On a condition of strong precompactness and the decay of periodic entropy solutions to scalar conservation laws

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Abstract

We propose a new sufficient non-degeneracy condition for the strong precompactness of bounded sequences satisfying the nonlinear first-order differential constraints. This result is applied to establish the decay property for periodic entropy solutions to multidimensional scalar conservation laws.

1 Introduction

Let Ω be an open domain in $\mathbb{R}^n$. We consider the sequence $u_k(x)$, $k \in \mathbb{N}$, bounded in $L^\infty(\Omega)$, which converges weakly-$*$ in $L^\infty(\Omega)$ to some function $u(x)$: $u_k \underset{k\to\infty}{\rightharpoonup} u$. Now let $\varphi(x,u) \in L^2_{loc}(\Omega, C(\mathbb{R},\mathbb{R}^n))$ be a Caratheodory vector-function (i.e. it is continuous with respect to u and measurable with respect to x) such that the functions

$$\alpha_M(x) = \max_{|u|\le M} |\varphi(x,u)| \in L^2_{loc}(\Omega) \quad \forall M > 0 \tag{1.1}$$

(here and below $|\cdot|$ stands for the Euclidean norm of a finite-dimensional vector).

By $\theta(\lambda)$ we shall denote the Heaviside function:

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

Suppose that for every $p \in \mathbb{R}$ the sequence of distributions

$$\operatorname{div}_x \left[\theta(u_k - p)(\varphi(x,u_k) - \varphi(x,p))\right] \text{ is precompact in } W^{-1}_{d,loc}(\Omega) \tag{1.2}$$

for some $d > 1$. Recall that $W^{-1}_{d,loc}(\Omega)$ is a locally convex space of distributions $u(x)$ such that $uf(x)$ belongs to the Sobolev space W^{-1}_d for all $f(x) \in C_0^\infty(\Omega)$.

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The topology in $W^{-1}_{d,loc}(\Omega)$ is generated by the family of semi-norms $u \to \|uf\|_{W^{-1}_d}$, $f(x) \in C_0^\infty(\Omega)$.

If the distributions $\operatorname{div}_x \varphi(x,k)$ are locally finite measures on Ω for all $k \in \mathbb{R}$, then the notion of entropy solutions (in Kruzhkov's sense) of the equation

$$\operatorname{div} \varphi(x,u) + \psi(x,u) = 0 \tag{1.3}$$

(with a Caratheodory source function $\psi(x,u) \in L^1_{loc}(\Omega, C(\mathbb{R}))$) is defined, see [15] and [16] (in the latter paper the more general ultra-parabolic equations are studied). As was shown in [16], assumption (1.2) is always satisfied for bounded sequences of entropy solutions of (1.3).

Our first result is the following strong precompactness property.

Theorem 1.1. *Suppose that for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the function $\lambda \to \xi \cdot \varphi(x,\lambda)$ is not constant in any vicinity of the point $u(x)$ (here and in the sequel "$\cdot$" denotes the inner product in $\mathbb{R}^n$). Then $u_k(x) \underset{k\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ (strongly).*

Theorem 1.1 extends the results of [15], where the strong precompactness property was established under the more restrictive non-degeneracy condition: *for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the function $\lambda \to \xi \cdot \varphi(x,\lambda)$ is not constant on nonempty intervals.*

The proof of Theorem 1.1 is based on a new localization principle for H-measure (with "continuous" indexes) corresponding to the sequence u_k, see Theorem 3.5 and its Corollary 3.6 below.

Using this theorem and results of [17], we will also derive the more precise criterion for the decay of periodic entropy solutions of scalar conservation laws

$$u_t + \operatorname{div}_x \varphi(u) = 0, \tag{1.4}$$

$u = u(t,x)$, $(t,x) \in \Pi = (0,+\infty) \times \mathbb{R}^n$. The flux vector $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u))$ is supposed to be merely continuous: $\varphi(u) \in C(\mathbb{R}, \mathbb{R}^n)$. Recall the definition of entropy solution to equation (1.4) in the Kruzhkov sense [7].

Definition 1.2. A bounded measurable function $u = u(t,x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1.4) if for all $k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x [\operatorname{sign}(u-k)(\varphi(u) - \varphi(k))] \le 0 \tag{1.5}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$).

As usual, condition (1.5) means that for all non-negative test functions $f = f(t,x) \in C_0^1(\Pi)$

$$\int_\Pi [|u-k| f_t + \operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] dtdx \ge 0.$$

As was shown in [13] (see also [14]), an e.s. $u(t,x)$ always admits a strong trace $u_0 = u_0(x) \in L^\infty(\mathbb{R}^n)$ on the initial hyperspace $t = 0$ in the sense of relation

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \quad \text{in } L^1_{loc}(\mathbb{R}^n), \tag{1.6}$$

that is, $u(t,x)$ is an e.s. to the Cauchy problem for equation (1.4) with initial data

$$u(0,x) = u_0(x). \tag{1.7}$$

Remark 1.3. It was also established in [13, Corollary 7.1] that, after possible correction on a set of null measure, an e.s. $u(t,x)$ is continuous on $[0,+\infty)$ as a map $t \mapsto u(t,\cdot)$ into $L^1_{loc}(\mathbb{R}^n)$. In the sequel we will always assume that this property is satisfied.

Suppose that the initial function u_0 is periodic with a lattice of periods L, i.e., $u_0(x+e) = u_0(x)$ a.e. on $\mathbb{R}^n$ for every $e \in L$ (we will call such functions L-periodic). Denote by $\mathbb{T}^n = \mathbb{R}^n/L$ the corresponding n-dimensional torus, and by L' the dual lattice $L' = \{ \xi \in \mathbb{R}^n \mid \xi \cdot x \in \mathbb{Z} \ \forall x \in L \}$. In the case under consideration when the flux vector is merely continuous the property of finite speed of propagation for initial perturbation may be violated, which, in the multidimensional situation $n > 1$, may even lead to the nonuniqueness of e.s. to Cauchy problem (1.4), (1.7), see examples in [8, 9]. But for a periodic initial function $u_0(x)$, an e.s. $u(t,x)$ of (1.4), (1.7) is unique (in the class of all e.s., not necessarily periodic) and space-periodic, the proof can be found in [12]. It is also shown in [12] that the mean value of e.s. over the period does not depend on time:

$$\int_{\mathbb{T}^n} u(t,x)dx = I \doteq \int_{\mathbb{T}^n} u_0(x)dx, \tag{1.8}$$

where dx is the normalized Lebesgue measure on $\mathbb{T}^n$. The following theorem generalizes the previous results of [3, 17].

Theorem 1.4. *Suppose that*

$$\forall \xi \in L', \xi \neq 0 \quad \text{the function } u \to \xi \cdot \varphi(u) \text{ is not affine on any vicinity of } I. \tag{1.9}$$

Then

$$\lim_{t\to+\infty} u(t,\cdot) = I = \int_{\mathbb{T}^n} u_0(x)dx \quad \text{in } L^1(\mathbb{T}^n). \tag{1.10}$$

Moreover condition (1.9) is necessary and sufficient for the decay property (1.10).

In the case $\varphi(u) \in C^2(\mathbb{R},\mathbb{R}^n)$ Theorem 1.4 was proved in [3]. As was noticed in [3, Remark 2.1], decay property (1.10) holds under the the weaker regularity requirement $\varphi(u) \in C^1(\mathbb{R},\mathbb{R}^n)$ but under the more restrictive assumption that for each $\xi \in L'$ I is not an interior point of the closure of the union of all open intervals, over which the function $\xi \cdot \varphi'(u)$ is constant. Let us demonstrate that condition (1.9) is less restrictive than this assumption even in the case $\varphi(u) \in C^1(\mathbb{R}, \mathbb{R}^n)$. Suppose that $n = 1$, $\varphi(u) \in C^1(\mathbb{R})$ is a primitive of the Cantor function, so that $\varphi'(u)$ is increasing, continuous, and maximal intervals, over which it remains constant, are exactly the connected component of the complement $\mathbb{R} \setminus K$ of the Cantor set $K \subset [0, 1]$. Since K has the empty interior the assumption of [3] is never satisfied while (1.9) holds for each $I \in K$.

2 Preliminaries

We need the concept of measure valued functions (Young measures). Recall (see [4, 20]) that a measure-valued function on Ω is a weakly measurable map $x \mapsto \nu_x$ of Ω into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of ν_x means that for each continuous function $g(\lambda)$ the function $x \to \langle \nu_x, g(\lambda) \rangle \doteq \int g(\lambda) d\nu_x(\lambda)$ is measurable on Ω.

A measure-valued function ν_x is said to be bounded if there exists $M > 0$ such that $\operatorname{supp} \nu_x \subset [-M, M]$ for almost all $x \in \Omega$.

