Generic Regularity of Conservative Solutions to a Nonlinear Wave Equation

Alberto Bressan[(*)] and Geng Chen[(**)],

(*) Department of Mathematics, Penn State University, University Park, Pa. 16802, U.S.A.
(**) School of Mathematics, Georgia Institute of Technology, Atlanta, Ga. 30332, U.S.A.

e-mails: bressan@math.psu.edu , gchen73@math.gatech.edu

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Abstract

The paper is concerned with conservative solutions to the nonlinear wave equation $u_{tt}-c(u)\big(c(u)u_x\big)_x = 0$. For an open dense set of $\mathcal{C}^3$ initial data, we prove that the solution is piecewise smooth in the t-x plane, while the gradient u_x can blow up along finitely many characteristic curves. The analysis is based on a variable transformation introduced in [7], which reduces the equation to a semilinear system with smooth coefficients, followed by an application of Thom's transversality theorem.

1 Introduction

Consider the quasilinear second order wave equation

$$u_{tt} - c(u)(c(u)u_x)_x \;=\; 0\,, \qquad\qquad t \in [0,T], \quad x \in \mathbb{R}\,. \tag{1.1}$$

On the wave speed c we assume

(A) The map $c : \mathbb{R} \mapsto \mathbb{R}_+$ is smooth and uniformly positive. The quotient $c'(u)/c(u)$ is uniformly bounded. Moreover, the following generic condition is satisfied:

$$c'(u) \;=\; 0 \qquad \Longrightarrow \qquad c''(u) \;\neq\; 0. \tag{1.2}$$

Notice that, by (1.2), the derivative $c'(u)$ vanishes only at isolated points.

The analysis in [7, 3] shows that, for any initial data

$$u(0,x) \;=\; u_0(x)\,, \qquad u_t(0,x) \;=\; u_1(x)\,, \tag{1.3}$$

with $u_0 \in H^1(\mathbb{R})$, $u_1 \in \mathbf{L}^2(\mathbb{R})$, the Cauchy problem admits a unique conservative solution $u = u(t,x)$, Hölder continuous in the t-x plane. We recall that conservative solutions satisfy an additional conservation law for the energy, so that the total energy

$$\mathcal{E}(t) \;=\; \frac{1}{2}\int [u_t^2 + c^2(u)u_x^2]\,dx$$

coincides with a constant for a.e. time t. A detailed construction of a global semigroup of these solutions, including more singular initial data, was carried out in [15].

In the present paper we study the structure of these solutions. Roughly speaking, we prove that, for generic smooth initial data (u_0, u_1), the solution is piecewise smooth. Its gradient u_x blows up along finitely many smooth curves in the t-x plane. Our main result is

Theorem 1. *Let the function $u \mapsto c(u)$ satisfy the assumptions* **(A)** *and let $T > 0$ be given. Then there exists an open dense set of initial data*

$$\mathcal{D} \;\subset\; \Big(\mathcal{C}^3(\mathbb{R}) \cap H^1(\mathbb{R})\Big) \times \Big(\mathcal{C}^2(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})\Big)$$

such that, for $(u_0, u_1) \in \mathcal{D}$, the conservative solution $u = u(t,x)$ of (1.1)-(1.3) is twice continuously differentiable in the complement of finitely many characteristic curves γ_i, within the domain $[0,T] \times \mathbb{R}$.

For the scalar conservation law in one space dimension, a well known result by Schaeffer [17] shows that generic solutions are piecewise smooth, with finitely many shocks on any bounded domain in the t-x plane. A similar result was proved by Dafermos and Geng [8], for a special 2×2 Temple class system of conservation laws. It remains an outstanding open problem to understand whether generic solutions to more general 2×2 systems (such as the p-system of isentropic gas dynamics) remain piecewise smooth, with finitely many shock curves.

The proof in [17] relies on the Hopf-Lax representation formula, while the proof in [8] is based on the analysis of solutions along characteristics. In the present paper we take a quite different approach, based on the representation of solutions in terms of a semilinear system introduced in [7]. In essence, the analysis in [7] shows that, after a suitable change of variables, the quantities

$$w \;\doteq\; 2\arctan(u_t + c(u)u_x), \qquad\qquad z \;\doteq\; 2\arctan(u_t - c(u)u_x),$$

satisfy a semilinear system of equations, w.r.t. new independent variables X, Y. See (2.16)–(2.20) in Section 2 for details. Since this system has smooth coefficients, starting with smooth initial data one obtains a globally defined smooth solution. To recover the singularities of the solution u of (1.1) in the original t-x plane, it now suffices to study the level sets

$$\{w(X,Y) = \pi\}\,, \qquad\qquad \{z(X,Y) = \pi\}\,. \tag{1.4}$$

Since w and z are smooth, the generic structure of these level sets can be analyzed by techniques of singularity theory [2, 9, 10, 14, 18], relying on Thom's transversality theorem. One should be aware that, while the map $(X,Y) \mapsto (t,x,u,w,z)$ is smooth, the inverse map $(t,x) \mapsto (X,Y)$ can have singularities. This variable transformation is indeed the source of singularities in the solution $u = u(t,x)$ of (1.1).

The present work was motivated by a research program aimed at the construction of a distance which renders Lipschitz continuous the semigroup of conservative solutions of (1.1). Toward this goal, one needs a dense set of piecewise smooth paths of solutions, whose weighted length can be controlled in time. In the final section of this paper we thus consider a 1-parameter family of initial data $\lambda \mapsto (u_0^\lambda, u_1^\lambda)$, with $\lambda \in [0,1]$. We show that it can be uniformly approximated by a second path of initial data $\lambda \mapsto (\tilde{u}_0^\lambda, \tilde{u}_1^\lambda)$, such that the corresponding solutions $\tilde{u}^\lambda = \tilde{u}^\lambda(t,x)$ of (1.1) are piecewise smooth in the domain $[0,T] \times \mathbb{R}$, for all except at most finitely values of $\lambda \in [0,1]$. An application of this result to the construction of a Lipschitz metric will appear in the forthcoming paper [4].

The remainder of the paper is organized as follows. In Section 2 we review the variable change introduced in [7] and derive the semilinear system used in the construction of conservative solutions to (1.1). In Section 3 we construct families of smooth solutions to the semilinear system, depending on parameters. By a transversality argument, in Section 4 we show that for almost all of these solutions the level sets (1.4) satisfy a number of generic properties. After these preliminaries, the proof of Theorem 1 is completed in Section 5. Finally, in Section 6 we prove a theorem on generic regularity for 1-parameter family of solutions.

For the nonlinear equation (1.1), the formation of singularities in finite time was first studied in [11]. Based on the representations [7, 5], a detailed asymptotic description of structurally stable singularities is given in [6], for conservative as well as dissipative solutions.

We conjecture that the regularity property stated in Theorem 1 should also hold for generic dissipative solutions of (1.1). However, in the dissipative case the corresponding semilinear system derived in [5] contains discontinuous terms, and smooth initial data do not yield globally smooth solutions. For this reason, the techniques used in this paper can no longer be applied. We remark that, at the present time, the uniqueness and continuous dependence of dissipative solutions to (1.1) has not yet been proved, for general initial data $(u_0, u_1) \in H^1(\mathbb{R}) \times \mathbf{L}^2(\mathbb{R})$.

2 Review of the main equations

Consider the variables

$$\begin{cases} R \doteq u_t + c(u)u_x\,, \\ S \doteq u_t - c(u)u_x\,, \end{cases} \tag{2.1}$$

so that

$$u_t = \frac{R+S}{2}\,, \qquad u_x = \frac{R-S}{2c}\,. \tag{2.2}$$

For a smooth solution of (1.1), these variables satisfy

$$\begin{cases} R_t - cR_x = \frac{c'}{4c}(R^2 - S^2), \\ S_t + cS_x = \frac{c'}{4c}(S^2 - R^2). \end{cases} \tag{2.3}$$

In addition, R^2 and S^2 satisfy the balance laws

$$\begin{cases} (R^2)_t - (cR^2)_x = \frac{c'}{2c}(R^2S - RS^2)\,, \\ (S^2)_t + (cS^2)_x = \frac{c'}{2c}(S^2R - SR^2)\,. \end{cases} \tag{2.4}$$

As a consequence, for smooth solutions the following quantity is conserved:

$$E \doteq \frac{1}{2}(u_t^2 + c^2u_x^2) = \frac{R^2+S^2}{4}\,. \tag{2.5}$$

One can think of R^2 and S^2 as the energy densities of backward and forward moving waves, respectively. Notice that these are not separately conserved. Indeed, by (2.4) energy can be exchanged between forward and backward waves.

It is well known that, even for smooth initial data, the quantities u_t, u_x can blow up in finite time [11]. To deal with possibly unbounded values of R, S, following [7] it is convenient to introduce a new set of dependent variables:

$$w \doteq 2\arctan R\,, \qquad\qquad z \doteq 2\arctan S\,. \tag{2.6}$$

Using (2.3), we obtain the equations

$$w_t - c\,w_x = \frac{2}{1+R^2}(R_t - c\,R_x) = \frac{c'}{2c}\frac{R^2-S^2}{1+R^2}\,, \tag{2.7}$$

$$z_t + c\,z_x = \frac{2}{1+S^2}(S_t + c\,S_x) = \frac{c'}{2c}\frac{S^2-R^2}{1+S^2}\,. \tag{2.8}$$

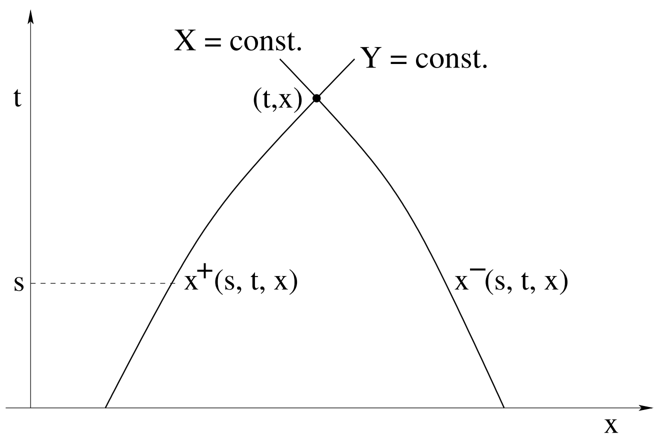

Figure 1: Characteristic curves. As new coordinates of the point (t,x) we choose the values $(X,Y) = \big(x^-(0,t,x)\,,\ -x^+(0,t,x)\big)$.

