

ON STOCHASTIC CONSERVATION LAWS AND MALLIAVIN CALCULUS

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ABSTRACT. For stochastic conservation laws driven by a semilinear noise term, we propose a generalization of the Kružkov entropy condition by allowing the Kružkov constants to be Malliavin differentiable random variables. Existence and uniqueness results are provided. Our approach sheds some new light on the stochastic entropy conditions put forth by Feng and Nualart [17] and Bauzet, Vallet, and Wittbold [3], and in our view simplifies some of the proofs.

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1. INTRODUCTION

Stochastic partial differential equations (stochastic PDEs) arise in many fields, such as biology, physics, engineering, and economics, in which random phenomena play a crucial role. Complex systems always contain some element of uncertainty. Uncertainty may arise in the system parameters, initial and boundary conditions, and external forcing processes. Moreover, in many situations there is incomplete or partial understanding of the governing physical laws, and many models are therefore best formulated using stochastic PDEs.

Recently there has been an interest in studying the effect of stochastic forcing on nonlinear conservation laws [3, 4, 9, 14, 21, 20, 17, 36, 35], with particular emphasis on existence and uniqueness questions (well-posedness). Deterministic conservation laws possess discontinuous (shock) solutions, and a weak formulation coupled with

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an appropriate entropy condition is required to ensure the well-posedness [23, 26]. The question of well-posedness gets somewhat more difficult by adding a stochastic source term, due to the interaction between noise and nonlinearity.

In a different direction, we also mention the recent works [24, 25] by Lions, Perthame, and Souganidis on conservation laws with (rough) stochastic fluxes.

To be more precise, we are interested in stochastic conservation laws driven by Gaussian noise of the following form:

$$(1.1) \quad \begin{cases} du(t, x) + \nabla \cdot f(u(t, x))dt = \int_Z \sigma(x, u(t, x), z)W(dt, dz), & (t, x) \in \Pi_T, \\ u(0, x) = u^0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $\Pi_T = (0, T) \times \mathbb{R}^d$, $T > 0$ is some fixed final time, $u^0 = u^0(x, \omega)$ is a given \mathcal{F}_0 -measurable random function, and the unknown $u = u(t, x, \omega)$ is a random (scalar) function. The flux function

$$(\mathcal{A}_f) \quad f : \mathbb{R} \rightarrow \mathbb{R}^d \text{ is assumed to be Lipschitz and } C^1(\mathbb{R}).$$

Concerning the source term, we let $Z \subseteq \mathbb{R}^n$ for some $n \geq 1$ and W be a space-time Gaussian white noise martingale random measure with respect to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ [39]. Its covariance measure is given by $dt \otimes d\mu$, where dt denotes Lebesgue measure on $[0, T]$ and μ is some deterministic σ -finite Borel measure on Z , that is, for measurable sets $A, B \subset Z$

$$E[W(t, A)W(t, B)] = t\mu(A \cap B).$$

The noise coefficient $\sigma : \mathbb{R}^d \times \mathbb{R} \times Z \rightarrow \mathbb{R}$ is a measurable function satisfying

$$(\mathcal{A}_\sigma) \quad \exists M \in L^2(Z) \text{ s.t. } \begin{cases} |\sigma(x, u, z) - \sigma(x, v, z)| \leq |u - v| M(z), \\ |\sigma(x, u, z)| \leq M(z)(1 + |u|), \end{cases}$$

for all $(x, z) \in \mathbb{R}^d \times Z$. Note that W induces a cylindrical Wiener process (with identity covariance operator) on $L^2(Z) = L^2(Z, \mathcal{B}(Z), \mu)$ which we also denote by W [30, § 7.1.2]. Let $G(u) : L^2(Z) \rightarrow L^2(\mathbb{R}^d, \phi)$ be defined by

