A_0 u = \tilde{a}(t,x) \cdot \nabla u = \frac{\partial}{\partial t} u + a(t,x) \cdot \nabla_x u$, $u = u(t,x) \in C^1_0(\mathbb{R}^{n+1})$, is essentially skew-adjoint.*

Proof. Let us consider the extended transport equation

$$v_t + \tilde{a}(s,x) \cdot \nabla_{s,x} v = v_t + v_s + a(s,x) \cdot \nabla_x v = 0, \tag{5.4}$$

where $v = v(t,s,x)$, $t > 0$, $(s,x) \in \mathbb{R}^{n+1}$. After the change $u(t,s,x) = v(t+s,t,x)$ we obtain the equation

$$u_t + a(t,x) \cdot \nabla_x u = 0,$$

which coincides with (5.3). Therefore, any g.s. of (5.3) (which necessarily admits some initial data (1.2)) satisfies the renormalization property if and only if this is true for g.s. of equation (5.4). By Theorem 5.3, the latter is equivalent to the essential skew-adjointness of the operator $\tilde{a}(t,x) \cdot \nabla u$. The proof is complete. □

6 Contraction semigroup, which provides g.s. and a criterion of the uniqueness

In this section we study the general case when the skew-symmetric operator A is not necessarily skew-adjoint. We proof that in this case there always exists a linear C_0-semigroup T_t such that $u(t,x) = T_t u_0$ is a g.s. of (1.1), (1.2), and $\|T_t u_0\|_2 \le \|u_0\|_2$ for all $u_0 \in L^2$ (i.e., T_t are contractions in L^2). Let $\tilde{A}$ be a maximal skew-symmetric extension of A. Then $A \subset \tilde{A} \subset -\tilde{A}^* \subset -A^*$. Denote by $d_+ = d_+(\tilde{A}) = \operatorname{codim} \operatorname{Im}(E+\tilde{A})$, $d_- = d_-(\tilde{A}) = \operatorname{codim} \operatorname{Im}(E-\tilde{A})$ the deficiency indexes of $\tilde{A}$ (generally, these are cardinal numbers). Since $\tilde{A}$ is a maximal skew-symmetric operator, either $d_+ = 0$ or $d_- = 0$. Let us define $B = -\tilde{A}$ if $d_+ = 0$, $B = \tilde{A}^*$ if $d_- = 0$ (observe that in the case $d_+ = d_- = 0$ the operator $\tilde{A}$ is skew-adjoint and $-\tilde{A} = \tilde{A}^*$).

Theorem 6.1. *The operator B generates the semigroup of contractions $T_t u = e^{Bt}$ on L^2 such that $u(t,x) = T_t u_0$ is a g.s. of (1.1), (1.2) for every initial data $u_0 \in L^2$. Moreover, in the case $d_+ = 0$ the operators T_t are isometric, that is $\|T_t u\|_2 = \|u\|_2$ $\forall u \in L^2$.*

Proof. If $d_+ = 0$ then $\mathrm{Im}(E+\tilde{A}) = L^2$ and the operator $B = -\tilde{A}$ is m-dissipative. By the Lumer-Phillips theorem it generates the semigroup of contractions on L^2. Moreover, in this case B is skew-symmetric and the operators $T_t = e^{Bt}$ are isometric. In the remaining case when $d_- = 0$ the operator $\tilde{A}$ is m-dissipative. Then (see [5]) the operator $B = A^*$ is also m-dissipative and generates the semigroup of contractions. Since $-\tilde{A} \subset \tilde{A}^* \subset A^*$, then $B \subset A^*$ and by virtue of Lemma 3.1 we conclude that the functions $u(t,x) = T_t u_0(x)$ are g.s. of (1.1), (1.2). □

The following statement gives the criterion of uniqueness of a contraction semigroups constructed in Theorem 6.1.