We now perform a further change of independent variables (Fig. 1). Consider the equations for the backward and forward characteristics:

$$\dot{x}^- = -\,c(u)\,, \qquad\qquad \dot{x}^+ = c(u)\,, \tag{2.9}$$

where the upper dot denotes a derivative w.r.t. time. The characteristics passing through the point (t,x) will be denoted by

$$s \mapsto x^-(s,t,x)\,, \qquad\qquad s \mapsto x^+(s,t,x)\,,$$

respectively. As coordinates (X,Y) of a point (t,x) we shall use the intersections of these characteristics with the x-axis, namely

$$X \doteq x^-(0,t,x)\,, \qquad\qquad Y \doteq -\,x^+(0,t,x)\,. \tag{2.10}$$

Of course this implies

$$X_t - c(u)X_x = 0\,, \qquad\qquad Y_t + c(u)Y_x = 0\,, \tag{2.11}$$

$$(X_x)_t - (c\,X_x)_x \;=\; 0\,, \qquad\qquad (Y_x)_t + (c\,Y_x)_x \;=\; 0\,. \tag{2.12}$$

For any smooth function f, using (2.11) one finds

$$\left\{\begin{array}{rclclcl} f_t + cf_x &=& f_X X_t + f_Y Y_t + cf_X X_x + cf_Y Y_x &=& (X_t + cX_x) f_X &=& 2cX_x f_X\,,\\[2mm] f_t - cf_x &=& f_X X_t + f_Y Y_t - cf_X X_x - cf_Y Y_x &=& (Y_t - cY_x) f_Y &=& -2cY_x f_Y\,. \end{array}\right. \tag{2.13}$$

We now introduce the further variables

$$p \;\doteq\; \frac{1+R^2}{X_x}\,, \qquad\qquad q \;\doteq\; \frac{1+S^2}{-Y_x}\,. \tag{2.14}$$

Notice that the above definitions imply

$$\frac{1}{X_x} \;=\; \frac{p}{1+R^2} \;=\; p\,\cos^2\frac{w}{2}\,, \qquad\qquad \frac{-1}{Y_x} \;=\; \frac{q}{1+S^2} \;=\; q\,\cos^2\frac{z}{2}\,. \tag{2.15}$$

Starting with the nonlinear equation (1.1), using X, Y as independent variables one obtains a semilinear hyperbolic system with smooth coefficients for the variables u, w, z, p, q, namely

$$\left\{\begin{array}{rcl} u_X &=& \frac{\sin w}{4c}\,p\,,\\[2mm] u_Y &=& \frac{\sin z}{4c}\,q\,, \end{array}\right. \tag{2.16}$$

$$\left\{\begin{array}{rcl} w_Y &=& \frac{c'}{8c^2}\,(\cos z - \cos w)\,q\,,\\[2mm] z_X &=& \frac{c'}{8c^2}\,(\cos w - \cos z)\,p\,, \end{array}\right. \tag{2.17}$$

$$\left\{\begin{array}{rcl} p_Y &=& \frac{c'}{8c^2}\,(\,\sin z - \sin w)\,pq\,,\\[2mm] q_X &=& \frac{c'}{8c^2}\,(\,\sin w - \sin z)\,pq\,. \end{array}\right. \tag{2.18}$$

The map $(X, Y) \mapsto (t, x)$ can be constructed as follows. Setting $f = x$, then $f = t$ in the two equations at (2.13), we find

$$\left\{\begin{array}{rcl} c &=& 2cX_x\,x_X\,,\\[2mm] -c &=& -2cY_x\,x_Y\,, \end{array}\right. \qquad\qquad \left\{\begin{array}{rcl} 1 &=& 2cX_x\,t_X\,,\\[2mm] 1 &=& -2cY_x\,t_Y\,, \end{array}\right.$$

respectively. Therefore, using (2.15) we obtain

$$\left\{\begin{array}{rclcl} x_X &=& \frac{1}{2X_x} &=& \frac{(1+\cos w)\,p}{4}\,,\\[2mm] x_Y &=& \frac{1}{2Y_x} &=& -\frac{(1+\cos z)\,q}{4}\,, \end{array}\right. \tag{2.19}$$

$$\left\{\begin{array}{rclcl} t_X &=& \frac{1}{2cX_x} &=& \frac{(1+\cos w)\,p}{4c}\,,\\[2mm] t_Y &=& \frac{1}{-2cY_x} &=& \frac{(1+\cos z)\,q}{4c}\,. \end{array}\right. \tag{2.20}$$

Given the initial data (1.3), the corresponding boundary data for (2.17)-(2.20) can be determined as follows. In the X-Y plane, consider the line

$$\gamma_0 \;=\; \{X + Y = 0\} \;\subset\; \mathbb{R}^2$$

parameterized as $x \mapsto (\overline{X}(x), \overline{Y}(x)) \doteq (x, -x)$. Along γ_0 we can assign the boundary data $(\overline{u}, \overline{w}, \overline{z}, \overline{p}, \overline{q})$ by setting

$$\overline{u} = u_0(x), \qquad \begin{cases} \overline{w} = 2\arctan R(0,x), \\ \overline{z} = 2\arctan S(0,x), \end{cases} \qquad \begin{cases} \overline{p} \equiv 1 + R^2(0,x), \\ \overline{q} \equiv 1 + S^2(0,x), \end{cases} \tag{2.21}$$

at each point $(x, -x) \in \gamma_0$. We recall that, at time $t = 0$, by (1.3) one has

$$\begin{aligned} R(0,x) &= (u_t + c(u)u_x)(0,x) = u_1(x) + c(u_0(x))u_{0,x}(x), \\ S(0,x) &= (u_t - c(u)u_x)(0,x) = u_1(x) - c(u_0(x))u_{0,x}(x). \end{aligned}$$

Remark 1. Since the semilinear system (2.17)–(2.20) has smooth coefficients, for smooth initial data all components of the solution remain smooth on the entire X-Y plane. As proved in [7], the quadratic terms in (2.18) (containing the product pq) account for transversal wave interactions and do not produce finite time blow up of the variables p, q. Moreover, if the values of p, q are uniformly positive along a line $\{X + Y = \kappa\}$, then they remain uniformly positive on compact sets of the X-Y plane. Throughout this paper, we always consider solutions of (2.17)–(2.20) where $p, q > 0$.

By expressing the solution $u(X, Y)$ in terms of the original variables (t, x), one obtains a solution of the Cauchy problem (1.1)-(1.3). Indeed, the following was proved in [7].

Lemma 1. *Let (u, w, z, p, q, x, t) be a smooth solution to the system (2.16)–(2.20), with $p, q > 0$. Then the set of points*

$$\Big\{(t(X,Y),\, x(X,Y),\, u(X,Y))\,; \quad (X,Y) \in \mathbb{R}^2\Big\} \tag{2.22}$$

is the graph of a conservative solution to the variational wave equation (1.1).

We observe that, while the functions

$$(X,Y) \mapsto u(X,Y), \qquad (X,Y) \mapsto \Lambda(X,Y) \doteq (t(X,Y),\, x(X,Y)) \tag{2.23}$$

are globally smooth, the map $\Lambda : \mathbb{R}^2 \mapsto \mathbb{R}^2$ may not have a smooth inverse. Indeed, Λ may not even be one-to-one. Therefore, the solution $u(t,x) = u(\Lambda^{-1}(t,x))$ can fail to be smooth. This happens precisely at points where the Jacobian matrix $D\Lambda$ is not invertible. By (2.19)-(2.20), singularities occur when $\cos w = -1$ or $\cos z = -1$.

Remark 2. The system (2.17)–(2.20) is invariant under translation by 2π in w and z. We can thus think of w, z as points in the quotient manifold $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Throughout the following we take advantage of this fact and regard a solution of (2.17)–(2.20) as a map $(X,Y) \mapsto (u, w, z, p, q, x, t)$ from $\mathbb{R}^2$ into $\mathbb{R} \times \mathbb{T} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Observe that we have the implications

$$\begin{aligned} w \neq \pi \quad &\Longrightarrow \quad \cos w > -1, \\ z \neq \pi \quad &\Longrightarrow \quad \cos z > -1. \end{aligned} \tag{2.24}$$

Remark 3. In general, many distinct solutions to the system (2.16)–(2.20) can yield the same solution $u = u(t, x)$ of (1.1).

Indeed, let $(u, w, z, p, q, x, t)(X, Y)$ be one particular solution. Let $\phi, \psi : \mathbb{R} \mapsto \mathbb{R}$ be two $\mathcal{C}^2$ bijections, with $\phi' > 0$ and $\psi' > 0$. Introduce the new independent and dependent variables $(\widetilde{X}, \widetilde{Y})$ and $(\tilde{u}, \tilde{w}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{x}, \tilde{t})$ by setting

$$X = \phi(\widetilde{X}), \qquad\qquad Y = \psi(\widetilde{Y}), \tag{2.25}$$

$$(\tilde{u}, \tilde{w}, \tilde{z}, \tilde{x}, \tilde{t})(\widetilde{X}, \widetilde{Y}) = (u, w, z, p, q, x, t)(X, Y), \tag{2.26}$$

$$\begin{cases} \tilde{p}(\widetilde{X}, \widetilde{Y}) = p(X, Y) \cdot \phi'(\widetilde{X}), \\[2mm] \tilde{q}(\widetilde{X}, \widetilde{Y}) = q(X, Y) \cdot \psi'(\widetilde{Y}). \end{cases} \tag{2.27}$$

Then, as functions of $(\widetilde{X}, \widetilde{Y})$, the variables $(\tilde{u}, \tilde{w}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{x}, \tilde{t})$ provide another solution of the same system (2.16)–(2.20). Moreover, by (2.26) the set

$$\Big\{ \Big(\tilde{t}(\widetilde{X}, \widetilde{Y}),\ \tilde{x}(\widetilde{X}, \widetilde{Y}),\ \tilde{u}(\widetilde{X}, \widetilde{Y}) \Big) ; \quad (\widetilde{X}, \widetilde{Y}) \in \mathbb{R}^2 \Big\} \tag{2.28}$$

coincides with the set (2.22). Hence it is the graph of the same solution u of (1.1). One can regard the variable transformation (2.25) simply as a relabeling of forward and backward characteristics, in the solution u. A detailed analysis of relabeling symmetries, in connection with the Camassa-Hom equation, can be found in [13].