Measure-valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $u(x) \in L^\infty(\Omega)$ and $\delta(\lambda - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are called *regular*. We identify these measure-valued functions and the corresponding functions $u(x)$, so that there is a natural embedding of $L^\infty(\Omega)$ into the set $\mathrm{MV}(\Omega)$ of bounded measure-valued functions on Ω.

Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem by L. Tartar [20].

Theorem 2.1. *Let $u_k(x) \in L^\infty(\Omega)$, $k \in \mathbb{N}$, be a bounded sequence. Then there exist a subsequence (we keep the notation $u_k(x)$ for this subsequence) and a bounded measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that*

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_k) \underset{k\to\infty}{\to} \langle \nu_x, g(\lambda) \rangle \quad \textit{weakly-}* \textit{ in } L^\infty(\Omega). \tag{2.1}$$

Besides, ν_x is regular, i.e., $\nu_x(\lambda) = \delta(\lambda - u(x))$ if and only if $u_k(x) \underset{k\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ (strongly).

We will essentially use in the sequel the variant of H-measures with "continuous indexes" introduced in [10]. This variant extends the original concept of H-measure invented by L. Tartar [21] and P. Gerárd [5] and it appears to be a powerful tool in nonlinear analysis.

Suppose $u_k(x)$ is a bounded sequence in $L^\infty(\Omega)$. Passing to a subsequence if necessary, we can suppose that this sequence converges to a bounded measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ in the sense of relation (2.1). We introduce the measures $\gamma_x^k(\lambda) = \delta(\lambda - u_k(x)) - \nu_x(\lambda)$ and the corresponding distribution functions $U_k(x,p) = \gamma_x^k((p,+\infty))$, $u_0(x,p) = \nu_x((p,+\infty))$ on $\Omega \times \mathbb{R}$. Observe that $U_k(x,p), u_0(x,p) \in L^\infty(\Omega)$ for all $p \in \mathbb{R}$, see [10, Lemma 2]. We define the set

$$E = E(\nu_x) = \left\{ \ p_0 \in \mathbb{R} \ \mid \ u_0(x,p) \underset{p \to p_0}{\to} u_0(x,p_0) \ \text{ in } L^1_{loc}(\Omega) \ \right\}.$$

As was shown in [10, Lemma 4], the complement $\mathbb{R} \setminus E$ is at most countable and if $p \in E$ then $U_k(x,p) \underset{k\to\infty}{\rightharpoonup} 0$ weakly-$*$ in $L^\infty(\Omega)$.

Let $F(u)(\xi)$, $\xi \in \mathbb{R}^n$, be the Fourier transform of a function $u(x) \in L^2(\mathbb{R}^n)$, $S = S^{n-1} = \{ \ \xi \in \mathbb{R}^n \ \mid \ |\xi| = 1 \ \}$ be the unit sphere in $\mathbb{R}^n$. Denote by $u \to \overline{u}$, $u \in \mathbb{C}$ the complex conjugation.

The next result was established in [10, Theorem 3], [11, Proposition 2, Lemma 2].

Proposition 2.2. *(i) There exists a family of locally finite complex Borel measures* $\{\mu^{pq}\}_{p,q\in E}$ *in* $\Omega \times S$ *and a subsequence* $U_r(x,p) = U_{k_r}(x,p)$ *such that for all* $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ *and* $\psi(\xi) \in C(S)$

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r(\cdot,p))(\xi)\overline{F(\Phi_2 U_r(\cdot,q))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi; \tag{2.2}$$

(ii) For any $p_1,\dots,p_l \in E$ *the matrix* $\{\mu^{p_ip_j}\}_{i,j=1}^l$ *is Hermitian and nonnegative definite, that is, for all* $\zeta_1,\dots,\zeta_l \in \mathbb{C}$ *the measure*

$$\sum_{i,j=1}^{l} \zeta_i\overline{\zeta_j}\mu^{p_ip_j} \ge 0.$$

We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the H-measure corresponding to the subsequence $u_r(x) = u_{k_r}(x)$.

As was demonstrated in [10], the H-measure $\mu^{pq} = 0$ for all $p,q \in E$ if and only if the subsequence $u_r(x)$ converges as $r \to \infty$ strongly (in $L^1_{loc}(\Omega)$).

Since $|U_k(x,p)| \le 1$, it readily follows from (2.2) and Plancherel's equality that $\mathrm{pr}_\Omega|\mu^{pq}| \le \mathrm{meas}$ for $p,q \in E$, where meas is the Lebesgue measure on Ω, and by $|\mu|$ we denote the variation of a Borel measure μ (this is the minimal of nonnegative Borel measures ν such that $|\mu(A)| \le \nu(A)$ for all Borel sets A). This implies the representation $\mu^{pq} = \mu_x^{pq}dx$ (the disintegration of H-measures). More exactly, choose a countable dense subset $D \subset E$. The following statement was proved in [11, Proposition 3], see also [15, Proposition 3].

Proposition 2.3. *There exists a family of complex finite Borel measures* $\mu_x^{pq} \in \mathrm{M}(S)$ *in the sphere* S *with* $p,q \in D$, $x \in \Omega'$, *where* Ω' *is a subset of* Ω *of full measure, such that* $\mu^{pq} = \mu_x^{pq}dx$, *that is, for all* $\Phi(x,\xi) \in C_0(\Omega \times S)$ *the function*

$$x \to \langle \mu_x^{pq}(\xi), \Phi(x,\xi)\rangle = \int_S \Phi(x,\xi)d\mu_x^{pq}(\xi)$$

is Lebesgue-measurable on Ω*, bounded, and*

$$\langle \mu^{pq}, \Phi(x,\xi)\rangle = \int_\Omega \langle \mu_x^{pq}(\xi), \Phi(x,\xi)\rangle dx.$$

Moreover, for $p, p', q \in D$*,* $p' > p$

$$\operatorname{Var} \mu_x^{pq} \doteq |\mu_x^{pq}|(S) \le 1 \quad \textit{and} \quad \operatorname{Var}(\mu_x^{p'q} - \mu_x^{pq}) \le 2\left(\nu_x((p,p'))\right)^{1/2}. \tag{2.3}$$

We choose a non-negative function $K(x) \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\int K(x)dx = 1$ and set $K_m(x) = m^n K(mx)$ for $m \in \mathbb{N}$. Clearly, the sequence of K_m converges in $\mathcal{D}'(\mathbb{R}^n)$ to the Dirac δ-function (that is, this sequence is an approximate unity). We define $\Phi_m(x) = (K_m(x))^{1/2}$. As was shown in [11, Remark 4] (see also [15, Remark 2(b)]), the measures μ_x^{pq} can be explicitly represented by the relation

$$\Phi(x)\langle \mu_x^{pq}, \psi(\xi)\rangle = \lim_{m\to\infty} \langle \mu_x^{pq}(y,\xi), \Phi(y)K_m(x-y)\psi(\xi)\rangle =$$
$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} F(\Phi\Phi_m U_r(\cdot,p))(\xi)\overline{F(\Phi_m U_r(\cdot,q))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi \tag{2.4}$$

for all $\psi(\xi) \in C(S)$, where $\Phi\Phi_m U_r^p(y) = \Phi(y)\Phi_m(x-y)U_r(y,p)$, $\Phi_m U_r^q(y) = \Phi_m(x-y)U_r(y,q)$, and $\Phi(y) \in L^2_{loc}(\Omega)$ be an arbitrary function such that x is its Lebesgue point.