$$(1.2) \quad G(u)h(x) = \int_Z \sigma(x, u(x), z)h(z) d\mu(z).$$

By (\mathcal{A}_σ) , G is a Lipschitz map from $L^2(\mathbb{R}^d, \phi)$ into $\mathcal{L}_2(L^2(Z), L^2(\mathbb{R}^d, \phi))$ with $\|G\|_{\text{Lip}} \leq \|M\|_{L^2(Z)}$, where $\mathcal{L}_2(L^2(Z), L^2(\mathbb{R}^d, \phi))$ denotes the space of Hilbert-Schmidt operators from $L^2(Z)$ into $L^2(\mathbb{R}^d, \phi)$, cf. Section 2. In this setting, (1.1) may be written as

$$du + \nabla \cdot f(u)dt = G(u)dW(t),$$

where the right-hand side is interpreted with respect to the cylindrical Wiener process [14]. In what follows, we will in general stick to the σ notation. We refer to [13] for a comparison of the stochastic integrals. The Malliavin calculus used later is developed with respect to the isonormal Gaussian process $W : H \rightarrow L^2(\Omega)$ defined by

$$(1.3) \quad W(h) = \int_0^T \int_Z h(s, z)W(ds, dz) = \int_0^T h(s)dW(s),$$

where H denotes the space $L^2([0, T] \times Z, \mathcal{B}([0, T]) \otimes \mathcal{B}(Z), dt \otimes d\mu)$. Concerning the notation and basic theory of Malliavin calculus we refer to [27].

When the noise term in (1.1) is additive (σ is independent of u), Kim [21] used Kružkov's entropy condition and proved the well-posedness of entropy solutions, see also Vallet and Wittbold [36]. When the noise term is additive, a change of variable turns (1.1) into a conservation law with random flux function and well-known "deterministic" techniques apply.

When the noise is multiplicative (i.e., σ depends on u), a simple adaptation of Kruřkov’s techniques fails to capture a specific “noise-noise” interaction term correlating two entropy solutions, and as a consequence they do not lead to the L^1 -contraction principle. This issue was resolved by Feng and Nualart [17], introducing an additional condition capturing the missing noise-noise interaction. These authors employ the Kruřkov entropy condition (on Itô form)

$$(1.4) \quad \begin{aligned} & \partial_t |u - c| + \partial_x [\text{sign}(u - c)(f(u) - f(c))] \\ & \leq \frac{1}{2} \text{sign}'(u - c) \sigma(u)^2 + \sigma(u) \partial_t W, \quad \forall c \in \mathbb{R}, \end{aligned}$$

which is understood in the distributional sense (and via an approximation of $\text{sign}(\cdot)$). Here, for the sake of simplicity, we take W to be an ordinary Brownian motion (Z is a point) and $d = 1$. The above family of inequalities, indexed over the “Kruřkov” constants c , is in [17] augmented with an additional condition related to certain substitution formulas [27, § 3.2.4], which enables the authors to recover the above mentioned interaction term and thus provide, for the first time, a general uniqueness result for stochastic conservation laws.

The additional condition proposed in [17] is rather technical and difficult to comprehend at first glance. Furthermore, the existence proof (passing to the limit in a sequence vanishing viscosity approximations) becomes increasingly difficult, with several added arguments revolving around fractional Sobolev spaces, estimates of the moments of increments, modulus of continuity of Itô processes, and the Garcia-Rodemich-Rumsey lemma.

Recently, Bauzet, Vallet, and Wittbold [3] provided a framework that uses the Kruřkov entropy inequalities (1.4) but bypasses the Feng-Nualart condition. Rather than comparing two entropy solutions directly, their uniqueness result compares the entropy solution against the vanishing viscosity solution, which is generated as the weak limit (as captured by the Young measure) of a sequence of solutions to stochastic parabolic equations with vanishing viscosity parameter. Although with this approach the existence proof becomes simple, many technical difficulties are added to the uniqueness proof. At this point, let us mention that Debussche and Vovelle [14] have provided an alternative well-posedness theory based on a kinetic formulation. The kinetic formulation avoids some of the difficulties alluded to above, thanks to the so-called entropy defect measure.