For future reference we observe that

$$\tilde{w}_{\widetilde{X}}(\widetilde{X}, \widetilde{Y}) = w_X(X, Y) \cdot \phi'(\widetilde{X}),$$

$$\tilde{w}_{\widetilde{X}\widetilde{X}}(\widetilde{X}, \widetilde{Y}) = w_{XX}(X, Y) \cdot [\phi'(\widetilde{X})]^2 + w_X(X, Y) \cdot \phi''(\widetilde{X}).$$

In particular, one has the equivalences

$$\begin{aligned} \tilde{w}_{\widetilde{X}}(\widetilde{X}, \widetilde{Y}) = 0 \qquad &\Longleftrightarrow \qquad w_X(X, Y) = 0, \\ \tilde{z}_{\widetilde{Y}}(\widetilde{X}, \widetilde{Y}) = 0 \qquad &\Longleftrightarrow \qquad z_Y(X, Y) = 0, \\ (\tilde{w}_{\widetilde{X}},\, \tilde{w}_{\widetilde{X}\widetilde{X}})(\widetilde{X}, \widetilde{Y}) = (0, 0) \qquad &\Longleftrightarrow \qquad (w_X, w_{XX})(X, Y) = (0, 0), \\ (\tilde{z}_{\widetilde{Y}},\, \tilde{z}_{\widetilde{Y}\widetilde{Y}})(\widetilde{X}, \widetilde{Y}) = (0, 0) \qquad &\Longleftrightarrow \qquad (z_Y, z_{YY})(X, Y) = (0, 0). \end{aligned} \tag{2.29}$$

2.1 Compatible boundary data

More generally, instead of (2.21) we can assign boundary data for the system (2.13)–(2.18) on a line $\gamma = \{X + Y = \kappa\}$. Namely:

$$u(s, \kappa - s) = \overline{u}(s), \qquad \begin{cases} w(s, \kappa - s) = \overline{w}(s), \\ z(s, \kappa - s) = \overline{z}(s), \end{cases} \qquad \begin{cases} p(s, \kappa - s) = \overline{p}(s), \\ q(s, \kappa - s) = \overline{q}(s), \end{cases} \tag{2.30}$$

for suitable smooth functions $\overline{u}, \overline{w}, \overline{z}, \overline{p}, \overline{q}$. If both identities in (2.16) hold, then

$$\frac{d}{ds}\overline{u}(s) \;=\; \frac{d}{ds}u(s,\kappa-s) \;=\; u_X - u_Y \;=\; \frac{\sin w}{4c}\,p - \frac{\sin z}{4c}\,q\,. \tag{2.31}$$

The boundary data should thus satisfy the compatibility condition

$$\frac{d}{ds}\overline{u}(s) \;=\; \frac{\sin \overline{w}(s)}{4c(\overline{u}(s))}\,\overline{p}(s) - \frac{\sin \overline{z}(s)}{4c(\overline{u}(s))}\,\overline{q}(s) \tag{2.32}$$

As remarked earlier, the system (2.16)–(2.18) is overdetermined. Indeed, the function $u = u(X,Y)$ could be recovered by either one of the identities in (2.16). We now prove that, if the compatibility condition (2.32) holds, then any smooth solution satisfying one of the identities in (2.16) satisfies the other as well.

Lemma 2. *Let u, w, z, p, q be smooth functions on $\mathbb{R}^2$ which satisfy (2.17)-(2.18) together with the boundary conditions (2.30) along the line $\gamma = \{X+Y=\kappa\}$. Assume that the compatibility condition (2.32) holds. Then one has*

$$u_Y \;=\; \frac{\sin z}{4c(u)}\,q \qquad \textit{for all}\;\; (X,Y)\in\mathbb{R}^2 \tag{2.33}$$

if and only if

$$u_X \;=\; \frac{\sin w}{4c(u)}\,p \qquad \textit{for all}\;\; (X,Y)\in\mathbb{R}^2. \tag{2.34}$$

Proof. Consider the smooth, strictly increasing function

$$\Phi(u) \;=\; \int_0^u 4c(s)\,ds\,.$$

Observe that the identities (2.33), (2.34) are equivalent respectively to

$$\Phi(u)_Y \;=\; \sin z\cdot q\,, \qquad\qquad \Phi(u)_X \;=\; \sin w\cdot p\,. \tag{2.35}$$

Assume that (2.33) holds. Then

$$\Phi(u(X,Y)) \;=\; \Phi(u(X,\,\kappa-X)) + \int_{\kappa-X}^{Y} [\sin z\cdot q](X,s)\,ds\,. \tag{2.36}$$

Differentiating w.r.t. X, and using the first equations in (2.17)-(2.18) together with the compatibility condition (2.32) we obtain

$$\begin{aligned}
\Phi(u)_X(X,Y) \;&=\; \Phi'(u)\cdot[u_X-u_Y](X,\kappa-X) + [\sin z\cdot q](X,\kappa-X)\\
&\qquad + \int_{\kappa-Y}^{Y}[\cos z\cdot z_X q + \sin z\cdot q_X](X,s)\,ds\\
&=\; [\sin w\cdot p](X,\kappa-X) + \int_{\kappa-Y}^{Y}\Big[\frac{c'(u)}{8c^2(u)}(1+\sin(z+w))pq\Big](X,s)\,ds\\
&=\; [\sin w\cdot p](X,\kappa-X) + \int_{\kappa-Y}^{Y}\frac{\partial}{\partial Y}[\sin w\cdot p](X,s)\,ds \;=\; [\sin w\cdot p](X,Y)\,.
\end{aligned} \tag{2.37}$$

We have thus proved that the second identity in (2.35) holds. This is equivalent to (2.34). The converse implication is proved in the same way. □

Next, consider initial data for t, x, on the curve $\gamma = \{X + Y = \kappa\}$, say

$$x(s, \kappa - s) \; = \; \overline{x}(s)\,, \qquad\qquad t(s, \kappa - s) \; = \; \overline{t}(s)\,. \tag{2.38}$$

Using (2.19)-(2.20) we derive the compatibility conditions

$$\frac{d}{ds}\overline{x}(s) \; = \; \frac{(1+\cos\overline{q}(s))\overline{p}(s) + (1+\cos\overline{z}(s))\overline{q}(s)}{4}\,, \tag{2.39}$$

$$\frac{d}{ds}\overline{t}(s) \; = \; \frac{(1+\cos\overline{w}(s))\overline{p}(s) - (1+\cos\overline{z}(s))\overline{q}(s)}{4c(\overline{u}(s))}\,. \tag{2.40}$$

Lemma 3. *Let $(u, w, z, p, q)(X, Y)$ be a solution of the system (2.16)–(2.18). Then there exists a solution $(t, x)(X, Y)$ of (2.19)-(2.20) with boundary data (2.38) if and only if the compatibility conditions (2.39)-(2.40) are satisfied.*

Proof. 1. Assume that the equations (2.19)-(2.20) are satisfied for all $(X, Y) \in \mathbb{R}^2$. In particular, they are satisfied along the curve $\gamma = \{X + Y = \kappa\}$. This implies

$$\frac{d}{ds}\,\overline{x}(s) \; = \; \frac{d}{ds}\,\overline{x}(s, \kappa - s) \; = \; [x_X - x_Y](s, \kappa - s) \; = \; \frac{(1+\cos w)p + (1+\cos z)q}{4}\,,$$

where the right hand side is evaluated at $(X, Y) = (x, \kappa - s)$. Hence (2.39) holds. The identity (2.40) is derived in the same way.

2. Next, assume that the compatibility conditions (2.39)-(2.40) are satisfied. To prove that (2.19) admits a solution, it then suffices to check that the differential form

$$\frac{(1+\cos w)p}{4}\,dX - \frac{(1+\cos z)q}{4}\,dY$$

is closed. This is true because

$$\begin{aligned}
\left[\frac{(1+\cos w)p}{4}\right]_Y \; &= \; -\frac{\sin w}{4} w_Y\, p + \frac{1+\cos w}{4}\,p_Y \\
&= \; -\frac{\sin w}{4} \cdot \frac{c'}{8c^2}(\cos z - \cos w)\,pq + \frac{1+\cos w}{4} \cdot \frac{c'}{8c^2}(\sin z - \sin w)\,pq \\
&= \; \frac{c'}{32c^2}\Big[(1+\cos w)\sin z - (1+\cos z)\sin w\Big]\,pq \; = \; \left[-\frac{(1+\cos z)q}{4}\right]_X.
\end{aligned} \tag{2.41}$$

Similarly, to prove that (2.20) admits a solution, it suffices to check that the differential form

$$\frac{(1+\cos w)p}{4c}\,dX + \frac{(1+\cos z)q}{4c}\,dY$$

is closed. This is true because

$$\begin{aligned}
\left[\frac{(1+\cos w)p}{4c}\right]_Y &= -\frac{\sin w}{4c}w_Y\, p - \frac{1+\cos w}{4c^2}c' u_Y\, p + \frac{1+\cos w}{4c}\, p_Y \\
= \; -\frac{\sin w}{4c}\cdot\frac{c'}{8c^2}&(\cos z - \cos w)\, pq - \frac{1+\cos w}{4c^2}c'\frac{\sin z}{4c}\, pq + \frac{1+\cos w}{4c}\cdot\frac{c'}{8c^2}(\sin z - \sin w)\, pq \\
= \; -\frac{c'}{32c^3}&\Big[(1+\cos w)\sin z + (1+\cos z)\sin w\Big]\, pq \;=\; \left[\frac{(1+\cos z)q}{4c}\right]_X .
\end{aligned} \tag{2.42}$$

□

Remark 4. Let a solution (u, w, z, p, q) of (2.16)–(2.18) be given. If we assign the values of t, x at a single point (X_0, Y_0), then by the compatibility conditions (2.39)-(2.40) and the equations (2.19)-(2.20) the functions $t(X, Y)$, $x(X, Y)$ are uniquely determined for all $(X, Y) \in \mathbb{R}^2$. Choosing different values of t, x at the point (X_0, Y_0) we obtain the same solution $u = u(t, x)$ of (1.1), up to a shift of coordinates in the t, x plane.

3 Families of perturbed solutions

Let a point (X_0, Y_0) be given and consider the line

$$\gamma \doteq \Big\{(X, Y)\,; \quad X + Y = \kappa\Big\}, \qquad\qquad \kappa \doteq X_0 + Y_0\,. \tag{3.1}$$

We can then arbitrarily assign the values of w, z, p, q at every point $(X, Y) \in \gamma$. Moreover, we can arbitrarily choose the values of u, x, t at the single point (X_0, Y_0). In turn, these choices uniquely determine functions u, x, t on γ which satisfy the compatibility conditions (2.32) and (2.39)-(2.40).

