Control of Biological Resources on Graphs

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Abstract

A biological resource is a population characterized by birth, aging and death, grown in order to produce a profit. The evolution of this system is described by a structured population model, modified to take into account the selection for reproduction or for the market. This selection is the control that has to be optimized in order to maximize the profit. First we prove the well posedness of the descriptive model. Then, the profit is shown to be Gâteaux differentiable with respect to the controls. Finally, we ensure that the maximal profit can be reached by means of Bang–Bang controls.

Keywords: Conservation Laws; Optimal Control Problems; Management of Biological Resources; Structured Population Dynamics.

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1 Introduction

In biological resource management, one typically rears/breeds a species up to a suitable stage, then part of the population is kept for reproduction, while the rest goes to the market. Call $J = J(t,a)$ the density of the juveniles at time t of age or size a. Then, according to the standard modeling of a biological population with age structure, see e.g. [1, 2, 19, 20], J solves

$$\partial_t J + \partial_a \left(g_J(t,a)\, J\right) = d_J(t,a)\, J\,. \tag{1.1}$$

Here, g_J is the usual growth function and $-d_J$ the mortality rate. When juveniles reach a given age/size $\bar a$, they are selected: a portion is directed to the market, while the others are kept for reproduction purposes. By $S = S(t,a)$ we denote the density of the individuals that are going to be sold, while $R = R(t,a)$ stands for the density of those reserved for reproduction. Similarly to above, we obtain the equations

$$\left\{\begin{array}{l} \partial_t S + \partial_a \left(g_S(t,a)\, S\right) = d_S(t,a)\, S \\ \partial_t R + \partial_a \left(g_R(t,a)\, R\right) = d_R(t,a)\, R\,. \end{array}\right. \tag{1.2}$$

A key role is played by the control parameter $\eta = \eta(t)$ specifying the ratio of the juveniles sent to the market at time t. Clearly, $1 - \eta(t)$ then stands for the fraction kept for reproduction. Hence, (1.2) needs to be coupled to (1.1) through the inflows at $\bar a$

$$\left\{\begin{array}{l} g_S(t,\bar a)\, S(t,\bar a+) = \eta(t)\, g_J(t,\bar a)\, J(t,\bar a-) \\ g_R(t,\bar a)\, R(t,\bar a+) = \left(1-\eta(t)\right)\, g_J(t,\bar a)\, J(t,\bar a-)\,. \end{array}\right. \tag{1.3}$$

In turn, the inflow to (1.1) is provided by the fertility of the R population, so that

$$g_J(t,0)\, J(t,0+) = \int_{\bar a}^{+\infty} w(\alpha)\, R(t,\alpha)\, \mathrm{d}\alpha\ ,$$

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where the function $w = w(a)$ describes the fertility of the adults R at age a.

We introduce now the selling procedure. It is reasonable to assume that the adults S can be sold at the predetermined ages/sizes $\bar{a}_1, \ldots, \bar{a}_N$, with $\bar{a} < \bar{a}_1 < \bar{a}_2 < \cdots < \bar{a}_N$, see Figure 1. The percentage of individuals sold at the age $\bar{a}_i$ is $\vartheta_i = \vartheta_i(t)$, with $\vartheta_i \in [0,1]$ for $i = 1, \ldots, N$. Therefore, the evolution of S needs to be modified inserting the conditions

$$S(t, \bar{a}_i+) = \big(1 - \vartheta_i(t)\big)\ S(t, \bar{a}_i-) \qquad \text{for} \qquad i = 1, \ldots, N\,. \tag{1.4}$$

Without any lack of generality, we may impose that $\vartheta_N = 1$, so that all individuals in the S population are sold within age/size $\bar{a}_N$. Remark that condition (1.4) at $\bar{a}_i$ prescribes that from the total salable individuals $S(t, \bar{a}_i-)$, the amount $\big(1 - \vartheta_i(t)\big)\ S(t, \bar{a}_i-)$ continues to be grown, while the amount $\vartheta_i(t)\, S(t, \bar{a}_i-)$ is sold and disappears from the future evolution of S as solution to (1.2).

The graph representing the above framework is in Figure 1.

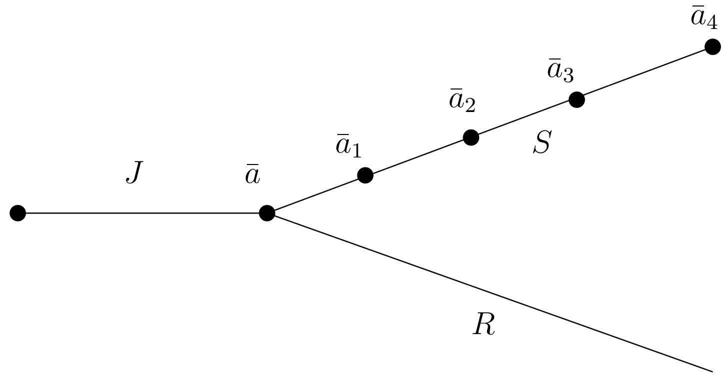


Figure 1: A graph representing a possible framework for the exploitation of biological resources: at the age/size $\bar{a}$, juveniles reach the adult stage, they are split into a part R used for reproduction and a part S which is sold at the ages/sizes $\bar{a}_1, \bar{a}_2, \ldots$.

The above description of the populations' evolution has to be completed with the economic part, specifying the cost, income and profit functions. A natural shape for the breeding cost is

$$\begin{aligned}\mathcal{C}(\eta, \vartheta; T) \quad = \quad & \int_0^T \int_0^{\bar{a}} C_J\left(t, a, J(t,a)\right) \mathrm{d}a\, \mathrm{d}t + \int_0^T \int_{\bar{a}}^{\bar{a}_N} C_S\left(t, a, S(t,a)\right) \mathrm{d}a\, \mathrm{d}t \\ & + \int_0^T \int_{\bar{a}}^{+\infty} C_R\left(t, a, R(t,a)\right) \mathrm{d}a\, \mathrm{d}t\ .\end{aligned} \tag{1.5}$$

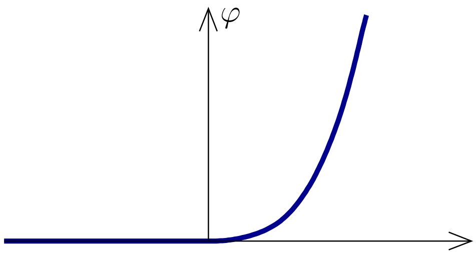
The quantity $C_u(t, a, u)$, for $u = J, S, R$, is the breeding cost of the population u, of age/size a, at time t.

The income $\mathcal{I}$ is due to the amount of individuals sold at the various ages/sizes $\bar{a}_i$. Call $P_i(t)$ the price at which an individual of age/size $\bar{a}_i$ is sold at time t. Therefore,

$$\mathcal{I}(\eta, \vartheta; T) = \sum_{i=1}^{N} \int_0^T P_i(t)\ \vartheta_i(t)\ S(t, \bar{a}_i-)\ \mathrm{d}t\ . \tag{1.6}$$

Note that setting $P_N < 0$ allows to comprehend the situation of unsold individuals causing an increase in the cost.

A particular role is played by the time horizon T. Clearly, as long as the profit $\mathcal{I} - \mathcal{C}$ does not depend on what happens after time T, an optimal strategy is likely to consist in selling all individuals before time T. This choice is not always reasonable and to avoid it we consider also a smooth penalization Φ whenever the total amount of juveniles at time T falls below, say, the initial value $\int_0^{\bar a} J_o(a)\,\mathrm{d}a$, such as for instance



$$\Phi(\eta, \vartheta; T) = \varphi\left(\int_0^{\bar a} \big(J_o(a) - J(T, a)\big)\,\mathrm{d}a\right) \tag{1.7}$$

for a suitable smooth function $\varphi \colon \mathbb{R} \to \mathbb{R}^+$.

The question that naturally arises is to find the strategies $\eta_\star$ and $\vartheta_\star$ that maximize the net profit $\mathcal{P}$, i.e., the difference of the income $\mathcal{I}$ minus the costs $\mathcal{C}$ and Φ:

$$\mathcal{P}(\eta, \vartheta; T) = \mathcal{I}(\eta, \vartheta; T) - \mathcal{C}(\eta, \vartheta; T) - \Phi(\eta, \vartheta; T)\,. \tag{1.8}$$

In mathematical terms, we are lead to the optimization problem

$$\text{find} \quad \begin{array}{l} \eta \in \mathbf{BV}([0,T];[0,1]) \\ \vartheta \in \mathbf{BV}([0,T];[0,1]^N) \end{array} \quad \text{to maximize} \quad \mathcal{P}(\eta, \vartheta; T) \tag{1.9}$$

for given $T > 0$ and for a fixed initial datum (J_o, S_o, R_o). The choice of the space $\mathbf{BV}$ is motivated, *a posteriori*, by Theorem 2.1 which ensures the well posedness of (1.1)–(1.2)–(1.3)–(1.4) under the condition that the total variation of η and ϑ be bounded.

Thanks to Theorem 2.1, suitable regularity conditions on the costs (1.5), (1.7) and on the income (1.6), a Weierstraß type argument can be used to ensure the existence of the optimal controls $\eta_\star$ and $\vartheta_\star$. We then proceed seeking information that may ease the actual search for these optimal controls.

Remark that the dependence of $\mathcal{P}$ from the η and ϑ is highly nonlinear, differently from [6]. The standard situation when dealing with conservation or balance laws is that solutions depend from data and parameters in a merely Lipschitz continuous way, see [10, Theorems 2.4 and 2.5] as well as Theorem 2.1 below. Here, we prove that the income $\mathcal{P}$ is Gâteaux differentiable with respect to the controls, see Theorem 2.5. This result fully justifies gradient methods in the search for $\eta_\star$ and $\vartheta_\star$. The case with η and ϑ both constant in time is settled in [15], where it is shown that the net profit is differentiable with respect to both (fixed) control parameters.

However, since explicit analytic forms for the directional derivatives of $\mathcal{P}$ are in general practically useless, we proceed considering *Bang–Bang* controls, i.e., controls that are piecewise constant on intervals and attain only the values 0 and 1. In optimal control theory, the notion of Bang–Bang controls is widely used, since optimal controls for several problems, such as minimum time problems, are indeed Bang–Bang; see [7, 21]. In our case, Theorem 2.8 ensures that, in the search for $\eta_\star$ and $\vartheta_\star$, considering Bang–Bang controls is sufficient. This information significantly shortens the numerical procedures that can be used to find the global minimum of $\mathcal{P}$.

The description of structured population models is a well established research area, refer for instance to the classical textbook [19], to the more recent [4, 13, 17, 20] and to the references therein. An introduction to optimal control in structured population models is found, for instance, in [3, Chapter 3], [4, Chapter 4] or [17, Section 7.3]. A different approach to natural resources modeling, based on viability theory, is in [13, Chapter 5]. Control problems on renewable equations, through approximations based on ordinary differential equations, are treated for instance in [9, 11, 16].

This paper is organized as follows: the next section deals with the analytic framework and presents the main well posedness and differentiability results. Technical details are deferred to Section 4.

2 Main Results

Let I be a real interval and $u\colon I \to \mathbb{R}$ be measurable. The following norms are used in the sequel

$$\|u\|_{\mathbf{L^1}(I;\mathbb{R})} = \int_I \left|u(t)\right| \mathrm{d}t \qquad \text{and} \qquad \|u\|_{\mathbf{L^\infty}(I;\mathbb{R})} = \operatorname{supess}_{t\in I} \left|u(t)\right| ,$$

while in the case $u\colon I \times J \to \mathbb{R}$, J being a real interval, we set

$$\|u\|_{\mathbf{C^0}(I;\mathbf{L^1}(J;\mathbb{R}))} = \sup_{t\in I} \int_J \left|u(t,x)\right| \mathrm{d}x \ . \tag{2.1}$$

Throughout, we fix the following notation:

$$N \in \mathbb{N}\setminus\{0\}\,, \quad 0 < \bar a = \bar a_0 < \bar a_1 < \ldots < \bar a_N\,, \quad \begin{array}{l} I_J = [0,\bar a]\,,\\ I_S = [\bar a, \bar a_N]\,,\\ I_R = [\bar a, +\infty[\ , \end{array} \quad I_T = \begin{array}{l}[0,T],\ \text{or}\\ [0,+\infty[\ ,\end{array} \tag{2.2}$$

the cases where T needs to be finite are explicitly signaled. As in [10], we consider the following model describing the evolution of the species J, S, R:

$$\left\{\begin{array}{ll}
\partial_t J + \partial_a \left(g_J(t,a)\, J\right) = d_J(t,a)\, J & (t,a) \in I_T \times I_J\\
\partial_t S + \partial_a \left(g_S(t,a)\, S\right) = d_S(t,a)\, S & (t,a) \in I_T \times \left(I_S \setminus \{\bar a_1, \ldots, \bar a_N\}\right)\\
\partial_t R + \partial_a \left(g_R(t,a)\, R\right) = d_R(t,a)\, R & (t,a) \in I_T \times I_R\\
g_J(t,0)\, J(t,0+) = \int_{\bar a}^{+\infty} w(\alpha)\, R(t,\alpha)\,\mathrm{d}\alpha & t \in I_T\\
g_S(t,\bar a)\, S(t,\bar a+) = \eta(t)\, g_J(t,\bar a)\, J(t,\bar a-) & t\in I_T\\
g_R(t,\bar a)\, R(t,\bar a+) = \left(1-\eta(t)\right)\, g_J(t,\bar a)\, J(t,\bar a-) & t \in I_T\\
S(t,\bar a_i+) = \left(1-\vartheta_i(t)\right)\, S(t,\bar a_i-) & t\in I_T, \quad i = 1,\ldots,N-1\\
J(0,a) = J_o(a) & a \in I_J\\
S(0,a) = S_o(a) & a \in I_S\\
R(0,a) = R_o(a) & a \in I_R\,.
\end{array}\right. \tag{2.3}$$

The following assumptions are required on (2.3):

(A) There exist positive $\check g, \hat g$ such that, for $u = J, S, R$,

$$\begin{array}{lll}
g_u \in \mathbf{C^1}(I_T \times I_u; [\check g, \hat g]) & \text{and} & \left\{\begin{array}{l} \sup_{t\in\mathbb{R}^+} \mathrm{TV}\left(g_u(t,\cdot)\right) < +\infty\,;\\ \sup_{t\in\mathbb{R}^+} \mathrm{TV}\left(\partial_x g_u(t,\cdot)\right) < +\infty\,;\end{array}\right.\\
d_u \in (\mathbf{C^1}\cap\mathbf{L^\infty})(I_T\times I_u;\mathbb{R}) & \text{and} & \sup_{t\in\mathbb{R}^+} \mathrm{TV}\left(d_u(t,\cdot)\right) < +\infty\,;
\end{array} \tag{2.4}$$

while the fertility function w satisfies $w \in \mathbf{C^1_c}([\bar a, +\infty[\,;\mathbb{R}^+)$.

