# Delta shocks and vacuum states for the isentropic magnetogasdynamics equations for Chaplygin gas as pressure and magnetic field vanish **

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**Abstract**

This paper is concerned with the Riemann problem for the isentropic Chaplygin gas magnetogasdynamics equations and the formation of delta shocks and vacuum states as pressure and magnetic field vanish. Firstly, the Riemann problem of the isentropic magnetogasdynamics equations for Chaplygin gas is solved analytically. Secondly, it is rigorously proved that, as both the pressure and the magnetic field vanish, the Riemann solution containing two shock waves tends to a delta shock solution to the transport equations, and the intermediate density between the two shocks tends to a weighted $\delta$-measure which forms the delta shock; while the Riemann solution containing two rarefaction waves tends to a two-contact-discontinuity solution to the transport equations, the termediate state between the two contact discontinuities is a vacuum state.

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**Keywords:** Isentropic magnetogasdynamics; Chaplygin gas; Riemann problem; Transport equations; Delta shock wave; Vacuum state

## 1. Introduction

In this paper, we are concerned with the system of conservation law governing the one-dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid subjected to a transverse magnetic field (see [10, 11]):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (p + \rho u^2 + B^2/2\mu)_x = 0, \end{cases} \tag{1.1}$$

where $\rho > 0$, $u$, $p$, $B$ and $\mu > 0$ represent the density, velocity, pressure, transverse magnetic field and magnetic permeability, respectively; $p$ and $B$ are known functions defined as

$$p = -\frac{k_1}{\rho} \tag{1.2}$$

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and $B = k_2\rho$, where $k_1$ and $k_2$ are positive constants. The independent variables $t$ and $x$ denote time and space, respectively. The adiabatic exponent in (1.2) can be viewed as $\gamma = -1$ by comparing with the state equation $p = k_1\rho^{\gamma}$ with $\gamma \geq 1$ for the polytropic gas. The gas (1.2) whose adiabatic constant $\gamma = -1$ is usually called as the Chaplygin gas.

For the isentropic Chaplygin gas Euler equations, Brenier [1] firstly studied the 1-D Riemann problem and obtained solutions with concentration when initial data belong to a certain domain in the phase plane. Furthermore, Guo, Sheng and Zhang [6] abandoned this constrain and constructively obtained the global solutions to the 1-D Riemann problem, in which the $\delta$-shock developed. Moreover, they also systematically studied the 2-D Riemann problem for isentropic Chaplygin gas equations. For the 2-D case, we can also refer to [9] in which D. Serre studied the interaction of the pressure waves for the 2-D isentropic irrotational Chaplygin gas and constructively proved the existence of transonic solutions for two cases, saddle and vortex of 2-D Riemann problem. Recently, Sheng, Wang and Yin [13] and Wang [15] studied the Riemann problem for the generalized Chaplygin gas and obtained the solutions to the Riemann problem and the interactions of elementary waves. The Riemann solutions to the transport equations in zero-pressure flow in gas dynamics were presented by Sheng and Zhang in [14], in which delta shocks and vacuum states appeared.

In related researchs of the $\delta$-shock waves, one very important and interesting topic is to study the phenomena of concentration and cavitation and the formation of $\delta$-shock waves and vacuum states in solutions. In earlier paper [4], Chen and Liu [4] studied the formation of $\delta$-shocks and vacuum states of the Riemann solutions to the isentropic Euler equations for polytropic gas as $\varepsilon \to 0$, in which they took the equation of state as $P = \varepsilon p$ for $p = \rho^{\gamma}/\gamma$ $(\gamma > 1)$. Further, they also obtained the same results for the Euler equations for nonisentropic fluids in [5]. The same problem for the the isentropic Euler equations for isothermal case was studied by Li [7], in which he proved that when temperature drops to zero, the solution containing two shock waves converges to the delta shock solution to the transport equations and the solution containing two rarefaction waves converges to the solution involving vacuum to the transport equations. Then, the results were extended to the relativistic Euler equations for polytropic gas by Yin and Sheng [17] and for Chaplygin gas by Yin and Song [18], the isentropic Euler equations for the generalized Chaplygin gas by Sheng, Wang and Yin [13] and for modified Chaplygin gas by Yang and Wang [16], the perturbed Aw-Rascle model by Shen and Sun [12], the isentropic magnetogasdynamics equations for polytropic gas by Shen [11], the generalized pressureless gas dynamics model with a scaled pressure term by Mitrovic and Nedeljkov [8], etc.

In this paper, we study the Riemann problem of the isentropic magnetogasdynamics equations for Chaplygin gas and the formation of delta shocks and vacuum states as pressure and magnetic field vanish. The organization of the paper is as follows. In Sections 2 and 3, the Riemann problems for the isentropic Chaplygin gas magnetogasdynamics equations and the transport equations are analyzed by characteristic analysis. In Sections 4 and 5, we investigate the formation of $\delta$-shocks and vacuum states of the Riemann solutions to the isentropic magnetogasdynamics equations for Chaplygin gas as pressure and magnetic field vanish.

## 2. Riemann problem for system (1.1)-(1.2)

In this section, we discuss the Riemann solutions of (1.1) and (1.2) with initial data

$$(\rho, u)(x,0) = (\rho_\pm, u_\pm), \qquad \pm x > 0, \tag{2.1}$$

where $\rho_\pm > 0$ and $u_\pm$ are arbitrary constants.

For smooth solution, system (1.1) is equivalent to

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ w^2/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0, \tag{2.2}$$

where $w = (c^2 + b^2)^{1/2}$ is the magneto-acoustic speed with $c = (p'(\rho))^{1/2}$ as the local sound speed and $b = (B^2(\rho)/\mu\rho)^{1/2}$ the Alfven speed. Here, prime denotes differentiation with respect to $\rho$. The eigenvalues of system (1.1) and (1.2) are

$$\lambda_1 = u - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}, \quad \lambda_2 = u + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}.$$

Therefore, system (1.1) and (1.2) is strictly hyperbolic for $\rho > 0$.