Based on this observation, we can construct several families of perturbations of a given solution of (2.16)–(2.18). The main goal of this section is to prove

Lemma 4. *Let the assumption* **(A)** *hold. Let (u, w, z, p, q) be a smooth solution of the system (2.16)-(2.18) and let a point $(X_0, Y_0) \in \mathbb{R}^2$ be given.*

(1) *If $(w, w_X, w_{XX})(X_0, Y_0) = (\pi, 0, 0)$, then there exists a 3-parameter family of smooth solutions $(u^\theta, w^\theta, z^\theta, p^\theta, q^\theta)$ of (2.16)-(2.18), depending smoothly on $\theta \in \mathbb{R}^3$, such that the following holds.*

(i) When $\theta = 0 \in \mathbb{R}^3$ one recovers the original solution, namely $(u^0, w^0, z^0, p^0, q^0) = (u, w, z, p, q)$.

(ii) At the point (X_0, Y_0), when $\theta = 0$ one has

$$\text{rank } D_\theta(w^\theta\,,\ w^\theta_X\,,\ w^\theta_{XX}) \;=\; 3\,. \tag{3.2}$$

(2) *If $(w, z, w_X)(X_0, Y_0) = (\pi, \pi, 0)$, then there exists a 3-parameter family of smooth solutions $(u^\theta, w^\theta, z^\theta, p^\theta, q^\theta)$ satisfying (i)-(ii) as above, with (3.2) replaced by*

$$\text{rank } D_\theta(w^\theta\,,\ z^\theta\,,\ w^\theta_X) \;=\; 3\,. \tag{3.3}$$

(3) *If $(w, w_X, c'(u))(X_0, Y_0) = (\pi, 0, 0)$, then there exists a 3-parameter family of smooth solutions $(u^\theta, w^\theta, z^\theta, p^\theta, q^\theta)$ satisfying (i)-(ii) as above, with (3.2) replaced by*

$$\text{rank } D_\theta(w^\theta\,,\ w^\theta_X\,,\ c'(u^\theta)) \;=\; 3\,. \tag{3.4}$$

For example (3.2) means that we can construct perturbed solutions, depending on parameters $\theta_1, \theta_2, \theta_3$, such that the Jacobian matrix

$$D_\theta(w^\theta\,,\ w^\theta_X\,,\ w^\theta_{XX}) \;=\; \begin{pmatrix} \frac{\partial}{\partial\theta_1} w & \frac{\partial}{\partial\theta_2} w & \frac{\partial}{\partial\theta_3} w \\ \frac{\partial}{\partial\theta_1} w_X & \frac{\partial}{\partial\theta_2} w_X & \frac{\partial}{\partial\theta_3} w_X \\ \frac{\partial}{\partial\theta_1} w_{XX} & \frac{\partial}{\partial\theta_2} w_{XX} & \frac{\partial}{\partial\theta_3} w_{XX} \end{pmatrix}, \tag{3.5}$$

computed at $\theta = 0$, has full rank at the point (X_0, Y_0).

3.1 Proof of Lemma 4

Let (u, w, z, p, q) be a $\mathcal{C}^\infty$ solution of the system (2.16)-(2.18). Given the point (X_0, Y_0), consider the line γ in (3.1) and let $(\overline{u}, \overline{w}, \overline{z}, \overline{p}, \overline{q})$ be the values of the solution along γ, as in (2.30).

For future use, we compute the values of w_X, w_{XX} at the point (X_0, Y_0). At any point $(s, \kappa - s) \in \gamma$ we have

$$w_X - w_Y \;=\; \overline{w}'(s)\,, \qquad\qquad z_X - z_Y \;=\; \overline{z}'(s)\,, \qquad\qquad q_X - q_Y \;=\; \overline{q}'(s)\,.$$

Here and in the sequel, a prime denotes derivative w.r.t. the parameter s along the curve γ. Using (2.17)-(2.18) we obtain

$$w_X(X_0, Y_0) \;=\; \overline{w}' + \frac{c'(\overline{u})}{8c^2(\overline{u})}(\cos\overline{z} - \cos\overline{w})\overline{q}\,, \tag{3.6}$$

$$z_Y(X_0, Y_0) \;=\; -\,\overline{z}' + \frac{c'(\overline{u})}{8c^2(\overline{u})}(\cos\overline{w} - \cos\overline{z})\overline{p}\,, \tag{3.7}$$

$$q_Y(X_0, Y_0) \;=\; -\,\overline{q}' + \frac{c'(\overline{u})}{8c^2(\overline{u})}(\sin\overline{w} - \sin\overline{z})\overline{pq}\,, \tag{3.8}$$

where all terms on the right hand sides are evaluated at $s = X_0$.

A further differentiation yields

$$\frac{d^2}{ds^2}\overline{w}(s) \;=\; \frac{d}{ds}[w_X(s, \kappa - s) - w_Y(s, \kappa - s)] \;=\; [w_{XX} + w_{YY} - 2w_{XY}](s, \kappa - s).$$

Using (2.16)–(2.18) together with (3.6)–(3.8) we obtain

$$
\begin{aligned}
& w_{YX}(X_0,Y_0)\\
&= \Big(\tfrac{c'(\overline{u})}{8c^2(\overline{u})}\Big)'\,\overline{u}_X\,(\cos\overline{z}-\cos\overline{w})\,\overline{q}+\tfrac{c'}{8c^2}\,(\overline{w}_X\sin\overline{w}-\overline{z}_X\sin\overline{z})\,\overline{q}+\tfrac{c'}{8c^2}\,(\cos\overline{z}-\cos\overline{w})\,\overline{q}_X\\
&= \Big(\tfrac{c'}{8c^2}\Big)'\,\tfrac{\sin\overline{w}}{4c}\,(\cos\overline{z}-\cos\overline{w})\,\overline{pq}\\
&\qquad +\tfrac{c'}{8c^2}\,\Big\{(\overline{w}'+\tfrac{c'}{8c^2}(\cos\overline{z}-\cos\overline{w})\overline{q})\sin\overline{w}-\tfrac{c'}{8c^2}(\cos\overline{w}-\cos\overline{z})\overline{p}\sin\overline{z}\Big\}\,\overline{q}\\
&\qquad +(\tfrac{c'}{8c^2})^2\,(\cos\overline{z}-\cos\overline{w})\,(\sin\overline{w}-\sin\overline{z})\overline{pq}\\
&\doteq f_1
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
& w_{YY}(X_0,Y_0)\\
&= \Big(\tfrac{c'(\overline{u})}{8c^2(\overline{u})}\Big)'\,\overline{u}_Y\,(\cos\overline{z}-\cos\overline{w})\,\overline{q}+\tfrac{c'}{8c^2}\,(\overline{w}_Y\sin\overline{w}-\overline{z}_Y\sin\overline{z})\,\overline{q}+\tfrac{c'}{8c^2}\,(\cos\overline{z}-\cos\overline{w})\,\overline{q}_Y\\
&= \Big(\tfrac{c'}{8c^2}\Big)'\,\tfrac{\sin\overline{z}}{4c}\,(\cos\overline{z}-\cos\overline{w})\,\overline{q}^2\\
&\qquad +\tfrac{c'}{8c^2}\,\Big\{\tfrac{c'}{8c^2}(\cos\overline{z}-\cos\overline{w})\overline{q}\sin\overline{w}-(\,-\overline{z}'+\tfrac{c'}{8c^2}(\cos\overline{w}-\cos\overline{z})\overline{p})\sin\overline{z}\Big\}\,\overline{q}\\
&\qquad +\tfrac{c'}{8c^2}\,(\cos\overline{z}-\cos\overline{w})\,(\,-\overline{q}'+\tfrac{c'}{8c^2}(\sin\overline{w}-\sin\overline{z})\overline{pq})\\
&\doteq f_2\,.
\end{aligned}
\tag{3.10}
$$

Hence

$$
w_{XX}(X_0,Y_0)\;=\;\overline{w}''+2f_1-f_2\,. \tag{3.11}
$$

We now construct families $(\overline{u}^\theta,\overline{q}^\theta,\overline{z}^\theta,\overline{p}^\theta,\overline{q}^\theta)$ of perturbations of the data (2.30) along the curve γ, so that at the point (X_0,Y_0) the matrices in (3.2)–(3.4) have full rank. These perturbations will have the form

$$
\left\{\begin{aligned}
\overline{w}^\theta(s) &= \overline{w}(s)+\textstyle\sum_{i=1}^3\theta_iW_i(s)\,,\\
\overline{z}^\theta(s) &= \overline{z}(s)+\textstyle\sum_{i=1}^3\theta_iZ_i(s)\,,
\end{aligned}\right.
\qquad
\left\{\begin{aligned}
\overline{p}^\theta(s) &= \overline{p}(s)+\textstyle\sum_{i=1}^3\theta_iP_i(s)\,,\\
\overline{q}^\theta(s) &= \overline{q}(s)+\textstyle\sum_{i=1}^3\theta_iQ_i(s)\,,
\end{aligned}\right.
\tag{3.12}
$$

for suitable functions $W_i,Z_i,P_i,Q_i\in\mathcal{C}^\infty_c$. Moreover, at the point $s=X_0$ we set

$$
\overline{u}^\theta(X_0)\;=\;\overline{u}(X_0)+\sum_{i=1,2,3}\theta_iU_i\,. \tag{3.13}
$$

In turn, the above definitions together with the compatibility conditions (2.32) determine the values of $\overline{u}^\theta(s)$ for all $s\in\mathbb{R}$. In particular, for each $\theta\in\mathbb{R}^3$ we obtain a unique solution of the semilinear system (2.16)–(2.18).

We observe that the functions W_i, Z_i, P_i, Q_i can be chosen arbitrarily. Hence at the point $s = X_0$, $\theta = 0$, we can arbitrarily assign all derivatives

$$\frac{d}{d\theta}\frac{d^k}{ds^k}\overline{w}^\theta, \qquad \frac{d}{d\theta}\frac{d^k}{ds^k}\overline{z}^\theta, \qquad \frac{d}{d\theta}\frac{d^k}{ds^k}\overline{p}^\theta, \qquad \frac{d}{d\theta}\frac{d^k}{ds^k}\overline{q}^\theta,$$

with $k = 0, 1, 2, \ldots$ Moreover, we can arbitrarily choose the quantity $\frac{d}{d\theta}\overline{u}^\theta(X_0)$, while all higher order derivatives $\frac{d}{d\theta}\frac{d^k}{ds^k}\overline{u}^\theta$, with $k \geq 1$, are then determined by the compatibility condition (2.32).