(ID) $J_o \in \mathbf{BV}(I_J;\mathbb{R})$, $S_o \in \mathbf{BV}(I_S;\mathbb{R})$ and $R_o \in (\mathbf{L^1}\cap\mathbf{BV})(I_R;\mathbb{R})$.

For later use, we introduce the following assumptions on the functions defining the net profit (1.8):

(P) $P_i \in \mathbf{L^\infty}(I_T;\mathbb{R})$ for $i = 1,\ldots,N$.

(C) For $u = J, S, R$, the function $C_u : I_T \times I_u \times \mathbb{R} \to \mathbb{R}$ satisfies

1. C_u is $(\mathcal{L}\otimes\mathcal{B})$-measurable, where $\mathcal{L}$, resp. $\mathcal{B}$, is the Lebesgue σ-algebra on $I_T \times I_u$, resp. the Borel σ-algebra on $\mathbb{R}^+$;
2. the function $s \to C_u(t,a,s)$ is upper semicontinuous and concave for a.e. $(t,a) \in I_T \times I_u$;
3. there exist $c \in \mathbf{L^1}\left(I_T \times I_u;\mathbb{R}\right)$ and $b\in\mathbb{R}$ such that
$$\left|C_u(t,a,s)\right| \le c(t,a) + b|s|$$
for a.e. $(t,a) \in I_T \times I_u$ and $s \in \mathbb{R}^+$.

(φ) $\varphi \in \mathbf{C^0}\left(\mathbb{R};\mathbb{R}^+\right)$.

2.1 Well Posedness and Gâteaux Differentiability

As a first step, we extend [10, Theorem 2.4, Theorem 2.5, and Proposition 2.6] to the case of time dependent controls $\eta = \eta(t)$ and $\vartheta = \vartheta(t)$.

Theorem 2.1. *Use the notation* (2.2) *and pose conditions **(A)**, **(ID)**. For any $\eta \in \mathbf{BV}(I_T;[0,1])$ and $\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$, system* (2.3) *admits a unique solution. Moreover,*

$$\text{if} \left\{\begin{array}{l} J_o \geq 0 \\ S_o \geq 0 \\ R_o \geq 0 \end{array}\right. \qquad \text{then} \qquad \left\{\begin{array}{l} J(t) \geq 0 \\ S(t) \geq 0 \\ R(t) \geq 0 \end{array}\right. \text{for all } t \in I_T$$

and there exists a function $\mathcal{K} \in \mathbf{C^0}(I_T;\mathbb{R}^+)$, with $\mathcal{K}(0)=0$, dependent only on g_J, g_S, g_R, d_J, d_S, d_R and w such that for any initial data (J'_o, S'_o, R'_o) and (J''_o, S''_o, R''_o) and for any controls η', η'', ϑ' and ϑ'', the corresponding solutions (J', S', R') and (J'', S'', R'') to (2.3) *satisfy the following stability estimate:*

$$\begin{aligned} & \left\| J'(t) - J''(t) \right\|_{\mathbf{L^1}(I_J;\mathbb{R})} + \left\| S'(t) - S''(t) \right\|_{\mathbf{L^1}(I_S;\mathbb{R})} + \left\| R'(t) - R''(t) \right\|_{\mathbf{L^1}(I_R;\mathbb{R})} \\ \leq \; & \mathcal{K}(t) \left(\left\| J'_o - J''_o \right\|_{\mathbf{L^1}(I_J;\mathbb{R})} + \left\| S'_o - S''_o \right\|_{\mathbf{L^1}(I_S;\mathbb{R})} + \left\| R'_o - R''_o \right\|_{\mathbf{L^1}(I_R;\mathbb{R})} \right) \\ & + t\, \mathcal{K}(t) \left(\left\| J'_o - J''_o \right\|_{\mathbf{L^\infty}(I_J;\mathbb{R})} + \left\| S'_o - S''_o \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} + \left\| R'_o - R''_o \right\|_{\mathbf{L^\infty}(I_R;\mathbb{R})} \right) \\ & + \mathcal{K}(t) \left(\left\| \eta' - \eta'' \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} + \left\| \vartheta' - \vartheta'' \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R}^N)} \right) . \end{aligned}$$

The proof is deferred to § 4.1. The following corollary is then immediate.

Corollary 2.2. *Use* (2.2) *and pose conditions **(A)**, **(ID)**. Then, for bounded T, the map*

$$\begin{array}{rccccccc} \mathcal{S}: & \mathbf{BV}(I_T;[0,1]) \times \mathbf{BV}(I_T;[0,1]^N) & \to & \mathbf{C^0}\Big(I_T; \mathbf{BV}(I_J;\mathbb{R}^+) & \times & \mathbf{BV}(I_S;\mathbb{R}^+) & \times & \mathbf{BV}(I_R;\mathbb{R}^+)\Big) \\ & \eta \qquad , \qquad \vartheta & \to & J & , & S & , & R \end{array} \tag{2.5}$$

where (J,S,R) is the solution to (2.3) *with controls (η,ϑ), is Lipschitz continuous with respect to the $\mathbf{L}^\infty$ norm in (η,ϑ) and to the norm* (2.1) *in (J,S,R).*

Clearly, under suitable regularity conditions on $C_J, C_S, C_R, P_i, \varphi$, the above result ensures the continuity of the functionals defined in (1.5), (1.6), (1.7) and, consequently, in (1.8) as functions of the control parameters η and ϑ.

We proceed proving the Gâteaux differentiability of the profit (1.8). With reference to (2.3), for $u = J, S, R$, introduce the following notation for the u-characteristic lines, with $t \in \mathbb{R}^+$ and $a, a_o \in I_u$:

$$\begin{array}{lll} t \to \mathcal{A}_u(t;t_o,a_o) & \text{is the solution to} & \left\{\begin{array}{l} \dot a = g_u(t,a) \\ a(t_o) = a_o \end{array}\right. \quad \text{and} \\ a \to \mathcal{T}_u(a;t_o,a_o) & \text{is its inverse, i.e.,} & \mathcal{A}_u\left(\mathcal{T}_u(a;t_o,a_o);t_o,a_o\right) = a \quad \text{for all } a \in I_u\,. \end{array} \tag{2.6}$$

Define T_n recursively for $n \in \mathbb{N}$ by

$$T_0 = 0 \quad \text{and} \quad \mathcal{A}_J(T_n;T_{n-1},0) = \bar a \quad \text{or, equivalently,} \quad \mathcal{T}_J(\bar a;T_{n-1},0) = T_n\,. \tag{2.7}$$

If g_J satisfies **(A)**, then the sequence T_n is well defined and $T_n \to +\infty$ as $n \to +\infty$. The interval $[T_{n-1}, T_n]$ is the time period when the juveniles of the n-th generation are born. Correspondingly, introduce also the following regions, representing the life span of the different generations, see Figure 2:

$$\begin{array}{rcll} G^i_J &=& \left\{(t,a) \in I_T \times I_J \colon \mathcal{T}_J(\bar a;t,a) \in [T_{i-1},T_i]\right\} & i \in \mathbb{N}\setminus\{0\} \\ G^0_S &=& \left\{(t,a) \in I_T \times I_S \colon a > \mathcal{A}_S(t;0,\bar a)\right\} & \\ G^i_S &=& \left\{(t,a) \in I_T \times I_S \colon a \in [\mathcal{A}_S(t;T_i,\bar a), \mathcal{A}_S(t;T_{i-1},\bar a)]\right\} & i \in \mathbb{N}\setminus\{0\} \\ G^0_R &=& \left\{(t,a) \in I_T \times I_R \colon a > \mathcal{A}_R(t;0,\bar a)\right\} & \\ G^i_R &=& \left\{(t,a) \in I_T \times I_R \colon a \in [\mathcal{A}_R(t;T_i,\bar a), \mathcal{A}_R(t;T_{i-1},\bar a)]\right\} & i \in \mathbb{N}\setminus\{0\} \end{array} \tag{2.8}$$

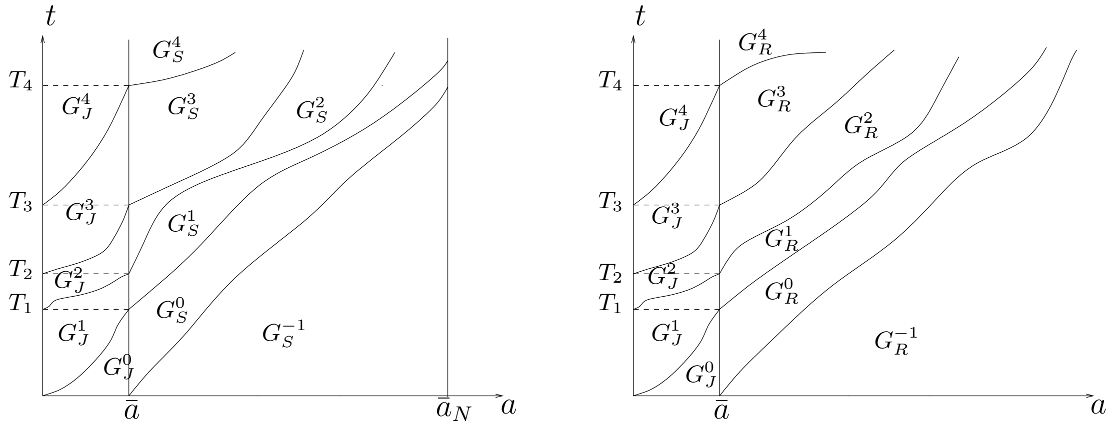


Figure 2: Times T_n and regions G_J^n, G_S^n, G_R^n defined in (2.7) and in (2.8). Left, with reference to the J and S populations, right with reference to the J and R populations. These regions identify the life span of the different generations, separately for the J, S and R populations.

The analytic expression of the solution to (2.3) is in principle available, though essentially unusable. Therefore, we provide below a representation of the variations ΔJ, ΔS, ΔR corresponding to variations $\Delta\eta$, $\Delta\vartheta$ in the controls η, ϑ. To simplify the notation, fix

- the functions defining (2.3), i.e., $g_J, g_S, g_R, d_J, d_S, d_R, w$;
- the initial datum (J_o, S_o, R_o);
- the controls η and ϑ,

and, for $u = J, S, R$, denote by

$$\Upsilon_u = \Upsilon_u(\tau, t, a) \qquad \text{with} \qquad \Upsilon_u \in \mathbf{BV_{loc}}(I_T \times I_T \times I_u; \mathbb{R})$$

a map whose actual value is unimportant in the present context, but it depends only on g_J, g_S, g_R, d_J, d_S, d_R, w, J_o, S_o, R_o and η and ϑ, but does *not* depend on $\Delta\eta$ or on $\Delta\vartheta$.

The following propositions, rather technical, provide the form of the dependence of the variation of the solution to (2.3) from the variation on the controls. The proofs are in § 4.1.

Proposition 2.3. *Use the notation* (2.2) *and pose conditions **(A)**, **(ID)**. Choose controls* $\eta \in \mathbf{BV}(I_T; [0,1])$ *and* $\vartheta \in \mathbf{BV}(I_T; [0,1]^N)$. *Call* (J, S, R), *respectively* $(J + \Delta J, S + \Delta S, R + \Delta R)$, *the solution to* (2.3) *with control* η, *respectively* $\eta + \Delta\eta$, *where*

$$\eta + \Delta\eta \in \mathbf{BV}(I_T; [0,1]) \qquad \textit{and} \qquad \operatorname{spt}\Delta\eta \subseteq \left]T_{h-1}, T_h\right],$$

for a fixed $h \in \mathbb{N}$ *such that* $T_h < T$. *Then,*

$$\Delta J(t,a) = \begin{cases} \displaystyle\int_{T_{h-1}}^{\min\{t,T_h\}} \Delta\eta(\tau)\ \Upsilon_J(\tau,t,a)\,\mathrm{d}\tau & (t,a) \in \left]T_{h-1}, T\right[\times I_J \\ 0 & (t,a) \in [0, T_{h-1}] \times I_J \end{cases}$$

$$\Delta S(t,a) = \begin{cases} \displaystyle\int_{T_{h-1}}^{T_h} \Delta\eta(\tau)\ \Upsilon_S(\tau,t,a)\,\mathrm{d}\tau & (t,a) \in \bigcup_{i=h+1}^{+\infty} G_S^i \\ \Delta\eta\left(\mathcal{T}_S(\bar a; t, a)\right) \Upsilon_S\left(\mathcal{T}_S(\bar a; t,a), t, a\right) & (t,a) \in G_S^h \\ 0 & (t,a) \in \bigcup_{i=0}^{h-1} G_S^i \end{cases}$$

$$\Delta R(t,a) = \begin{cases} \displaystyle\int_{T_{h-1}}^{T_h} \Delta\eta(\tau)\ \Upsilon_R(\tau,t,a)\,\mathrm{d}\tau & (t,a) \in \bigcup_{i=h+1}^{+\infty} G_R^i \\ \Delta\eta\left(\mathcal{T}_R(\bar a; t, a)\right) \Upsilon_R\left(\mathcal{T}_R(\bar a; t,a), t, a\right) & (t,a) \in G_R^h \\ 0 & (t,a) \in \bigcup_{i=0}^{h-1} G_R^i \end{cases}$$

where $\Upsilon_J, \Upsilon_S, \Upsilon_R$ are independent from $\Delta\eta$.

Proposition 2.4. *Use the notation* (2.2), (2.7), (2.8) *and pose conditions **(A)**, **(ID)**. Choose $\eta \in \mathbf{BV}(I_T;[0,1])$ and $\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$. Call (J,S,R), respectively $(J+\Delta J, S+\Delta S, R+\Delta R)$, the solution to* (2.3) *with control ϑ, respectively $\vartheta + \Delta\vartheta$, where*

$$\vartheta + \Delta\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$$

Then,

$$\begin{array}{rcl} \Delta J(t,a) & = & 0 \qquad (t,a) \in I_T \times I_J \\ \Delta S(t,a) & = & \left\{ \begin{array}{lll} 0 & (t,a) \in I_T \times]\bar a, \bar a_1[& \\ \Upsilon_S^i(t,a)\, M_\vartheta^i(\Delta\vartheta_1, \ldots, \Delta\vartheta_i) & (t,s) \in I_T \times]\bar a_i, \bar a_{i+1}[& i = 1, \ldots, N \end{array} \right. \\ \Delta R(t,a) & = & 0 \qquad (t,a) \in I_T \times I_R \end{array}$$

where Υ_S^i is independent from ϑ and $\Delta\vartheta$, the function M_ϑ^i is linear in each argument with coefficients dependent on $\vartheta_1, \ldots, \vartheta_i$, for $i = 1, \ldots, N$.

Below, for simplicity, we choose, as time interval where the profit is maximized, an interval of the form $[0, T_n]$ for some $n \in \mathbb{N} \setminus \{0\}$.