The corresponding right eigenvectors are

$$\overrightarrow{r_1} = (-\rho, \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}})^T, \quad \overrightarrow{r_2} = (\rho, \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}})^T.$$

By simple calculation, we get

$$\nabla\lambda_i \cdot \overrightarrow{r_i} = \frac{3k_2^2\rho}{2\mu\sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}} \neq 0, \quad i = 1, 2.$$

Therefore, both the characteristic fields are genuinely nonlinear.

Since system (1.1), (1.2) and the Riemann data (2.1) are invariant under stretching of coordinates: $(x,t) \to (\alpha x, \alpha t)$ ($\alpha$ is constant), we seek the self-similar solution

$$(\rho, u)(x,t) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}.$$

Then Riemann problem (1.1), (1.2) and (2.1) is reduced to the following boundary value problem of ordinary differential equations:

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + \big(-\frac{k_1}{\rho} + \rho u^2 + \frac{(k_2\rho)^2}{2\mu}\big)_\xi = 0, \end{cases} \tag{2.3}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

For any smooth solution, system (2.3) can be written as

$$\begin{pmatrix} u - \xi & \rho \\ -\xi u + \frac{k_1}{\rho^2} + u^2 + \frac{k_2^2\rho}{\mu} & -\xi\rho + 2\rho u \end{pmatrix} \begin{pmatrix} \rho_\xi \\ u_\xi \end{pmatrix} = 0. \tag{2.4}$$

It provides either general solutions (constant states)

$$(\rho, u)(\xi) = \text{constant} \quad (\rho > 0)$$

or singular solutions called the rarefaction waves $R_1$ and $R_2$ which denote, respectively, 1-rarefaction waves and 2-rarefaction waves,

$$R_1: \begin{cases} \xi = \lambda_1 = u - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}, \\ u - u_- = -\int_{\rho_-}^{\rho} \frac{\sqrt{\frac{k_1}{s^2}+\frac{k_2^2 s}{\mu}}}{s} ds, \quad \rho < \rho_-, \end{cases} \tag{2.5}$$

and

$$R_2: \begin{cases} \xi = \lambda_2 = u + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}, \\ u - u_- = \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{k_1}{s^2}+\frac{k_2^2 s}{\mu}}}{s} ds, \quad \rho > \rho_-. \end{cases} \tag{2.6}$$

Differentiating the second equation of (2.5) with respect to $\rho$ yields $u_\rho = -\frac{\sqrt{\frac{k_1}{\rho^2}+\frac{k_2^2\rho}{\mu}}}{\rho} < 0$, and subsequently,

$$u_{\rho\rho} = \frac{\frac{4k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}{2\rho^2\sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}} > 0,$$

which mean that the 1-rarefaction wave curve $R_1$ is monotonic decreasing and convex in the $(\rho, u)$ plane $(\rho > 0)$. Similarly, from the second equation of (2.6), we have $u_\rho > 0$ and $u_{\rho\rho} < 0$, which mean that the 2-rarefaction wave curve $R_2$ is monotonic increasing and concave in the $(\rho, u)$ plane $(\rho > 0)$. In addition, it can be verified that $\lim_{\rho\to 0^+} u = +\infty$ for the 1-rarefaction wave curve $R_1$, which implies that $R_1$ has the $u$-axis as its asymptotic line. It can also be proved that $\lim_{\rho\to+\infty} u = +\infty$ for the 2-rarefaction wave curve $R_2$.

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot conditions hold:

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + [-\frac{k_1}{\rho} + \rho u^2 + \frac{(k_2\rho)^2}{2\mu}] = 0, \end{cases} \tag{2.7}$$

where $[\rho] = \rho - \rho_-$, etc. Solving (2.7), we obtain two shock waves $S_1$ and $S_2$

$$S_1: \begin{cases} \sigma = u_- - \rho\sqrt{\frac{1}{\rho\rho_-}\left(\frac{k_1}{\rho\rho_-} + \frac{k_2^2(\rho+\rho_-)}{2\mu}\right)}, \\ u = u_- - \sqrt{\frac{1}{\rho\rho_-}\left(\frac{k_1}{\rho\rho_-} + \frac{k_2^2(\rho+\rho_-)}{2\mu}\right)}(\rho - \rho_-), \quad \rho > \rho_-, \end{cases} \tag{2.8}$$

$$S_2: \begin{cases} \sigma = u_- + \rho\sqrt{\frac{1}{\rho\rho_-}\left(\frac{k_1}{\rho\rho_-} + \frac{k_2^2(\rho+\rho_-)}{2\mu}\right)}, \\ u = u_- + \sqrt{\frac{1}{\rho\rho_-}\left(\frac{k_1}{\rho\rho_-} + \frac{k_2^2(\rho+\rho_-)}{2\mu}\right)}(\rho - \rho_-), \quad \rho < \rho_-. \end{cases} \tag{2.9}$$

Differentiating the second equation of (2.8) with respect to $\rho$, for $\rho > \rho_-$ we have

$$u_\rho = -\frac{1}{2\sqrt{\frac{1}{\rho\rho_-}\left(\frac{k_1}{\rho\rho_-} + \frac{k_2^2(\rho+\rho_-)}{2\mu}\right)}}\left(\frac{2k_1}{\rho_-\rho^3} + \frac{k_2^2}{\rho_-\mu} + \frac{k_2^2}{2\rho\mu} + \frac{k_2^2\rho_-}{2\rho^2\mu}\right) < 0,$$

which mean that the 1-shock curve $S_1$ is monotonic decreasing in the $(\rho, u)$ plane $(\rho > 0)$. Similarly, from the second equation of (2.9), for $\rho < \rho_-$ we have $u_\rho > 0$, which mean that the 2-shock wave curve $S_2$ is monotonic increasing in the $(\rho, u)$ plane $(\rho > 0)$. In addition, it is easily derived from (2.9) that

$\lim\limits_{\rho\to 0^+} u = -\infty$ for the 2-shock curve $S_2$, which implies that $S_2$ has the $u$-axis as its asymptotic line. It can also be derived from (2.8) that $\lim\limits_{\rho\to +\infty} u = -\infty$ for the 1-shock curve $S_1$.