1. To achieve (3.2), we choose perturbations $(\overline{u}^{\theta_i}, \overline{w}^{\theta_i}, \overline{z}^{\theta_i}, \overline{p}^{\theta_i}, \overline{q}^{\theta_i})$, $i = 1, 2, 3$, so that the Jacobian matrix of first order derivatives w.r.t. $\theta_1, \theta_2, \theta_3$, computed at $s = X_0$ and $\theta = 0$, is given by

$$D_\theta \begin{pmatrix} \overline{u} \\ \overline{w} \\ \overline{z} \\ \overline{z}' \\ \overline{w}' \\ \overline{w}'' \\ \overline{p} \\ \overline{q} \\ \overline{q}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At the point (X_0, Y_0), by (3.6) and (3.11) this yields

$$D_\theta \begin{pmatrix} w \\ w_X \\ w_{XX} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

Notice that, for the third family of perturbations (corresponding to the third column), the first order variations of f_1 and f_2 in (3.9)-(3.10) both vanish at (X_0, Y_0). This achieves (3.2).

2. To achieve (3.3), we choose perturbations $(\overline{u}^{\theta_i}, \overline{w}^{\theta_i}, \overline{z}^{\theta_i}, \overline{p}^{\theta_i}, \overline{q}^{\theta_i})$, $i = 1, 2, 3$, so that at $s = X_0$ and $\theta = 0$ one has

$$D_\theta \begin{pmatrix} \overline{u} \\ \overline{w} \\ \overline{z} \\ \overline{w}' \\ \overline{p} \\ \overline{q} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.14}$$

At the point (X_0, Y_0), by (3.6) this yields

$$D_\theta \begin{pmatrix} w \\ z \\ w_X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}. \tag{3.15}$$

Hence (3.3) holds.

3. Finally, we construct three families of perturbations satisfying (3.4). If at (X_0, Y_0) we have $c'(u(X_0, Y_0)) = 0$, then the assumption **(A)** implies

$$c''(u(X_0, Y_0)) \neq 0. \tag{3.16}$$

To achieve (3.4), we choose three families of perturbations such that, at $s = X_0$ and $\theta = 0$,

$$D_\theta \begin{pmatrix} \overline{u} \\ \overline{w} \\ \overline{z} \\ \overline{w}' \\ \overline{p} \\ \overline{q} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

At the point (X_0, Y_0), by (3.6) and the first equation in (2.17), this yields

$$D_\theta \begin{pmatrix} w \\ w_X \\ c'(u) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & c''(u) \end{pmatrix} \tag{3.18}$$

This achieves (3.4). □

4 Generic solutions of the semilinear system

In this section we study smooth solutions to the semilinear system (2.17)–(2.20), determining the generic structure of the level sets $\{(X,Y)\,;\ \ w(X,Y) = \pi\}$ and $\{(X,Y)\,;\ \ z(X,Y) = \pi\}$.

Lemma 5. *Let the function $u \mapsto c(u)$ satisfy the assumptions* **(A)** *and consider a compact domain of the form*

$$\Gamma \doteq \Big\{(X,Y)\,;\ \ |X| + |Y| \leq M\Big\}. \tag{4.1}$$

Call $\mathcal{S}$ the family of all $\mathcal{C}^2$ solutions to the system (2.16)–(2.18), with $p, q > 0$ for all $(X,Y) \in \mathbb{R}^2$. Moreover, call $\mathcal{S}' \subset \mathcal{S}$ the subfamily of all solutions (u, w, z, p, q) such that, for $(X,Y) \in \Gamma$, none of the following values is attained:

$$\begin{cases} (w, w_X, w_{XX}) = (\pi, 0, 0), \\ (z, z_Y, z_{YY}) = (\pi, 0, 0), \end{cases} \tag{4.2}$$

$$\begin{cases} (w, z, w_X) = (\pi, \pi, 0), \\ (w, z, z_Y) = (\pi, \pi, 0), \end{cases} \tag{4.3}$$

$$\begin{cases} (w, w_X, c'(u)) = (\pi, 0, 0), \\ (z, z_Y, c'(u)) = (\pi, 0, 0). \end{cases} \tag{4.4}$$

Then $\mathcal{S}'$ is a relatively open and dense subset of $\mathcal{S}$, in the topology induced by $\mathcal{C}^2(\Gamma)$.

Some words of explanation are in order (Fig. 2). Asking that the values in (4.2) are never attained is equivalent to the implications

$$\begin{cases} w = \pi \ \text{ and } \ w_X = 0 \quad \Longrightarrow \quad w_{XX} \neq 0, \\ z = \pi \ \text{ and } \ z_Y = 0 \quad \Longrightarrow \quad z_{YY} \neq 0. \end{cases} \tag{4.5}$$

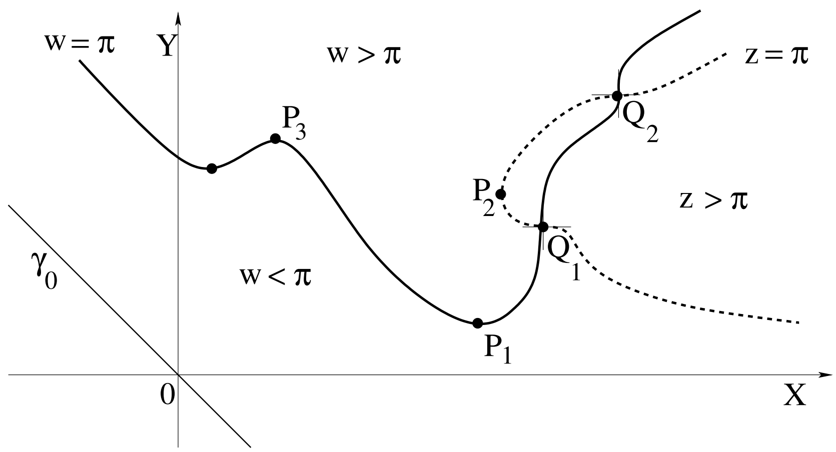

Figure 2: Two level sets $\{w = \pi\}$ and $\{z = \pi\}$, in a generic solution of (2.16)–(2.18). At P_1, P_2 one has $w = \pi$, $w_X = 0$ while the generic conditions imply $w_Y \neq 0$, $w_{XX} \neq 0$. At the points Q_1, Q_2 where the two singular curves cross, by (2.17) one has $w_Y = z_X = 0$, while the generic conditions imply $w_X \neq 0$, $z_Y \neq 0$. Hence the two curves have a perpendicular intersection.

Writing the level curves in the form $\{w(X,Y) = \pi\} = \{Y = \varphi(X)\}$ and $\{z(X,Y) = \pi\} = \{X = \psi(Y)\}$, this imposes some restrictions at the points where $\varphi' = 0$ or $\psi' = 0$.

Asking that the values in (4.3) are never attained is equivalent to the implication

$$[w = \pi \quad \text{and} \quad z = \pi] \qquad \Longrightarrow \qquad [w_X \neq 0 \quad \text{and} \quad z_Y \neq 0]. \tag{4.6}$$

This imposes restrictions at points where two level curves $\{w = \pi\}$ and $\{z = \pi\}$ cross each other.

Finally, the lemma states the existence of a perturbed solution such that values (4.4) are never attained. To understand the meaning of this condition, consider a solution which never attains any of the values in (4.3)-(4.4). In this case, by (2.24) the conditions $w = \pi$ and $w_X = 0$ together imply

$$w_Y = \frac{c'(u)}{8c^2(u)}(\cos z + 1)q \neq 0\,.$$

This is equivalent to the implication

$$w = \pi \qquad \Longrightarrow \qquad (w_X, w_Y) \neq (0,0).$$

By the implicit function theorem, the level set $\{w = \pi\}$ is then the union of regular curves in the X-Y plane (restricted to the domain Γ). Similarly, the level set $\{z = \pi\}$ will be a union of regular curves.

We shall give a proof of Lemma 5, using Lemma 4 together with Thom's transversality theorem. For readers' convenience, we first review some basic definitions [2, 12, 18].

Definition (map transverse to a submanifold). Let $f : X \mapsto Y$ be a smooth map of manifolds and let W be a submanifold of Y. We say that f is *transverse* to W at a point $p \in X$, and write $f \pitchfork_p W$, if

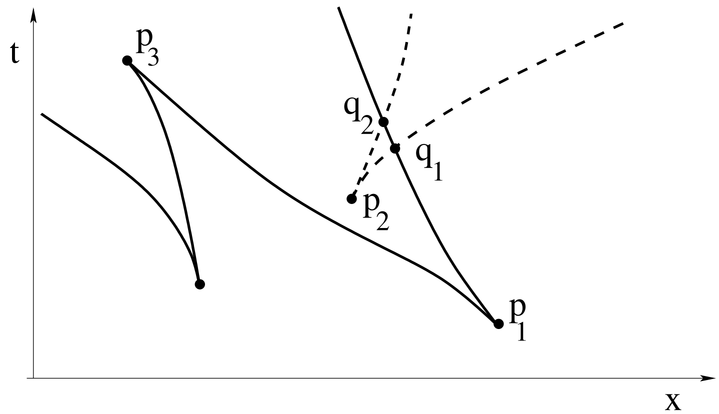

Figure 3: A generic solution $u = u(t,x)$ of (1.1) with smooth initial data remains smooth outside finitely many singular points and finitely many singular curves where $u_x \to \pm\infty$. The curves where u is singular are the images of the curves where $w = \pi$ or $z = \pi$ in Fig. 2, under the map $(t,x) = \Lambda(X,Y)$ at (2.23). Here $p_i = \Lambda(P_i)$ are points where singular curves originate or terminate, while $q_j = \Lambda(Q_j)$ are points where two singular curves cross.

- either $f(p) \notin W$,
- or else $f(p) \in W$ and $T_{f(p)}Y = (df)_p(T_pX) + T_{f(p)}W$.

Here T_pX denotes the tangent space to X at the point $p \in X$, while T_qY and T_qW denote respectively the tangent spaces to Y and to W at the point $q \in W \subset Y$. Finally, $(df)_p : T_pX \mapsto T_{f(p)}Y$ denotes the differential of the map f at the point p.