Theorem 2.5. *Use the notation* (2.2) *and pose conditions **(A)**, **(ID)**. Choose $\eta \in \mathbf{BV}(I_T;[0,1])$ and $\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$. Moreover, with reference to* (1.5), (1.6), (1.7) *and* (1.8),

($\mathcal{C}$) *$C_J \in \mathbf{L}^\infty(I_T \times I_J \times \mathbb{R}^+;\mathbb{R})$, $C_S \in \mathbf{L}^\infty(I_T \times I_S \times \mathbb{R}^+;\mathbb{R})$ and $C_R \in \mathbf{L}^\infty(I_T \times I_R \times \mathbb{R}^+;\mathbb{R})$ admit a constant $C > 0$ such that for all $t \in I_T$ and $J, S, R, \Delta J, \Delta S, \Delta R \in \mathbb{R}$,*

$$\begin{array}{rll} \left| C_J(t,a,J+\Delta J) - \left[C_J(t,a,J) + \partial_J C_J(t,a,J)\,\Delta J\right] \right| \le C\ (\Delta J)^2 & & (t,a) \in I_T \times I_J \\ \left| C_S(t,a,S+\Delta S) - \left[C_S(t,a,S) + \partial_S C_S(t,a,S)\,\Delta S\right] \right| \le C\ (\Delta S)^2 & & (t,a) \in I_T \times I_S \\ \left| C_R(t,a,R+\Delta R) - \left[C_R(t,a,R) + \partial_R C_R(t,a,R)\,\Delta R\right] \right| \le C\ (\Delta R)^2 & & (t,a) \in I_T \times I_R \end{array}$$

and there exists $\lambda \in \mathbf{L^1}(I_R;\mathbb{R})$ such that $\left|C_R(t,a,R)\right| \le \lambda(a)$ for all $(t,a) \in I_T \times I_R$.

($\mathbf{\Phi}$) *$\varphi \in \mathbf{C^1}(\mathbb{R};\mathbb{R})$.*

Then, the functionals $\mathcal{C}$, Φ, $\mathcal{I}$ and $\mathcal{P}$ defined in (1.5), (1.6), (1.7), *and* (1.8) *are Gâteaux differentiable in η and ϑ in any directions $\Delta\eta$ and $\Delta\vartheta$ such that $\eta + \Delta\eta \in \mathbf{BV}(I_T;[0,1])$, $\operatorname{spt}\Delta\eta \subseteq]T_{h-1}, T_h]$ for some $h \ge 1$ and $\vartheta + \Delta\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$.*

The proof is deferred to § 4.1.

2.2 Bang–Bang Controls

This paragraph is devoted to show that the supremum of the net profit $\mathcal{P}$, defined in (1.8), can be arbitrarily approximated by using the class of *Bang–Bang controls*; see [5, 14]. Here, by a Bang–Bang control we mean a step function, i.e., a finite sum of characteristic functions of intervals, taking values in the set $\{0,1\}$. In classical optimal control theory, various problems, especially minimum time problems, admit Bang–Bang controls as optimal ones; see [21]. In this setting, instead, it is not clear if optimal controls are indeed Bang–Bang. However they can be used in order to approximate the optimal cost.

To this aim, we first obtain further information on the regularity of the maps $\mathcal{S}$ defined in (2.5). Indeed, Corollary 2.2 proves the continuity of $\mathcal{S}$ in the *strong* topology, whereas Bang–Bang controls may approximate any control only in the *weak$*$* topology of $\mathbf{L}^\infty$.

Proposition 2.6. *Use the notation* (2.2) *and pose conditions **(A)**, **(ID)**. Fix* $\bar\eta \in \mathbf{BV}(I_T;[0,1])$, $\bar\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$, $i \in \{1,\dots,n\}$, $\iota \in \{1,\dots,N\}$ *and for any* $\eta_i \in \mathbf{BV}([T_{i-1},T_i];[0,1])$, $\vartheta_\iota \in \mathbf{BV}(I_T;[0,1])$ *denote*

$$\widehat{\eta}_i(t) = \left\{ \begin{array}{ll} \bar\eta(t) & \text{for } t \in I_T \setminus [T_{i-1},T_i]\,, \\ \eta_i(t) & \text{for } t \in [T_{i-1},T_i]\,, \end{array} \right. \qquad \widehat{\vartheta}_\iota = \left\{ \begin{array}{ll} \bar\vartheta_j & \text{for } j \in \{1,\dots,N\} \setminus \{\iota\}\,, \\ \vartheta_\iota & \text{for } j = \iota\,. \end{array} \right. \tag{2.9}$$

Then, using $\mathcal{S}$ *as in (2.5), the maps*

$$\begin{array}{rcrcl} \mathcal{S}^i_\eta & : & \mathbf{BV}([T_{i-1},T_i];[0,1]) & \to & \mathbf{C^0}\left(I_T; \mathbf{BV}(I_J;\mathbb{R}^+) \times \mathbf{BV}(I_S;\mathbb{R}^+) \times \mathbf{BV}(I_R;\mathbb{R}^+)\right) \\ & & \eta_i & \to & \mathcal{S}(\widehat{\eta}_i, \bar\vartheta) \end{array}$$

$$\begin{array}{rcrcl} \mathcal{S}^\iota_\vartheta & : & \mathbf{BV}(I_T;[0,1]) & \to & \mathbf{C^0}\left(I_T; \mathbf{BV}(I_J;\mathbb{R}^+) \times \mathbf{BV}(I_S;\mathbb{R}^+) \times \mathbf{BV}(I_R;\mathbb{R}^+)\right) \\ & & \vartheta_\iota & \to & \mathcal{S}(\bar\eta, \widehat{\vartheta}_\iota) \end{array}$$

are both sequentially continuous with respect to the weak$*$ topology on $\eta_i \in \mathbf{L}^\infty([T_{i-1},T_i];\mathbb{R})$, *on* $\vartheta_\iota \in \mathbf{L}^\infty(I_T;\mathbb{R})$ *and to the weak topology on* $(J,S,R) \in \mathbf{L^1}(I_T \times I_J;\mathbb{R}) \times \mathbf{L^1}(I_T \times I_S;\mathbb{R}) \times \mathbf{L^1}(I_T \times I_R;\mathbb{R})$.

The proof is in § 4.2.

The next technical result ensures that Bang–Bang controls in $\lambda\colon I_T \to \{0,1\}$ can approximate any (measurable) control $\lambda\colon I_T \to [0,1]$ with respect to the weak$*$ topology in $\mathbf{L}^\infty(I_T;[0,1])$.

Proposition 2.7. *Fix* $T > 0$. *Let* $\lambda \in \mathbf{L}^\infty(I_T;[0,1])$. *Then, there exists a sequence of Bang–Bang controls*

$$\lambda_m = \sum_{k=1}^{N_m} \chi_{[a^k_m, b^k_m[} \qquad \text{with } 0 \le a^1_m < b^1_m < a^2_m < b^2_m < \dots < a^m_m < b^m_m \le T$$

such that $\lambda_m \overset{*}{\rightharpoonup} \lambda$ *in* $\mathbf{L}^\infty(I_T;[0,1])$.

The proof is deferred to § 4.2.

The following Theorem states that in the search for optimality, considering Bang–Bang controls is sufficient.

Theorem 2.8. *Use the notation* (2.2) *and pose conditions **(A)**, **(ID)**, **(P)**, **(C)**, and **(φ)**. Define*

$$\mathcal{P}_\star(T) = \sup \left\{ \mathcal{P}(\eta,\vartheta;T) \colon \begin{array}{l} \eta \in \mathbf{BV}(I_T;[0,1]) \\ \vartheta \in \mathbf{BV}(I_T;[0,1]^N) \end{array} \right\}.$$

For every $\varepsilon > 0$ *there exists a Bang–Bang control* $(\eta_\varepsilon, \vartheta_\varepsilon)$ *such that*

$$\mathcal{P}(\eta_\varepsilon, \vartheta_\varepsilon; T) \ge \mathcal{P}_\star(T) - \varepsilon\,.$$

3 An Optimization Procedure

The numerical procedure developed to actually find optimal controls and profits consists of two parts: the integration of the partial differential equations (2.3) and the maximization of the profit (1.8).

The former can be accomplished on the basis of any of the numerical methods for balance laws currently available. In the general form (2.3), due to the dependence of the growth functions on t and x, the classical Lax-Friedrichs method [18, Section 4.6] can be an effective choice. To deal with the source term, we use the classical operator splitting method, see for instance [18, Chapter 17]. Whenever the growth functions g_J, g_S, g_R and mortality functions d_J, d_S, d_R are constant, the explicit expressions of the solutions to the partial differential equations in (2.3) are available and, obviously, faster and more precise computations are possible.

Once the solution is available, the computations of the cost (1.5), of the income (1.6) and of the penalization (1.7) are straightforward.

We use the following iterative procedure to find an optimal control and the corresponding maximal profit. Fix a finite mesh $\mathcal{M} \subset [0,1]$ for the control variable. Call Δt the time step used in the numerical solution of the partial differential equations in (2.3). Fix a positive $n \in \mathbb{N}$ and assume for simplicity that $T = k\, n\, \Delta t$, with $k \in \mathbb{N}$. Then, call $\mathbf{PC}(n\,\Delta t; \mathcal{M})$ the set of functions defined on $[0,T]$, attaining values in $\mathcal{M}$ and constant on each of the intervals $I_1 = [0, n\,\Delta t]$, $I_2 = [n\,\Delta t, 2n\,\Delta t]$, ..., $I_k = [(k-1)n\,\Delta t, T]$.

We seek controls η and $\vartheta_1, \ldots, \vartheta_{N-1}$ in $\mathbf{PC}(n\,\Delta t; \mathcal{M})$ that maximize $\mathcal{P}$ as defined in (1.8). To this aim, as *Step* 1 fix an arbitrary first guess η^1, $\vartheta^1 \equiv (\vartheta_1^1, \ldots, \vartheta_{N-1}^1)$ of the controls. We proceed iteratively as follows: given a choice η^i, ϑ^i resulting from *Step* i, introduce a random permutation $p^i \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$. Define $\eta_*^{i+1,p^i(1)} \in \mathcal{M}$ and $\vartheta_*^{i+1,p^i(1)} \in \mathcal{M}^{N-1}$, so that

$$\begin{aligned}
(\eta_*^{i+1,p^i(1)}, \vartheta_*^{i+1,p^i(1)}) &= \operatorname*{argmax}_{\tilde\eta \in \mathcal{M},\, \tilde\vartheta \in \mathcal{M}^{N-1}} \mathcal{P}\left(\eta^i \chi_{[0,T]\setminus I_{p^i(1)}} + \tilde\eta\, \chi_{I_{p^i(1)}}, \vartheta^i \chi_{[0,T]\setminus I_{p^i(1)}} + \tilde\vartheta\, \chi_{I_{p^i(1)}}; T\right) \\
\widehat{\eta}^1 &= \eta^i \chi_{[0,T]\setminus I_{p^i(1)}} + \eta_*^{i+1,p^i(1)} \chi_{I_{p^i(1)}} \\
\widehat{\vartheta}^1 &= \vartheta^i \chi_{[0,T]\setminus I_{p^i(1)}} + \vartheta_*^{i+1,p^i(1)} \chi_{I_{p^i(1)}}
\end{aligned}$$

and, recursively for $j = 2, \ldots, k$,

$$\begin{aligned}
(\eta_*^{i+1,p^i(j)}, \vartheta_*^{i+1,p^i(j)}) &= \operatorname*{argmax}_{\tilde\eta \in \mathcal{M},\, \tilde\vartheta \in \mathcal{M}^{N-1}} \mathcal{P}\left(\widehat{\eta}\, \chi_{[0,T]\setminus I_{p^i(1)}} + \tilde\eta\, \chi_{I_{p^i(1)}}, \widehat{\vartheta}\, \chi_{[0,T]\setminus I_{p^i(1)}} + \tilde\vartheta\, \chi_{I_{p^i(1)}}; T\right) \\
\widehat{\eta}^j &= \widehat{\eta}^{j-1} \chi_{[0,T]\setminus I_{p^i(j)}} + \eta_*^{i+1,p^i(j)} \chi_{I_{p^i(j)}} \\
\widehat{\vartheta}^j &= \widehat{\vartheta}^{j-1} \chi_{[0,T]\setminus I_{p^i(j)}} + \vartheta_*^{i+1,p^i(1)} \chi_{I_{p^i(j)}},
\end{aligned}$$

obtaining at *Step* $(i+1)$ the control parameters

$$\eta^{i+1} = \widehat{\eta}^k \quad \text{and} \quad \vartheta^{i+1} = \widehat{\vartheta}^k .$$

Remark that the present method applies to the search for both a bang–bang as well as a general control. Indeed, in the former case it is sufficient to set $\mathcal{M} = \{0,1\}$. In the latter case, a natural choice is that $\mathcal{M}$ consists of a number (80 in Table 1) of control values uniformly distributed in $[0,1]$.

As an example, consider problem (2.3) with

$$\begin{array}{llllc}
I_J = [0,1] & g_J(t,a) = 1.5 & d_J(t,a) = 0 & N = 1 & \\
I_S = [1,2] & g_S(t,a) = 1 & d_S(t,a) = 0 & \bar a = 1 & T = 4\,, \quad w(\alpha) = 4\,, \\
I_R = [1,2] & g_R(t,a) = 2 & d_R(t,a) = 0.5 & \bar a_1 = 2 &
\end{array} \tag{3.1}$$

and initial datum

$$J_o(a) = 1\,, \quad S_o(a) = 0\,, \quad R_o(a) = 0\,, \tag{3.2}$$

while the cost and profit functions in (1.8) are defined through

$$\begin{array}{llll}
C_J(t,a,j) = 0, & & & \\
C_S(t,a,s) = 0, & P_1(t) = 10 & \text{and} & \varphi(\alpha) = 40\, \max\{0, -\alpha\}\,. \\
C_R(t,a,r) = 5r, & & &
\end{array} \tag{3.3}$$

With the choices above, problem (2.3) can be explicitly integrated and the cost (1.8) can be computed. Numerically, an optimal bang–bang control was obtained, see Figure 3. To compare the corresponding costs and computation times, see Table 1.