In the phase plane ($\rho > 0$, $u \in \mathbf{R}$), through point $(\rho_-, u_-)$, we draw the elementary wave curves $R_1$, $R_2$, $S_1$ and $S_2$, respectively. Then the phase plane is divided into four regions I, II, III and IV$(\rho_-, u_-)$ (see Fig. 1).

By the analysis method in phase plane, for any given state $(\rho_+, u_+)$, one can construct the Riemann solutions as follows:

(1) $(\rho_+, u_+) \in I(\rho_-, u_-) : R_1 + R_2$;

(2) $(\rho_+, u_+) \in II(\rho_-, u_-) : R_1 + S_2$;

(3) $(\rho_+, u_+) \in III(\rho_-, u_-) : S_1 + R_2$;

(4) $(\rho_+, u_+) \in IV(\rho_-, u_-) : S_1 + S_2$.

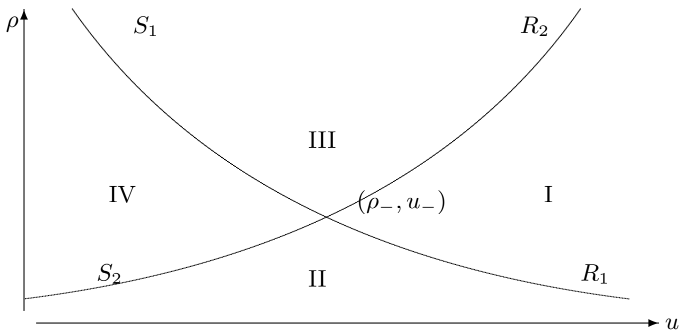


**Fig. 1. Curves of elementary waves.**

Thus we have proved the following result

**Theorem 1.** For Riemann problem (1.1), (1.2) and (2.1), there exists a unique entropy solution, which consists of shock waves, rarefaction waves, and constant states.

# 3. Riemann problem for the transport equations

The Riemann solutions to the transport equations in zero-pressure flow were presented by Sheng and Zhang in [14]. The Riemann problem to the transport equations are

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0 \end{cases} \tag{3.1}$$

with initial data

$$(\rho, u)(x, 0) = (\rho_\pm, u_\pm), \qquad \pm x > 0. \tag{3.2}$$

The system has a double eigenvalue

$$\lambda = u$$

and only one right eigenvector

$$\overrightarrow{r} = (r, 0)^T.$$

By a direct calculation,

$$\nabla\lambda \cdot \overrightarrow{r} \equiv 0.$$

Thus (3.1) is nonstrictly hyperbolic and $\lambda$ is linearly degenerate.

As usual, we seek the self-similar solution

$$(\rho, u)(x, t) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}.$$

Then Riemann problem (3.1) and (3.2) is reduced to the following boundary value problem of ordinary differential equations:

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2)_\xi = 0, \end{cases} \tag{3.3}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

For any smooth solution, system (3.3) can be written as

$$\begin{pmatrix} u - \xi & \rho \\ 0 & \rho(u - \xi) \end{pmatrix} \begin{pmatrix} \rho_\xi \\ u_\xi \end{pmatrix} = 0. \tag{3.4}$$

It provides either the general solution (constant state)

$$(\rho, u)(\xi) = \text{constant} \quad (\rho \neq 0)$$

or the singular solution

$$\begin{cases} \rho = 0, \\ u = \xi, \end{cases} \tag{3.5}$$

which is called the vacuum state (see [14]), where $u(\xi)$ is an arbitrary smooth function.

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot condition holds:

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + [\rho u^2] = 0, \end{cases} \tag{3.6}$$

where $[q] = q_+ - q_-$ denotes the jump of $q$ across the discontinuity. By solving (3.6), we obtain

$$J : \xi = \sigma = u_-(= \lambda_-) = u_+(= \lambda_+), \tag{3.7}$$

which is a contact discontinuity. It is a slip line and just the characteristic of solutions on both its sides in $(x, t)$-plane.

The Riemann problem (3.1) and (3.2) can be solved by contact discontinuities, vacuum or delta shock wave connecting two constant states $(u_\pm, v_\pm)$.

For the case $u_- < u_+$, there is no characteristic passing through the region $u_-t < x < u_+t$ and the vacuum appears in this region. The solution can be expressed as

$$(\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), & -\infty < x < u_-, \\ (0, \xi), & u_- \leq \xi \leq u_+, \\ (\rho_+, u_+), & u_+ < \xi < +\infty. \end{cases} \tag{3.8}$$

For the case $u_- = u_+$, it is easy to see that the constant states $(\rho_\pm, u_\pm)$ can be connected by a contact discontinuity.