Theorem 6.2. *A contraction semigroups T_t, which provides g.s. $T_t u_0$, is unique if and only if A is a maximal skew-symmetric operator.*

Proof. If the skew-symmetric operator A is not maximal (that is, $d_+(A), d_-(A) > 0$), then there exist different maximal skew-symmetric extensions $\tilde{A}_1, \tilde{A}_2$, such that $d_+(\tilde{A}_1) = d_+(\tilde{A}_2)$, $d_-(\tilde{A}_1) = d_-(\tilde{A}_2)$. Then m-dissipative operators B_1, B_2 corresponding to $\tilde{A}_1$, $\tilde{A}_2$ are different. By the Hille-Yosida theorem they generates different semigroups. Therefore, the uniqueness assumption implies that A is a maximal skew-symmetric operator. Conversely, suppose that the operator A is maximal and T_t is a contraction semigroup in L^2, which provides g.s. of problem (1.1), (1.2). Then, by the Lumer-Phillips theorem, the infinitesimal generator C of this semigroup is m-dissipative (maximal dissipative) and by Lemma 3.1 $C \subset A^*$. Since also $-A \subset A^*$, we see that $Cx = -Ax$ $\forall x \in D(C) \cap D(A)$. This allows to define the linear operator $\tilde{C}$ on $D(\tilde{C}) = D(C) + D(A)$, setting $\tilde{C}w = Cu - Av$ if $w = u + v$, $u \in D(C)$, $v \in D(A)$. If $w = u_1 + v_1 = u_2 + v_2$, where $u_1, u_2 \in D(C)$, $v_1, v_2 \in D(A)$, then $u_1 - u_2 = v_2 - v_1 \in D(C) \cap D(A)$ and $C(u_1 - u_2) = -A(v_2 - v_1)$, which implies the equality $Cu_1 - Av_1 = Cu_2 - Av_2$, showing that the value $\tilde{C}w$ does not depend on a representation $w = u + v$, $u \in D(C)$, $v \in D(A)$. Thus, the operator $\tilde{C}$ is well-defined and by the construction $C \subset \tilde{C}$, $-A \subset \tilde{C}$. If $w = u+v$, where $u \in D(C)$, $v \in D(A)$, then

$$(\tilde{C}w, w)_2 = (Cu - Av, u + v)_2 = (Cu, u)_2 - (Av, v)_2 + (v, Cu)_2 - (Av, u)_2$$
$$= (Cu, u)_2 - (Av, v)_2 + (v, A^* u)_2 - (Av, u)_2 = (Cu, u)_2, \quad (6.1)$$

where we use that $C \subset A^*$ and the relations $(Av, u)_2 = (v, A^* u)_2$, $(Av, v) = 0$ (we recall that A is skew-symmetric). Since the operator C is dissipative, then $(Cu, u)_2 \le 0$ (see [5]) and it follows from (6.1) that $(\tilde{C}w, w)_2 \le 0$ for all $w \in D(\tilde{C})$. This means that $\tilde{C}$ is a dissipative operator. But $C \subset \tilde{C}$ while C is a maximal dissipative operator. Therefore, $C = \tilde{C}$ and in particular $D(\tilde{C}) =$

$D(C)+D(A)=D(C)$. Hence, $D(A)\subset D(C)$ and $-A\subset C\subset A^*$. We recall that A is a maximal skew-symmetric operator, so that either $d_+(A)=0$ or $d_-(A)=0$. In the first case $\mathrm{Im}(E+A)=L^2$, that is, $-A$ is m-dissipative operator. From the relation $-A\subset C$ it now follows that $C=-A=B$. In the second case $\mathrm{Im}(E-A)=L^2$ and A is an m-dissipative operator. By the known property (see [5]) A^* is an m-dissipative operator as well. Since operator C is also m-dissipative, it follows from the relation $C\subset A^*$ that $C=A^*=B$. In both cases C coincides with the operator B from Theorem 6.1. This, in turn, implies the uniqueness of the semigroup T_t. □

Now we are ready to prove part (ii) of main Theorem 1.1 claiming that the uniqueness of any g.s. holds if and only if the operator A is skew-adjoint that, in turn, is equivalent to the renormalization property. It is clear that the renormalization property for every g.s. implies the uniqueness. The inverse statement is a consequence of the following theorem.