From this representation (with $\Phi \equiv 1$) and Proposition 2.2(ii) it follows that for all $p_1, \ldots, p_l \in D$, $x \in \Omega'$, $\zeta_1, \ldots, \zeta_l \in \mathbb{C}$ the measure

$$\mu = \sum_{i,j=1}^{l} \zeta_i \overline{\zeta_j} \mu_x^{p_i p_j} \ge 0. \tag{2.5}$$

Indeed, for every nonnegative $\psi(\xi) \in C(S)$

$$< \mu(\xi), \psi(\xi) >= \lim_{m\to\infty}\left\langle \sum_{i,j=1}^{l} \zeta_i\overline{\zeta_j}\mu^{p_i p_j}(y,\xi), K_m(x-y)\psi(\xi)\right\rangle \ge 0.$$

This, in particular implies, that $\mu_x^{pp} \ge 0$, $\mu_x^{qp} = \overline{\mu_x^{pq}}$, and for every Borel set $A \subset S$

$$|\mu_x^{pq}|(A) \le \left(\mu_x^{pp}(A)\mu_x^{qq}(A)\right)^{1/2} \tag{2.6}$$

(see [11, 15]). For completeness we provide below the simple proof of (2.6). In view of (2.5) (with l=2) the matrix $M = \begin{pmatrix} \mu_x^{pp}(A) & \mu_x^{pq}(A) \\ \mu_x^{qp}(A) & \mu_x^{qq}(A) \end{pmatrix}$ is Hermitian and nonnegative definite. Therefore,

$$\mu_x^{pp}(A)\mu_x^{qq}(A) - |\mu_x^{pq}(A)|^2 = \mu_x^{pp}(A)\mu_x^{qq}(A) - \mu_x^{pq}(A)\mu_x^{qp}(A) = \det M \ge 0.$$

By Young's inequality for any positive constant c and all Borel sets $A \subset S$

$$|\mu_x^{pq}(A)| \le (\mu_x^{pp}(A)\mu_x^{qq}(A))^{1/2} \le \frac{c}{2}\mu_x^{pp}(A) + \frac{1}{2c}\mu_x^{qq}(A).$$

Since $\mu = \frac{c}{2}\mu_x^{pp} + \frac{1}{2c}\mu_x^{qq}$ is nonnegative Borel measure, it follows from this inequality that the variation $|\mu_x^{pq}| \le \mu$. This implies that

$$|\mu_x^{pq}|(A) \le \frac{c}{2}\mu_x^{pp}(A) + \frac{1}{2c}\mu_x^{qq}(A) \quad \forall c > 0. \tag{2.7}$$

It is easily computed that

$$\inf_{c>0}\left(\frac{c}{2}\mu_x^{pp}(A) + \frac{1}{2c}\mu_x^{qq}(A)\right) = (\mu_x^{pp}(A)\mu_x^{qq}(A))^{1/2}$$

and (2.6) follows from (2.7).

3 Localization principles and the strong precompactness property

Lemma 3.1. *For each $p, q \in \mathbb{R}$, $x \in \Omega'$ there exist one-sided limits in the space* $\mathrm{M}(S)$ *of finite Borel measures on S (with the standard norm* $\operatorname{Var}\mu$*):*

$$\begin{aligned} &\mu_x^{p'q'} \to \mu_x^{pq+} \quad \text{as } (p',q') \to (p,q), \quad p',q' \in D, p' > p, q' > q, \\ &\mu_x^{p'q'} \to \mu_x^{pq-} \quad \text{as } (p',q') \to (p,q), \quad p',q' \in D, p' < p, q' < q. \end{aligned}$$

Moreover, $\operatorname{Var}\mu^{pq\pm} \le 1$ *and for every Borel set $A \subset S$ and each $p_i \in \mathbb{R}$, $i = 1,\dots,l$ the matrices $\{\mu_x^{p_ip_j\pm}(A)\}_{i,j=1}^l$ are Hermitian and nonnegative definite, that is, the measures*

$$\sum_{i,j=1}^{l} \zeta_i\overline{\zeta_j}\mu_x^{p_ip_j\pm} \ge 0 \tag{3.1}$$

for all complex $\zeta_i \in \mathbb{C}$, $i = 1,\dots,l$.

Proof. Let $x \in \Omega'$, $p, q \in \mathbb{R}$, $p_1, q_1, p_2, q_2 \in D$, $p_2 > p_1 > p$, $q_2 > q_1 > q$. Then, in view of (2.3) and the equality $\mu_x^{qp} = \overline{\mu_x^{pq}}$,

$$\begin{aligned} \operatorname{Var}(\mu_x^{p_2q_2} - \mu_x^{p_1q_1}) \le 2\nu_x((p_1,p_2)) + 2\nu_x((q_1,q_2)) \le \\ 2\nu_x((p,p_2)) + 2\nu_x((q,q_2)) \underset{(p_2,q_2)\to(p,q)}{\to} 0. \end{aligned}$$

By the Cauchy criterion, this implies that there exists a limit μ_x^{pq+} in $\mathrm{M}(S)$ as $(p',q') \to (p,q)$, $p',q' \in D$, $p' > p$, $q' > q$. Similarly, for each $p_1, q_1, p_2, q_2 \in D$ such that $p_2 < p_1 < p$, $q_2 < q_1 < q$

$$\begin{aligned} \operatorname{Var}(\mu_x^{p_2q_2} - \mu_x^{p_1q_1}) \le 2\nu_x((p_2,p_1)) + 2\nu_x((q_2,q_1)) \le \\ 2\nu_x((p_2,p)) + 2\nu_x((q_2,q)) \underset{(p_2,q_2)\to(p,q)}{\to} 0, \end{aligned}$$

which implies existence of a left-sided limit μ_x^{pq-} in $\mathrm{M}(S)$ as $(p',q') \to (p,q)$, $p',q' \in D$, $p' < p$, $q' < q$. By Proposition 2.3 $\operatorname{Var} \mu_x^{p'q'} \le 1$, which implies in the limits as $p' \to p\pm$, $q' \to q\pm$ that $\operatorname{Var} \mu_x^{pq\pm} \le 1$. Finally, for every $p_i' \in D$, $\zeta_i \in \mathbb{C}$, $i = 1, \dots, l$ the measures

$$\sum_{i,j=1}^{l} \zeta_i \overline{\zeta_j} \mu_x^{p_i' p_j' \pm} \ge 0.$$

In the limits as $p_i' \to p_i\pm$ this implies (3.1). □

Corollary 3.2. *Let $p, q \in \mathbb{R}$, $x \in \Omega'$. Then for every Borel set $A \subset S$*

$$|\mu_x^{pq+}|(A) \le \left(\mu_x^{pp+}(A)\mu_x^{qq+}(A)\right)^{1/2}, \ |\mu_x^{pq-}|(A) \le \left(\mu_x^{pp-}(A)\mu_x^{qq-}(A)\right)^{1/2}. \quad (3.2)$$

Proof. Relations (3.2) follow from (3.1) in the same way as in the proof of inequality (2.6) above. □

Remark 3.3. By continuity of μ_x^{pq} with respect to variables $p, q \in D$, we see that for $p \in D$

$$\mu_x^{pq\pm} = \lim_{q' \to q\pm} \lim_{p' \to p\pm} \mu_x^{p'q'} = \lim_{q' \to q\pm} \mu_x^{pq'} \text{ in } \mathrm{M}(S).$$

Analogously, if $q \in D$, then

$$\mu_x^{pq\pm} = \lim_{p' \to p\pm} \mu_x^{p'q} \text{ in } \mathrm{M}(S).$$

If the both indices $p, q \in D$, then evidently $\mu_x^{pq\pm} = \mu_x^{pq}$.

Now we suppose that $f(y,\lambda) \in L^2_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$ is a Caratheodory vector-function on $\Omega \times \mathbb{R}$. In particular,

$$\forall M > 0 \quad \|f(x,\cdot)\|_{M,\infty} = \max_{|\lambda| \le M} |f(x,\lambda)| = \alpha_M(x) \in L^2_{loc}(\Omega). \quad (3.3)$$

Since the space $C(\mathbb{R}, \mathbb{R}^n)$ is separable with respect to the standard locally convex topology generated by seminorms $\|\cdot\|_{M,\infty}$, then, by the Pettis theorem (see [6], Chapter 3), the map $x \to F(x) = f(x,\cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ is strongly measurable and in view of estimate (3.3) we see that $|F(x)|^2 \in L^1_{loc}(\Omega, C(\mathbb{R}))$. In particular (see [6], Chapter 3), the set Ω_f of common Lebesgue points of the maps $F(x), |F(x)|^2$ has full measure. As was demonstrated in [15], for $x \in \Omega_f$

$$\lim_{m \to \infty} \int K_m(x-y) \|F(x) - F(y)\|^2_{M,\infty} dy = 0 \quad \forall M > 0. \quad (3.4)$$

Clearly, each $x \in \Omega_f$ is a common Lebesgue point of all functions $x \to f(x,\lambda)$, $\lambda \in \mathbb{R}$. Let $\Omega'' = \Omega' \cap \Omega_f$, $\gamma_x^r(\lambda) = \delta(\lambda - u_r(x)) - \nu_x(\lambda)$.

Suppose that $x \in \Omega''$, $p \in \mathbb{R}$, H_+, H_- are the minimal linear subspaces of $\mathbb{R}^n$, containing supports of the measures μ_x^{pp+}, μ_x^{pp-}, respectively. We fix $q \in D$ and introduce for $p' \in D$ the function

$$I_r(y,p') = \int f(y,\lambda)(\theta(\lambda - p') - \theta(\lambda - q)) d\gamma_y^r(\lambda) \in L^2_{loc}(\Omega). \quad (3.5)$$

Proposition 3.4. *Assume that $q > p$ and $f(x,\lambda) \in H_+^\perp$ for all $\lambda \in \mathbb{R}$. Then*

$$\lim_{p' \to p+} \lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi = 0 \quad (3.6)$$

for all $\psi(\xi) \in C(S)$. Analogously, if $q < p$ and $f(x,\lambda) \in H_-^\perp$ $\forall \lambda \in \mathbb{R}$, then $\forall \psi(\xi) \in C(S)$

$$\lim_{p' \to p-} \lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi = 0. \quad (3.7)$$

Here $\Phi_m = \Phi_m(x-y) = \sqrt{K_m(x-y)}$ and $I_r(y,p'), U_r(y,p')$ are functions of the variable $y \in \Omega$.