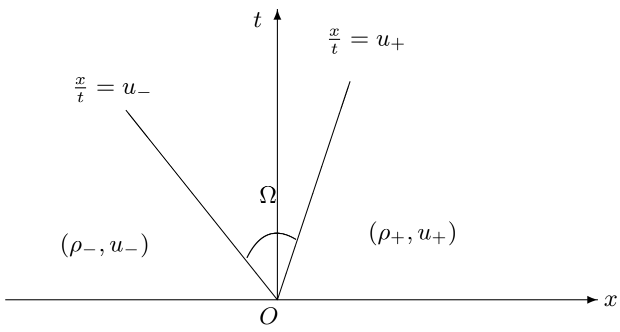


**Fig. 2. Characteristics overlap domain.**

For the case $u_- > u_+$, the characteristic lines originating from the origin will overlap in a domain $\Omega$, as shown in Fig. 2. So, singularity must happen in $\Omega$. It is easy to know that the singularity is impossible to be a jump with finite amplitude because the Rankine-Hugoniot condition is not satisfied on the bounded jump. In other words, there is no solution which is piecewise smooth and bounded. Motivated by [14], we seek solutions with delta distribution at the jump.

To do so, a two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{(t(s), x(s)) : a < s < b\}$ is defined by

$$\langle w(s)\delta_L, \varphi\rangle = \int_a^b w(s)\varphi(t(s), x(s))ds \tag{3.9}$$

for any $\varphi \in C_0^\infty(R \times R_+)$.

Let us consider a solution of (3.1) and (3.2) of the form

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < \sigma t, \\ (w(t)\delta(x - \sigma t), \sigma), & x = \sigma t, \\ (\rho_+, u_+), & x > \sigma t, \end{cases} \tag{3.10}$$

where $\sigma$ is a constant, $w(t) \in C^1[0, +\infty)$, and $\delta(\cdot)$ is the standard Dirac measure. $x(t)$, $w(t)$ and $\sigma$ are the location, weight and velocity of the delta shock, respectively. Then the following generalized

Rankine-Hugoniot conditions hold:

$$\begin{cases} \frac{dx(t)}{dt} = \sigma, \\ \frac{dw(t)}{dt} = \sigma[\rho] - [\rho u], \\ \frac{d(w(t)\sigma)}{dt} = \sigma[\rho u] - [\rho u^2], \end{cases} \tag{3.11}$$

where $[\rho] = \rho_+ - \rho_-$, with initial data

$$(x, w)(0) = (0, 0). \tag{3.12}$$

In addition, to guarantee uniqueness, the delta shock wave should satisfy the entropy condition:

$$u_+ < \sigma < u_-.$$

Solving the system of simple ordinary differential equations (3.11) with initial data (3.12), we have, when $\rho_- = \rho_+$,

$$x(t) = \frac{1}{2}(u_- + u_+)t, \quad w(t) = (\rho_- u_- - \rho_+ u_+)t,$$
$$\sigma = \frac{1}{2}(u_- + u_+);$$

when $\rho_- \neq \rho_+$,

$$x(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}t, \quad w(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)t,$$
$$\sigma = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}.$$

# 4. Formation of $\delta$-shocks

In this section, we study the formation of $\delta$-shock waves in the Riemann solutions of system (1.1) and (1.2) in the case $(\rho_+, u_+) \in IV(\rho_-, u_-)$ with $u_- > u_+$ as both the pressure and the magnetic field vanish.

## 4.1. Limit behavior of Riemann solutions as $k_1, k_2 \to 0$

When $(\rho_+, u_+) \in IV(\rho_-, u_-)$, for each pair of fixed $k_1 > 0$ and $k_2 > 0$, suppose that $(\rho_*, u_*)$ is the intermediate state connected with $(\rho_-, u_-)$ by a 1-shock $S_1$ with speed $\sigma_1$ and $(\rho_+, u_+)$ by a 2-shock $S_2$ with speed $\sigma_2$. Then it follows

$$S_1 : \begin{cases} \sigma_1 = u_- - \rho_* \sqrt{\frac{1}{\rho_*\rho_-}\left(\frac{k_1}{\rho_*\rho_-} + \frac{k_2^2(\rho_*+\rho_-)}{2\mu}\right)}, \\ u_* = u_- - \sqrt{\frac{1}{\rho_*\rho_-}\left(\frac{k_1}{\rho_*\rho_-} + \frac{k_2^2(\rho_*+\rho_-)}{2\mu}\right)}(\rho_* - \rho_-), \quad \rho_* > \rho_-, \end{cases} \tag{4.1}$$

$$S_2 : \begin{cases} \sigma_2 = u_* + \rho_+ \sqrt{\frac{1}{\rho_+\rho_*}\left(\frac{k_1}{\rho_+\rho_*} + \frac{k_2^2(\rho_++\rho_*)}{2\mu}\right)}, \\ u_+ = u_* + \sqrt{\frac{1}{\rho_+\rho_*}\left(\frac{k_1}{\rho_+\rho_*} + \frac{k_2^2(\rho_++\rho_*)}{2\mu}\right)}(\rho_+ - \rho_*), \quad \rho_+ < \rho_*. \end{cases} \tag{4.2}$$

The addition of (4.1) and (4.2) gives

$$u_- - u_+ = \sqrt{\frac{1}{\rho_-} - \frac{1}{\rho_*}}\sqrt{k_1(\frac{1}{\rho_-} - \frac{1}{\rho_*}) + \frac{k_2^2(\rho_*^2 - \rho_-^2)}{2\mu}}$$

$$+\sqrt{\frac{1}{\rho_+} - \frac{1}{\rho_*}}\sqrt{k_1(\frac{1}{\rho_+} - \frac{1}{\rho_*}) + \frac{k_2^2(\rho_*^2 - \rho_+^2)}{2\mu}}, \quad \rho_* > \rho_\pm. \tag{4.3}$$

For any given $\rho_\pm > 0$, if $\lim\limits_{k_1,k_2\to 0} \rho_* = M \in [\max(\rho_-, \rho_+), +\infty)$, then by taking the limit $k_1, k_2 \to 0$ in (4.3), we have $u_- - u_+ = 0$, which contradicts with $u_- > u_+$. Therefore, $\lim\limits_{k_1,k_2\to 0} \rho_* = +\infty$. Letting $k_1, k_2 \to 0$ in (4.3), we obtain the following result.