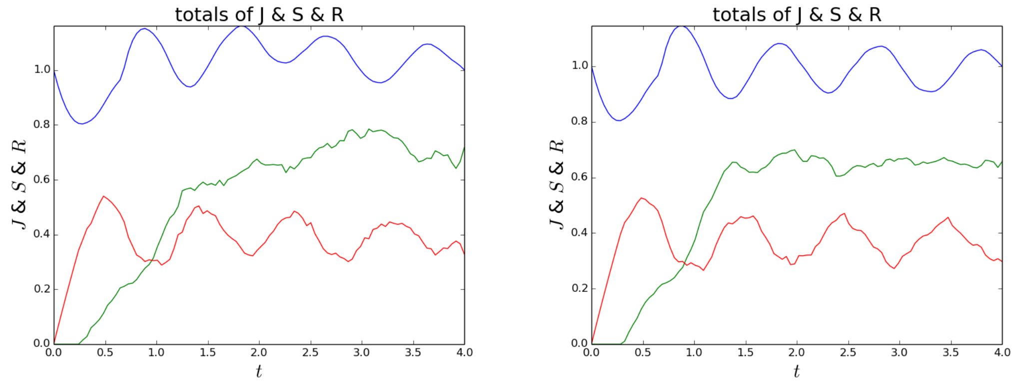


Figure 3: Graphs of the total amounts of the three populations: $\int J$ in blue, $\int S$ in green and $\int R$ in red in the solution to (2.3) with parameters (3.1)–(3.2)–(3.3), after 4 optimization steps as in Table1, right. Left, with a bang-bang control and, right, with a general one.

	2 steps		4 steps	
	Bang–Bang	No Bang–Bang	Bang–Bang	No Bang–Bang
$\mathcal{C}$ in (1.5)	7.638	7.282	7.666	7.306
$\mathcal{I}$ in (1.6)	17.602	17.191	17.675	17.219
Φ in (1.7)	0.000	0.000	0.000	0.000
$\mathcal{P}$ in (1.8)	9.964	9.909	10.009	9.914
$\Delta t \times 10^4$	3.125	3.125	3.125	3.125
n° control switches	800	800	800	800
n° control values	2	40	2	40
Computation time	81 min	1753 min	160 min	3488 min

Table 1: Results and computational time of the optimization problem (1.9) with parameters (3.1)–(3.2)–(3.3), using bang–bang or general controls, with 2 or 4 steps.

A major reduction in the computational time is obtained thanks to the restriction of the search for the optimal controls to bang-bang ones. Moreover, using a fine mesh for the control parameters (40 values between 0 and 1) does not ensure a gain in the total income. Partly, this is also due to both the optimization procedure described above and to the random choices of the optimization intervals.

4 Technical Details

To simplify the notation, for $u = J, S, R$ we denote below

$$\psi_u(t_1, t_2, a) = \exp \int_{t_1}^{t_2} \Big(d_u \left(s, \mathcal{A}_u(s; t_2, a)\right) - \partial_a g_u \left(s, \mathcal{A}_u(s; t_2, a)\right) \Big) \, \mathrm{d}s \tag{4.1}$$

where the map $t \to \mathcal{A}_u\left(t, t_o, a_o\right)$ is defined in (2.6). As in [10, 20], we recall that the initial – boundary value problem for the renewal equation

$$\begin{cases} \partial_t u + \partial_a \left(g_u(t,a)\, u\right) = d_u(t,a)\, u & t \geq 0 \\ u(0,a) = u_o(a) & a \geq a_u \\ g_u(t, a_u)\, u(t, a_u+) = b(t) & \end{cases} \tag{4.2}$$

admits a unique solution that can be explicitly computed integrating along characteristics as

$$u(t,a) = \begin{cases} u_o\left(\mathcal{A}_u(0;t,a)\right)\ \psi_u(0,t,a) & a \geq \mathcal{A}_u(t;0,a_u) \\ \frac{b\left(\mathcal{T}_u(a_u;t,a)\right)}{g_u\left(\mathcal{T}_u(a_u;t,a),a_u\right)}\ \psi_u\big(\mathcal{T}_u(a_u;t,a),t,a\big) & a < \mathcal{A}_u(t;0,a_u)\,, \end{cases} \tag{4.3}$$

where the maps $t \to \mathcal{A}_u(t,t_o,a_o)$ and $a \to \mathcal{T}_u(a;t_o,a_o)$ are defined as in (2.6). By the standard theory of ordinary differential equations, e.g. [7, Section 2.3], we also have:

$$\begin{aligned} \partial_t \mathcal{A}_u(t;t_o,a_o) &= g_u\left(t,\mathcal{A}_u(t;t_o,a_o)\right) && (4.4)\\ \partial_{t_o} \mathcal{A}_u(t;t_o,a_o) &= -g_u(t_o,a_o)\,\exp\int_{t_o}^{t} \partial_a g_u\left(s,\mathcal{A}_u(s;t_o,a_o)\right)\mathrm{d}s && (4.5)\\ \partial_{a_o} \mathcal{A}_u(t;t_o,a_o) &= \exp\int_{t_o}^{t} \partial_a g_u\left(s,\mathcal{A}_u(s;t_o,a_o)\right)\mathrm{d}s\,. && (4.6) \end{aligned}$$

4.1 Proofs Related to § 2.1

Lemma 4.1. *Use the notation (2.2) and let g_S, d_S satisfy* **(A)** *for $u = S$. For every $\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$, $b_S \in \mathbf{BV}(I_T;\mathbb{R})$ and for any initial datum $S_o \in \mathbf{BV}(I_S;\mathbb{R})$, the initial boundary value problem*

$$\begin{cases} \partial_t S + \partial_a\left(g_S(t,a)\,S\right) = d_S(t,a)\,S & (t,a) \in I_T \times \left(I_S \setminus \{\bar a_1,\ldots,\bar a_N\}\right) \\ g_S(t,\bar a+)\,S(t,\bar a+) = b_S(t) & t \in I_T \\ S(t,\bar a_i+) = \left(1-\vartheta_i(t)\right)S(t,\bar a_i-) & t \in I_T\,,\quad i=1,\ldots,N \\ S(0,a) = S_o(a) & a \in I_S \end{cases} \tag{4.7}$$

admits a unique solution. If $b_S \geq 0$ and $S_o \geq 0$, then also $S \geq 0$. Moreover, if $\vartheta',\vartheta'' \in \mathbf{BV}(I_T,[0,1])$, $b'_S, b''_S \in \mathbf{BV}(I_T;\mathbb{R})$ and $S'_o, S''_o \in \mathbf{BV}(I_S;\mathbb{R})$, then the solutions S' and S'' to

$$\begin{cases} \partial_t S + \partial_a\left(g_S(t,a)\,S\right) = d_S(t,a)\,S \\ g_S(t,\bar a+)\,S(t,\bar a+) = b'_S(t) \\ S(t,\bar a_i+) = \left(1-\vartheta'_i(t)\right)S(t,\bar a_i-) \\ S(0,a) = S'_o(a) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t S + \partial_a\left(g_S(t,a)\,S\right) = d_S(t,a)\,S \\ g_S(t,\bar a+)\,S(t,\bar a+) = b''_S(t) \\ S(t,\bar a_i+) = \left(1-\vartheta''_i(t)\right)S(t,\bar a_i-) \\ S(0,a) = S''_o(a) \end{cases}$$

satisfy for all $t \in I_T$ the estimates:

$$\begin{aligned} \left\|S'(t)-S''(t)\right\|_{\mathbf{L^1}(I_S;\mathbb{R})} \leq{}& C\left(\left\|S'_o\right\|_{\mathbf{L}^\infty(I_S;\mathbb{R})} + \left\|b'_S\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})}\right)\left\|\vartheta'-\vartheta''\right\|_{\mathbf{L^1}([0,t];\mathbb{R})} e^{Ct} \\ &+ \left(\left\|S'_o - S''_o\right\|_{\mathbf{L^1}(I_S;\mathbb{R})} + \frac{1}{\check g}\left\|b'_S - b''_S\right\|_{\mathbf{L^1}([0,t];\mathbb{R})}\right)e^{Ct} \\ \left\|S'(t)-S''(t)\right\|_{\mathbf{L}^\infty(I_S;\mathbb{R})} \leq{}& C\left(\left\|S'_o\right\|_{\mathbf{L}^\infty(I_S;\mathbb{R})} + \left\|b'_S\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})}\right)\left\|\vartheta'-\vartheta''\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})} e^{Ct} \\ &+ \left(\left\|S'_o - S''_o\right\|_{\mathbf{L}^\infty(I_S;\mathbb{R})} + \frac{1}{\check g}\left\|b'_S - b''_S\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})}\right)e^{Ct} \end{aligned}$$

where C is a positive constant depending only on N, g_S and d_S.

Proof. Denote by S_* the solution to (4.7) obtained setting $\vartheta_i(t) = 0$ for all $t \in I_T$ and all $i = 1,\ldots,N$. A direct application of (4.3) allows to write the general solution to (4.7) as

$$\begin{aligned} S(t,a) &= S_*(t,a) \prod_{i \in \Theta(t,a)} \left(1-\vartheta_i(\mathcal{T}(\bar a_i;t,a))\right) \qquad \text{where} \\ \Theta(t,a) &= \left\{i \in \{1,\ldots,N\} \colon \exists t_* \in [0,t] \text{ with } \mathcal{A}_S(t;t_*,\bar a_i) = a\right\}. \end{aligned}$$

To obtain the stability estimate, we use [10, Lemma 2.2], so that

$$
\begin{aligned}
\left\| S'_*(t) - S''_*(t) \right\|_{\mathbf{L^1}(I_S;\mathbb{R})} & \leq \left(\left\| S'_o - S''_o \right\|_{\mathbf{L^1}(I_S;\mathbb{R})} + \frac{1}{\check g} \left\| b'_S - b''_S \right\|_{\mathbf{L^1}([0,t];\mathbb{R})} \right) e^{Ct} \\
\left\| S'_*(t) - S''_*(t) \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} & \leq \left(\left\| S'_o - S''_o \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} + \frac{1}{\check g} \left\| b'_S - b''_S \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} \right) e^{Ct} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \left\| S'(t) - S''(t) \right\|_{\mathbf{L^1}(I_S;\mathbb{R})} \\
= & \int_{I_S} \left| S'(t,a) - S''(t,a) \right| \mathrm{d}a \\
\leq & \int_{I_S} \left| S'_*(t,a) \right| \left| \prod_{i \in \Theta(t,a)} \left(1 - \vartheta'_i \left(\mathcal{T}(\bar a_i; t, a)\right)\right) - \prod_{i \in \Theta(t,a)} \left(1 - \vartheta''_i \left(\mathcal{T}(\bar a_i; t, a)\right)\right) \right| \mathrm{d}a \\
& + \int_{I_S} \left| S'_*(t,a) - S''_*(t,a) \right| \prod_{i \in \Theta(t,a)} \left(1 - \vartheta''_i \left(\mathcal{T}(\bar a_i; t, a)\right)\right) \mathrm{d}a \\
\leq & C \left\| S'_*(t) \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} \int_{I_S} \sum_{i \in \Theta(t,a)} \left| \vartheta' \left(\mathcal{T}(\bar a_i; t, a)\right) - \vartheta'' \left(\mathcal{T}(\bar a_i; t, a)\right) \right| \mathrm{d}a + \left\| S'_*(t) - S''_*(t) \right\|_{\mathbf{L^1}(I_S;\mathbb{R})} \\
\leq & C \left(\left(\left\| S'_o \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} + \left\| b'_S \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} \right) \left\| \vartheta' - \vartheta'' \right\|_{\mathbf{L^1}([0,t];\mathbb{R})} \right) e^{Ct} \\
& + \left(\left\| S'_o - S''_o \right\|_{\mathbf{L^1}(I_S;\mathbb{R})} + \frac{1}{\check g} \left\| b'_S - b''_S \right\|_{\mathbf{L^1}([0,t];\mathbb{R})} \right) e^{Ct} .
\end{aligned}
$$

Passing to the $\mathbf{L}^\infty$ norm:

$$
\begin{aligned}
& \left| S'(t,a) - S''(t,a) \right| \\
\leq & \left| S'_*(t,a) \right| \left| \prod_{i \in \Theta(t,a)} \left(1 - \vartheta'_i \left(\mathcal{T}(\bar a_i; t, a)\right)\right) - \prod_{i \in \Theta(t,a)} \left(1 - \vartheta''_i \left(\mathcal{T}(\bar a_i; t, a)\right)\right) \right| \\
& + \left| S'_*(t,a) - S''_*(t,a) \right| \prod_{i \in \Theta(t,a)} \left(1 - \vartheta''_i \left(\mathcal{T}(\bar a_i; t, a)\right)\right) \\
\leq & C \left\| S'_*(t) \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} \left\| \vartheta' - \vartheta'' \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} + \left\| S'_*(t) - S''_*(t) \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} \\
\leq & C \left(\left\| S'_o \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} + \left\| b'_S \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} \right) \left\| \vartheta' - \vartheta'' \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} e^{Ct} \\
& + \left(\left\| S'_o - S''_o \right\|_{\mathbf{L^\infty}(I_S;\mathbb{R})} + \frac{1}{\check g} \left\| b'_S - b''_S \right\|_{\mathbf{L^\infty}([0,t];\mathbb{R})} \right) e^{Ct}
\end{aligned}
$$

completing the proof. □

Proof of Theorem 2.1. The specific structure of (2.3) allows for a treatment simpler than the general one adopted in [10, Theorem 2.4]. The positivity of the solution directly follows from [10, Proposition 2.6].