Proof. Note that starting from some index m the supports of the functions $\Phi_m(x-y)$ lie in some compact subset B of Ω. Without loss of generality we can assume that $\operatorname{supp}\Phi_m \subset B$ for all $m \in \mathbb{N}$. Let

$$\tilde{I}_r(y,p') = \int f(x,\lambda)(\theta(\lambda-p') - \theta(\lambda-q))d\gamma_y^r(\lambda) \in L^2_{loc}(\Omega),$$

$M = \sup\limits_{r\in\mathbb{N}} \|u_r\|_\infty$. Then $\operatorname{supp}\gamma_y^r \subset [-M,M]$, and

$$|I_r(y,p') - \tilde{I}_r(y,p')| \le \int |f(y,\lambda) - f(x,\lambda)| d|\gamma_y^r|(\lambda) \le 2\|F(y) - F(x)\|_{M,\infty}.$$

By Plancherel's identity

$$\left|\int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi - \right.$$
$$\left.\int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m \tilde{I}_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi\right| =$$
$$\left|\int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m (I_r(\cdot,p') - \tilde{I}_r(\cdot,p')))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi\right| \le$$
$$\|\psi\|_\infty \|\Phi_m(I_r(\cdot,p') - \tilde{I}_r(\cdot,p'))\|_2 \|\Phi_m U_r(\cdot,p')\|_2 \le$$
$$\|\psi\|_\infty \|\Phi_m(I_r(\cdot,p') - \tilde{I}_r(\cdot,p'))\|_2 \le$$
$$2\|\psi\|_\infty \left(\int K_m(x-y)\|F(y) - F(x)\|^2_{M,\infty} dy\right)^{1/2}.$$

Here we take account of the equality

$$\|\Phi_m\|_2 = \left(\int_\Omega K_m(x-y)dy\right)^{1/2} = 1.$$

From the above estimate and (3.4) it follows that

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m \tilde{I}_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| = 0. \quad (3.8)$$

Observe that the function $\tilde{f}(\lambda) = f(x,\lambda) \in C(\mathbb{R}, H_+^\perp)$ is continuous and does not depend on y. Therefore for any $\varepsilon > 0$ there exists a piece-wise constant vector-valued function $g(\lambda)$ of the form $g(\lambda) = \sum_{i=1}^{k} v_i\theta(\lambda - p_i)$, where $v_i \in H_+^\perp$, $p = p_1 < p_2 < \cdots < p_k = q$ such that $\|\tilde{f}\chi - g\|_\infty \le \varepsilon$ on $\mathbb{R}$. Here $\chi(\lambda) = \theta(\lambda-p)-\theta(\lambda-q)$. Moreover, by the density of D, we may suppose that $p_i \in D$ for $i > 1$. We define for $p' \in D \cap (p, p_2)$

$$J_r(y,p') = \int g(\lambda)\theta(\lambda - p')d\gamma_y^r(\lambda).$$

Using again Plancherel's identity and the fact that

$$|\tilde{I}_r(y,p') - J_r(y,p')| = \left|\int (\tilde{f}\cdot\chi - g)(\lambda)\theta(\lambda-p')d\gamma_y^r(\lambda)\right| \le \int |(\tilde{f}\cdot\chi - g)(\lambda)|d|\gamma_y^r|(\lambda) \le 2\varepsilon,$$

we obtain

$$\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m \tilde{I}_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m J_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| =$$
$$\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m(\tilde{I}_r(\cdot,p') - J_r(\cdot,p')))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| \le$$
$$\|\Phi_m(\tilde{I}_r(\cdot,p') - J_r(\cdot,p'))\|_2\cdot\|\Phi_m U_r(\cdot,p')\|_2\cdot\|\psi\|_\infty \le 2\|\psi\|_\infty\varepsilon \quad (3.9)$$

for all $\psi(\xi) \in C(S)$. Since

$$J_r(y,p') = \int\left(\sum_{i=1}^{k} v_i\theta(\lambda - p_i')\right)d\gamma_y^r(\lambda) = \sum_{i=1}^{k} v_i U_r(y,p_i'),$$

where $p_i' = \max(p_i,p') \in D$, it follows from (2.4) with account of Remark 3.3 that

$$\lim_{p'\to p+}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m J_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p')(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi =$$
$$\lim_{p'\to p+}\sum_{i=1}^{k}\langle\mu_x^{p_i'p'}, (v_i\cdot\xi)\psi(\xi)\rangle = \sum_{i=1}^{k}\langle\mu_x^{p_ip+}, (v_i\cdot\xi)\psi(\xi)\rangle = 0. \quad (3.10)$$

The last equality is a consequence of the inclusion $\operatorname{supp}\mu_x^{p_ip+} \subset \operatorname{supp}\mu_x^{pp+} \subset H_+$ (because of Corollary 3.2) combined with the relation $v_i \perp H_+$. By (3.8), (3.9) and (3.10), we have

$$\varlimsup_{p'\to p+}\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\left|\int_{\mathbb{R}^n}\frac{\xi}{|\xi|}\cdot F(\Phi_m I_r(\cdot,p'))(\xi)\overline{F(\Phi_m U_r(\cdot,p'))(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| \le \text{const}\cdot\varepsilon,$$

and it suffices to observe that $\varepsilon > 0$ can be arbitrary to complete the proof of (3.6). The proof of relation (3.7) is similar to the proof of (3.6) and is omitted. $\square$

Now we assume that the sequence u_k satisfies constraints (1.2). We choose a subsequence u_r and the corresponding H-measure $\mu^{pq} = \mu_x^{pq}dx$. Assume that $x \in \Omega'' = \Omega' \cap \Omega_\varphi$, $p_0 \in \mathbb{R}$. As above, let H_+, H_- be the minimal linear subspaces of $\mathbb{R}^n$ containing $\operatorname{supp}\mu_x^{p_0p_0+}$, $\operatorname{supp}\mu_x^{p_0p_0-}$, respectively.

Theorem 3.5 (localization principle)**.** *There exists a positive δ such that $(\varphi(x,\lambda) - \varphi(x,p))\cdot\xi = 0$ for all $\xi \in H_+$, $\lambda \in [p_0, p_0+\delta]$ and all $\xi \in H_-$, $\lambda \in [p_0-\delta, p_0]$.*

Proof. The proof is analogous to the proof of [15, Theorem 4] (if $d = 2$), for arbitrary $d > 1$ see the proof of [16, Theorem 4] (where the more general case of ultra-parabolic constraints was treated). For completeness we provide the details. Observe firstly that in view of (1.2) the sequence of distributions

$$\mathcal{L}_p^r(y) = \operatorname{div}_y\left(\int\theta(\lambda-p)(\varphi(y,\lambda)-\varphi(y,p))d\gamma_y^r(\lambda)\right)\underset{r\to\infty}{\to} 0 \text{ in } W_{d,loc}^{-1}(\Omega). \quad (3.11)$$

For $p,q \in D$, $q > p > p_0$ we consider the sequence of distributions

$$\mathcal{L}_q^r - \mathcal{L}_p^r = \operatorname{div}_y\,(Q_r^p(y)), \quad r \in \mathbb{N},$$

where the vector-valued functions $Q_r^p(y)$ (for fixed $q \in D$) are as follows:

$$\begin{aligned} Q_r^p(y) = &\int(\varphi(y,\lambda)-\varphi(y,q))\theta(\lambda-q)d\gamma_y^r(\lambda) - \\ &\int(\varphi(y,\lambda)-\varphi(y,p))\theta(\lambda-p)d\gamma_y^r(\lambda) = \\ &\int(\varphi(y,q)-\varphi(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda) - \\ &\int(\varphi(y,q)-\varphi(y,p))\theta(\lambda-p)d\gamma_y^r(\lambda) = \\ &\int(\varphi(y,q)-\varphi(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda) - (\varphi(y,q)-\varphi(y,p))U_r(y,p); \end{aligned} \quad (3.12)$$

here $\chi(\lambda) = \theta(\lambda-p)-\theta(\lambda-q)$ is the indicator function of the segment $(p,q]$. As was already noted, $\operatorname{div}_y\,(Q_r^p(y))\underset{r\to\infty}{\to} 0$ in $W_{d,loc}^{-1}(\Omega)$ and if $\Phi(y) \in C_0^\infty(\Omega)$ then

$$\operatorname{div}_y\,(Q_r^p\Phi(y))\underset{r\to\infty}{\to} 0 \text{ in } W_d^{-1}. \quad (3.13)$$

Using the Fourier transformation, from (3.13) we obtain

$$|\xi|^{-1}\xi \cdot F(Q_r^p\Phi)(\xi) = F(g_r), \quad g_r \underset{r\to\infty}{\to} 0 \ \text{ in } L^d(\mathbb{R}^n) \tag{3.14}$$

(see [15, 16] for details).