The purpose of our work is to propose a slight modification of the Kruřkov entropy condition (1.4) that will shed some new light on [17], and also [3]. To this end, we recall that a uniqueness proof for entropy solutions is based on a technique known as “doubling of variables”. Suppose that v is another entropy solution of (1.1) with initial condition v^0 . The key idea is to consider v as a function of a different set of variables, say $v = v(s, y)$, and then for each fixed $(s, y) \in \Pi_T$, take $c = v(s, y)$ in the entropy condition for u . In the case that u and v are stochastic fields, $v(s, y)$ is no longer a constant, but rather a random variable. Hence it seems natural to utilize an entropy condition in which the Kruřkov parameters c in (1.4) are random variables rather than constants.

Let us do an informal derivation of an entropy condition based on this idea. As above, we let W be an ordinary Brownian motion and $d = 1$. For each fixed $\varepsilon > 0$, suppose u^ε is a sufficiently regular solution of the stochastic parabolic equation

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \sigma(u^\varepsilon) \partial_t W + \varepsilon \partial_x^2 u^\varepsilon,$$

where the time derivative is understood in the sense of distributions. We apply the *anticipating* Itô formula (Theorem 6.7) to $|u^\varepsilon - V|$, with V being an arbitrary

Malliavin differentiable random variable. Taking expectations, we obtain

$$\begin{aligned} & E [\partial_t |u^\varepsilon - V| + \partial_x(\text{sign}(u^\varepsilon - V)(f(u^\varepsilon) - f(V)))] \\ & \quad + E [\text{sign}'(u^\varepsilon - V)\sigma(u^\varepsilon)D_t V] - \frac{1}{2}E [\text{sign}'(u^\varepsilon - V)\sigma^2(u^\varepsilon)] \\ & \quad = \varepsilon E [\text{sign}(u^\varepsilon - V)\partial_x^2 u^\varepsilon], \end{aligned}$$

where $D_t V$ is the Malliavin derivative of V at t . As

$$\varepsilon E [\text{sign}(u^\varepsilon - V)\partial_x^2 u^\varepsilon] \leq \varepsilon E [\partial_x^2 |u^\varepsilon - V|],$$

it follows that

$$\begin{aligned} & E [\partial_t |u^\varepsilon - V| + \partial_x(\text{sign}(u^\varepsilon - V)(f(u^\varepsilon) - f(V)))] \\ & \quad + E [\text{sign}'(u^\varepsilon - V)\sigma(u^\varepsilon)D_t V] - \frac{1}{2}E [\text{sign}'(u^\varepsilon - V)\sigma^2(u^\varepsilon)] \\ & \quad \leq \varepsilon E [\partial_x^2 |u^\varepsilon - V|]. \end{aligned}$$

Suppose $u^\varepsilon \rightarrow u$ in a suitable sense as $\varepsilon \downarrow 0$. Then the limit u ought to satisfy

$$(1.5) \quad \begin{aligned} & E [\partial_t |u - V| + \partial_x(\text{sign}(u - V)(f(u) - f(V)))] \\ & \quad + E [\text{sign}'(u - V)\sigma(u)D_t V] - \frac{1}{2}E [\text{sign}'(u - V)\sigma^2(u)] \leq 0, \end{aligned}$$

which is the entropy condition that we propose should replace (1.4).

At least informally, it is easy to see why this entropy condition implies the L^1 contraction principle. Let $u = u(t, x)$ and $v = v(s, y)$ be two solutions satisfying the entropy condition (1.5). Suppose u, v are both Malliavin differentiable and spatially regular. The entropy condition for u yields

$$\begin{aligned} & E [\partial_t |u - v| + \partial_x(\text{sign}(u - v)(f(u) - f(v)))] \\ & \quad + E [\text{sign}'(u - v)\sigma(u)D_t v] - \frac{1}{2}E [\text{sign}'(u - v)\sigma^2(u)] \leq 0. \end{aligned}$$

Similarly, the entropy condition of v yields

$$\begin{aligned} & E [\partial_s |v - u| + \partial_y(\text{sign}(v - u)(f(v) - f(u)))] \\ & \quad + E [\text{sign}'(v - u)\sigma(v)D_s u] - \frac{1}{2}E [\text{sign}'(v - u)\sigma^2(v)] \leq 0. \end{aligned}$$