Once (J, S, R) is known at time $t = T_n$ (for $n \geq 0$), the solution to (2.3) can then be constructed on $]T_n, T_{n+1}]$ through the following three steps:

1. Define J for $t \in \,]T_n, T_{n+1}]$ and $a \in [\mathcal{A}_J(t; T_n, 0), \bar a]$ through

$$
J(t,a) = J\left(T_n, \mathcal{A}_J(T_n; t, a)\right) \; \psi_J(T_n, t, a) .
$$

2. Define S for $(t,a) \in \,]T_n, T_{n+1}] \times I_S$, by using Lemma 4.1, as the solution to

$$\left\{\begin{array}{ll}\partial_t S + \partial_a \left(g_S(t,a)\, S\right) = d_S(t,a)\, S & (t,a) \in \,]T_n, T_{n+1}] \times \left(I_S \setminus \{\bar a_1, \ldots, \bar a_N\}\right) \\ g_S(t, \bar a+)\, S(t, \bar a+) = \eta(t)\, g_j(t, \bar a)(t)\, J(t, \bar a-) & t \in \,]T_n, T_{n+1}] \\ S(t, \bar a_i+) = \left(1 - \vartheta_i(t)\right) S(t, \bar a_i-) & t \in \,]T_n, T_{n+1}] \,, \quad i = 1, \ldots, N \\ S(T_n+, a) = S(T_n-, a) & a \in I_S \,.\end{array}\right.$$

3. Define R for $(t,a) \in \,]T_n, T_{n+1}] \times I_R$ as solution to

$$\left\{\begin{array}{ll}\partial_t R + \partial_a \left(g_R(t,a)\, R\right) = d_R(t,a)\, R & (t,a) \in \,]T_n, T_{n+1}] \times I_R \\ g_R(t, \bar a)\, R(t, \bar a+) = \left(1 - \eta(t)\right)\, g_J(t, \bar a)\, J(t, \bar a-) & t \in \,]T_n, T_{n+1}] \\ R(T_n+, a) = R(T_n-, a) & a \in I_R \,.\end{array}\right.$$

4. Define J for $t \in \,]T_n, T_{n+1}]$ and $a \in [0, \mathcal{A}_J(t; T_n, 0)]$ through

$$J(t,a) = \frac{\int_{I_R} w(\alpha)\, R\left(\mathcal{T}_J(\alpha; T_n, 0), \alpha\right) \mathrm{d}\alpha}{g_J\left(\mathcal{T}_J(\alpha; T_n, 0)\right)}\, \psi_J\left(\mathcal{T}_J(\alpha; T_n, 0), t, a\right) \,.$$

The proof is completed using the estimates [10, Formulæ (2.15) and (2.18)]. □

Lemma 4.2. *Use the notation* (2.2) *and pose conditions **(A)**, **(ID)**. For any controls* $\eta \in \mathbf{BV}(\mathbb{R}^+; [0,1])$ *and* $\vartheta \in \mathbf{BV}(I_T; [0,1]^N)$,

$$\begin{array}{rcll} J(t,a) & = & J(\tau, 0)\, \psi_J(\tau, t, a) & \left\{\begin{array}{l}(t,a) \in G_J^i \,,\ i \geq 2 \\ \quad \tau = \mathcal{T}_J(0; t, a)\end{array}\right. \\ S(t,a) & = & \eta_i(\tau)\, \frac{g_J(\tau, \bar a)}{g_S(\tau, \bar a)}\, J(\tau, \bar a)\, \psi_S(\tau, t, a) & \left\{\begin{array}{l}(t,a) \in G_S^i \,,\ i \geq 1 \\ \quad \tau = \mathcal{T}_S(\bar a; t, a) \\ \quad a \in [\bar a, \bar a_1[\end{array}\right. \\ S(t,a) & = & \left(1 - \vartheta_j(\tau)\right)\, S(\tau, \bar a_j)\, \psi_S(\tau, t, a) & \left\{\begin{array}{l}(t,a) \in G_S^i \,,\ i \geq 1 \\ \quad \tau = \mathcal{T}_S(\bar a_j; t, a) \\ \quad a \in \left[\bar a_j, \bar a_{j+1}\right[\\ \quad j = 2, \ldots, N-1\end{array}\right. \\ R(t,a) & = & \left(1 - \eta_i(\tau)\right)\, \frac{g_J(\tau, \bar a)}{g_R(\tau, \bar a)}\, J(\tau, \bar a)\, \psi_R(\tau, t, a) & \left\{\begin{array}{l}(t,a) \in G_R^i \,,\ i \geq 1 \\ \quad \tau = \mathcal{T}_R(\bar a; t, a)\end{array}\right.\end{array}$$

The proof is an immediate consequence of (2.3) and (4.3).

Proof of Proposition 2.3. Let $t \in \,]T_{n-1}, T_n]$ and proceed by induction on h and n.

***h* = 1 and *n* = 1**: If $a \in [\mathcal{A}_J(t; 0, 0), \bar a]$, then $\Delta J(t,a) = 0$. Otherwise, by Lemma 4.2, for $a \in [0, \mathcal{A}_J(t; 0, 0)]$,

$$\begin{array}{rl} & \Delta J(t,a) \\ = & \Delta J\left(\mathcal{T}_J(0; t, a), 0\right)\ \psi_J\left(\mathcal{T}_J(0; t, a), t, a\right) \\ = & \displaystyle\int_{I_R} w(\alpha)\, \Delta R\left(\mathcal{T}_J(0; t, a), \alpha\right)\, \mathrm{d}\alpha\ \frac{\psi_J\left(\mathcal{T}_J(0; t, a), t, a\right)}{g_J\left(\mathcal{T}_J(0; t, a), 0\right)} \\ = & \displaystyle -\int_{\bar a}^{\mathcal{A}_R(\mathcal{T}_J(0;t,a);0,\bar a)} w(\alpha)\, \Delta\eta\big(\mathcal{T}_R(\bar a; \mathcal{T}_J(0; t, a), \alpha)\big)\ \frac{g_J\left(\mathcal{T}_R(\bar a; \mathcal{T}_J(0; t, a), \alpha), \bar a\right)}{g_R\left(\mathcal{T}_R(\bar a; \mathcal{T}_J(0; t, a), \alpha), \bar a\right)} \\ & \displaystyle \times J\left(\mathcal{T}_R(\bar a; \mathcal{T}_J(0; t, a), \alpha), \bar a\right)\ \psi_R(\mathcal{T}_R(\bar a; \mathcal{T}_J(0; t, a), \alpha), \mathcal{T}_J(0; t, a), \alpha)\ \mathrm{d}\alpha\ \frac{\psi_J\left(\mathcal{T}_J(0; t, a), t, a\right)}{g_J\left(\mathcal{T}_J(0; t, a), 0\right)}\end{array}$$

$$= \int_0^{\min\{t,T_1\}} \Delta\eta(\tau)\ \Upsilon_J^1(\tau,t,a)\,\mathrm{d}\tau$$

where we used the change of variable $\tau = \mathcal{T}_R(\bar a;\mathcal{T}_J(0;t,a),\alpha)$ and Υ_J^1 is a suitable function in $(\mathbf{BV}\cap\mathbf{L}^\infty)([T_0,T_1]^2\times I_J;\mathbb{R}^+)$.

$\boldsymbol{h=1}$ **and** $\boldsymbol{n>1}$: Proceed by induction on n. Let $(t,a)\in\,]T_{n-1},T_n]\times I_J$ and consider first the case $(t,a)\in G_J^n$. Then,

$$\begin{aligned}
\Delta J(t,a) &= \Delta J\left(T_{n-1},\mathcal{A}_J(T_{n-1};t,a)\right)\ \psi_J\left(T_{n-1},t,a\right)\\
&= \int_{T_0}^{T_1} \Delta\eta(\tau)\ \Upsilon_J^1\left(\tau,T_{n-1},\mathcal{A}_J(T_{n-1};t,a)\right)\mathrm{d}\tau\ \ \psi_J\left(T_{n-1},t,a\right)\\
&= \int_{T_0}^{T_1} \Delta\eta(\tau)\ \Upsilon_J^1(\tau,t,a)\,\mathrm{d}\tau\ ,
\end{aligned}$$

where the inductive assumption was used. In the case $(t,a)\in G_J^{n+1}$, then,

$$\begin{aligned}
\Delta J(t,a) &= \Delta J\left(\mathcal{T}_J(0;t,a),0\right)\ \psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)\\
&= \int_{I_J} w(\alpha)\ \Delta R\left(\mathcal{T}_J(0;t,a),\alpha\right)\mathrm{d}\alpha\ \ \frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
&= \int_{\bar a}^{\mathcal{A}_R(\mathcal{T}_J(0;t,a);\bar a,T_0)} w(\alpha)\ \Delta R\big(\mathcal{T}_J(0;t,a),\alpha\big)\,\mathrm{d}\alpha\ \ \frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
&= \int_0^{\mathcal{T}_J(0;t,a)} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\Delta R\left(\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\\
&\qquad\times\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\ ,
\end{aligned}$$

where we use the substitution $\tau = \mathcal{T}_R(\bar a;\mathcal{T}_J(0;t,a),\alpha)$ or equivalently $\alpha = \mathcal{A}_R\left(\mathcal{T}_J\left(0;t,a\right);\tau,\bar a\right)$. Therefore the difference $\Delta J(t,a)$ can be written as

$$\begin{aligned}
&\Delta J(t,a)\\
=\ & \int_{T_0}^{T_1} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\Delta R\left(\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
+\ & \sum_{i=2}^{n-1}\int_{T_{i-1}}^{T_i} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\Delta R\left(\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
+\ & \int_{T_{n-1}}^{\mathcal{T}_J(0;t,a)} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\Delta R\left(\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
=\ & -\int_{T_0}^{T_1} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\Delta\eta(\tau)\frac{g_J(\tau,\bar a)}{g_R(\tau,\bar a)}J\left(\tau,\bar a\right)\\
&\times\psi_R\left(\tau,\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
+\ & \sum_{i=2}^{n-1}\int_{T_{i-1}}^{T_i} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\eta(\tau)\frac{g_J(\tau,\bar a)}{g_R(\tau,\bar a)}\Delta J\left(\tau,\bar a\right)\\
&\times\psi_R\left(\tau,\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
+\ & \int_{T_{n-1}}^{\mathcal{T}_J(0;t,a)} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\eta(\tau)\frac{g_J(\tau,\bar a)}{g_R(\tau,\bar a)}\Delta J\left(T_{n-1},\mathcal{A}_J\left(T_{n-1};\tau,\bar a\right)\right)
\end{aligned}$$

$$\times\psi_J\left(T_{n-1},\tau,\bar a\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}.$$

Using the inductive hypothesis, we obtain that $\Delta J(t,a)$ can be written as

$$\begin{aligned}
&\Delta J(t,a)\\
= \; & -\int_{T_0}^{T_1} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\Delta\eta(\tau)\frac{g_J(\tau,\bar a)}{g_R(\tau,\bar a)}J\left(\tau,\bar a\right)\\
&\times\psi_R\left(\tau,\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
+ \; & \sum_{i=2}^{n-1}\int_{T_{i-1}}^{T_i} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\eta(\tau)\frac{g_J(\tau,\bar a)}{g_R(\tau,\bar a)}\int_{T_0}^{T_1}\Delta\eta(r)\Upsilon_J\left(r,\tau,\bar a\right)\mathrm{d}r\\
&\times\psi_R\left(\tau,\mathcal{T}_J(0;t,a),\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
+ \; & \int_{T_{n-1}}^{\mathcal{T}_J(0;t,a)} w\left(\mathcal{A}_R(\mathcal{T}_J(0;t,a);\tau,\bar a)\right)\eta(\tau)\frac{g_J(\tau,\bar a)}{g_R(\tau,\bar a)}\int_{T_0}^{T_1}\Delta\eta(r)\,\Upsilon_J\left(r,T_{n-1},\mathcal{A}_J\left(T_{n-1};\tau,\bar a\right)\right)\mathrm{d}r\\
&\times\psi_J\left(T_{n-1},\tau,\bar a\right)\frac{d\alpha}{d\tau}\,\mathrm{d}\tau\,\frac{\psi_J\left(\mathcal{T}_J(0;t,a),t,a\right)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\\
= \; & \int_{T_0}^{T_1}\Delta\eta(r)\,\Upsilon_J\left(r,t,a\right)\mathrm{d}r\,,
\end{aligned}$$

completing the case $\boldsymbol{h=1}$ **and** $\boldsymbol{n>1}$.

$\boldsymbol{h>1}$ **and** $\boldsymbol{n<h}$: Clearly, $\Delta J(t,a)=0$.

$\boldsymbol{h>1}$ **and** $\boldsymbol{n=h}$: This case can be treated exactly as the case $h=n=1$.

$\boldsymbol{h>1}$ **and** $\boldsymbol{n>h}$: The translation $t\to t-T_h$ along the time axis allows to reduce to the computations related to the already considered case $h=1$ and $n>1$.

Consider now the population S. If $(t,a)\in G_S^i$ with $i<h$, then $\Delta S(t,a)=0$. Otherwise, if $(t,a)\in G_S^h$, then by Lemma 4.2, in the case $a\in[\bar a,\bar a_1[$

$$\begin{aligned}
\Delta S(t,a) &= \Delta S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\psi_S\left(\mathcal{T}_S(\bar a;t,a),t,a\right)\\
&= \Delta\eta\left(\mathcal{T}_S(\bar a;t,a)\right)\,\frac{g_J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}\,J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\psi_S\left(\mathcal{T}_S(\bar a;t,a),t,a\right)\\
&= \Delta\eta\left(\mathcal{T}_S(\bar a;t,a)\right)\,\Upsilon_S\left(\mathcal{T}_S(\bar a;t,a),t,a\right)\,,
\end{aligned}$$

where $\mathcal{T}_S(\bar a;t,a)\in[T_{h-1},T_h]$. In the case $(t,a)\in G_S^i$ with $i>h$ and $a\in[\bar a,\bar a_1[$, by Lemma 4.2,

$$\begin{aligned}
\Delta S(t,a) &= \Delta S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\psi_S\left(\mathcal{T}_S(\bar a;t,a),t,a\right)\\
&= \eta\left(\mathcal{T}_S(\bar a;t,a)\right)\,\frac{g_J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}\,\Delta J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\psi_S\left(\mathcal{T}_S(\bar a;t,a),t,a\right)\\
&= \eta\left(\mathcal{T}_S(\bar a;t,a)\right)\,\frac{g_J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}\int_{T_{h-1}}^{T_h}\Delta\eta(\tau)\,\Upsilon_J(\tau,t,a)\,\mathrm{d}\tau\,\psi_S\left(\mathcal{T}_S(\bar a;t,a),t,a\right)\\
&= \int_{T_{h-1}}^{T_h}\Delta\eta(\tau)\,\Upsilon_S(\tau,t,a)\,\mathrm{d}\tau\,.
\end{aligned}$$

Whenever $(t,a)\in G_S^h$ and $a\in\left[\bar a_j,\bar a_{j+1}\right[$ with $j\geq 1$, by Lemma 4.2,

$$\begin{aligned}
\Delta S(t,a) &= \Big(1-\vartheta_j\left(\mathcal{T}_S(\bar a_j;t,a)\right)\Big)\,\Delta S\left(\mathcal{T}_S(\bar a_j;t,a),\bar a_j\right)\\
&= \Upsilon_S\left(\mathcal{T}_S(\bar a;t,a),t,a\right)\,\Delta S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\,.
\end{aligned}$$

The proof is completed treating the population R in a similar way. □

Proof of Proposition 2.4. Clearly, $\Delta J = \Delta R = 0$ by construction, as also $\Delta S(t,a) = 0$ for $a \in \left]\bar a, \bar a_1\right[$. In the case $a \in \left]\bar a_i, \bar a_{i+1}\right[$, apply (4.3) and (1.4). □

Proof of Theorem 2.5. Call (J,S,R), respectively $(J+\Delta^\varepsilon J, S+\Delta^\varepsilon S, R+\Delta^\varepsilon R)$, the solution to (2.3) with the control η, respectively $\eta+\varepsilon\Delta\eta$, with $\varepsilon \in \left]0,1\right[$. Denote for $u = J,S,R$

$$\bar{\mathcal{C}}_u(\eta) = \int_0^{T_n} \int_{I_u} C_u\left(t,a,u(t,a)\right) \mathrm{d}a\,\mathrm{d}t \,.$$