Theorem 6.3. *Assume that any g.s. of problem (1.1), (1.2) with $u_0\in L^2$ is unique in the class of g.s. with bounded $\|u(t,\cdot)\|_2$. Then these g.s. satisfy the renormalization property and, therefore, the operator A is skew-adjoint.*

Proof. It is clear that the uniqueness assumption implies the uniqueness of a contraction semigroups T_t, which provides g.s. By Theorem 6.2 the operator A is maximal skew-symmetric, that is, one of its deficiency indexes d_+ or d_- is zero. In view of Theorem 6.1 in the case $d_+=0$ the semigroup T_t consists of isometric embeddings. Therefore, the g.s. $u=u(t,x)=T_tu_0(x)$ satisfies the property: $\|u(t,\cdot)\|_2=\|u_0\|_2$. Let $\tilde u=\tilde u(t,x)$ be a weak limit of a subsequence of g.s. $u_k(t,x)$ to the approximate problem (3.1), (1.2). Since $\|u_k(t,\cdot)\|_2=\|u_0\|_2$, then

$$\|u_k\|_{L^2(\Pi_T)}=\sqrt{T}\|u_0\|_2=\|u\|_{L^2(\Pi_T)}, \tag{6.2}$$

where $\Pi_T=(0,T)\times\mathbb{R}^n$. Since u, $\tilde u$ are g.s. of the same problem (1.1), (1.2), then by the uniqueness assumption $u=\tilde u$. Hence $u_k\rightharpoonup u$ as $k\to\infty$ weakly in $L^2(\Pi_T)$ while in view of (6.2) $\|u\|_{L^2(\Pi_T)}=\|u_k\|_{L^2(\Pi_T)}$ for all $k\in\mathbb{N}$. By the known property of weak convergence we conclude that $u_k\to u$ as $k\to\infty$ strongly in $L^2(\Pi_T)$ for all $T>0$. As in the proof of Corollary 4.1, this implies that u is a renormalized solution of (1.1), (1.2). Thus, requirement (R) is fulfilled and by Theorem 3.1 the operator A is skew-adjoint.

Now we consider the case when $d_-=0$. In this case the operator $-A$ generates the semigroup S_t of isometries in L^2. We choose $T>0$ and set

$$u=u(t,x)=\begin{cases} v(T-t,x), & 0\le t<T,\\ \bar u(t-T,x), & t\ge T,\end{cases}$$

where $v(t,x)=S_tv_0(x)$ and $\bar u=\bar u(t,x)$ is a g.s. of (1.1), (1.2) with initial data $v_0\in L^2$. It is easy to verify that $u(t,x)$ is a g.s. of problem (1.1), (1.2) with the initial function $u_0 = \tilde{u}(T, \cdot)$. By the uniqueness of this g.s. $u = \tilde{u}$, where, as above, $\tilde{u} = \tilde{u}(t, x)$ is a weak limit of the sequence $u_k(t, x)$ of g.s. to approximate problem (3.1), (1.2). We see that

$$\|\tilde{u}\|_{L^2(\Pi_T)} = \|u\|_{L^2(\Pi_T)} = \|v\|_{L^2(\Pi_T)} = \sqrt{T}\|u_0\|_2 = \|u_k\|_{L^2(\Pi_T)} \ \forall k \in \mathbb{N}.$$