Suppose that $t > s$. Then $D_t v(s) = 0$ as v is adapted (to the underlying filtration). Adding the last two equations we obtain

$$\begin{aligned} & E [(\partial_t + \partial_s)|u - v| + (\partial_x + \partial_y)(\text{sign}(u - v)(f(u) - f(v)))] \\ & \quad + E [\text{sign}'(u - v)\sigma(v)D_s u] - \frac{1}{2}E [\text{sign}'(u - v)(\sigma^2(u) + \sigma^2(v))] \leq 0. \end{aligned}$$

Completing the square yields

$$(1.6) \quad \begin{aligned} & E [(\partial_t + \partial_s)|u - v| + (\partial_x + \partial_y)(\text{sign}(u - v)(f(u) - f(v)))] \\ & \quad + E [\text{sign}'(u - v)\sigma(v)(D_s u - \sigma(u))] - \underbrace{\frac{1}{2}E [\text{sign}'(u - v)(\sigma(u) - \sigma(v))^2]}_{=0} \leq 0. \end{aligned}$$

Next we write

$$D_s u(t) - \sigma(u(t)) = (D_s u(t) - \sigma(u(s))) + (\sigma(u(s)) - \sigma(u(t))),$$

and attempt to send $t \downarrow s$. The second term tends to zero almost everywhere. Formally, for fixed s , we observe that $D_s u(t)$ satisfies the initial value problem

$$(1.7) \quad \begin{cases} dw + \partial_x(f'(u)w) dt = (\sigma'(u)w) dW(t), & (t > s), \\ w(s) = \sigma(u(s)). \end{cases}$$

And so, concluding that

$$D_s u(t) \rightarrow \sigma(u(s)), \quad \text{as } t \downarrow s,$$

amounts to showing that the solution to (1.7) satisfies the initial condition (in some weak sense). Given this result, the L^1 contraction property follows from (1.6) in a standard way:

$$(1.8) \quad \frac{d}{dt} E \left[\|u(t) - v(t)\|_{L^1(\mathbb{R})} \right] \leq 0.$$

The above argument is hampered by an obstacle; namely, that the Malliavin differentiability of an entropy solution seems hard to establish. This can be seen related to the *discontinuous* coefficient $f'(u)$ in the stochastic continuity equation (1.7), making it difficult to establish the existence of a (properly defined) weak solution. In the deterministic context, continuity (and related transport) equations with low-regularity coefficients have been an active area of research in recent years, see for example [1, 5, 6] (and also [18] for a particular stochastic setting). Continuity equations arise in many applications, such as fluid mechanics. They also appear naturally when one linearizes a nonlinear conservation law $u_t + f(u)_x = 0$ into $w_t + (f'(u)w)_x = 0$, see references in [1, 5, 6]. The present work shows that stochastic continuity equations arise naturally as well, through linearisation (by the Malliavin derivative) of stochastic conservation laws driven by semilinear noise. However, the study of such equations is beyond the present paper and is left for future work.

As alluded to above, to make the L^1 contraction argument rigorous we would need to know that at least one of the two entropy solutions being compared, is Malliavin differentiable. To avoid this nontrivial issue, we shall employ a more indirect approach, motivated by [3], comparing one entropy solution against the solution of the viscous problem linked to the other entropy solution, relying on weak compactness in the space of Young measures for the viscous approximation. The Malliavin differentiability of the viscous solution is then established and its Malliavin derivative is shown to satisfy a linear stochastic parabolic equation, with an initial condition fulfilled in the weak sense. Given these results, the proof of the L^1 contraction property follows as outlined above.

Finally, we mention that the approach developed herein appears useful in the study of error estimates for numerical approximations of stochastic conservation laws, whenever the approximation is Malliavin differentiable. It seems to us that this Malliavin differentiability is indeed often available. Furthermore, the approach may be extended so as to cover strongly degenerate parabolic equations with Lévy noise, cf. [4], [34]. It also constitutes a starting point for developing a well-posedness theory for stochastic conservation laws with random, possibly anticipating initial data. Note however that this seems to depend on the Malliavin differentiability of the entropy solution (Lemma 5.3 is no longer applicable).