**Lemma 1.**

$$\lim_{k_1,k_2\to 0} k_2^2\rho_*^2 = \frac{2\mu\rho_-\rho_+(u_- - u_+)^2}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}. \tag{4.4}$$

**Lemma 2.**

$$\lim_{k_1,k_2\to 0} u_* = \lim_{k_1,k_2\to 0} \sigma_1 = \lim_{k_1,k_2\to 0} \sigma_2 = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma, \tag{4.5}$$

$$\lim_{k_1,k_2\to 0} \int_{\sigma_1 t}^{\sigma_2 t} \rho_* dx = (\sigma[\rho] - [\rho u])t = \sqrt{\rho_-\rho_+}(u_- - u_+)t = w(t). \tag{4.6}$$

**Proof.** Letting $k_1, k_2 \to 0$ in (4.1) and noting Lemma 4.1, we have

$$\lim_{k_1,k_2\to 0} u_* = u_- - \lim_{k_1,k_2\to 0} \sqrt{\frac{1}{\rho_-} - \frac{1}{\rho_*}}\sqrt{k_1(\frac{1}{\rho_-} - \frac{1}{\rho_*}) + \frac{k_2^2(\rho_*^2 - \rho_-^2)}{2\mu}}$$

$$= u_- - \sqrt{\frac{1}{\rho_-}}\sqrt{\frac{\rho_-\rho_+(u_- - u_+)^2}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}} = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma. \tag{4.7}$$

From the first equation of (4.1), by Lemma 4.1, we obtain

$$\lim_{k_1,k_2\to 0} \sigma_1 = u_- - \lim_{k_1,k_2\to 0} \sqrt{\frac{k_1}{\rho_-^2} + \frac{k_2^2\rho_*^2(\frac{1}{\rho_-} + \frac{1}{\rho_*})}{2\mu}}$$

$$= u_- - \sqrt{\frac{\rho_+(u_- - u_+)^2}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}} = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma. \tag{4.8}$$

From (4.2) and (4.7), we can easily get

$$\lim_{k_1,k_2\to 0} \sigma_2 = \lim_{k_1,k_2\to 0} u_* + \lim_{k_1,k_2\to 0} \sqrt{\frac{k_1}{\rho_*^2} + \frac{k_2^2\rho_+^2(\frac{1}{\rho_*} + \frac{1}{\rho_+})}{2\mu}} = \sigma. \tag{4.9}$$

Thus it can be seen from (4.8) and (4.9) that when $k_1, k_2 \to 0$, the two shocks $S_1$ and $S_2$ will coincide whose velocities are identical with that of the delta shock wave of the transport equations with the same Riemann initial data $(\rho_\pm, u_\pm)$.

Using the Rankine-Hugoniot conditions (2.7) for $S_1$ and $S_2$, we have

$$\begin{cases} \sigma_1(\rho_* - \rho_-) = \rho_* u_* - \rho_- u_-, \\ \sigma_2(\rho_+ - \rho_*) = \rho_+ u_+ - \rho_* u_*. \end{cases} \tag{4.10}$$

Then from (4.8) and (4.9) it follows that

$$\lim_{k_1,k_2\to 0} (\sigma_1 - \sigma_2)\rho_* = \lim_{k_1,k_2\to 0} (\rho_+ u_+ - \rho_- u_- + \sigma_1\rho_- - \sigma_2\rho_+) = [\rho u] - \sigma[\rho]. \tag{4.11}$$

This implies that

$$\lim_{k_1,k_2\to 0}\int_{\sigma_1 t}^{\sigma_2 t}\rho_* dx=(\sigma[\rho]-[\rho u])t=\sqrt{\rho_-\rho_+}(u_- - u_+)t=w(t). \tag{4.12}$$

The proof is completed.

**Remark 1.** From the above results, it can be seen that the limit of the Riemann solution of system (1.1) and (1.2) as $k_1, k_2 \to 0$ in the case $(\rho_+, u_+) \in IV(\rho_-, u_-)$ is just the delta shock solution of (3.1)-(3.2) when $u_- > u_+$.

## 4.2. $\delta$-shocks and concentration

Now, we give the following results which give a very nice depiction of the limit in the case $(\rho_+, u_+) \in IV(\rho_-, u_-)$.

**Theorem 2.** Let $u_- > u_+$ and $(\rho_+, u_+) \in IV(\rho_-, u_-)$. For any fixed $k_1, k_2 > 0$, assuming that $(\rho, u)$ is a solution containing two shocks $S_1$ and $S_2$ of (1.1)-(1.2) with Riemann initial data (2.1), constructed in Section 2, it is obtained that as $k_1, k_2 \to 0$, $(\rho, u)$ converges in the sense of distributions, and the limit functions $\rho$ and $\rho u$ are the sums of a step function and a $\delta$-measure with weights

$$(\sigma[\rho]-[\rho u])t \quad \text{and} \quad (\sigma[\rho u]-[\rho u^2])t,$$

respectively, which form a delta shock wave solution of (3.1) with the same Riemann initial data $(\rho_\pm, u_\pm)$.