We say that f is *transverse* to W, and write $f \pitchfork W$, if $f \pitchfork_p W$ for every $p \in X$.

In the special case where $W = \{y\}$ consists of a single point, $f \pitchfork W$ if and only if y is a regular value of f, in the following sense.

Definition (regular value). Let $f : X \mapsto Y$ be a smooth map of manifolds. A point $y \in Y$ is a *regular value* if, for every $p \in X$ such that $f(p) = y$, one has

$$T_{f(p)}Y \;=\; (df)_p(T_pX).$$

Transversality Theorem. *Let X, Θ, and Y be smooth manifolds, W a submanifold of Y. Let $\theta \mapsto \phi^\theta$ be a smooth map which to each $\theta \in \Theta$ associates a function $\phi^\theta \in \mathcal{C}^\infty(X,Y)$, and define $\Phi : X \times \Theta \mapsto Y$ by setting $\Phi(x,\theta) = \phi^\theta(x)$. If $\Phi \pitchfork W$ then the set $\{\theta \in \Theta, ; \quad \phi^\theta \pitchfork W\}$ is dense in Θ.*

For a proof, see [2, 12].

4.1 Proof of Lemma 5.

1. We shall use the representation

$$S' \;=\; S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6\,, \tag{4.7}$$

where $S_1, \ldots, S_6 \subset \mathcal{S}$ are the families of solutions for which one of the six values listed in (4.2)–(4.4) is never attained on Γ. For example, $\mathcal{S}_1$ is the set of all solutions such that

$$(w, w_X, w_{XX})(X,Y) \neq (\pi, 0, 0) \qquad \text{for all } (X,Y) \in \Gamma, \tag{4.8}$$

while $\mathcal{S}_6$ is the set of all solutions such that

$$(z, z_Y, c'(u))(X,Y) \neq (\pi, 0, 0) \qquad \text{for all } (X,Y) \in \Gamma. \tag{4.9}$$

Since Γ is a compact domain, it is clear that each $\mathcal{S}_i$ is a relatively open subset of $\mathcal{S}$, in the topology of $\mathcal{C}^2(\Gamma)$. In the remainder of the proof we will show that each S_i is dense on $\mathcal{S}$.

2. Let (u, w, z, p, q) be any $\mathcal{C}^2$ solution of (2.16)–(2.18), with $p, q > 0$. By a smooth approximation of the data along the line $\gamma_0 = \{X + Y = 0\}$, it is not restrictive to assume that $u, w, z, p, q \in \mathcal{C}^\infty(\mathbb{R}^2)$. We begin by looking at the first condition in (4.2).

Given any point $(X_0, Y_0) \in \Gamma$, two cases can occur.

CASE 1: $(w, w_X, w_{XX})(X_0, Y_0) \neq (\pi, 0, 0)$. In this case, by continuity, there exists a neighborhood $\mathcal{N}$ of (X_0, Y_0) in the X-Y plane where we still have $(w, w_X, w_{XX}) \neq (\pi, 0, 0)$.

CASE 2: $(w, w_X, w_{XX})(X_0, Y_0) = (\pi, 0, 0)$. In this case, by Lemma 4 we can find a 3-parameter family of solutions $(u^\theta, w^\theta, z^\theta, p^\theta, q^\theta)$ such that the 3×3 Jacobian matrix of the map

$$(\theta_1, \theta_2, \theta_3) \mapsto \Big(w^\theta(X,Y),\ w^\theta_X(X,Y), w^\theta_{XX}(X,Y) \Big) \tag{4.10}$$

has rank 3 at the point (X_0, Y_0), when $\theta = 0$. By continuity, this matrix still has rank 3 on a neighborhood $\mathcal{N}$ of (X_0, Y_0), for θ sufficiently close to zero.

We now choose finitely many points (X_i, Y_i), $i = 1, \ldots, n$, such that the corresponding open neighborhoods $\mathcal{N}_{(X_i, Y_i)}$ cover the compact set Γ. Call $n_\mathcal{I}$ the cardinality of the set of indices

$$\mathcal{I} \doteq \{ i\,;\ \ (w, w_X, w_{XX})(X_i, Y_i) = (\pi, 0, 0) \} \tag{4.11}$$

so that CASE 2 applies, and set $N = 3n_\mathcal{I}$.

3. Let $\Omega \supset \Gamma$ be an open set contained in the union of the neighborhoods $\mathcal{N}_{(X_i, Y_i)}$, and call $B_\varepsilon \doteq \{\theta \in \mathbb{R}^N\,;\ \ |\theta| < \varepsilon\}$ the open ball of radius ε in $\mathbb{R}^N$.

We shall construct a family $(u^\theta, w^\theta, z^\theta, p^\theta, q^\theta)$ of smooth solutions to (2.16)–(2.18), such that the map

$$(X, Y, \theta) \mapsto \Big(w^\theta(X,Y),\ w^\theta_X(X,Y),\ w^\theta_{XX}(X,Y) \Big) \tag{4.12}$$

from $\Omega \times B_\varepsilon$ into $\mathbb{R}^3$ has $(\pi, 0, 0)$ as a regular value. Toward this goal, we need to combine perturbations based at possibly different points (X_i, Y_i) into a single N-parameter family of perturbed solutions.

Let $(u, w, z, p, q)(X,Y)$ be a solution to the system (2.16)–(2.18). For each $k = 1, \ldots, N$, let a point (X_k, Y_k) be given, together with a number $U_k \in \mathbb{R}$ and functions $W_k, Z_k, P_k, Q_k \in \mathcal{C}^\infty_c(\mathbb{R})$. By the previous analysis, a 1-parameter family of perturbed solutions to (2.16)–(2.18) is then determined as follows. For $|\varepsilon| < \varepsilon_k$ sufficiently small, let

$$(u^\varepsilon, w^\varepsilon, z^\varepsilon, p^\varepsilon, q^\varepsilon) \doteq \Psi^\varepsilon_k(u, w, z, p, q) \tag{4.13}$$

be the unique solution of (2.16)–(2.18) with data assigned on the line $\gamma_k \doteq \{X+Y = X_k+Y_k\}$ by setting

$$u^\varepsilon(X_k, Y_k) \;=\; u(X_k, Y_k) + \varepsilon U_k\,,$$

while for $(X, Y) \in \gamma_k$

$$w^\varepsilon \;=\; w + \varepsilon W_k\,, \qquad z^\varepsilon \;=\; z + \varepsilon Z_k\,, \qquad p^\varepsilon \;=\; p + \varepsilon P_k\,, \qquad q^\varepsilon \;=\; q + \varepsilon Q_k\,.$$

Given $(\theta_1, \ldots, \theta_N)$, a perturbation of the original solution (u, w, z, p, q) is defined as the composition of N perturbations:

$$(u^\theta, w^\theta, z^\theta, p^\theta, q^\theta) \;\doteq\; \Psi_N^{\theta_N} \circ \cdots \circ \Psi_1^{\theta_1}(u, w, z, p, q). \tag{4.14}$$

4. At each point (X_i, Y_i) with $i \in \mathcal{I}$, we can apply Lemma 4 and obtain three 1-parameter families of perturbed solutions so that the Jacobian matrix (4.10) has rank 3 on $\mathcal{N}_{(X_i,Y_i)}$, for all θ small enough.

Combining all these perturbations, we obtain an N-parameter family of solutions such that the value $(\pi, 0, 0)$ is a regular value for the map (4.12), from $\Omega \times B_\varepsilon$ into $\mathbb{R}^3$.

By the transversality theorem, for a.e. θ the value $(\pi, 0, 0)$ is a regular value for the map $(X, Y) \;\mapsto\; \Big(w^\theta(X, Y)\,,\; w^\theta_X(X, Y)\,,\; w^\theta_{XX}(X, Y)\Big)$ from Ω into $\mathbb{R}^3$. Since Ω has dimension 2, for a.e. θ the corresponding solution $(u^\theta, w^\theta, z^\theta, p^\theta, q^\theta)$ has the property that

$$(w^\theta(X, Y)\,,\; w^\theta_X(X, Y)\,,\; w^\theta_{XX}(X, Y)) \;\neq\; (\pi, 0, 0)$$

for all $(X, Y) \in \Gamma$. This proves that the set $\mathcal{S}_1$ of solutions for which (4.8) holds is dense on $\mathcal{S}$.

5. Repeating the above construction, we obtain that each $\mathcal{S}_i$, $i = 1, \ldots, 6$, is a relatively open, dense subset of $\mathcal{S}$. By (4.7), the intersection S' is is a relatively open, dense subset of $\mathcal{S}$. $\square$

5 Proof of Theorem 1.

Consider the product space

$$\mathcal{U} \;\doteq\; \Big(\mathcal{C}^3(\mathbb{R}) \cap H^1(\mathbb{R})\Big) \times \Big(\mathcal{C}^2(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})\Big) \tag{5.1}$$

with norm

$$\|(u_0, u_1)\|_{\mathcal{U}} \;\doteq\; \|u_0\|_{\mathcal{C}^3} + \|u_0\|_{H^1} + \|u_1\|_{\mathcal{C}^2} + \|u_1\|_{\mathbf{L}^2}\,.$$

Given initial data $(\hat{u}_0, \hat{u}_1) \;\in\; \mathcal{U}$, consider the open ball

$$B_\delta \;\doteq\; \Big\{(u_0, u_1) \in \mathcal{U}\,;\;\; \|(u_0, u_1) - (\hat{u}_0, \hat{u}_1)\|_{\mathcal{U}} \;<\; \delta\Big\}. \tag{5.2}$$

Theorem 1 will be proved by showing that, for any $(\hat{u}_0, \hat{u}_1) \;\in\; \mathcal{U}$ there exists a radius $\delta > 0$ and an open dense subset $\widehat{\mathcal{D}} \subseteq B_\delta$, with the following property: For every initial data $(u_0, u_1) \in \widehat{\mathcal{D}}$, the conservative solution $u = u(t, x)$ of (1.1)-(1.3) is twice continuously differentiable in the complement of finitely many characteristic curves γ_i, within the domain $[0, T] \times \mathbb{R}$.