The remaining part of the paper is organized as follows: We present the solution framework and gather some preliminary results in Section 2. Well-posedness results for the viscous approximations are provided in Section 3. Furthermore, we establish the Malliavin differentiability of these approximations and show that the Malliavin derivative can be cast as the solution of a linear stochastic parabolic equation. The question of (weak) satisfaction of the initial condition is addressed. Sections 4 and

5 supply detailed proofs for the existence and uniqueness of Young measure-valued entropy solutions. Finally, some basic results are collected in Section 6.

2. ENTROPY SOLUTIONS

Under the assumption $\sigma(x, 0, z) = 0$, the ordinary L^p spaces ($2 \leq p < \infty$) constitute a natural choice for (1.1). Without this assumption, it turns out that a certain class of weighted L^p spaces are better suited. For non-negative ϕ we define

$$\|u\|_{p,\phi} := \left(\int_{\mathbb{R}^d} |u(x)|^p \phi(x) dx \right)^{1/p}.$$

The relevant weights, denoted by \mathfrak{N} , consist of non-zero $\phi \in C^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for which there is a constant C_ϕ such that $|\nabla\phi(x)| \leq C_\phi\phi(x)$. The weighted L^p -space associated with ϕ is denoted by $L^p(\mathbb{R}^d, \phi)$.

To see that \mathfrak{N} is non-empty, consider $\phi_N(x) = (1 + |x|^2)^{-N}$ for $N \in \mathbb{N}$. Then we claim that $\phi_N \in \mathfrak{N}$ for all $N \geq d$. To this end, observe that

$$\nabla\phi_N(x) = -2N \frac{x}{1 + |x|^2} \phi_N(x),$$

so $|\nabla\phi_N(x)| \leq 2N\phi_N(x)$. Furthermore

$$\int_{\mathbb{R}^d} \phi_N(x) dx = \int_0^\infty \int_{\partial B(0,r)} \left(\frac{1}{1+r^2} \right)^N dS(r) dr \leq d\alpha(d)\pi/2.$$

Another function belonging to \mathfrak{N} is $\phi_\lambda(x) = \exp(-\lambda\sqrt{1 + |x|^2})$ for $\lambda > 0$ [38].

The fact that $\phi \in L^1(\mathbb{R}^d)$ yields $L^q(\mathbb{R}^d, \phi) \subset L^p(\mathbb{R}^d, \phi)$ for all $1 \leq p < q$. Indeed, with $r = q/p$ and $s = q/(q-p)$, observe that $r^{-1} + s^{-1} = 1$. Take $h = |u(t)|^p \phi^{1/r}$ and $g = \phi^{1/s}$ and apply Hölder's inequality to $\int |hg| dx$. This yields

$$(2.1) \quad \int_{\mathbb{R}^d} |u(x)|^p \phi dx \leq \left(\int_{\mathbb{R}^d} |u(x)|^q \phi(x) dx \right)^{p/q} \left(\int_{\mathbb{R}^d} \phi(x) dx \right)^{1-p/q},$$

or equivalently $\|u\|_{p,\phi} \leq \|u\|_{q,\phi} \|\phi\|_{L^1(\mathbb{R}^d)}^{1/p-1/q}$.

We shall also make use of the weighted L^∞ -norm

$$\|h\|_{\infty,\phi^{-1}} := \sup_{x \in \mathbb{R}^d} \{ |h(x)| \phi^{-1}(x) \}, \quad h \in C(\mathbb{R}^d).$$

Note that any compactly supported $h \in C(\mathbb{R}^d)$ is bounded in this norm, for $\phi \in \mathfrak{N}$. The norm is convenient due to the inequality $\|u\|_{p,\phi} \leq \|u\|_{p,h} \|h\|_{\infty,\phi^{-1}}$.