Using $\boldsymbol{(\mathcal{C})}$, with Υ_J as in Proposition 2.3 and by Theorem 2.1, compute:

$$\begin{aligned}
&\left|\left(\bar{\mathcal{C}}_J(\eta+\varepsilon\Delta\eta) - \bar{\mathcal{C}}_J(\eta)\right) - \varepsilon \int_0^{T_n}\int_{I_J}\int_{T_{h-1}}^{T_h} \partial_J C_J\left(t,a,J(t,a)\right)\ \Delta\eta(\tau)\ \Upsilon_J(\tau,t,a)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}t\right| \\
= & \left|\int_0^{T_n}\int_{I_J}\left[C_J\left(t,a,(J+\Delta^\varepsilon J)(t,a)\right) - C_J\left(t,a,J(t,a)\right)\right.\right. \\
& \qquad \left.\left. -\varepsilon\int_{T_{h-1}}^{T_h}\partial_J C_J\left(t,a,J(t,a)\right)\Delta\eta(\tau)\Upsilon_J(\tau,t,a)\,\mathrm{d}\tau\right]\mathrm{d}a\,\mathrm{d}t\right| \\
\leq & \left|\int_0^{T_n}\int_{I_J}\partial_J C_J\left(t,a,J(t,a)\right)\left(\Delta^\varepsilon J(t,a) - \varepsilon\int_{T_{h-1}}^{T_h}\Delta\eta(\tau)\Upsilon_J(\tau,t,a)\,\mathrm{d}\tau\right)\mathrm{d}a\,\mathrm{d}t\right| \\
& + C\left|\int_0^{T_n}\int_{I_J}\left(\Delta^\varepsilon J(t,a)\right)^2\mathrm{d}a\,\mathrm{d}t\right| \\
\leq & \ C\,\varepsilon^2\int_0^{T_n}\int_{I_J}\left(\int_{T_{h-1}}^{T_h}\Delta\eta(\tau)\ \Upsilon_J(\tau,t,a)\,\mathrm{d}\tau\right)^2\mathrm{d}a\,\mathrm{d}t \\
\leq & \ C\,\varepsilon^2\ \left(\|\Delta\eta\|_{\mathbf{L}^\infty([T_{h-1},T_h];\mathbb{R})}\right)^2\int_0^{T_n}\int_{I_J}\left(\int_{T_{h-1}}^{T_h}\Upsilon_J(\tau,t,a)\,\mathrm{d}\tau\right)^2\mathrm{d}a\,\mathrm{d}t
\end{aligned}$$

proving that in the limit $\varepsilon \to 0+$, we have

$$\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left(\bar{\mathcal{C}}_J(\eta+\varepsilon\Delta\eta) - \bar{\mathcal{C}}_J(\eta)\right) = \int_0^{T_n}\int_{I_J}\int_{T_{h-1}}^{T_h}\partial_J C_J\left(t,a,\tilde J(t,a)\right)\ \Delta\eta(\tau)\ \Upsilon_J(\tau,t,a)\,\mathrm{d}\tau\,\mathrm{d}a\,\mathrm{d}t\,,$$

which shows the Gâteaux differentiability of $\bar{\mathcal{C}}_J$.

The computations for $\bar{\mathcal{C}}_S$ are entirely similar. The computations for $\bar{\mathcal{C}}_R$ can be reduced to the ones above, noting that $a \to \Delta^\varepsilon R(t,a)$ vanishes outside the bounded interval $[0, \mathcal{A}_R(t;\bar a,0)]$.

Finally, the Gâteaux differentiability of the profit with respect to ϑ directly follows from Proposition 2.4. □

4.2 Proofs Related to § 2.2

Lemma 4.3. *Use the notation* (2.2) *and pose conditions* ***(A)**, **(ID)**. Fix* $i \in \mathbb{N}$. *For all* $k \in \mathbb{N}$, *let* $\eta_i^k \in \mathbf{BV}([T_{i-1},T_i];[0,1])$ *be such that* $\eta_i^k \overset{*}{\rightharpoonup} \eta_i$ *in* $\mathbf{L}^\infty([T_{i-1},T_i];\mathbb{R})$, *as* $k \to +\infty$. *Call* (J,S,R), *respectively* (J_k,S_k,R_k), *the solution to* (2.3) *corresponding to* $\widehat{\eta}_i$, *respectively to* $\widehat{\eta}_i^k$, *as defined in (2.9). Then, for* $j = i, i+1, i+2, \ldots$, *in the limit* $k \to +\infty$

$$\begin{array}{rcll}
J_k(t,a) & \to & J(t,a) & \textit{for all } t \in \left]T_{j-1},T_j\right] \textit{ and for a.e. } a \in [0,\bar a]\,, \\
S_k(t) & \rightharpoonup & S(t) & \textit{weakly in } \mathbf{L}^1(I_S;\mathbb{R}) \textit{ for all } t \in \left]T_{j-1},T_j\right]\,, \\
S_k(\cdot,a-) & \rightharpoonup & S(\cdot,a-) & \textit{weakly in } \mathbf{L}^1([T_{j-1},T_j];\mathbb{R}) \textit{ for all } a \in I_S\,, \\
R_k(t) & \rightharpoonup & R(t) & \textit{weakly in } \mathbf{L}^1(I_R;\mathbb{R}) \textit{ for all } t \in \left]T_{j-1},T_j\right]\,, \\
S_k & \rightharpoonup & S & \textit{weakly in } \mathbf{L}^1([T_{j-1},T_j]\times I_S;\mathbb{R})\,, \\
R_k & \rightharpoonup & R & \textit{weakly in } \mathbf{L}^1([T_{j-1},T_j]\times I_R;\mathbb{R})\,.
\end{array} \tag{4.8}$$

Proof. By construction,

$$(J_k, S_k, R_k)(t) = (J, S, R)(t) \quad \text{for all } t \in [0, T_{i-1}] .$$

We now prove by induction on $j = i, i+1, i+2, \ldots$ that, as $k \to +\infty$, the convergences in (4.8) hold. To this aim, fix

$$\begin{array}{rcl}(\varphi_J, \varphi_S, \varphi_R) & \in & \Big(\mathbf{L^1}(I_T \times I_J; \mathbb{R}) \times \mathbf{L^1}(I_T \times I_S; \mathbb{R}) \times \mathbf{L^1}(I_T \times I_R; \mathbb{R})\Big)^* \\ & = & \mathbf{L}^\infty(I_T \times I_J; \mathbb{R}) \times \mathbf{L}^\infty(I_T \times I_S; \mathbb{R}) \times \mathbf{L}^\infty(I_T \times I_R; \mathbb{R}) ; \\ (\Phi_J, \Phi_S, \widetilde{\Phi}_S, \Phi_R) & \in & \Big(\mathbf{L^1}(I_J; \mathbb{R}) \times \mathbf{L^1}(I_S; \mathbb{R}) \times \mathbf{L^1}(I_T; \mathbb{R}) \times \mathbf{L^1}(I_R; \mathbb{R})\Big)^* \\ & = & \mathbf{L}^\infty(I_J; \mathbb{R}) \times \mathbf{L}^\infty(I_S; \mathbb{R}) \times \mathbf{L}^\infty(I_T; \mathbb{R}) \times \mathbf{L}^\infty(I_R; \mathbb{R}) .\end{array}$$

Let $j = i$. If $t \in]T_{i-1}, T_i]$, then by (4.3) and with the notation (4.1),

$$S_k(t,a) = \begin{cases} \frac{\eta_i^k\left(\mathcal{T}_S(\bar a;t,a)\right) g_J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right) J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)} \psi_S\big(\mathcal{T}_S(\bar a;t,a),t,a\big) & a \in [\bar a, \mathcal{A}_S(t;T_{i-1},\bar a)] \\ S(t,a) & a > \mathcal{A}_S(t;T_{i-1},\bar a) \end{cases}$$

$$R_k(t,a) = \begin{cases} \frac{\left[1-\eta_i^k\left(\mathcal{T}_R(\bar a;t,a)\right)\right] g_J\left(\mathcal{T}_R(\bar a;t,a),\bar a\right) J\left(\mathcal{T}_R(\bar a;t,a),\bar a\right)}{g_R\left(\mathcal{T}_R(\bar a;t,a),\bar a\right)} \psi_R\big(\mathcal{T}_R(\bar a;t,a),t,a\big) & a \in [\bar a, \mathcal{A}_R(t;T_{i-1},\bar a)] \\ R(t,a) & a > \mathcal{A}_R(t;T_{i-1},\bar a) \end{cases}$$

and analogous expressions hold for S and R. Therefore, using (4.5),

$$\begin{aligned} & \int_{I_S} \big(S_k(t,a) - S(t,a)\big)\, \Phi_S(a)\, \mathrm{d}a \\ = & \int_{\bar a}^{\mathcal{A}_S(t;T_{i-1},\bar a)} \frac{\Big(\eta_i^k\left(\mathcal{T}_S(\bar a;t,a)\right) - \eta_i\left(\mathcal{T}_S(\bar a;t,a)\right)\Big) g_J\big(\mathcal{T}_S(\bar a;t,a),\bar a\big)\, J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)} \\ & \qquad \times \psi_S\big(\mathcal{T}_S(\bar a;t,a),t,a\big)\, \Phi_S(a)\, \mathrm{d}a \\ = & \int_{T_{i-1}}^{t} \Big(\eta_i^k(\tau) - \eta_i(\tau)\Big)\, g_J(\tau,\bar a)\, J(\tau,\bar a)\, \psi_S\big(\tau,t,\mathcal{A}_S(t;\tau,\bar a)\big)\, \Phi_S\left(\mathcal{A}_S(t;\tau,\bar a)\right) \\ & \qquad \times \exp\left(\int_\tau^t \partial_a g_S\left(s, \mathcal{A}_S(s;\tau,\bar a)\right) \mathrm{d}s\right) \mathrm{d}\tau \\ \to & \ 0 \quad \text{as } k \to +\infty \end{aligned}$$

since the map

$$\tau \to g_J(\tau,\bar a)\ J(\tau,\bar a)\ \psi_S\big(\tau,t,\mathcal{A}_S(t;\tau,\bar a)\big)\ \Phi_S\left(\mathcal{A}_S(t;\tau,\bar a)\right)\ \exp\left(\int_\tau^t \partial_a g_S\left(s,\mathcal{A}_S(s;\tau,\bar a)\right) \mathrm{d}s\right)$$

is in $\mathbf{L}^\infty([T_{i-1},T_i];\mathbb{R}) \subset \mathbf{L^1}([T_{i-1},T_i];\mathbb{R})$, proving that $S_k(t) \rightharpoonup S(t)$ weakly in $\mathbf{L^1}(I_S;\mathbb{R})$. The convergence $R_k(t) \rightharpoonup R(t)$ weakly in $\mathbf{L^1}(I_R;\mathbb{R})$ is proved analogously.

Pass now to

$$\begin{aligned} & \int_{T_{i-1}}^{T_i} \int_{I_S} \big(S_k(t,a) - S(t,a)\big)\, \varphi_S(t,a)\, \mathrm{d}a\, \mathrm{d}t \\ = & \int_{T_{i-1}}^{T_i} \int_{\bar a}^{\mathcal{A}_S(t;T_{i-1},\bar a)} \frac{\Big(\eta_i^k\left(\mathcal{T}_S(\bar a;t,a)\right) - \eta_i\left(\mathcal{T}_S(\bar a;t,a)\right)\Big) g_J\big(\mathcal{T}_S(\bar a;t,a),\bar a\big)\, J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)} \\ & \qquad \times \psi_S\big(\mathcal{T}_S(\bar a;t,a),t,a\big)\, \varphi_S(t,a)\, \mathrm{d}a\, \mathrm{d}t \\ = & \int_{T_{i-1}}^{T_i} \int_{T_{i-1}}^{t} \Big(\eta_i^k(\tau) - \eta_i(\tau)\Big)\, g_J(\tau,\bar a)\, J(\tau,\bar a)\, \psi_S\big(\tau,t,\mathcal{A}_S(t;\tau,\bar a)\big)\, \varphi_S\left(t,\mathcal{A}_S(t;\tau,\bar a)\right) \end{aligned}$$

$$
\begin{aligned}
& \times \exp\left(\int_\tau^t \partial_a g_S\left(s, \mathcal{A}_S(s;\tau,\bar a)\right) \mathrm{d}s\right) \mathrm{d}\tau \, \mathrm{d}t \\
= & \int_{T_{i-1}}^{T_i} \left(\eta_i^k(\tau) - \eta_i(\tau)\right) \int_\tau^{T_i} g_J(\tau,\bar a)\, J(\tau,\bar a)\, \psi_S\big(\tau, t, \mathcal{A}_S(t;\tau,\bar a)\big)\, \varphi_S\left(t, \mathcal{A}_S(t;\tau,\bar a)\right) \\
& \times \exp\left(\int_\tau^t \partial_a g_S\left(s, \mathcal{A}_S(s;\tau,\bar a)\right) \mathrm{d}s\right) \mathrm{d}t \, \mathrm{d}\tau \\
\to & \;\; 0 \quad \text{as } k \to +\infty\,,
\end{aligned}
$$

since the map

$$
\tau \to \int_\tau^{T_i} g_J(\tau,\bar a)\, J(\tau,\bar a)\, \psi_S\big(\tau, t, \mathcal{A}_S(t;\tau,\bar a)\big)\, \varphi_S\left(t, \mathcal{A}_S(t;\tau,\bar a)\right) \exp\left[\int_\tau^t \partial_a g_S\left(s, \mathcal{A}_S(s;\tau,\bar a)\right) \mathrm{d}s\right] \mathrm{d}t
$$

is in $\mathbf{L}^\infty([T_{i-1},T_i];\mathbb{R}) \subset \mathbf{L}^\mathbf{1}([T_{i-1},T_i];\mathbb{R})$, proving that $S_k \rightharpoonup S$ in $\mathbf{L}^\mathbf{1}(I_T \times I_S;\mathbb{R})$. The convergence $R_k \rightharpoonup R$ in $\mathbf{L}^\mathbf{1}(I_T \times I_R;\mathbb{R})$ is proved analogously.