As was shown in the first part of our proof, this implies the strong convergence $u_k \underset{k\to\infty}{\to} u$ in $L^2(\Pi_T)$ and, therefore, the renormalization property. By the latter we find that $v(t, x)$ is a renormalized solution of the Cauchy problem for the equation $v_t - \mathrm{div} av = 0$ with initial data v_0 (we also take into account that $T > 0$ is arbitrary). Thus, requirement (R) for this equation is satisfied and by Theorem 3.1 we conclude that the operator $-A$ is skew-adjoint. This, in turn, implies that A is a skew-adjoint operator. By Theorem 5.1 we see that any g.s. of (1.1), (1.2) is a renormalized solution of this problem as well. The proof is complete. $\square$

7 Generalized characteristics

We assume that the operator A is skew-adjoint. By Theorem 5.1 for every $u_0(x) \in L^\infty = L^\infty(\mathbb{R}^n)$ there exists a unique g.s. $u(t, x) \in L^\infty(\Pi)$ of the problem (1.1), (1.2), and this g.s. is a renormalized solution as well. It is clear that $\|u\|_\infty \le M \doteq \|u_0\|_\infty$ (this can be derived from the renormalization property. Indeed, $v = (|u| - M)^+$ ia a g.s. of (1.1), (1.2) with initial data $(|u_0| - M)^+ = 0$, which implies that $v = 0$, i.e., $|u| \le M$). As readily follows from the definition of g.s. and the renormalization property, the functions $t \to p(u(t, \cdot))$ are weakly continuous on some set of full measure for every $p(u) \in C(\mathbb{R})$, which implies that the map $t \to u(t, \cdot)$ is strongly continuous in $L^1_{loc}(\mathbb{R}^n)$. In particular, after possible correction of u on the set of null measure, we may and will assume that the functions $u(t, \cdot) \in L^\infty$ are well-defined for all $t \ge 0$ and depend continuously on t (in the space $L^1_{loc}(\mathbb{R}^n)$). Let $u_1 = u_1(t, x)$, $u_2 = u_2(t, x)$ be g.s. of problem (1.1), (1.2) with initial functions $u_{01} = u_{01}(x)$, $u_{02} = u_{02}(x)$, respectively. Then, by the renormalization property $u_1u_2 = [(u_1 + u_2)^2 - u_1^2 - u_2^2]/2$ is a g.s. of (1.1), (1.2) with the initial data $u_{01}u_{02} = [(u_{01} + u_02)^2 - u_{01}^2 - u_{02}^2]/2$. Hence, the map $T_t(u_0) = u(t, \cdot)$ is a homomorphism of the algebra L^∞: $T_t(uv) = T_tuT_tv$ for all $u, v \in L^\infty(\mathbb{R}^n)$. Obviously, the semigroup T_t can be extended to the group T_t of isomorphisms of L^∞. These isomorphisms generate the corresponding homeomorphisms $y_t : \mathcal{S} \to \mathcal{S}$ of the spectrum $\mathcal{S}$ of C^*-algebra L^∞, so that

$$\widehat{u(t, \cdot)}(X) = \widehat{u}_0(y_t(X)) \quad \text{for all } X \in \mathcal{S}, \tag{7.1}$$

where $\widehat{u} \in C(\mathcal{S})$ denotes the Gelfand transform of $u \in L^\infty$: $\widehat{u}(X) = \langle X, u\rangle$ (recall that $\mathcal{S}$ consists on multiplicative functionals $X : L^\infty \to \mathbb{C}$). Denote by $x_t : \mathcal{S} \to \mathcal{S}$ the inverse homeomorphism $x_t = y_t^{-1}$. Then (7.1) can be written as

$$\widehat{u(t, \cdot)}(x_t(X_0)) = \widehat{u}_0(X_0) \quad \forall X_0 \in \mathcal{S},$$

that is, $\widehat{u(t,\cdot)}$ remains constant on the curve $X(t) = x_t(X_0)$, $t \in \mathbb{R}$. It is natural to call this curve the generalized characteristic of equation (1.1). In other words, $X(t)$ can be considered as a generalized solution to characteristic system (2.1) (extended to $\mathcal{S}$) with initial data $X(0) = X_0$.