**Proof.** Let $\xi = x/t$. Then for any fixed $k_1 > 0$ and $k_2 > 0$, the Riemann solution to the isentropic magnetogasdynamics equations for Chaplygin gas (1.1)-(1.2) can be written as

$$(\rho,u)(\xi)=\begin{cases}(\rho_-,u_-), & \xi<\sigma_1,\\ (\rho_*,u_*), & \sigma_1<\xi<\sigma_2,\\ (\rho_+,u_+), & \xi>\sigma_2,\end{cases} \tag{4.13}$$

which satisfies the following weak formulations:

$$\int_{-\infty}^{+\infty}(\xi-u(\xi))\rho(\xi)\psi'(\xi)d\xi+\int_{-\infty}^{+\infty}\rho(\xi)\psi(\xi)d\xi=0 \tag{4.14}$$

and

$$\int_{-\infty}^{+\infty}(\xi-u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi-\int_{-\infty}^{+\infty}\Big(-\frac{k_1}{\rho(\xi)}+\frac{k_2^2(\rho(\xi))^2}{2\mu}\Big)\psi'(\xi)d\xi+\int_{-\infty}^{+\infty}\rho(\xi)u(\xi)\psi(\xi)d\xi=0 \tag{4.15}$$

for any test function $\psi \in C_0^\infty(-\infty, +\infty)$.

The first integral on the left-hand side of (4.15) can be decomposed into

$$\Big\{\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\sigma_2}+\int_{\sigma_2}^{+\infty}\Big\}(\xi-u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi. \tag{4.16}$$

The sum of the first and the last terms in (4.16) is

$$\int_{-\infty}^{\sigma_1}(\xi-u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi+\int_{\sigma_2}^{+\infty}(\xi-u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi$$

$$=\rho_-u_-\sigma_1\psi(\sigma_1)-\rho_-u_-^2\psi(\sigma_1)-\rho_-u_-\int_{-\infty}^{\sigma_1}\psi(\xi)d\xi$$
$$-\rho_+u_+\sigma_2\psi(\sigma_2)+\rho_+u_+^2\psi(\sigma_2)-\rho_+u_+\int_{\sigma_2}^{+\infty}\psi(\xi)d\xi. \tag{4.17}$$

Taking the limit $k_1,k_2\to 0$ in (4.17) leads to

$$\lim_{k_1,k_2\to 0}\left(\int_{-\infty}^{\sigma_1}+\int_{\sigma_2}^{+\infty}\right)(\xi-u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi$$
$$=([\rho u^2]-\sigma[\rho u])\psi(\sigma)-\int_{-\infty}^{+\infty}(\rho_0u_0)(\xi-\sigma)\cdot\psi(\xi)d\xi, \tag{4.18}$$

where $(\rho_0u_0)(\xi)=\rho_-u_-+[\rho u]H(\xi)$ and $H$ is the Heaviside function.

For the second term in (4.16), integrating by parts again, we obtain

$$\int_{\sigma_1}^{\sigma_2}(\xi-u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi=\int_{\sigma_1}^{\sigma_2}(\xi-u_*)\rho_*u_*\psi'(\xi)d\xi$$
$$=-\rho_*u_*^2(\psi(\sigma_2)-\psi(\sigma_1))+\rho_*u_*(\sigma_2\psi(\sigma_2)-\sigma_1\psi(\sigma_1))-\rho_*u_*\int_{\sigma_1}^{\sigma_2}\psi(\xi)d\xi$$
$$=-u_*\rho_*(\sigma_2-\sigma_1)\left(\frac{\psi(\sigma_2)-\psi(\sigma_1)}{\sigma_2-\sigma_1}u_*-\frac{\sigma_2\psi(\sigma_2)-\sigma_1\psi(\sigma_1)}{\sigma_2-\sigma_1}+\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(\xi)d\xi\right). \tag{4.19}$$

Taking the limit $k_1,k_2\to 0$ in (4.19), noting (4.11) and the fact that both $\psi\in C_0^\infty(-\infty,+\infty)$ and $\lim\limits_{k_1,k_2\to 0}u_*=\lim\limits_{k_1,k_2\to 0}\sigma_1=\lim\limits_{k_1,k_2\to 0}\sigma_2=\sigma$, we deduce that

$$\lim_{k_1,k_2\to 0}\int_{\sigma_1}^{\sigma_2}(\xi-u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi=\sigma([\rho u]-\sigma[\rho])(\sigma\psi'(\sigma)-\sigma\psi'(\sigma)-\psi(\sigma)+\psi(\sigma))=0. \tag{4.20}$$

Similarly, the first integral on the left-hand side of (4.15) can be decomposed into three parts as

$$-\left\{\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\sigma_2}+\int_{\sigma_2}^{+\infty}\right\}\Big(-\frac{k_1}{\rho(\xi)}+\frac{k_2^2(\rho(\xi))^2}{2\mu}\Big)\psi'(\xi)d\xi, \tag{4.21}$$

which equals to

$$\int_{-\infty}^{\sigma_1}\Big(\frac{k_1}{\rho_-}-\frac{k_2^2\rho_-^2}{2\mu}\Big)\psi'(\xi)d\xi+\int_{\sigma_1}^{\sigma_2}\Big(\frac{k_1}{\rho_*}-\frac{k_2^2\rho_*^2}{2\mu}\Big)\psi'(\xi)d\xi+\int_{\sigma_2}^{+\infty}\Big(\frac{k_1}{\rho_+}-\frac{k_2^2\rho_+^2}{2\mu}\Big)\psi'(\xi)d\xi$$
$$=\Big(\frac{k_1}{\rho_-}-\frac{k_2^2\rho_-^2}{2\mu}\Big)\psi(\sigma_1)+\frac{k_1}{\rho_*}(\psi(\sigma_2)-\psi(\sigma_1))-\frac{k_2^2\rho_*^2}{2\mu}(\psi(\sigma_2)-\psi(\sigma_1))-\Big(\frac{k_1}{\rho_+}-\frac{k_2^2\rho_+^2}{2\mu}\Big)\psi(\sigma_2). \tag{4.22}$$

Taking the limit $k_1,k_2\to 0$ in (4.22), by Lemmas 4.1-4.2, we have

$$\lim_{k_1,k_2\to 0}\int_{-\infty}^{+\infty}\Big(\frac{k_1}{\rho(\xi)}-\frac{k_2^2(\rho(\xi))^2}{2\mu}\Big)\psi'(\xi)d\xi=0. \tag{4.23}$$

Summarizing (4.18), (4.20) and (4.23) leads to

$$\lim_{k_1,k_2\to 0}\int_{-\infty}^{+\infty}((\rho u)(\xi)-(\rho_0u_0)(\xi-\sigma))\psi(\xi)d\xi=(\sigma[\rho u]-[\rho u^2])\psi(\sigma), \tag{4.24}$$

which is true for any $\psi \in C_0^\infty(-\infty,+\infty)$.