1. Let $(\hat{u}_0, \hat{u}_1) \in \mathcal{U}$ be given. By the definition of the space $\mathcal{U}$ in (5.1), as $|x| \to \infty$ we have

$$\hat{u}_0(x) \to 0, \qquad \hat{u}_{0,x}(x) \to 0, \qquad \hat{u}_1(x) \to 0. \tag{5.3}$$

Hence the corresponding functions R, S in (2.1) satsfy

$$R(0,x) \ \to \ 0, \qquad\qquad S(0,x) \ \to \ 0.$$

From (2.3), it follows that the functions R, S remain uniformly bounded on a domain of the form $\{(t,x)\,;\ \ t \in [0,T], \quad |x| \geq r\}$, for r sufficiently large. More generally, we can choose $\delta > 0$ such that, for every initial data $(u_0, u_1) \in B_\delta$, the corresponding solution $u(t,x)$ remains twice continuously differentiable on the outer domain

$$\Big\{(t,x)\,; \quad t \in [0,T], \quad |x| \geq \rho\Big\}, \tag{5.4}$$

for some $\rho > 0$ sufficiently large. Its singularities can thus occur only on the compact domain $[0,T] \times [-\rho,\, \rho]$.

The subset $\widehat{D} \subset B_\delta$ is now defined as follows. $(u_0, u_1) \in \widehat{D}$ if $(u_0, u_1) \in B_\delta$ and moreover, for the corresponding solution (u, w, z, p, q) of (2.16)–(2.20) with boundary data (2.21), the values (4.2)–(4.4) are never attained, for any (X, Y) such that

$$\big(t(X,Y),\, x(X,Y)\big) \ \in \ [0,T] \times [-\rho,\, \rho]. \tag{5.5}$$

It is important to observe that, by (2.29), the above condition is independent of the relabeling (2.25).

2. For any $(u_0, u_1) \in B_\delta$ we now consider the corresponding solution (t, x, u, w, z, p, q) of the system (2.16)–(2.20), with boundary data as in (2.21). Let Λ be the map at (2.23) and let Γ be the square with side $2M$ in the X-Y plane, as in (4.1).

By choosing M large enough, and by possibly shrinking the radius δ, we can achieve the inclusion

$$[0,T] \times [-\rho, \rho] \ \subset \ \Lambda(\Gamma)\,, \tag{5.6}$$

for every $(u_0, u_1) \in B_\delta$.

3. We begin by proving that $\widehat{\mathcal{D}}$ is open, in the topology of $\mathcal{C}^3 \times \mathcal{C}^2$. Indeed, consider initial data $(u_0, u_1) \in \widehat{\mathcal{D}}$ and let $(u_0^\nu, u_1^\nu)_{\nu \geq 1}$ be a sequence of initial data converging to (u_0, u_1).

Assume, by contradiction, that $(u_0^\nu, u_1^\nu) \notin \widehat{\mathcal{D}}$ for all $\nu \geq 1$. To fix the ideas, let (X^ν, Y^ν) be points at which the corresponding solutions $(u^\nu, w^\nu, z^\nu, p^\nu, q^\nu)$ satisfy

$$(w^\nu, w_X^\nu, w_{XX}^\nu)(X^\nu, Y^\nu) \ = \ (\pi,\, 0,\, 0), \qquad (t^\nu, x^\nu)(X^\nu, Y^\nu) \ \in \ [0,T] \times [-\rho,\, \rho], \tag{5.7}$$

for all $\nu \geq 1$. By (5.6), since the domain Γ in (4.1) is compact, by possibly taking a subsequence we can assume $(X^\nu, Y^\nu) \to (\overline{X}, \overline{Y})$. By continuity, this implies

$$(w, w_X, w_{XX})(\overline{X}, \overline{Y}) \ = \ (\pi,\, 0,\, 0), \qquad\qquad (t,x)(\overline{X},\, \overline{Y}) \ \in \ [0,T] \times [-\rho,\, \rho],$$

contradicting the assumption $(u_0, u_1) \in \widehat{\mathcal{D}}$.

The other cases in (4.2)–(4.4) are handled in the same way. This proves that $\widehat{\mathcal{D}}$ is open.

4. Next, we claim that $\widehat{D}$ is dense in B_δ. Indeed, let $(u_0, u_1) \in B_\delta$ be given. By an arbitrarily small perturbation (measured in the norm of $\mathcal{U}$), we can assume that $u_0, u_1 \in \mathcal{C}^\infty$.

Using Lemma 5, we can construct a sequence of solutions $(u^\nu, w^\nu, z^\nu, p^\nu, q^\nu, x^\nu, t^\nu)$ of (2.16)–(2.20) such that:

(i) For every bounded set $\Omega \subset \mathbb{R}^2$ and any $k \geq 1$, the $\mathcal{C}^k$ norm of the difference satisfies

$$\lim_{\nu\to\infty} \Big\|(u^\nu - u,\ w^\nu - w,\ z^\nu - z,\ p^\nu - p,\ q^\nu - q,\ x^\nu - x,\ t^\nu - t)\Big\|_{\mathcal{C}^k(\Omega)} = 0. \qquad (5.8)$$

(ii) For every $\nu \geq 1$, the values in (4.2)–(4.4) are never attained, for any $(X, Y) \in \Gamma$.

Consider the corresponding solutions $u^\nu(t, x)$ of (1.1), with graph

$$\Big\{(u^\nu(X,Y),\ t^\nu(X,Y),\ x^\nu(X,Y))\,;\ \ (X,Y) \in \mathbb{R}^2\Big\} \subset \mathbb{R}^3.$$

For $t = 0$, by (5.8) the corresponding sequence of initial values satisfies

$$\lim_{\nu\to\infty} \|u^\nu(0,\cdot) - u_0\|_{\mathcal{C}^k([a,b])} = 0, \qquad \lim_{\nu\to\infty} \|u^\nu_t(0,\cdot) - u_1\|_{\mathcal{C}^k([a,b])} = 0, \qquad (5.9)$$

for every bounded interval $[a, b]$.

Next, consider a cutoff function $\eta \in \mathcal{C}^\infty_c$ such that

$$\begin{aligned} \eta(x) &= 1 \qquad \text{if } |x| \leq r, \\ \eta(x) &= 0 \qquad \text{if } |x| \geq r+1, \end{aligned} \qquad (5.10)$$

with $r >> \rho$ sufficiently large. For every $\nu \geq 1$, consider the initial data

$$\tilde u^\nu_0 \doteq \eta u^\nu_0 + (1-\eta)u_0\,, \qquad \tilde u^\nu_1 \doteq \eta u^\nu_1 + (1-\eta)u_1\,.$$

By (5.9) we have

$$\lim_{\nu\to\infty} \|(\tilde u^\nu_0 - u_0,\ \tilde u^\nu_1 - u_1)\|_{\mathcal{U}} = 0. \qquad (5.11)$$

Moreover, if $r > 0$ was chosen large enough, we have

$$\tilde u^\nu(t,x) = u^\nu(t,x) \qquad \text{for all } (t,x) \in [0,T] \times [-\rho, \rho]\,,$$

while $\tilde u^\nu$ remains $\mathcal{C}^2$ on the outer domain (5.4). The above implies $(\tilde u^\nu_0, \tilde u^\nu_1) \in \widehat{\mathcal{D}}$ for all $\nu \geq 1$ sufficiently large, proving that $\widehat{\mathcal{D}}$ is dense on B_δ.

5. To complete the proof we need to show that, for every initial data $(u_0, u_1) \in \widehat{\mathcal{D}}$, the solution $u(t, x)$ of (1.1) is piecewise $\mathcal{C}^2$ on the domain $[0, T] \times \mathbb{R}$.

By the previous arguments, we already know that u is $\mathcal{C}^2$ on the outer domain (5.4). It thus remains to study the singularities of u on the inner domain $[0, T] \times [-\rho,\, \rho]$. For this purpose, call $(u, w, z, p, q, t, x)(X, Y)$ the corresponding solution of (2.16)–(2.20), with boundary data as in (2.21). By (5.6), every point of the inner domain is contained in the image of the square Γ in (4.1).

Consider a point $(X_0, Y_0) \in \Gamma$. Two cases can occur.

CASE 1: $w(X_0, Y_0) \neq \pi$ and $z(X_0, Y_0) \neq \pi$. By (2.19)-(2.20) it follows

$$\det \begin{pmatrix} x_X & x_Y \\ t_X & t_Y \end{pmatrix} = \frac{(1+\cos w)p}{4} \cdot \frac{(1+\cos z)p}{4c} + \frac{(1+\cos z)q}{4} \cdot \frac{(1+\cos w)p}{4c} > 0.$$

Hence the map $(X, Y) \mapsto (x, t)$ is locally invertible in a neighborhood of (X_0, Y_0). We can thus conclude that the function u is $\mathcal{C}^2$ in a neighborhood of the point $(t(X_0, Y_0), x(X_0, Y_0))$.

CASE 2: $w(X_0, Y_0) = \pi$. In this case we have either $w_X(X_0, Y_0) \neq 0$, or else by (2.17)

$$w_Y(X_0, Y_0) = \frac{c'(u)}{8c^2(u)}(\cos z + 1)q \neq 0. \tag{5.12}$$

Indeed, we always have $c(u) > 0$ and $q > 0$. Moreover, by construction the values $(w, z, w_X) = (\pi, \pi, 0)$ and $(w, w_X, c'(u)) = (\pi, 0, 0)$ are never attained in Γ. This implies (5.12).

By the implicit function theorem, we thus conclude that the sets

$$S^w \doteq \{(X, Y) \in \Gamma\,;\;\; w(X, Y) = \pi\}, \qquad\qquad S^z \doteq \{(X, Y) \in \Gamma\,;\;\; z(X, Y) = \pi\} \tag{5.13}$$

are the union of finitely many $\mathcal{C}^2$ curves.

The set of points (t, x) where u is singular coincides with the image of the two sets S^w, S^z under the $\mathcal{C}^2$ map

$$(X, Y) \mapsto \Lambda(X, Y) = (t(X, Y),\; x(X, Y)).$$

6. To complete the proof, we study in more detail the images of the singular sets S^w, S^z.

By (4.5) there can be only finitely many points inside Γ where $w = \pi$ and $w_X = 0$, say $P_i = (X_i, Y_i)$, $i = 1, \ldots, m$. Moreover, by (4.6), at a point $(X_0, Y_0) \in S^w \cap S^z$ we have

$$w_X \neq 0, \qquad w_Y = 0, \qquad z_X = 0, \qquad z_Y \neq 0.$$

Therefore, as shown in Fig. 2, the two curves $\{w = \pi\}$ and $\{z = \pi\}$ intersect perpendicularly. As a consequence, inside the compact set Γ, there can be only finitely many such intersection points, say $Q_i = (X_i', Y_i')$, $i = 1, \ldots, n$.