Let us describe the spectrum $\mathcal{S}$. The below characterization of $\mathcal{S}$ is rather well-known but we cannot find the appropriate references and, therefore, give the description of $\mathcal{S}$ in details. First of all, we introduce the notion of essential ultrafilter.

We call sets $A, B \subset \mathbb{R}^n$ equivalent: $A \sim B$ if $\mu(A \triangle B) = 0$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference and μ is the outer Lebesgue measure. Let $\mathfrak{F}$ be a filter in $\mathbb{R}^n$. This filter is called *essential* if from the conditions $A \in \mathfrak{F}$ and $B \sim A$ it follows that $B \in \mathfrak{F}$. It is clear that an essential filter cannot include sets of null measure, since such sets are equivalent to $\emptyset$. Using Zorn's lemma, one can prove that any essential filter is contained in a maximal essential filter. Maximal essential filters are called essential ultrafilters.

Lemma 7.1. *Let $\mathfrak{U}$ be an essential ultrafilter. Then for each $A \subset \mathbb{R}^n$ either $A \in \mathfrak{U}$ or $\mathbb{R}^n \setminus A \in \mathfrak{U}$.*

Proof. Assuming that $A \notin \mathfrak{U}$, we introduce

$$\mathfrak{F} = \{\ B \subset \mathbb{R}^n \mid B \cup A \in \mathfrak{U}\ \}.$$

Obviously, $\mathfrak{F}$ is an essential filter, $\mathbb{R}^n \setminus A \in \mathfrak{F}$, and $\mathfrak{U} \le \mathfrak{F}$. Since the filter $\mathfrak{U}$ is maximal, we obtain that $\mathfrak{U} = \mathfrak{F}$. Hence, $\mathbb{R}^n \setminus A \in \mathfrak{U}$. The proof is complete. $\square$

The property indicated in Lemma 7.1 is the characteristic property of ultrafilters, see for example, [6]. Therefore, we obtain the following statement.

Corollary 7.1. *Any essential ultrafilter is an ultrafilter, i.e. a maximal element in a set of all filters.*

Lemma 7.2. *Let $\mathfrak{U}$ be an essential ultrafilter, and $f(x)$ be a bounded function in $\mathbb{R}^n$. Then there exists $\lim\limits_{\mathfrak{U}} f(x)$. If a function $g(x) = f(x)$ almost everywhere on $\mathbb{R}^n$, then there exists $\lim\limits_{\mathfrak{U}} g(x) = \lim\limits_{\mathfrak{U}} f(\xi)$.*

Proof. By Corollary 7.1 $\mathfrak{U}$ is an ultrafilter. By the known properties of ultrafilters, the image $f_*\mathfrak{U}$ is an ultrafilter on the compact $[-M, M]$, where $M = \sup|f(x)|$, and this ultrafilter converges to some point $y \in [-M, M]$. Therefore, $\lim\limits_{\mathfrak{U}} f(x) = \lim f_*\mathfrak{U} = y$. Further, suppose that a function $g = f$ a.e. on $\mathbb{R}^n$. Then the set $E = \{x \in \mathbb{R}^n \mid g(x) \neq f(x)\ \}$ has null Lebesgue measure. Let V be a neighborhood of y. Then $g^{-1}(V) \supset f^{-1}(V) \setminus E$. By the convergence of the ultrafilter $f_*\mathfrak{U}$ the set $f^{-1}(V) \in \mathfrak{U}$. Since $\mathfrak{U}$ is an essential ultrafilter while $f^{-1}(V) \setminus E \sim f^{-1}(V)$, then $f^{-1}(V) \setminus E \in \mathfrak{U}$. This set is contained in $g^{-1}(V)$, and we claim that $g^{-1}(V) \in \mathfrak{U}$. Since V is an arbitrary neighborhood of y, we conclude that $\lim\limits_{\mathfrak{U}} g(x) = y$. The proof is complete. $\square$

By the statement of Lemma 7.2, the functional $f \to \lim\limits_{\mathfrak{U}} f(\xi)$ is well-defined on $L^\infty(\mathbb{R}^n)$ and it is a linear multiplicative functional on $L^\infty(\mathbb{R}^n)$. In other words, this functional belongs to the spectrum $\mathcal{S}$ of algebra $L^\infty(\mathbb{R}^n)$. Let us demonstrate that, conversely, any linear multiplicative functional on $L^\infty(\mathbb{R}^n)$ coincides with the limit along some essential ultrafilter.