Denote by \mathcal{E} the set of non-negative convex functions in $C^2(\mathbb{R})$ with $S(0) = 0$, S' bounded, and S'' compactly supported. Suppose $Q : \mathbb{R}^2 \rightarrow \mathbb{R}^d$ satisfies

$$\partial_1 Q(u, c) = S'(u - c) f'(u) \quad (u, c \in \mathbb{R}),$$

where $S \in \mathcal{E}$. Then we call $(S(\cdot - c), Q(\cdot, c))$ an *entropy/entropy-flux pair* (indexed over $c \in \mathbb{R}$). For short, we say that (S, Q) is in \mathcal{E} if S is in \mathcal{E} . For $(S, Q) \in \mathcal{E}$, $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, and $V \in \mathbb{D}^{1,2}$, we define the functional

$$\begin{aligned} \mathbf{Ent}[(S, Q), \varphi, V](u) &:= E \left[\int_{\mathbb{R}^d} S(u^0(x) - V) \varphi(0, x) dx \right] \\ &+ E \left[\iint_{\Pi_T} S(u - V) \partial_t \varphi + Q(u, V) \cdot \nabla \varphi dx dt \right] \\ &- E \left[\iint_{\Pi_T} \int_Z S''(u - V) \sigma(x, u, z) D_{t,z} V \varphi d\mu(z) dx dt \right] \\ &+ \frac{1}{2} E \left[\iint_{\Pi_T} \int_Z S''(u - V) \sigma(x, u, z)^2 \varphi d\mu(z) dx dt \right], \end{aligned}$$

where $D_{t,z}V$ is the Malliavin derivative of V at $(t, z) \in [0, T] \times Z$. We denote by $\mathbb{D}^{1,2}$ the space of Malliavin differentiable random variables in $L^2(\Omega)$ with Malliavin derivative in $L^2(\Omega; L^2([0, T] \times Z))$ [27, p. 27].

We claim that **Ent** is well-defined whenever $V \in \mathbb{D}^{1,2}$, $u \in L^2([0, T] \times \Omega; L^2(\mathbb{R}^d, \phi))$,

$$\|\varphi(t)\|_{\infty, \phi^{-1}}, \|\partial_t \varphi(t)\|_{\infty, \phi^{-1}}, \|\nabla \varphi(t)\|_{\infty, \phi^{-1}} \quad \text{are bounded on } [0, T]$$

(any $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ meets these criteria). To this end, observe that the first three terms are bounded due to the Lipschitz condition on S . Indeed,

$$(2.2) \quad |Q(u, V)| = \left| \int_V^u S'(u - V) \partial f(z) dz \right| \leq \|S\|_{\text{Lip}} \|f\|_{\text{Lip}} |u - V|,$$

and so

$$(2.3) \quad \begin{aligned} \left| E \left[\iint_{\Pi_T} Q(u, V) \cdot \nabla \varphi dx dt \right] \right| &\leq \|S\|_{\text{Lip}} \|f\|_{\text{Lip}} E \left[\iint_{\Pi_T} (|u| + |V|) |\nabla \varphi| dx dt \right] \\ &\leq \|S\|_{\text{Lip}} \|f\|_{\text{Lip}} \int_0^T E \left[\|u(t)\|_{1, \phi} \right] \|\nabla \varphi(t)\|_{\infty, \phi^{-1}} + E[|V|] \|\nabla \varphi(t)\|_{L^1(\mathbb{R}^d)} dt, \end{aligned}$$

which is finite. The terms involving σ is most easily seen to be well defined as a consequence of the fact that the Hilbert Schmidt norm of $G(u)$ (cf. (1.2)) is bounded. To simplify the notation, we let $HS = \mathcal{L}_2(L^2(Z); L^2(\mathbb{R}^d, \phi))$. Due to assumption (\mathcal{A}_σ) and Remark 2.1,

$$\begin{aligned} \|G(u)\|_{HS}^2 &= \int_{\mathbb{R}^d} \int_Z \sigma^2(x, u(x), z) \phi(x) d\mu(z) dx \\ &\leq 2 \|M\|_{L^2(Z)}^2 \int_{\mathbb{R}^d} (1 + |u(x)|^2) \phi(x) dx. \end{aligned}$$