Let $\psi(\xi) \in C^\infty(S)$. By the known Marcinkiewicz multiplier theorem (cf. [19, Chapter 4]) $\psi(\xi/|\xi|)$ is a Fourier multiplier in L^s for all $s > 1$. This implies that

$$\overline{F(U_r(\cdot,p)\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) = \overline{F(h_r)}(\xi), \tag{3.15}$$

where the sequence h_r is bounded in $L^{d'}$, $d' = d/(d-1)$.

By (3.14), (3.15) we obtain

$$\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(Q_r^p\Phi)(\xi)\overline{F(U_r(\cdot,p)\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) = \int_{\mathbb{R}^n} g_r(x)\overline{h_r(x)}dx \to 0$$

as $r \to \infty$, or in view of (3.12),

$$\lim_{r\to\infty}\left\{\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(U(\cdot,p)f\Phi)(\xi)\overline{F(U_r(\cdot,p)\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \right.$$
$$\left.\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(V_r(\cdot,p)\Phi)(\xi)\overline{F(U_r(\cdot,p)\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right\} = 0, \tag{3.16}$$

where

$$f(y) = \varphi(y,q) - \varphi(y,p) \ \text{ and } \ V_r(y,p) = \int(\varphi(y,q) - \varphi(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda).$$

Obviously, (3.16) remains valid for merely continuous $\psi(\xi)$. We set in (3.16) $\Phi(y) = \Phi_m(x-y)$, where the functions Φ_m were defined in section 2, and pass to the limit as $m \to \infty$, $p \to p_0+$. By (2.4) with $\Phi(y) = \varphi(y,q) - \varphi(y,p)$ and Lemma 3.1, we obtain

$$\lim_{p\to p_0+}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(U_r(\cdot,p)f\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi =$$
$$\lim_{p\to p_0+}(\varphi(x,q) - \varphi(x,p)) \cdot \langle\mu_x^{pp}, \xi\psi(\xi)\rangle = (\varphi(x,q) - \varphi(x,p_0)) \cdot \langle\mu_x^{p_0p_0+}, \xi\psi(\xi)\rangle,$$

therefore

$$(\varphi(x,q) - \varphi(x,p_0)) \cdot \langle\mu_x^{p_0p_0+}, \xi\psi(\xi)\rangle =$$
$$\lim_{p\to p_0+}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}\xi \cdot F(V_r(\cdot,p)\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi. \tag{3.17}$$

Let π_1 and π_2 be the orthogonal projections of $\mathbb{R}^n$ onto the subspaces H_+ and $H_+^\perp$, respectively; let $\tilde{\varphi}(x,\lambda) = \pi_1(\varphi(x,\lambda))$, $\bar{\varphi}(x,\lambda) = \pi_2(\varphi(x,\lambda))$. Recall that

H_+ is the smallest subspace containing $\operatorname{supp}\mu_x^{p_0p_0+}$. This readily implies that $\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle\in H_+$. Hence

$$(\varphi(x,q)-\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle=(\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle. \quad (3.18)$$

Further, $V_r(y,p)=\pi_1(V_r(y,p))+\pi_2(V_r(y,p))$ and

$$\pi_1(V_r(y,p))=\int(\tilde\varphi(y,q)-\tilde\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda),$$

$$\pi_2(V_r(y,p))=\int(\bar\varphi(y,q)-\bar\varphi(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda).$$

Observe that

$$\pi_2(V_r(y,p))=I_r(y,p),$$

where the function $I_r(y,p)$ is defined in (3.5) (with p' replaced by p) for a vector-function $f(y,\lambda)=\bar\varphi(y,q)-\bar\varphi(y,\lambda)\in H_+^\perp$. By Proposition 3.4 we obtain

$$\lim_{p\to p_0+}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\pi_2(V_r(y,p))\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi=0. \quad (3.19)$$

Let $\tilde V_r(y,p)=\pi_1(V_r(y,p))$. From (3.17), in view of (3.18) and (3.19), we see that

$$(\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle=$$
$$\lim_{p\to p_0+}\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}|\xi|^{-1}\xi\cdot F(\tilde V_r(\cdot,p)\Phi_m)(\xi)\overline{F(U_r(\cdot,p)\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi,$$

which in turn, by Bunyakovskii inequality and Plancherel's equality, gives us the estimate

$$\left|(\tilde\varphi(x,q)-\tilde\varphi(x,p_0))\cdot\langle\mu_x^{p_0p_0+},\xi\psi(\xi)\rangle\right|\le$$
$$\varlimsup_{p\to p_0+}\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|\tilde V_r(\cdot,p)\Phi_m\|_2\cdot\|U_r(\cdot,p)\Phi_m\|_2\cdot\|\psi\|_\infty\le$$
$$\varlimsup_{p\to p_0+}\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\|\tilde V_r(\cdot,p)\Phi_m\|_2\cdot\|\psi\|_\infty. \quad (3.20)$$

Next, for $M_q(y)=\max_{\lambda\in[p_0,q]}|\tilde\varphi(y,q)-\tilde\varphi(y,\lambda)|$

$$|\tilde V_r(y,p)|\le M_q(y)\left|\int\chi(\lambda)d\left(\nu_y^r(\lambda)+\nu_y^0(\lambda)\right)\right|=$$
$$M_q(y)(u_r(y,p)-u_r(y,q)+u_0(y,p)-u_0(y,q)).$$

In view of the elementary inequality $(a+b)^2\le 2(a^2+b^2)$ and the relation $0\le u_r(y,p)-u_r(y,q)\le 1$, $r\in\mathbb{N}\cup\{0\}$, we have

$$\|\tilde V_r(\cdot,p)\Phi_m\|_2^2\le 2\int_\Omega(M_q(y))^2\big((u_r(y,p)-u_r(y,q))^2+$$
$$(u_0(y,p)-u_0(y,q))^2\big)K_m(x-y)dy\le$$
$$2\int_\Omega(M_q(y))^2(u_r(y,p)-u_r(y,q)+$$
$$u_0(y,p)-u_0(y,q))K_m(x-y)dy. \quad (3.21)$$

Since $p, q \in D \subset E$, then

$$u_r(y,p) - u_r(y,q) \rightharpoonup u_0(y,p) - u_0(y,q)$$

as $r \to \infty$ in the weak-$*$ topology of $L^\infty(\Omega)$ and from (3.21) we now obtain the estimate

$$\varlimsup_{r\to\infty} \|\tilde{V}_r(\cdot,p)\Phi_m\|_2^2 \le 4\int_\Omega (M_q(y))^2(u_0(y,p) - u_0(y,q))K_m(x-y)dy,$$

from which, passing to the limit as $m \to \infty$, we obtain

$$\varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|\tilde{V}_r(\cdot,p)\Phi_m\|_2^2 \le 4(M_q(x))^2(u_0(x,p_0) - u_0(x,p)). \tag{3.22}$$

Here we bear in mind that by the definition of Ω' (see, for instance, [15, Proposition 3]) x is a Lebesgue point of the functions $u_0(y,p_0)$, $u_0(y,p)$. It is also used that $x \in \Omega_\varphi$ is a Lebesgue point of the function $(M_q(y))^2$ as well (this easily follows from the fact that x is a Lebesgue point of the maps $y \to \varphi(y,\cdot)$, $y \to |\varphi(y,\cdot)|^2$ into the spaces $C(\mathbb{R},\mathbb{R}^n)$, $C(\mathbb{R})$, respectively). From (3.22) in the limit as $p \to p_0$ it follows that

$$\varlimsup_{p\to p_0} \varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|\tilde{V}_r(\cdot,p)\Phi_m\|_2^2 \le 4(M_q(x))^2(u_0(x,p_0) - u_0(x,q)). \tag{3.23}$$

In view of (3.20) and (3.23),

$$\begin{gathered} |(\tilde{\varphi}(x,q) - \tilde{\varphi}(x,p_0)) \cdot \langle \mu_x^{p_0p_0+}, \xi\psi(\xi)\rangle| \le 2\|\psi\|_\infty M_q(x)\omega(q), \\ \omega(q) = (u_0(x,p_0) - u_0(x,q))^{1/2} = (\nu_x(p_0,q])^{1/2} \underset{q\to p_0}{\to} 0. \end{gathered} \tag{3.24}$$

It is clear that the set of vectors of the form $\langle \mu_x^{p_0p_0+}, \xi\psi(\xi)\rangle$, with real $\psi(\xi) \in C(S)$ spans the subspace H_+. Hence we can choose functions $\psi_i(\xi) \in C(S)$, $i = 1,\dots,l$ such that the vectors $v_i = \langle \mu_x^{p_0p_0+}, \xi\psi_i(\xi)\rangle$ make up an algebraic basis in H_+.