Theorem 7.1. *For each $X \in \mathcal{S}$ there exists an essential ultrafilter $\mathfrak{U}$ such that*

$$\langle X, f \rangle = \lim_{\mathfrak{U}} f(x) \quad \forall f \in L^\infty(\mathbb{R}^n). \tag{7.2}$$

Proof. We denote by $\chi_B = \chi_B(x)$ the indicator function of measurable set $B \subset \mathbb{R}^n$, and define

$$\mathfrak{F} = \{ \ A \subset \mathbb{R}^n \mid \langle X, \chi_B \rangle = 1 \ \text{ for some measurable } B \subset A \ \}.$$

It is directly verified that $\mathfrak{F}$ is an essential filter. Let us show that for every $f(x) \in L^\infty(\mathbb{R}^n)$ there exists $\lim\limits_{\mathfrak{F}} f(x)$. Let $\lambda = \langle X, f \rangle$, $\varepsilon > 0$,

$$V = V_\varepsilon = \{ \ x \in \mathbb{R}^n \mid |f(x) - \lambda| < \varepsilon \ \},$$

$\overline{V} = \mathbb{R}^n \setminus V$. It is clear that V is a measurable set. We are going to prove that $\langle X, \chi_V \rangle = 1$. We define the function

$$g(x) = \begin{cases} 1/(f(x) - \lambda) & , \quad x \in \overline{V}, \\ 0 & , \quad x \in V. \end{cases}$$

Since $|f(x) - \lambda| \geq \varepsilon$ on the set $\overline{V}$, then $g(x) \in L^\infty(\mathbb{R}^n)$ and, evidently, $g(x)(f(x) - \lambda) = \chi_{\overline{V}}$. Therefore,

$$\langle X, \chi_{\overline{V}} \rangle = \langle X, g \rangle (\langle X, f \rangle - \lambda) = 0.$$

This implies that

$$\langle X, \chi_V \rangle = \langle X, 1 - \chi_{\overline{V}} \rangle = 1 - \langle X, \chi_{\overline{V}} \rangle = 1,$$

as was to be proved. Hence, $V = V_\varepsilon \in \mathfrak{F}$ for all $\varepsilon > 0$, which means that $\lim\limits_{\mathfrak{F}} f(x) = \lambda = \langle X, f \rangle$. Notice that the latter relation holds for every $f \in L^\infty(\mathbb{R}^n)$. Let $\mathfrak{U}$ be an essential ultrafilter such that $\mathfrak{F} \subset \mathfrak{U}$. Then relation (7.2) is fulfilled. $\square$

Notice that the essential ultrafilter indicated in Theorem 7.1 is not unique, but it belongs to a unique equivalence class corresponding to the relation

$$\mathfrak{U}_1 \sim \mathfrak{U}_2 \ \Leftrightarrow \ \lim_{\mathfrak{U}_1} f = \lim_{\mathfrak{U}_2} f \ \forall f \in L^\infty(\mathbb{R}^n) \tag{7.3}$$

on the set of essential ultrafilters.