Boundedness of the last term follows as

$$(2.4) \quad \begin{aligned} \left| E \left[\iint_{\Pi_T} \int_Z S''(u - V) \sigma(x, u, z)^2 \varphi d\mu(z) dx dt \right] \right| \\ \leq \|S''\|_\infty E \left[\int_0^T \|\varphi(t)\|_{\infty, \phi^{-1}} \|G(u(t))\|_{HS}^2 dt \right]. \end{aligned}$$

By (2.1), the sub-multiplicativity of the Hilbert Schmidt norm, and the Hölder inequality, it follows that

$$(2.5) \quad \begin{aligned} \left| E \left[\iint_{\Pi_T} \int_Z S''(u - V) \sigma(x, u, z) D_{t,z} V \varphi d\mu(z) dx dt \right] \right| \\ \leq \|S''\|_\infty E \left[\iint_{\Pi_T} \|\varphi(t)\|_{\infty, \phi^{-1}} \left| \int_Z \sigma(x, u, z) D_{t,z} V d\mu(z) \right| \phi dx dt \right] \\ \leq \|S''\|_\infty \|\phi\|_{L^1(\mathbb{R}^d)}^{1/2} E \left[\int_0^T \|\varphi(t)\|_{\infty, \phi^{-1}} \|G(u(t)) D_t V\|_{2, \phi} dt \right] \\ \leq \|S''\|_\infty \|\phi\|_{L^1(\mathbb{R}^d)}^{1/2} E \left[\int_0^T \|\varphi(t)\|_{\infty, \phi^{-1}}^2 \|G(u(t))\|_{HS}^2 dt \right]^{1/2} \\ \times \|DV\|_{L^2(\Omega; L^2([0, T] \times Z))} < \infty. \end{aligned}$$

Remark 2.1. Let A be an integral operator with kernel $a = a(x, z)$. In the above calculations, we applied the relation

$$\|A\|_{HS}^2 = \int_{\mathbb{R}^d} \int_Z a^2(x, z) \phi(x) d\mu(z) dx.$$

To see why this holds, recall that $\|A\|_{\text{HS}}^2 = \text{tr}(A^*A)$. Now

$$(A^*Ah)(z) = \int_{\mathbb{R}^d} a(x, z) \left(\int_Z a(x, z')h(z') d\mu(z') \right) \phi(x) dx.$$

Hence, A^*A is an integral operator with kernel $k \in L^2(Z^2)$ defined by

$$k(z, z') = \int_{\mathbb{R}^d} a(x, z)a(x, z')\phi(x) dx,$$

and so

$$\text{tr}(A^*A) = \int_Z k(z, z) d\mu(z) = \int_{\mathbb{R}^d} \int_Z a^2(x, z)\phi(x) d\mu(z) dx.$$

Let \mathcal{P} denote the predictable σ -algebra on $[0, T] \times \Omega$ with respect to $\{\mathcal{F}_t\}$ [10, § 2.2]. In general we are working with equivalence classes of functions with respect to the measure $dt \otimes dP$. The equivalence class u is said to be *predictable* if it has a version \tilde{u} that is \mathcal{P} -measurable. In some of the arguments, to avoid picking versions, we consider the completion of \mathcal{P} with respect to $dt \otimes dP$, denoted by \mathcal{P}^* . We recall that any jointly measurable and adapted process is \mathcal{P}^* -measurable, see [10, Theorem 3.7].

Definition 2.1 (Entropy solution). An entropy solution $u = u(t, x; \omega)$ of (1.1) with initial condition $u^0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$ is a function satisfying:

- (i) u is a predictable process in $L^2([0, T] \times \Omega; L^2(\mathbb{R}^d, \phi))$.
- (ii) For any random variable $V \in \mathbb{D}^{1,2}$, any entropy/entropy-flux pair (S, Q) in \mathcal{E} , and all nonnegative test functions $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$,

$$\mathbf{Ent}[(S, Q), \varphi, V](u) \geq 0.$$

Here $L^2([0, T] \times \Omega; L^2(\mathbb{R}^d, \phi))$ is the Lebesgue-Bochner space, see Section 6.4.