By (3.24), for $\psi(\xi) = \psi_i(\xi)$, $i = 1,\dots,l$, we obtain

$$|(\tilde{\varphi}(x,q) - \tilde{\varphi}(x,p_0)) \cdot v_i| \le c_i\omega(q)M_q(x), \quad c_i = \text{const},$$

and since v_i, $i = 1,\dots,l$ is a basis in H_+, these estimates show that

$$\begin{gathered} |\tilde{\varphi}(x,q) - \tilde{\varphi}(x,p_0)| \le c\omega(q)M_q(x) = \\ c\omega(q) \max_{\lambda\in[p_0,q]} |\tilde{\varphi}(x,q) - \tilde{\varphi}(x,\lambda)|, \quad c = \text{const}. \end{gathered} \tag{3.25}$$

We take $q = p_0 + \delta$, where $\delta > 0$ is so small that $2c\omega(q) = \varepsilon < 1$. Then, in view of (3.25),

$$|\tilde{\varphi}(x,q) - \tilde{\varphi}(x,p_0)| \le \frac{\varepsilon}{2} \max_{\lambda\in[p_0,p]} |\tilde{\varphi}(x,q) - \tilde{\varphi}(x,\lambda)|, \tag{3.26}$$

and since $\varphi(x,q)$ is continuous with respect to q and the set D is dense, the estimate (3.26) holds for all $q \in [p_0, p_0+\delta]$.

We claim that now $\tilde{\varphi}(x,p) = \tilde{\varphi}(x,p_0)$ for $p \in [p_0, p_0+\delta]$. Indeed, assume that for $p' \in [p_0, p_0+\delta]$

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)|.$$

Then for $\lambda \in [p_0, p']$ we have

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \le |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| + \\ |\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \le 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|$$

and

$$\max_{\lambda\in[p_0,p']} |\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \le 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|.$$

We now derive from (3.26) with $p = p'$ that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \le \varepsilon|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|,$$

and since $\varepsilon < 1$, this implies that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{\lambda\in[p_0,p_0+\delta]} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| = 0.$$

We conclude that $\varphi(x,\lambda) - \varphi(x,p_0) \in H_+^\perp$ for all $\lambda \in [p_0, p_0+\delta]$, i.e., $(\varphi(x,\lambda) - \varphi(x,p_0)) \cdot \xi = 0$ on the segment $[p_0, p_0+\delta]$ for all $\xi \in H_+$.

To prove that for some sufficiently small $\delta > 0$ $(\varphi(x,\lambda) - \varphi(x,p_0)) \cdot \xi = 0$ on the segment $[p_0-\delta, p_0]$ for all $\xi \in H_-$, we take $p,q \in D$, $q < p < p_0$ and repeat the reasonings used in the first part of the proof. As a result, we obtain the relation similar to (3.24)

$$|(\tilde{\varphi}(x,q) - \tilde{\varphi}(x,p_0)) \cdot \langle \mu_x^{p_0p_0-}, \xi\psi(\xi)\rangle| \le 2\|\psi\|_\infty M_q(x)\omega(q),$$

where

$$M_q(x) = \max_{\lambda\in[q,p_0]} |\tilde{\varphi}(y,q) - \tilde{\varphi}(y,\lambda)|,$$
$$\omega(q) = \lim_{p\to p_0-} (u_0(x,q) - u_0(x,p))^{1/2} = (\nu_x(q,p_0))^{1/2} \underset{q\to p_0}{\to} 0.$$

This relation readily implies the desired statement $(\varphi(x,\lambda) - \varphi(x,p_0)) \cdot \xi = 0$ on the segment $[p_0-\delta, p_0]$ for all $\xi \in H_-$, where δ is sufficiently small.

The proof is complete. □

Corollary 3.6. *Let $x \in \Omega''$, $[a,b]$ be the minimal segment, containing* $\operatorname{supp}\nu_x$ *and $p_0 \in (a,b)$. Then, in the notations of Theorem 3.5,* $\operatorname{supp}\mu_x^{p_0p_0+} \cap \operatorname{supp}\mu_x^{p_0p_0-} \ne \emptyset$ *and for all $\xi \in H_+ \cap H_-$, $\xi \ne 0$ the function $\xi \cdot \varphi(x,\lambda)$ is constant in a vicinity of p_0.*

Proof. First, note that since $x \in \Omega'' \subset \Omega'$ is a Lebesgue point of the functions $u_0(\cdot, p)$ for all $p \in D$ while D is dense, the distribution function $u_0(x,\lambda) = \nu_x((\lambda, +\infty))$ is uniquely defined by the relation $u_0(x,\lambda) = \sup\limits_{p\in D, p>\lambda} u_0(x,p)$. In particular, the measure ν_x is well-defined at the point x.

The statement that the function $\lambda \to \xi \cdot \varphi(x,\lambda)$ is constant in a vicinity of p_0 for all $\xi \in H_+ \cap H_-$, $\xi \neq 0$ readily follows from the assertion of Theorem 3.5. Hence, we only need to show that $\operatorname{supp} \mu_x^{p_0p_0+} \cap \operatorname{supp} \mu_x^{p_0p_0-} \neq \emptyset$. We assume to the contrary that $S_+ \cap S_- = \emptyset$, where $S_\pm = \operatorname{supp} \mu_x^{p_0p_0\pm}$. Denote $C_+ = S \setminus S_+$, $C_- = S \setminus S_-$, Then $S = C_+ \cup C_-$, $\mu_x^{p_0p_0+}(C_+) = \mu_x^{p_0p_0-}(C_-) = 0$. Therefore, by relation (2.6), for all $p, q \in D$, $p < p_0 < q$

$$\operatorname{Var} \mu_x^{pq} = |\mu_x^{pq}|(S) \le |\mu_x^{pq}|(C_+) + |\mu_x^{pq}|(C_-) \le \\ (\mu_x^{pp}(C_+)\mu_x^{qq}(C_+))^{1/2} + (\mu_x^{pp}(C_-)\mu_x^{qq}(C_-))^{1/2} \le (\mu_x^{qq}(C_+))^{1/2} + (\mu_x^{pp}(C_-))^{1/2},$$

where we use that $\mu_x^{pp}(A) \le \mu_x^{pp}(S) \le 1$ for all $p \in D$ and every Borel set $A \subset S$, see (2.3). It follows from the obtained estimate and Lemma 3.1 that

$$\lim_{p\to p_0-} \lim_{q\to p_0+} \operatorname{Var} \mu_x^{pq} \le \left(\mu_x^{p_0p_0+}(C_+)\right)^{1/2} + \left(\mu_x^{p_0p_0-}(C_-)\right)^{1/2} = 0.$$

Thus,

$$\mu_x^{pq} \to 0 \;\text{ in } \mathrm{M}(S) \;\text{ as } p \to p_0-, q \to p_0+ . \tag{3.27}$$

On the other hand, by (2.4)

$$\mu_x^{pq}(S) = \lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_m U_r(\cdot,p))(\xi)\overline{F(\Phi_m U_r(\cdot,q))(\xi)} d\xi = \\ \lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} U_r(y,p) U_r(y,q) K_m(x-y) dy. \tag{3.28}$$

Observe that $U_r(x,\lambda) = \theta(u_r(x) - \lambda) - u_0(x,\lambda)$. Since $U_r(\cdot,p) \underset{r\to\infty}{\rightharpoonup} 0$ for all $p \in D$ and $(\theta(u_r(y) - p) - 1)\theta(u_r(y) - q) \equiv 0$, we find

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} U_r(y,p) U_r(y,q) K_m(x-y) dy = \\ \lim_{r\to\infty} \int_{\mathbb{R}^n} (U_r(y,p) - 1) U_r(y,q) K_m(x-y) dy = \\ \lim_{r\to\infty} \int_{\mathbb{R}^n} (\theta(u_r(y) - p) - 1 - u_0(y,p))(\theta(u_r(y) - q) - u_0(y,q)) K_m(x-y) dy = \\ \lim_{r\to\infty} \int_{\mathbb{R}^n} [(1 - \theta(u_r(y) - p)) u_0(y,q) - u_0(y,p)(\theta(u_r(y) - q) - u_0(y,q))] K_m(x-y) dy \\ = \int_{\mathbb{R}^n} (1 - u_0(y,p)) u_0(y,q) K_m(x-y) dy.$$