By Theorem 7.1 any generalized characteristic $X(t) = x_t(X_0)$ can be described as a curve $\mathfrak{U}(t)$ on a set of essential ultrafilters

We call an ultrafilter $\mathfrak{U}$ bounded if it contains a bounded set. It is clear that a bounded ultrafilter $\mathfrak{U}$ contains some compact set K. Then $\mathfrak{U}|_K = \{ B \in \mathfrak{U} \mid B \subset K \}$ is an ultrafilter on the compact K and, therefore, it converges to some element $y \in K$. Then $y = \lim \mathfrak{U}$. We have established that any bounded ultrafilter on $\mathbb{R}^n$ converges. Notice that, conversely, if an ultrafilter $\mathfrak{U}$ converges, $y = \lim \mathfrak{U}$, then $\mathfrak{U}$ contains all neighborhoods of y and, therefore, is bounded.

By Theorem 7.1 any generalized characteristic $X(t) = x_t(X_0)$ can be described as a curve $\mathfrak{U}(t)$, $t \in \mathbb{R}$ on a set of essential ultrafilters, which is uniquely defined up to the equivalence (7.3). We complete this section by the following result.

Theorem 7.2. *Let $\mathfrak{U}(t)$, $t \in \mathbb{R}$, be a generalized characteristic. Assume that the essential ultrafilter $\mathfrak{U}(t_0)$ is bounded for some $t_0 \in \mathbb{R}$. Then $\mathfrak{U}(t)$ is bounded for all $t \in \mathbb{R}$, and the curve $x(t) = \lim \mathfrak{U}(t)$, $t \in \mathbb{R}$, is Lipschitz: $|x(t)-x(t_0)| \le N|t-t_0|$.*

Proof. Since the ultrafilter $\mathfrak{U}(t_0)$ is bounded, there exists the limit $x(t_0) = \lim \mathfrak{U}(t_0)$. Therefore, for every $\varepsilon > 0$ the ball

$$V_\varepsilon = \{ x \in \mathbb{R}^n \mid |x - x(t_0)| < \varepsilon \} \in \mathfrak{U}(t_0).$$

Denote by $u_0(x)$ the indicator function of this ball and let $u(t,x) \in L^\infty(\mathbb{R}^{n+1})$ be the unique g.s. of equation (1.1) satisfying the Cauchy condition $u(t_0,x) = u_0(x)$. As readily follows from the statements of Proposition 1.1, $u(t,x) = 0$ for $|x - x(t_0)| \ge \varepsilon + N|t-t_0|$. By the definition of generalized characteristics

$$u(t,x) = \lim_{\mathfrak{U}(t)} u(t,\cdot) = \lim_{\mathfrak{U}(t_0)} u_0 = 1. \tag{7.4}$$

Let us show that the ball

$$V_{\varepsilon+N|t-t_0|} = \{ x \in \mathbb{R}^n \mid |x - x(t_0)| < \varepsilon + N|t-t_0| \} \in \mathfrak{U}(t).$$

Otherwise, its complement $\overline{V_{\varepsilon+N|t-t_0|}} \in \mathfrak{U}(t)$. Since $u(t,x) = 0$ on this set, we claim that $\lim\limits_{\mathfrak{U}(t)} u(t,\cdot) = 0$. This contradicts (7.4), therefore, we conclude that $V_{\varepsilon+N|t-t_0|} \in \mathfrak{U}(t)$. Hence, the ultrafilter $\mathfrak{U}(t)$ is bounded and $x(t) \doteq \lim \mathfrak{U}(t)$ lays in the closure of $V_{\varepsilon+N|t-t_0|}$. This implies that $|x(t)-x(t_0)| \le \varepsilon + N|t-t_0|$. Since $\varepsilon > 0$ is arbitrary, we conclude that $|x(t) - x(t_0)| \le N|t-t_0|$. $\square$

Remark that the curves $x = x(t) = \lim \mathfrak{U}(t)$, $t \in \mathbb{R}$ can be treated as the projection of a generalized characteristic $\mathfrak{U}(t)$ on the “physical” space $\mathbb{R}^n$. In some sense $x(t)$ can be interpreted as a solution of characteristic system (2.1). As opposed to classic solutions, $x(t)$ is not uniquely determined by $(t_0, x(t_0))$, actually it is determined by a point $(t_0, \mathfrak{U}(t_0))$.

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