Moreover, by (2.3) and (4.3),

$$
J_k(t,a) = \begin{cases} \frac{\int_{\bar a}^{+\infty} w(\alpha)\, R_k\left(\mathcal{T}_J(0;t,a),\alpha\right)\mathrm{d}\alpha}{g_J\left(\mathcal{T}_J(0;t,a),0\right)}\, \psi_J\big(\mathcal{T}_J(0;t,a),t,0\big) & a \in [0, \mathcal{A}_J(t;T_{i-1},0)] \\ J(t,a) & a \in [\mathcal{A}_J(t;T_{i-1},0), \bar a] \end{cases}
$$

so that, by the weak convergence $R_k(t) \rightharpoonup R(t) \in \mathbf{L}^\mathbf{1}(I_R;\mathbb{R})$, as $k \to +\infty$,

$$
J_k(t,a) \to J(t,a) \qquad \text{for all } t \in \left]T_{i-1},T_i\right] \text{ and for a.e. } a \in [0,\bar a]\,.
$$

Let $j > i$. If $t \in \left]T_{j-1},T_j\right]$,

$$
\begin{aligned}
S_k(t,a) &= \begin{cases} \frac{\bar\eta\left(\mathcal{T}_S(\bar a;t,a)\right) g_J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right) J_k\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)} \psi_S\big(\mathcal{T}_S(\bar a;t,a),t,a\big) & a \in [\bar a, \mathcal{A}_S(t;T_{j-1},\bar a)] \\ S_k\left(T_{j-1}, \mathcal{A}_S(T_{j-1};t,a)\right)\ \psi_S(T_{j-1},t,a) & a > \mathcal{A}_S(t;T_{j-1},\bar a) \end{cases} \\
R_k(t,a) &= \begin{cases} \frac{\left[1-\bar\eta\left(\mathcal{T}_R(\bar a;t,a)\right)\right] g_J\left(\mathcal{T}_R(\bar a;t,a),\bar a\right) J_k\left(\mathcal{T}_R(\bar a;t,a),\bar a\right)}{g_R\left(\mathcal{T}_R(\bar a;t,a),\bar a\right)} \psi_R\big(\mathcal{T}_R(\bar a;t,a),t,a\big) & a \in [\bar a, \mathcal{A}_R(t;T_{i-1},\bar a)] \\ R_k\left(T_{j-1}, \mathcal{A}_R(T_{j-1};t,a)\right)\ \psi_R(T_{j-1},t,a) & a > \mathcal{A}_R(t;T_{j-1},\bar a) \end{cases}
\end{aligned}
$$

Similarly to above, using (4.3) and (4.6)

$$
\begin{aligned}
& \int_{I_S} \big(S_k(t,a) - S(t,a)\big)\, \Phi_S(a)\, \mathrm{d}a \\
= & \int_{\bar a}^{\mathcal{A}_S(t;T_{j-1},\bar a)} \frac{\bar\eta\left(\mathcal{T}_S(\bar a;t,a)\right) g_J\big(\mathcal{T}_S(\bar a;t,a),\bar a\big) \Big(J_k\left(\mathcal{T}_S(\bar a;t,a),\bar a\right) - J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\Big)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)} \\
& \qquad \times \psi_S\big(\mathcal{T}_S(\bar a;t,a),t,a\big)\ \Phi_S(a)\, \mathrm{d}a \\
& + \int_{\mathcal{A}_S(t;T_{j-1},\bar a)}^{+\infty} \Big(S_k\left(T_{j-1}, \mathcal{A}_S(T_{j-1};t,a)\right) - S\left(T_{j-1}, \mathcal{A}_S(T_{j-1};t,a)\right)\Big)\, \psi_S(T_{j-1},t,a)\, \Phi_S(a)\, \mathrm{d}a \\
= & \int_{\bar a}^{\mathcal{A}_S(t;T_{j-1},\bar a)} \frac{\bar\eta\left(\mathcal{T}_S(\bar a;t,a)\right) g_J\big(\mathcal{T}_S(\bar a;t,a),\bar a\big)\ (J_k - J)\Big(T_{j-1}, \mathcal{A}_J\left(T_{j-1};\mathcal{T}_S(\bar a;t,a),\bar a\right)\Big)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)} \\
& \qquad \times \psi_J\left(T_{j-1}, \mathcal{T}_S(\bar a;t,a),\bar a\right) \psi_S\big(\mathcal{T}_S(\bar a;t,a),t,a\big)\ \Phi_S(a)\, \mathrm{d}a \\
& + \int_{\bar a}^{\mathcal{A}_S(t;T_{i-1},\bar a)} \Big(S_k\left(T_{j-1},\alpha\right) - S\left(T_{j-1},\alpha\right)\Big)\, \psi_S\left(T_{j-1},t,\mathcal{A}_S(t;T_{j-1},\alpha)\right) \\
& \qquad \times \exp\left(\int_{T_{j-1}}^t \partial_a g_S\left(s, \mathcal{A}_S(s;T_{j-1},\alpha)\right) \mathrm{d}s\right)\, \Phi_S\left(\mathcal{A}_S(t;T_{j-1},\alpha)\right) \mathrm{d}\alpha
\end{aligned}
$$

$$\to \quad 0 \quad \text{as } k \to +\infty\,,$$

since by the inductive hypothesis $J_k(T_{j-1}) \to J(T_{j-1})$ a.e. in $[0, \bar a]$ and the map

$$\alpha \to \psi_S\ \left(T_{j-1}, t, \mathcal{A}_S(t; T_{j-1}, \alpha)\right) \exp\left(\int_{T_{j-1}}^t \partial_a g_S\left(s, \mathcal{A}_S(s; T_{j-1}, \alpha)\right) \mathrm{d}s\right) \Phi_S\big(\mathcal{A}_S(t; T_{j-1}, \alpha)\big)$$

is in $\mathbf{L}^\infty(I_S; \mathbb{R})$. The convergence $R_k(t) \rightharpoonup R(t)$ weakly in $\mathbf{L^1}(I_R; \mathbb{R})$ is proved analogously.

Repeating computations similar to the latter ones, we have:

$$\begin{aligned}
&\int_{T_{j-1}}^{T_j} \int_{I_S} \big(S_k(t,a) - S(t,a)\big)\, \varphi_S(t,a)\, \mathrm{d}a\, \mathrm{d}t \\
= &\int_{T_{j-1}}^{T_j} \int_{\bar a}^{\mathcal{A}_S(t;T_{j-1},\bar a)} \frac{\bar\eta\left(\mathcal{T}_S(\bar a; t, a)\right) g_J\big(\mathcal{T}_S(\bar a; t,a), \bar a\big) \Big(J_k\left(\mathcal{T}_S(\bar a;t,a), \bar a\right) - J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\Big)}{g_S\left(\mathcal{T}_S(\bar a; t, a), \bar a\right)} \\
&\qquad \times \psi_S\big(\mathcal{T}_S(\bar a;t,a), t, a\big)\ \varphi_S(t,a)\, \mathrm{d}a\, \mathrm{d}t \\
&+ \int_{T_{j-1}}^{T_j} \int_{\mathcal{A}_S(t;T_{j-1},\bar a)}^{+\infty} \Big(S_k\left(T_{j-1}, \mathcal{A}_S(T_{j-1};t,a)\right) - S\left(T_{j-1}, \mathcal{A}_S(T_{j-1};t,a)\right)\Big) \\
&\qquad \times \psi_S(T_{j-1}, t, a)\, \varphi_S(t,a)\, \mathrm{d}a\, \mathrm{d}t \\
= &\int_{T_{j-1}}^{T_j} \int_{\bar a}^{\mathcal{A}_S(t;T_{j-1},\bar a)} \frac{\bar\eta\left(\mathcal{T}_S(\bar a; t, a)\right) g_J\big(\mathcal{T}_S(\bar a; t,a), \bar a\big)\ (J_k - J)\Big(T_{j-1}, \mathcal{A}_J\left(T_{j-1}; \mathcal{T}_S(\bar a;t,a), \bar a\right)\Big)}{g_S\left(\mathcal{T}_S(\bar a; t, a), \bar a\right)} \\
&\qquad \times \psi_J\left(T_{j-1}, \mathcal{T}_S(\bar a;t,a), \bar a\right) \psi_S\big(\mathcal{T}_S(\bar a;t,a), t, a\big)\ \varphi_S(t,a)\, \mathrm{d}a\, \mathrm{d}t \\
&+ \int_{T_{j-1}}^{T_j} \int_{\bar a}^{\mathcal{A}_S(t;T_{i-1},\bar a)} \Big(S_k\left(T_{j-1}, \alpha\right) - S\left(T_{j-1},\alpha\right)\Big) \psi_S\left(T_{j-1}, t, \mathcal{A}_S(t; T_{j-1}, \alpha)\right) \\
&\qquad \times \exp\left(\int_{T_{j-1}}^t \partial_a g_S\left(s, \mathcal{A}_S(s;T_{j-1},\alpha)\right) \mathrm{d}s\right) \varphi_S\left(t, \mathcal{A}_S(t;T_{j-1},\alpha)\right) \mathrm{d}\alpha\, \mathrm{d}t \\
\to &\quad 0 \quad \text{as } k \to +\infty\,,
\end{aligned}$$

by the inductive hypothesis. Indeed, the convergence $J_k(T_{j-1}) \to J(T_{j-1})$ a.e. in $[0, \bar a]$ ensures that the former integral above vanishes by the Dominated Convergence Theorem. The weak convergence $S_k(T_{j-1}) \rightharpoonup S(T_{j-1})$ in $\mathbf{L^1}(I_S; \mathbb{R})$ and the fact that the map

$$\alpha \to \psi_S\ \left(T_{j-1}, t, \mathcal{A}_S(t; T_{j-1}, \alpha)\right) \exp\left(\int_{T_{j-1}}^t \partial_a g_S\left(s, \mathcal{A}_S(s; T_{j-1}, \alpha)\right) \mathrm{d}s\right) \varphi_S\big(t, \mathcal{A}_S(t; T_{j-1}, \alpha)\big)$$

is in $\mathbf{L}^\infty(I_S; \mathbb{R})$, proves that also the latter integral above vanishes, again by the Dominated Convergence Theorem. This shows that $S_k \rightharpoonup S$ weakly in $\mathbf{L^1}(\big]T_{j-1}, T_j\big] \times I_S; \mathbb{R})$. The convergence $R_k \rightharpoonup R$ weakly in $\mathbf{L^1}(\big]T_{j-1}, T_j\big] \times I_R; \mathbb{R})$ is proved analogously.

Moreover, by (2.3) and (4.3),

$$J_k(t,a) = \begin{cases} \frac{\int_{\bar a}^{+\infty} w(\alpha)\ R_k\left(\mathcal{T}_J(0;t,a), \alpha\right) \mathrm{d}\alpha}{g_J\left(\mathcal{T}_J(0;t,a), 0\right)}\ \psi_J\big(\mathcal{T}_J(0;t,a), t, 0\big) & a \in [0, \mathcal{A}_J(t; T_{j-1}, 0)] \\ J_k\left(T_{j-1}, \mathcal{A}_J(T_{j-1}; t, a)\right) \psi_J(T_{j-1}, t, a) & a \in [\mathcal{A}_J(t;T_{j-1},0), \bar a]\,. \end{cases}$$

The inductive hypothesis ensures the weak convergence $R_k(t) \rightharpoonup R(t) \in \mathbf{L^1}(I_R; \mathbb{R})$ for $t \le T_{j-1}$ and the a.e. convergence $J_k(T_{j-1}) \to J(T_{j-1})$. Hence,

$$J_k(t,a) \to J(t,a) \qquad \text{for all } t \in \left]T_{i-1}, T_i\right] \text{ and for a.e. } a \in [0, \bar a]\,.$$

Finally fix $a \in I_S$ and define the set of indices

$$\mathcal{J} = \left\{ j \in \{1, \ldots, N\} : \bar a_j < a \right\}.$$

Using again (4.5), we have

$$\begin{aligned}
&\int_{I_T} \big(S_k(t,a-) - S(t,a-)\big)\, \widetilde{\Phi}_S(t)\,\mathrm{d}t \\
= &\int_{\min\left\{T,\mathcal{T}_S(a;0,\bar a)\right\}}^{T} \big(S_k(t,a-) - S(t,a-)\big)\, \widetilde{\Phi}_S(t)\,\mathrm{d}t \\
= &\int_{\min\left\{T,\mathcal{T}_S(a;0,\bar a)\right\}}^{T} \Big[\eta^k\left(\mathcal{T}_S(\bar a;t,a)\right) J_k\left(\mathcal{T}_S(\bar a;t,a),\bar a\right) - \eta\left(\mathcal{T}_S(\bar a;t,a)\right) J\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)\Big] \\
&\qquad \times \frac{g_J\big(\mathcal{T}_S(\bar a;t,a),\bar a\big)}{g_S\left(\mathcal{T}_S(\bar a;t,a),\bar a\right)} \prod_{j\in\mathcal{J}} \Big(1-\vartheta_j\left(\mathcal{T}_S\left(\bar a_j;t,a\right)\right)\Big)\, \psi_S\big(\mathcal{T}_S(\bar a;t,a),t,a\big)\,\widetilde{\Phi}_S(t)\,\mathrm{d}t \\
= &\int_0^{\max\left\{0,\mathcal{T}_S(\bar a;T,a)\right\}} \Big[\eta^k(\tau)J_k(\tau,\bar a) - \eta(\tau)J(\tau,\bar a)\Big] \frac{g_J(\tau,\bar a)}{g_S\left(\mathcal{T}_S(a;\tau,\bar a),a\right)} \psi_S\big(\tau,\mathcal{T}_S\left(a;\tau,\bar a\right),a\big) \\
&\times \widetilde{\Phi}_S\left(\mathcal{T}_S\left(a;\tau,\bar a\right)\right) \prod_{j\in\mathcal{J}} \Big(1-\vartheta_j\left(\mathcal{T}_S\left(\bar a_j;\tau,\bar a\right)\right)\Big) \exp\left(\int_\tau^{\mathcal{T}_S(a;\tau,\bar a)} \partial_a g_S\left(s,\mathcal{A}_S(s;\tau,\bar a)\right)\mathrm{d}s\right)\mathrm{d}\tau \\
\to\ & 0 \quad \text{as } k\to+\infty
\end{aligned}$$

since, $J_k(\tau,\bar a)\to J(\tau,\bar a)$ pointwise and consequently strongly in $\mathbf{L^1}\left(\left[0,\max\left\{0,\mathcal{T}_S\left(\bar a;T,a\right)\right\}\right];\mathbb{R}\right)$ by the Dominated Convergence Theorem, $\eta^k \rightharpoonup \eta$ in $\mathbf{L^1}\left(\left[0,\max\left\{0,\mathcal{T}_S\left(\bar a;T,a\right)\right\}\right];\mathbb{R}\right)$ and the map

$$\begin{aligned}
\tau \longrightarrow &\frac{g_J(\tau,\bar a)\,\psi_S\big(\tau,t,\mathcal{A}_S(t;\tau,\bar a)\big)\,\Phi_S\left(\mathcal{A}_S(t;\tau,\bar a)\right)}{g_S\left(\mathcal{T}_S(a;\tau,\bar a),a\right)} \exp\left(\int_\tau^t \partial_a g_S\left(s,\mathcal{A}_S(s;\tau,\bar a)\right)\mathrm{d}s\right) \\
&\times \prod_{j\in\mathcal{J}} \Big(1-\vartheta_j\left(\mathcal{T}_S\left(\bar a_j;\tau,\bar a\right)\right)\Big)
\end{aligned}$$

is in $\mathbf{L}^\infty\left(\left[0,\max\left\{0,\mathcal{T}_S\left(\bar a;T,a\right)\right\}\right];\mathbb{R}\right) \subset \mathbf{L^1}\left(\left[0,\max\left\{0,\mathcal{T}_S\left(\bar a;T,a\right)\right\}\right];\mathbb{R}\right)$, proving that the sequence $S_k(\cdot,a-)$ weakly converges to $S(\cdot,a-)$ in $\mathbf{L^1}(I_T;\mathbb{R})$. This completes the proof. □

Proof of Proposition 2.6. The weak sequential continuity of S^i_η follows from Lemma 4.3.