In the limit as $m \to \infty$ this yields

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} U_r(y,p)U_r(y,q)K_m(x-y)dy =$$
$$\lim_{m\to\infty}\int_{\mathbb{R}^n}(1-u_0(y,p))u_0(y,q)K_m(x-y)dy = (1-u_0(x,p))u_0(x,q).$$

Here we take into account that x is a Lebesgue point of the functions $u_0(y,p)$, $u_0(y,q)$. By (3.27), (3.28) we find

$$0 = \lim_{p\to p_0-}\lim_{q\to p_0+}\mu_x^{pq}(S) =$$
$$\lim_{p\to p_0-}\lim_{q\to p_0+}(1-u_0(x,p))u_0(x,q) = \nu_x((-\infty,p_0))\nu_x((p_0,+\infty)) > 0,$$

since $a < p_0 < b$ and $[a,b]$ is the minimal segment containing $\operatorname{supp}\nu_x$. The obtained contradiction implies that $S_+ \cap S_- \neq \emptyset$ and completes the proof. □

Now we are ready to prove Theorem 1.1.

Proof. Let $u_r = u_{k_r}$ be a subsequence of u_k chosen in accordance with Proposition 2.2. In particular, this subsequence converges to a measure-valued function $\nu_x \in \mathrm{MV}(\Omega)$. In view of (2.1) for a.e. $x \in \Omega$

$$u(x) = \int \lambda d\nu_x(\lambda). \tag{3.29}$$

We define the set of full measure $\Omega'' \subset \Omega$ and the minimal segment $[a(x),b(x)]$, containing $\operatorname{supp}\nu_x$, $x \in \Omega''$. In view of (3.29) $u(x) \in (a(x),b(x))$ whenever $a(x) < b(x)$. By Corollary 3.6 the function $\xi\cdot\varphi(x,\cdot)$ is constant in a vicinity of $u(x)$ for some vector $\xi \neq 0$. But this contradicts to the assumption of Theorem 1.1. Therefore, $a(x) = b(x) = u(x)$ for a.e. $x \in \Omega$. This means that $\nu_x(\lambda) = \delta(\lambda - u(x))$. By Theorem 2.1 the subsequence $u_r \to u$ as $r \to \infty$ in $L^1_{loc}(\Omega)$. Finally, since the limit function $u(x)$ does not depend of the choice of a subsequence u_r, we conclude that the original sequence $u_k \to u$ in $L^1_{loc}(\Omega)$ as $k \to \infty$. The proof is complete. □

4 Decay property

This section is devoted to the proof of Theorem 1.4. Suppose that $u(t,x)$ is a unique e.s. to problem (1.4), (1.7) with the periodic initial data $u_0(x)$. By Remark 1.3 we can assume that $u(t,x) \in C([0,+\infty), L^1(\mathbb{T}^n))$ (after possible correction on a set of null measure). We consider the sequence $u_k(t,x) = u(kt,kx)$, $k \in \mathbb{N}$, consisting of e.s. of (1.4). As was firstly shown in [2], the decay property (1.10) is equivalent to the strong convergence $u_r(t,x) \underset{r\to\infty}{\to} I = \text{const}$ in $L^1_{loc}(\Pi)$ of a subsequence $u_r = u_{k_r}(t,x)$. As follows from [17, Lemma 3.2(i)], $u_r \rightharpoonup u^*$, where

$u^* = u^*(t)$ is a weak-$*$ limit of the sequence $a_0(k_r t)$, where $a_0(t) = \int_{\mathbb{T}^n} u(t,x)dx$. Since $u(t,x)$ is an e.s. of (1.4), this function is constant: $a_0(t) \equiv I = \int_{\mathbb{T}^n} u_0(x)dx$, in view of (1.8). Therefore, $u_r \rightharpoonup I$ as $r \to \infty$ (actually, the original sequence $u_k \rightharpoonup I$ as $k \to \infty$).

Let μ^{pq}, $p,q \in E$, be the H-measure corresponding to a subsequence $u_r = u_{k_r}(t,x)$. Recall that $\mu^{pq} = \mu^{pq}(t,x,\tau,\xi) \in \mathrm{M}_{loc}(\Pi \times S)$, where

$$S = \{ \ \hat{\xi} = (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid |\hat{\xi}|^2 = \tau^2 + |\xi|^2 = 1 \ \}$$

is a unit sphere in the dual space $\mathbb{R}^{n+1}$ (the variable τ corresponds to the time variable t).

By [17, Theorem 3.1] the following localization principle holds

$$\operatorname{supp} \mu^{pq} \subset \Pi \times S_0,$$

where

$$S_0 = \{ \ \hat{\xi}/|\hat{\xi}| \mid \hat{\xi} = (\tau,\xi) \neq 0, \tau \in \mathbb{R}, \ \xi \in L' \ \}.$$

As was demonstrated in Proposition 2.3, $\mu^{pq} = \mu^{pq}_{t,x} dtdx$ for all $p,q \in D$, where $D \subset E$ is a countable dense subset and measures $\mu^{pq}_{t,x} \in \mathrm{M}(S)$, are defined for all (t,x) belonging to a set of full measure $\Pi' \subset \Pi$. Obviously, the identity

$$\langle \mu^{pp}, \Phi(t,x,\hat{\xi}) \rangle = \int_{\Pi} \langle \mu^{pp}_{t,x}(\hat{\xi}), \Phi(t,x,\hat{\xi}) \rangle dtdx, \tag{4.1}$$

$\Phi(t,x,\hat{\xi}) \in C_0(\Pi \times S)$, remains valid also for compactly supported Borel functions Φ. Taking $\Phi = \phi(t,x)h(\hat{\xi})$, where $\phi(t,x) \in C_0(\Pi)$, $\phi(t,x) \geq 0$ while $h(\hat{\xi})$ is an indicator function of the set $S \setminus S_0$, we derive from (4.1) that

$$\int_{\Pi} \mu^{pp}_{t,x}(S \setminus S_0)\phi(t,x)dtdx = 0$$

and since $\mu^{pp}_{t,x} \geq 0$ and $\phi(t,x) \in C_0(\Pi)$ is arbitrary nonnegative function, it follows from this identity that $\mu^{pp}_{t,x}(S \setminus S_0) = 0$ for all $p \in D$, $(t,x) \in \Pi'$. By relation (2.6) we claim that, more generally, $|\mu^{pq}_{t,x}|(S \setminus S_0) = 0$ for all $p,q \in D$, $(t,x) \in \Pi'$. Finally, in view of Lemma 3.1, we find that $|\mu^{pq\pm}_{t,x}|(S \setminus S_0) = 0$, that is,

$$\operatorname{supp} \mu^{pq\pm}_{t,x} \subset S_0 \ \forall p,q \in \mathbb{R}, (t,x) \in \Pi'. \tag{4.2}$$

Further, $u_r(t,x)$ is a sequence of entropy solutions of (1.4). Therefore (see for instance [16]) the sequences

$$\operatorname{div} [\theta(u_r - p)(\hat{\varphi}(u_r) - \hat{\varphi}(p))] = ((u_r - p)^+)_t + \operatorname{div}_x [\theta(u_r - p)(\varphi(u_r) - \varphi(p))]$$

are compact in $H^{-1}_{d,loc}(\Pi)$ for some $d > 1$ and all $p \in \mathbb{R}$, where $\hat{\varphi}(u) = (u, \varphi(u)) \in C(\mathbb{R}, \mathbb{R}^{n+1})$, and we use the notation $v^+ = \max(v,0)$.

Denote by $\nu_{t,x} \in \mathrm{MV}(\Pi)$ the limit measure valued function for a sequence u_r, and by $[a(t,x), b(t,x)]$ the minimal segment containing $\operatorname{supp} \nu_{t,x}$.