As done previously, we can obtain the limit for the first integral on the left-hand side of (4.14) as

$$\lim_{k_1,k_2\to 0}\int_{-\infty}^{+\infty}(\xi-u(\xi))\rho(\xi)\psi'(\xi)d\xi = ([\rho u]-\sigma[\rho])\psi(\sigma)-\int_{-\infty}^{\sigma}\rho_-\psi(\xi)d\xi-\int_{\sigma}^{+\infty}\rho_+\psi(\xi)d\xi$$
$$= ([\rho u]-\sigma[\rho])\psi(\sigma)-\int_{-\infty}^{+\infty}\rho_0(\xi-\sigma)\psi(\xi)d\xi, \tag{4.25}$$

where $\rho_0(\xi)=\rho_-+[\rho]H(\xi)$. Then returning to the formulation (4.14), we have

$$\lim_{k_1,k_2\to 0}\int_{-\infty}^{+\infty}(\rho(\xi)-\rho_0(\xi-\sigma))\psi(\xi)d\xi = (\sigma[\rho]-[\rho u])\psi(\sigma), \tag{4.26}$$

which is true for any $\psi \in C_0^\infty(-\infty,+\infty)$.

Finally, we study the limits of $\rho$ and $\rho u$ as $k_1,k_2\to 0$, by tracing the time-dependence of weights of the $\delta$-measure. Let $\phi(x,t)\in C_0^\infty((-\infty,+\infty)\times[0,+\infty))$, then we have

$$\lim_{k_1,k_2\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho(x/t)\phi(x,t)dxdt = \lim_{k_1,k_2\to 0}\int_0^{+\infty}t\left(\int_{-\infty}^{+\infty}\rho(\xi)\phi(\xi t,t)d\xi\right)dt. \tag{4.27}$$

Regarding $t$ as a parameter and applying (4.26), one can easily see that

$$\lim_{k_1,k_2\to 0}\int_{-\infty}^{+\infty}\rho(\xi)\phi(\xi t,t)d\xi = \int_{-\infty}^{+\infty}\rho_0(\xi-\sigma)\phi(\xi t,t)d\xi+(\sigma[\rho]-[\rho u])\phi(\sigma t,t)$$
$$=\frac{1}{t}\int_{-\infty}^{+\infty}\rho_0\left(\frac{x}{t}-\sigma\right)\phi(x,t)dx+(\sigma[\rho]-[\rho u])\phi(\sigma t,t), \tag{4.28}$$

Substituting (4.28) into (4.27) and noting $\rho_0\left(\frac{x}{t}-\sigma\right)=\rho_0(x-\sigma t)$, we have

$$\lim_{k_1,k_2\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho(x/t)\phi(x,t)dxdt = \int_0^{+\infty}\int_{-\infty}^{+\infty}\rho_0(x-\sigma t)\phi(x,t)dxdt$$
$$+\int_0^{+\infty}t(\sigma[\rho]-[\rho u])\phi(\sigma t,t)dt. \tag{4.29}$$

By definition (3.9), the last term on the right-hand side of (4.29) equals to $\langle w_1(t)\delta_S,\phi(\cdot,\cdot)\rangle$, where

$$w_1(t)=(\sigma[\rho]-[\rho u])t.$$

With the same reason as before, we arrive at

$$\lim_{k_1,k_2\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho(x/t)u(x/t)\phi(x,t)dxdt = \int_0^{+\infty}\int_{-\infty}^{+\infty}(\rho_0u_0)(x-\sigma t)\phi(x,t)dxdt$$
$$+\int_0^{+\infty}t(\sigma[\rho u]-[\rho u^2])\phi(\sigma t,t)dt. \tag{4.30}$$

The last term on the right-hand side of (4.30) equals to $\langle w_2(t)\delta_S,\phi(\cdot,\cdot)\rangle$, where

$$w_2(t)=(\sigma[\rho u]-[\rho u^2])t.$$

The proof is completed.

# 5. Formation of vacuum states

In this section, we study the formation of vacuum states in the Riemann solutions of system (1.1) and (1.2) in the case $(\rho_+, u_+) \in I(\rho_-, u_-)$ with $u_- < u_+$ and $\rho_\pm > 0$ as both the pressure and the magnetic field vanish. In this case, we know that the Riemann solution consists of a backward rarefaction wave $R_1$, a forward rarefaction wave $R_2$ and an intermediate state $(\rho_*, u_*)$ besides two constant states $(\rho_\pm, u_\pm)$, which are as follows

$$R_1 : \begin{cases} \xi = \lambda_1 = u - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}, \\ u = u_- - \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds, \quad \rho_* \le \rho \le \rho_-, \end{cases} \tag{5.1}$$

and

$$R_2 : \begin{cases} \xi = \lambda_2 = u + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}, \\ u = u_+ + \int_{\rho_+}^{\rho} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds, \quad \rho_* \le \rho \le \rho_+. \end{cases} \tag{5.2}$$

From (5.1) and (5.2), we can derive

$$u_+ - u_- = \int_{\rho_*}^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds + \int_{\rho_*}^{\rho_+} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds, \quad \rho_* \le \rho_\pm. \tag{5.3}$$