After removing these finitely many points P_i, Q_i, we can thus write $\mathcal{S}^w$ as a finite union of curves γ_j of the form

$$\gamma_j = \{(X, Y)\,;\;\; X = \phi_j(Y), \;\; a_i < Y < b_j\}. \tag{5.14}$$

for suitable functions γ_j of class $\mathcal{C}^2$. We claim that the image of $\Lambda(\gamma_j)$ is a $\mathcal{C}^2$ curve in the t-x plane. To prove this, it suffices to show that, on the open interval $]a_j, b_j[$ the differential of the map

$$Y \mapsto (x(\phi_j(Y), Y),\; t(\phi_j(Y), Y))$$

does not vanish. This is true because, by (2.20)

$$\frac{d}{dY} t(\phi_j(Y), Y) = t_X \cdot \phi_j' + t_Y = 0 \cdot \phi_j' + \frac{(1+\cos z)q}{4c(u)} > 0.$$

Indeed, $z \neq \pi$ while $c(u), q > 0$.

As shown in Fig. 3, restricted to the inner domain $[0,T]\times[-\rho,\,\rho]$ in the t-x plane, the singular set $\Lambda(S^w)$ is thus the union of the finitely many points

$$\begin{aligned} p_i &= \Lambda(P_i), \qquad i=1,\ldots,m,\\ q_i &= \Lambda(Q_i), \qquad i=1,\ldots,n, \end{aligned}$$

together with finitely many $\mathcal{C}^2$ curves $\Gamma_j \doteq \Lambda(\gamma_j)$. The same representation is valid for the image $\Lambda(S^z)$. This concludes the proof of Theorem 1. □

6 One-parameter families of solutions

In this section we study families of conservative solutions $u = u(t,x,\lambda)$ of (1.1) depending on an additional parameter $\lambda \in [0,1]$. We thus consider a 1-parameter family of initial data

$$u(0,x,\lambda) \;=\; u_0(x,\lambda), \qquad\qquad u_t(0,x,\lambda) \;=\; u_1(x,\lambda), \tag{6.1}$$

smoothly depending on the additional parameter $\lambda \in [0,1]$. More precisely, these paths of initial data will lie in the space

$$\mathcal{X} \;\doteq\; \Big(\mathcal{C}^3([0,1]\times\mathbb{R})\cap \mathbf{L}^\infty([0,1];H^1(\mathbb{R}))\Big)\times\Big(\mathcal{C}^2([0,1]\times\mathbb{R})\cap \mathbf{L}^\infty([0,1];\,\mathbf{L}^2(\mathbb{R}))\Big). \tag{6.2}$$

In particular, the map $(x,\lambda)\mapsto u_0(x,\lambda)$ is three times continuously differentiable and the H^1 norm of $u_0(\cdot,\lambda)$ is uniformly bounded for all λ. Moreover, the map $(x,\lambda)\mapsto u_1(x,\lambda)$ is two times continuously differentiable and the $\mathbf{L}^2$ norm of $u_1(\cdot,\lambda)$ is uniformly bounded for all λ.

By an adaptation of the previous arguments one obtains

Theorem 2. *Let the wave speed $c(u)$ satisfy the assumptions* **(A)** *and let $T>0$ be given. Then, for any 1-parameter family of initial data $(\hat u_0,\hat u_1)\in\mathcal{X}$ and any $\varepsilon>0$, there exists a perturbed family $(x,\lambda)\mapsto(u_0,u_1)(x,\lambda)$ such that*

$$\Big\|(u_0-\hat u_0\,,\;u_1-\hat u_1)\Big\|_{\mathcal{X}} \;<\; \varepsilon\,, \tag{6.3}$$

and moreover the following holds. For all except at most finitely many $\lambda\in[0,1]$, the conservative solution $u=u(t,x;\lambda)$ of (1.1) is smooth in the complement of finitely many points P_i and finitely many $\mathcal{C}^2$ curves γ_j in the domain $[0,T]\times\mathbb{R}$.

Toward a proof, we shall need

Lemma 6. *Let the function $u\mapsto c(u)$ satisfy the assumptions* **(A)**, *and let any $M>0$ be given.*

Then there exists a dense set of paths of initial data $\mathcal{D}\subset\mathcal{X}$ such that, if $(x,\lambda)\mapsto(u_0,u_1)(x,\lambda)$ lies in $\mathcal{D}$, then the corresponding solutions (t,x,u,w,z,p,q) of (2.17)–(2.20) with boundary data as in (2.21) have the following properties. On the domain Γ in (4.1) one has *(i) The map* $(X,Y,\lambda)\mapsto(w,w_X,w_{XX})$ *is transversal to the point* $(\pi,0,0)$.

(ii) The map $(X,Y,\lambda)\mapsto(z,z_Y,z_{YY})$ *is transversal to the point* $(\pi,0,0)$.

(iii) The map $(X,Y,\lambda)\mapsto(w,z,w_X)$ *is transversal to the point* $(\pi,\pi,0)$.

(iv) The map $(X,Y,\lambda)\mapsto(w,z,z_Y)$ *is transversal to the point* $(\pi,\pi,0)$.

(v) The map $(X,Y,\lambda)\mapsto(w,w_X,c'(u))$ *is transversal to the point* $(\pi,0,0)$.

(vi) The map $(X,Y,\lambda)\mapsto(z,z_Y,c'(u))$ *is transversal to the point* $(\pi,0,0)$.

Proof of Lemma 6. Consider any point (X_0,Y_0,λ_0). Then, there exist 3-parameter families of perturbed initial data (u_0^θ,u_1^θ), $\theta\in\mathbb{R}^3$ such that the properties **(1)–(3)** in Lemma 4 hold. Indeed, it suffices to repeat all the arguments in the proof of Lemma 4 regarding λ_0 as a constant. For a fixed $\lambda=\lambda_0$, the perturbations in (3.12) are thus functions of s only, constant w.r.t. λ.

Combining these perturbations, as in the proof of Lemma 5, we obtain a map $(X,Y,\lambda,\theta)\mapsto(u,w,z,p,q)$ for which all transversality conditions (i)–(vi) are satisfied. By the transversality theorem, for a.e. θ the corresponding map $(X,Y,\lambda)\mapsto(u^\theta,w^\theta,z^\theta,p^\theta,q^\theta)$ satisfies the same transversality conditions. This achieves the proof. $\square$

Proof of Theorem 2. As in the proof of Theorem 1, we first choose ρ large enough so that all our solutions will be $\mathcal{C}^2$ for (t,x) in the outer domain (5.4).

For each $\lambda\in[0,1]$, we denote by $(u,w,z,p,q,x,t)(X,Y,\lambda)$ the corresponding solution of the semilinear system (2.16)-(2.20). We choose M sufficiently large such that, for all $\lambda\in[0,1]$, the inner domain $[0,T]\times[-\rho,\,\rho]$ is contained in the image

$$\Lambda^\lambda(\Gamma)\;=\;\Big\{\big(t(X,Y,\lambda),\;x(X,Y,\lambda)\big)\,;\;\;|X|+|Y|\le M\Big\}.$$

By performing an arbitrarily small perturbation of the initial path of solutions we obtain a second path $\lambda\mapsto u(\cdot,\lambda)$ such that, in the corresponding solution $(u,w,z,p,q,x,t)(X,Y,\lambda)$, the transversality relations (i)–(vi) in Lemma 6 hold.

Since the variables $(X,Y,\lambda)\in\Gamma\times[0,1]$ range on a compact, three dimensional set, this implies that the values in (i)–(vi) are attained only at finitely many points, say (X_i,Y_i,λ_i), $i=1,\ldots,n$. Hence, for $\lambda\notin\{\lambda_1,\ldots,\lambda_n\}$, the solution $(t,x,u,w,z,p,q)(\cdot,\cdot,\lambda)$ does not attain any of the values in (i)–(vi), for $(X,Y)\in\Gamma$. As shown in steps **5-6** in the proof of Theorem 1, the corresponding solution $u=u(t,x;\,\lambda)$ is then piecewise smooth on the inner domain $[0,T]\times[-\rho,\,\rho]$. $\square$

Remark 5. For a given solution $u=u(t,x)$, define its *singular set* as

$$S^u\;\doteq\;\{(t,x)\,;\;\;u\text{ is not }\mathcal{C}^2\text{ on any neighborhood of }(t,x)\}.$$

In the above construction, one can regard $\lambda_1,\ldots,\lambda_n$ as *bifurcation values*, where the structure of the singular set changes (Fig. 4). On the other hand, for $\lambda\notin\{\lambda_1,\ldots,\lambda_n\}$ the solution $u(\cdot,\cdot;\lambda)$ is *structurally stable*. A small perturbation of the initial data does not change the topology of the singular set. Based on the present analysis, we speculate that a theory of generic structural stability and a global classification of solutions to (1.1) can be developed, in analogy to the classical theory for ODEs [1, 16].

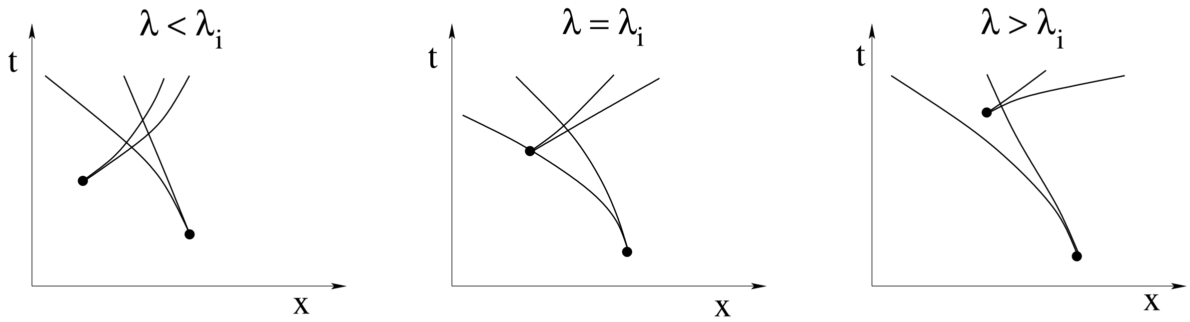

Figure 4: The singular set for a solution $u(t, x; \lambda)$. When the parameter λ crosses one of the critical values λ_i, the topology of the singular set changes.

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