Suppose that $(t,x) \in \Pi'$, $a(t,x) < b(t,x)$. Then $I = \int \lambda d\nu_{t,x}(\lambda) \in (a(t,x), b(t,x))$. By Corollary 3.6 we find that there exists $\hat{\xi} = (\tau, \xi) \in \operatorname{supp} \mu_{t,x}^{II+} \cap \operatorname{supp} \mu_{t,x}^{II-}$ and $\delta > 0$ such that the function

$$\lambda \to \hat{\xi} \cdot \hat{\varphi}(\lambda) = \tau u + \xi \cdot \varphi(u) = c = \text{const} \tag{4.3}$$

on the interval $V = \{\lambda \mid |\lambda - I| < \delta\}$. By (4.2) $\hat{\xi} \in S_0$, which implies that we can assume that $\xi \in L'$ in (4.3). Evidently, $\xi \neq 0$ (otherwise, $\tau u \equiv c$ on V for $\tau \neq 0$). Hence the function $\xi \cdot \varphi(u) = c - \tau u$ is affine, which contradicts (1.9). Thus, $a(t,x) = b(t,x) = I$ for a.e. $(t,x) \in \Pi$. We conclude that $\nu_{t,x}(\lambda) = \delta(\lambda - I)$ an by Theorem 2.1 the sequence $u_r \to I$ as $r \to \infty$ strongly (in $L^1_{loc}(\Pi)$). As was mentioned above (one can simply repeat the conclusive part of the proof of Theorem 1.1 in [17]), this implies (1.10).

Conversely, if the assumption (1.9) fails, we can find $\xi \in L'$, $\xi \neq 0$, and constants $a, b \in \mathbb{R}$ such that $\xi \cdot \varphi(\lambda) \equiv au + b$ on a segment $[I - \delta, I + \delta]$, $\delta > 0$. Then, as is easily verified, the function

$$u(t,x) = I + \delta \sin(2\pi(\xi \cdot x - at))$$

is the e.s. of (1.4), (1.7) with initial data $u_0(x) = I + \delta \sin(2\pi(\xi \cdot x))$. It is clear that $u_0(x)$ is L-periodic and $\int_{\mathbb{T}^n} u_0(x)dx = I$, but the e.s. $u(t,x)$ does not satisfy the decay property.

Example. Let $n = 1$, $\varphi(u) = |u|$. Let $u = u(t,x)$ be an e.s. of the problem

$$u_t + (|u|)_x = 0, \quad u(0,x) = u_0(x), \tag{4.4}$$

where $u_0(x) \in L^\infty(\mathbb{R})$ is a nonconstant periodic function with a period l (for a constant $u_0 \equiv c$ the e.s. $u \equiv c$ and the decay property is evident). Notice that no previous results [2, 3, 17] can help to answer the question whether the decay property is satisfied. However, as follows from Theorem 1.4, if $I = \frac{1}{l} \int_0^l u_0(x)dx = 0$, then the decay property holds: $\int_0^l |u(t,x)|dx \to 0$ as $t \to \infty$. Actually, the condition $\int_0^l u_0(x)dx = 0$ is also necessary for the decay property (1.10). Indeed, $u(t,x) = u_0(x \mp t)$ if $\pm u_0(x) \geq 0$ (then $\pm I > 0$), and the decay property is evidently violated. In the remaining case when u_0 changes sign we define the functions $u_+(t,x) = v_+(x-t)$, $u_-(t,x) = v_-(x+t)$, where $v_+(x) = \max(u_0(x), 0) \geq 0$, $v_-(x) = \min(u_0(x), 0) \leq 0$. Note that this functions take zero values on sets of positive measures. By the construction, $v_-(x) \leq u_0(x) \leq v_+(x)$ and $u_\pm(t,x)$ are e.s. of (4.4) with initial data $v_\pm(x)$. In view of the known property of monotone dependence of e.s. on initial data $u_-(t,x) \leq u(t,x) \leq u_+(t,x)$ a.e. on Π. These inequality can be written in the form

$$u(t, x - t) \geq v_-(x), \quad u(t, x + t) \leq v_+(x). \tag{4.5}$$

Assuming that $u(t,x)$ satisfies the decay property, we find, with the help of x-periodicity of $u(t,\cdot)$, that

$$\int_0^l |u(t,x\pm t)-I|dx = \int_0^l |u(t,x)-I|dx \to 0 \quad \text{as } t\to+\infty,$$

that is, the functions $u(t,x\pm t) \underset{t\to+\infty}{\to} I$ in $L^1([0,l])$. Passing to the limit as $t\to+\infty$ in (4.5), we find that $v_-(x)\le I\le v_+(x)$ for a.e. $x\in\mathbb{R}$. The latter is possible only if $I=0$. We conclude that the decay property holds only in the case $I=0$.

Remark 4.1. Theorem 1.4 can be extended to more general case of almost periodic initial data (in the Besicovitch sense [1]). Repeating the arguments of [18], we arrive at the following analogue of Theorem 1.4.

Theorem 4.2. *Let M_0 be the additive subgroup of $\mathbb{R}^n$ generated by the spectrum of u_0. Assume that for all $\xi\in M_0$, $\xi\ne 0$ the function $\xi\cdot\varphi(\lambda)$ is not affine in any vicinity of $I=\fint_{\mathbb{R}^n} u_0(x)$. Then the e.s. $u(t,x)$ of (1.4), (1.7) satisfies the decay property*

$$\lim_{t\to+\infty}\fint_{\mathbb{R}^n}|u(t,x)-I|dx=0.$$

Here $\fint_{\mathbb{R}^n} v(x)dx$ denotes the mean value of an almost periodic function $v(x)$ (see [1]).

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References

[1] A. S. Besicovitch, Almost Periodic Functions. Cambridge University Press, 1932.

[2] G.-Q. Chen, H. Frid, Decay of entropy solutions of nonlinear conservation laws, Arch. Rational Mech. Anal. 146 (2) (1999) 95–127.

[3] C. M. Dafermos, Long time behavior of periodic solutions to scalar conservation laws in several space dimensions, SIAM J. Math. Anal. 45:4 (2013), 2064–2070.

[4] R. J. DiPerna, Measure-valued solutions to conservation laws, Arch. Rational Mech. Anal. 88 (1985) 223–270.

[5] P. Gerárd, Microlocal defect measures, Comm. Partial Diff. Equat. 16 (1991) 1761–1794.

[6] E. Hille, R.S. Phillips, Functional analysis and semi-groups. Providence, 1957.

[7] S. N. Kruzhkov, First order quasilinear equations in several independent variables, Mat. Sb. 81 (1970) 228–255, English transl. in Math. USSR Sb. 10 (1970) 217–243.

[8] S. N. Kruzhkov, E. Yu. Panov, First-order conservative quasilinear laws with an infinite domain of dependence on the initial data, Dokl. Akad. Nauk SSSR 314 (1990) 79–84, English transl. in Soviet Math. Dokl. 42 (1991) 316–321.

[9] S. N. Kruzhkov, E. Yu. Panov, Osgood's type conditions for uniqueness of entropy solutions to Cauchy problem for quasilinear conservation laws of the first order, Ann. Univ. Ferrara Sez. VII (N.S.) 40 (1994) 31–54.

[10] E. Yu. Panov, On sequences of measure-valued solutions of first-order quasilinear equations, Mat. Sb. 185 (2) (1994) 87–106, English transl. in Russian Acad. Sci. Sb. Math. 81 (1) (1995) 211–227.

[11] E. Yu. Panov, Property of strong precompactness for bounded sets of measure valued solutions of a first-order quasilinear equation, Mat. Sb. 190 (3) (1999) 109–128, English transl. in Russian Acad. Sci. Sb. Math. 190 (3) (1999) 427–446.

[12] E. Yu. Panov, A remark on the theory of generalized entropy sub- and supersolutions of the Cauchy problem for a first-order quasilinear equation, Differ. Uravn. 37 (2) (2001) 252–259, English transl. in Differ. Equ. 37 (2) (2001) 272–280.

[13] E. Yu. Panov, Existence of strong traces for generalized solutions of multidimensional scalar conservation laws, J. Hyperbolic Differ. Equ. 2 (4) (2005) 885–908.

[14] E. Yu. Panov, Existence of strong traces for quasi-solutions of multidimensional conservation laws, J. Hyperbolic Differ. Equ. 4 (4) (2007) 729–770.

[15] E. Yu. Panov, Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux, Arch. Rational Mech. Anal. 195 (2) (2010) 643–673.

[16] E.Yu. Panov, Ultra-parabolic equations with rough coefficients. Entropy solutions and strong precompactness property, J. Mathematical Sciences 159 (2) (2009) 180–228.

[17] E. Yu. Panov, On decay of periodic entropy solutions to a scalar conservation law, Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire 30:6 (2013), 997-1007.

[18] E. Yu. Panov, On the Cauchy problem for scalar conservation laws in the class of Besicovitch almost periodic functions: global well-posedness and decay property, arXiv:1406.4838v2, 19 Jun 2014.

[19] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, N.J. (1970).

[20] L. Tartar, Compensated compactness and applications to partial differential equations, Nonlinear analysis and mechanics: Heriot. Watt Symposium, vol. 4 (Edinburgh 1979), Res. Notes Math. 39 (1979) 136–212.

[21] L. Tartar, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Edinburgh. Sect. A. 115 (3-4) (1990) 193–230.