To prove the weak sequential continuity of $\mathcal{S}^\iota_\vartheta$ with respect to ϑ_ι, using the notation above, simply note that

$$S_k(t,a) = \begin{cases} \Big(1-\vartheta^k_\iota\left(\mathcal{T}_S(\bar a_\iota;t,a)\right)\Big) S\left(\mathcal{T}_S(\bar a_\iota;t,a),\bar a_\iota-\right) & a\in[\bar a_\iota,\mathcal{A}_S(t;0,\bar a_\iota)] \\ S(t,a) & \text{otherwise} \end{cases}$$

while clearly $J_k \equiv J$ and $R_k \equiv R$. Therefore,

$$S_k(t,a) - S(t,a) = \begin{cases} \Big[\vartheta^k_\iota\left(\mathcal{T}_S(\bar a_\iota;t,a)\right) - \vartheta_\iota\left(\mathcal{T}_S(\bar a_\iota;t,a)\right)\Big] S\left(\mathcal{T}_S(\bar a_\iota;t,a),\bar a_\iota-\right) & a\in[\bar a_\iota,\mathcal{A}_S(t;0,\bar a_\iota)] \\ 0 & \text{otherwise} \end{cases}$$

and the weak sequential continuity immediately follows. □

Proof of Proposition 2.7. For $m\in\mathbb{N}\setminus\{0\}$, partition I_T through the points $\frac{k}{m}T$, for $k = 0,1,2,\ldots,m$. Define

$$\tau^k_m = \frac{k}{m}T + \int_{\frac{k}{m}T}^{\frac{k+1}{m}T} \lambda(\tau)\,\mathrm{d}\tau \qquad \text{and} \qquad \lambda_m = \sum_{k=0}^{m-1} \chi_{\left[\frac{k}{m}T,\tau^k_m\right[} \tag{4.9}$$

so that $\tau_m^k \in [\frac{k}{m}T, \frac{k+1}{m}T]$, $\lambda_m(I_T) \subseteq \{0,1\}$ and

$$\int_{\frac{k}{m}T}^{\frac{k+1}{m}T} \lambda_m(\tau)\,\mathrm{d}\tau = \int_{\frac{k}{m}T}^{\frac{k+1}{m}T} \lambda(\tau)\,\mathrm{d}\tau \qquad \text{for } k = 0,\ldots,m-1\,. \tag{4.10}$$

We now check that for all $\varphi \in \mathbf{L^1}(I_T;\mathbb{R})$, $\lim_{m\to+\infty} \int_{I_T} \big(\lambda_m(\tau) - \lambda(\tau)\big)\,\varphi(\tau)\,\mathrm{d}\tau = 0$.

Assume first that $\varphi \in \mathbf{C^0_c}(I_T;\mathbb{R})$. Fix $\varepsilon > 0$. By Heine–Cantor Theorem, φ is uniformly continuous and there exists an $m \in \mathbb{N}$ such that $\left|\varphi(t') - \varphi(t'')\right| < \varepsilon$ whenever $\tau', \tau'' \in I_T$ and $\left|\tau' - \tau''\right| < 1/m$. Then, using (4.9) and (4.10),

$$\begin{aligned}
& \int_{I_T} \big(\lambda_m(\tau) - \lambda(\tau)\big)\,\varphi(\tau)\,\mathrm{d}\tau \\
= & \sum_{k=0}^{m-1} \left(\int_{\frac{k}{m}T}^{\tau_m^k} \big(1-\lambda(\tau)\big)\,\varphi(\tau)\,\mathrm{d}\tau - \int_{\tau_m^k}^{\frac{k+1}{m}T} \lambda(\tau)\varphi(\tau)\,\mathrm{d}\tau \right) \\
\le & \sum_{k=0}^{m-1} \left(\left(\max_{[\frac{k}{m}T, \frac{k+1}{m}T]} \varphi\right) \int_{\frac{k}{m}T}^{\tau_m^k} \big(1-\lambda(\tau)\big)\,\mathrm{d}\tau \; - \left(\min_{[\frac{k}{m}T, \frac{k+1}{m}T]} \varphi\right) \int_{\tau_m^k}^{\frac{k+1}{m}T} \lambda(\tau)\,\mathrm{d}\tau \right) \\
= & \sum_{k=0}^{m-1} \left(\max_{[\frac{k}{m}T, \frac{k+1}{m}T]} \varphi - \min_{[\frac{k}{m}T, \frac{k+1}{m}T]} \varphi \right) \int_{\frac{k}{m}T}^{\tau_m^k} \big(1-\lambda(\tau)\big)\,\mathrm{d}\tau \\
\le & \ \varepsilon \int_{I_T} \big(1-\lambda(\tau)\big)\,\mathrm{d}\tau \\
\le & \ \varepsilon\, T\,.
\end{aligned}$$

An entirely analogous procedure leads to

$$\begin{aligned}
& \int_{I_T} \big(\lambda_m(\tau) - \lambda(\tau)\big)\,\varphi(\tau)\,\mathrm{d}\tau \\
\ge & \sum_{k=0}^{m-1} \left(\left(\min_{[\frac{k}{m}T, \frac{k+1}{m}T]} \varphi\right) \int_{\frac{k}{m}T}^{\tau_m^k} \big(1-\lambda(\tau)\big)\,\mathrm{d}\tau \; - \left(\max_{[\frac{k}{m}T, \frac{k+1}{m}T]} \varphi\right) \int_{\tau_m^k}^{\frac{k+1}{m}T} \lambda(\tau)\,\mathrm{d}\tau \right) \\
\ge & \ -\varepsilon\, T\,,
\end{aligned}$$

completing the case of a continuous φ.

Let now $\varphi \in \mathbf{L^1}(I_T;\mathbb{R})$ and fix $\varepsilon > 0$. Then, by [8, Theorem 4.12] there exists a map $\varphi_\varepsilon \in \mathbf{C^0_c}(I_T;\mathbb{R})$ such that $\|\varphi - \varphi_\varepsilon\|_{\mathbf{L^1}(I_T;\mathbb{R})} < \varepsilon$. Then, for a sufficiently large m, the computations above ensure that

$$\begin{aligned}
& \left| \int_{I_T} \big(\lambda_m(\tau) - \lambda(\tau)\big)\ \varphi(\tau)\,\mathrm{d}\tau \right| \\
\le & \left| \int_{I_T} \big(\lambda_m(\tau) - \lambda(\tau)\big)\ \varphi_\varepsilon(\tau)\,\mathrm{d}\tau \right| + \int_{I_T} \left|\lambda_m(\tau) - \lambda(\tau)\right| \left|\varphi(\tau) - \varphi_\varepsilon(\tau)\right|\,\mathrm{d}\tau \\
\le & \ 2\,\varepsilon\, T
\end{aligned}$$

completing the proof. $\square$

Proof of Theorem 2.8. Assume for simplicity that $T = T_n$ for some $n \in \mathbb{N}$. Fix $\bar\eta \in \mathbf{BV}(I_T;[0,1])$, $\bar\vartheta \in \mathbf{BV}(I_T;[0,1]^N)$, and, recalling the notation (2.9), define, for $i \in \{1,\ldots,n\}$ and for $\iota \in \{1,\ldots,N\}$, the maps

$$\begin{array}{rcrcl}
\mathcal{P}^i_\eta & : & \mathbf{BV}([T_{i-1},T_i];[0,1]) & \to & \mathbb{R} \\
 & & \eta_i & \to & \mathcal{P}(\widehat{\eta}_i, \bar\vartheta)
\end{array}
\qquad
\begin{array}{rcrcl}
\mathcal{P}^\iota_\vartheta & : & \mathbf{BV}(I_T;[0,1]) & \to & \mathbb{R} \\
 & & \vartheta_\iota & \to & \mathcal{P}(\bar\eta, \widehat{\vartheta}_\iota)\,.
\end{array}$$

It is sufficient to prove that they are sequentially lower semicontinuous with respect to the weak$*$ topology on $\mathbf{L}^\infty([T_{i-1},T_i];\mathbb{R})$ and on $\mathbf{L}^\infty(I_T;\mathbb{R})$.

1. **Lower semicontinuity of $\mathcal{P}^i_\eta$.** Consider a sequence $\eta^k_i \overset{*}{\rightharpoonup} \bar\eta$ in $\mathbf{L}^\infty\left([T_{i-1},T_i];\mathbb{R}\right)$. By Proposition 2.6 and by Lemma 4.3, we have that as $k\to+\infty$,

$$\begin{array}{lclllcll} J_k & \rightharpoonup & J & \mathbf{L^1}(I_T\times I_J;\mathbb{R}) & J_k(T,a) & \to & J(T,a) & \text{pointwise} \\ S_k & \rightharpoonup & S & \mathbf{L^1}(I_T\times I_S;\mathbb{R}) & S_k(\cdot,\bar a_j-) & \rightharpoonup & S(\cdot,a-) & \mathbf{L^1}(I_T;\mathbb{R}),\ a\in I_S\,, \\ R_k & \rightharpoonup & R & \mathbf{L^1}(I_T\times I_R;\mathbb{R}) & & & & \end{array}$$

where $(J_k,S_k,R_k)=\mathcal{S}(\hat\eta^k_i,\bar\vartheta)$ and $(J,S,R)=\mathcal{S}\left(\bar\eta,\bar\vartheta\right)$. Therefore, by **(C)** and [12, Example 1.23], we have that

$$\liminf_{k\to+\infty}\sum_{u=J,S,R}-\int_{I_T}\int_{I_u}C_u\left(t,a,u_k(t,a)\right)\mathrm{d}a\,\mathrm{d}t\geq\sum_{u=J,S,R}-\int_{I_T}\int_{I_u}C_u\left(t,a,u(t,a)\right)\mathrm{d}a\,\mathrm{d}t$$

and, by **(P)**,

$$\lim_{k\to+\infty}\sum_{j=1}^N\int_{I_T}P_j(t)\bar\vartheta_j(t)S_k(t,\bar a_j-)\,\mathrm{d}t=\sum_{j=1}^N\int_{I_T}P_j(t)\bar\vartheta_j(t)S(t,\bar a_j-)\,\mathrm{d}t$$

and, by **($\boldsymbol{\varphi}$)**,

$$\lim_{k\to+\infty}\varphi\left(\int_{I_J}\left[J_o(a)-J_k(T,a)\right]\mathrm{d}a\right)=\varphi\left(\int_{I_J}\left[J_o(a)-J(T,a)\right]\mathrm{d}a\right)$$

proving that $\mathcal{P}^i_\eta$ is sequentially lower semicontinuous.

2. **Lower semicontinuity of $\mathcal{P}^\iota_\vartheta$.** For $\iota\in\{1,\ldots,N\}$, consider a sequence $\vartheta^k_\iota \overset{*}{\rightharpoonup} \bar\vartheta_\iota$ in $\mathbf{L}^\infty(I_T;\mathbb{R})$. Denoting with $(J_k,S_k,R_k)=\mathcal{S}(\hat\eta^k_i,\bar\vartheta)$ and $(J,S,R)=\mathcal{S}\left(\bar\eta,\bar\vartheta\right)$, we have that $J_k=J$, $R_k=R$, and $S_k(t,a)=S(t,a)$ for every $t\in I_T$ and every $a\in I_S$ with $a<\bar a_\iota$. Hence,

$$\begin{aligned}\sum_{u=J,R}\int_{I_T}\int_{I_u}C_u\left(t,a,u_k(t,a)\right)\mathrm{d}a\,\mathrm{d}t &= \sum_{u=J,R}\int_{I_T}\int_{I_u}C_u\left(t,a,u(t,a)\right)\mathrm{d}a\,\mathrm{d}t\\ \sum_{j=1}^{\iota-1}\int_{I_T}P_j(t)\bar\vartheta_j(t)S_k(t,\bar a_j-)\,\mathrm{d}t &= \sum_{j=1}^{\iota-1}\int_{I_T}P_j(t)\bar\vartheta_j(t)S(t,\bar a_j-)\,\mathrm{d}t\\ \varphi\left(\int_{I_J}\left[J_o(a)-J_k(T,a)\right]\mathrm{d}a\right) &= \varphi\left(\int_{I_J}\left[J_o(a)-J(T,a)\right]\mathrm{d}a\right)\end{aligned}$$

for every $k\in\mathbb{N}$. Moreover, by **(P)**,

$$\begin{aligned}\lim_{k\to+\infty}\int_{I_T}P_\iota(t)\bar\vartheta^k_\iota(t)S_k(t,\bar a_\iota-)\,\mathrm{d}t &= \lim_{k\to+\infty}\int_{I_T}P_\iota(t)\bar\vartheta^k_\iota(t)S(t,\bar a_\iota-)\,\mathrm{d}t\\ &= \int_{I_T}P_\iota(t)\bar\vartheta_\iota(t)S(t,\bar a_\iota-)\,\mathrm{d}t\,.\end{aligned}$$

By Proposition 2.6, we deduce that $S_k\rightharpoonup S$ in $\mathbf{L}^\infty(I_T\times I_S;\mathbb{R})$ as $k\to+\infty$, and this convergence implies, by **(C)** and by [12, Example 1.23], that

$$\liminf_{k\to+\infty}-\int_{I_T}\int_{I_S}C_S\left(t,a,S_k(t,a)\right)\mathrm{d}a\,\mathrm{d}t\geq-\int_{I_T}\int_{I_S}C_S\left(t,a,S(t,a)\right)\mathrm{d}a\,\mathrm{d}t\,.$$

Finally, noting as in the proof of Proposition 2.6, that

$$S_k(t,a) = \begin{cases} \left(1 - \vartheta_\iota^k \left(\mathcal{T}_S(\bar a_\iota; t, a)\right)\right) S\left(\mathcal{T}_S(\bar a_\iota; t, a), \bar a_\iota -\right) & a \in [\bar a_\iota, \mathcal{A}_S(t; 0, \bar a_\iota)] \\ S(t,a) & \text{otherwise,} \end{cases}$$

we have that $S_k(\cdot, \bar a_j) \rightharpoonup S_k(\cdot, \bar a_j)$ in $\mathbf{L}^\infty(I_T; \mathbb{R})$ for every $j \in \{\iota + 1, \ldots, N\}$ and so, by **(P)**,

$$\lim_{k \to +\infty} \sum_{j=\iota+1}^{N} \int_{I_T} P_j(t) \bar\vartheta_j(t) S_k(t, \bar a_j -)\, \mathrm{d}t = \sum_{j=\iota+1}^{N} \int_{I_T} P_j(t) \bar\vartheta_j(t) S(t, \bar a_j -)\, \mathrm{d}t\ ,$$

proving that $\mathcal{P}_\vartheta^\iota$ is sequentially lower semicontinuous.

This completes the proof. □

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