For any given $\rho_\pm > 0$, if $\lim\limits_{k_1, k_2 \to 0} \rho_* = K \in (0, \min(\rho_-, \rho_+)]$, then by

$$\begin{aligned} \int_{\rho_*}^{\rho} \frac{\sqrt{A + \frac{B}{s^2}}}{s} ds = & -\sqrt{A + \frac{B}{\rho^2}} + \sqrt{A} \ln\left(\sqrt{A + \frac{B}{\rho^2}} + \sqrt{A}\right) + \sqrt{A} \ln \rho \\ & + \sqrt{A + \frac{B}{\rho_*^2}} - \sqrt{A} \ln\left(\sqrt{A + \frac{B}{\rho_*^2}} + \sqrt{A}\right) - \sqrt{A} \ln \rho_*, \quad A > 0, \end{aligned} \tag{5.4}$$

it follows that

$$\begin{aligned} 0 \le \int_{\rho_*}^{\rho_-} & \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds \le \int_{\rho_*}^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 \rho_-}{\mu}}}{s} ds \\ = & \sqrt{\frac{k_1}{\rho_*^2} + \frac{k_2^2 \rho_-}{\mu}} - \sqrt{\frac{k_2^2 \rho_-}{\mu}} \ln\left(\sqrt{\frac{k_1}{\rho_*^2} + \frac{k_2^2 \rho_-}{\mu}} + \sqrt{\frac{k_2^2 \rho_-}{\mu}}\right) - \sqrt{\frac{k_2^2 \rho_-}{\mu}} \ln \rho_* \\ & - \sqrt{\frac{k_1}{\rho_-^2} + \frac{k_2^2 \rho_-}{\mu}} + \sqrt{\frac{k_2^2 \rho_-}{\mu}} \ln\left(\sqrt{\frac{k_1}{\rho_-^2} + \frac{k_2^2 \rho_-}{\mu}} + \sqrt{\frac{k_2^2 \rho_-}{\mu}}\right) + \sqrt{\frac{k_2^2 \rho_-}{\mu}} \ln \rho_- \to 0, \quad \text{as } k_1, k_2 \to 0. \end{aligned} \tag{5.5}$$

Therefore, by the squeeze theorem in multivariable calculus, we arrive at

$$\lim_{k_1, k_2 \to 0} \int_{\rho_*}^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds = 0. \tag{5.6}$$

Similarly, we can obtain that

$$\lim_{k_1, k_2 \to 0} \int_{\rho_*}^{\rho_+} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds = 0. \tag{5.7}$$

Combining (5.3), (5.6) and (5.7), we have $u_- - u_+ = 0$, which contradicts with $u_- < u_+$. Therefore, $\lim\limits_{k_1, k_2 \to 0} \rho_* = 0$, which implies that a vacuum occurs. From (5.1), one can see that

$$u_- - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} \leq \lambda_1 = u_- - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} + \int_\rho^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{\mu}}}{s} ds$$

$$\leq u_- - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} + \int_\rho^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2\rho_-}{\mu}}}{s} ds, \quad \rho_* \leq \rho \leq \rho_- \tag{5.8}$$

It can be derived from (5.4) that

$$u_- - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} + \int_\rho^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2\rho_-}{\mu}}}{s} ds$$

$$= u_- - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho_-}{\mu}} - \sqrt{\frac{k_2^2\rho_-}{\mu}} \ln\left(\sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho_-}{\mu}} + \sqrt{\frac{k_2^2\rho_-}{\mu}}\right) - \sqrt{\frac{k_2^2\rho_-}{\mu}} \ln\rho$$

$$- \sqrt{\frac{k_1}{\rho_-^2} + \frac{k_2^2\rho_-}{\mu}} + \sqrt{\frac{k_2^2\rho_-}{\mu}} \ln\left(\sqrt{\frac{k_1}{\rho_-^2} + \frac{k_2^2\rho_-}{\mu}} + \sqrt{\frac{k_2^2\rho_-}{\mu}}\right) + \sqrt{\frac{k_2^2\rho_-}{\mu}} \ln\rho_-. \tag{5.9}$$

The uniform boundedness of $\rho(\xi)$ with respect to $k_1, k_2$ in this case leads to

$$\lim_{k_1,k_2\to 0}\left(u_- - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} + \int_\rho^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2\rho_-}{\mu}}}{s} ds\right) = \lim_{k_1,k_2\to 0}\left(u_- - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}\right) = u_-. \tag{5.10}$$

Then, by the squeeze theorem in multivariable calculus, we have $\lim\limits_{k_1,k_2\to 0} \lambda_1 = u_-$. Similarly, we can obtain that

$$\lim_{k_1,k_2\to 0} \lambda_2 = u_+ \text{ and } \lim_{k_1,k_2\to 0} u(\xi) = \xi, \quad \text{for } \xi \in (u_-, u_+). \tag{5.11}$$

Then from above we have proved the following results.

**Theorem 3.** In the case $(\rho_+, u_+) \in I(\rho_-, u_-)$ with $u_- < u_+$, as $k_1, k_2 \to 0$, the vacuum state occurs and two rarefaction waves $R_1$ and $R_2$ become two contact discontinuities $u = u_-$ and $u = u_+$, respectively, connecting the constant states $(\rho_\pm, u_\pm)$ with the vacuum ($\rho = 0$).

**Theorem 4.** In the case $(\rho_+, u_+) \in I(\rho_-, u_-)$ with $u_- < u_+$, as $k_1, k_2 \to 0$, the limit of the Riemann solution of (1.1) and (1.2) with initial data (2.1) is just the Riemann solution of the transport equations (3.1) for zero pressure flow with the same initial data, which contains two contact discontinuities $\xi = x/t = u_\pm$ and a vacuum state besides two constant states.

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