Remark 2.2 (Weak solutions). By Theorem 5.1 and the proof of Theorem 4.1, the $(\varepsilon \downarrow 0)$ limit of the viscous approximation (3.1) is an entropy solution in the sense of Definition 2.1. It follows that any entropy solution is a weak solution. *Informally* this is a consequence of the integration by parts formula for Malliavin derivatives. To illustrate this, let $d = 1$, W be an ordinary Brownian motion, and suppose that u is a Malliavin differentiable and spatially regular entropy solution. We want to outline an informal argument showing that u is a (strong) solution of (1.1). Let

$$(u)_+ = \begin{cases} u & \text{for } u > 0, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad \text{sign}_+(u) = \begin{cases} 1 & \text{for } u > 0, \\ 0 & \text{else,} \end{cases}$$

so that $(u)_+' = \text{sign}_+(u)$. The entropy inequality yields

$$\begin{aligned} & E \left[\partial_t(u - V)_+ + \partial_x(\text{sign}_+(u - V)(f(u) - f(V))) \right] \\ & + E \left[\text{sign}'_+(u - V) \sigma(u) D_t V \right] - \frac{1}{2} E \left[\text{sign}'_+(u - V) \sigma^2(u) \right] \leq 0. \end{aligned}$$

Note that

$$\sigma(u) D_t V - \frac{1}{2} \sigma^2(u) = -\sigma(u)(\sigma(u) - D_t V) + \frac{1}{2} \sigma^2(u).$$

By (1.7) and the chain rule, $D_t \text{sign}_+(u - V) = \text{sign}'_+(u - V)(\sigma(u) - D_t V)$. Hence,

$$\begin{aligned} & E \left[\partial_t(u - V)_+ + \partial_x(\text{sign}_+(u - V)(f(u) - f(V))) \right] \\ & - E \left[D_t \text{sign}_+(u - V) \sigma(u) \right] + \frac{1}{2} E \left[\text{sign}'_+(u - V) \sigma^2(u) \right] \leq 0 \end{aligned}$$

The integration by parts formula of Malliavin calculus yields

$$E \left[D_t \text{sign}_+(u - V) \sigma(u) \right] = E \left[\text{sign}_+(u - V) \sigma(u) \partial_t W \right],$$

and so

$$\begin{aligned} E [\partial_t(u - V)_+ + \partial_x(\text{sign}_+(u - V)(f(u) - f(V)))] \\ \leq E [\text{sign}_+(u - V)\sigma(u)\partial_t W]. \end{aligned}$$

Suppose there is a Malliavin differentiable random variable V satisfying

$$\begin{cases} u - V > 0 & \text{for } \omega \in A, \\ u - V < 0 & \text{else,} \end{cases}$$

for any $A \in \mathcal{F}_T$. Then

$$\partial_t u + \partial_x f(u) \leq \sigma(u)\partial_t W.$$

The opposite inequality follows by considering $(\cdot)^-$.

Let us fix some notation. For $n = 1, 2, \dots$, we will denote by J^n a non-negative, smooth function satisfying

$$\text{supp}(J^n) \subset B(0, 1), \quad \int_{\mathbb{R}^n} J^n(x) dx = 1, \quad \text{and } J^n(x) = J^n(-x),$$

for all $x \in \mathbb{R}^n$. For any $r > 0$ we let $J_r^n(x) = \frac{1}{r^n} J^n(\frac{x}{r})$. For $n = 1$ we let $J_r^+(x) = J_r(x - r)$ and note that $\text{supp}(J_r^+) \subset (0, 2r)$. As the value of n is understood from the context, we will write $J = J^n$.

According to Theorem 4.1 and Theorem 5.1, if $u^0 \in L^p(\Omega; L^p(\mathbb{R}^d, \phi))$, then the entropy solution belongs to $L^p(\Omega \times [0, T]; L^p(\mathbb{R}^d, \phi))$ for any $2 \leq p < \infty$. As a consequence of the entropy inequality we obtain the following:

Proposition 2.1. *Let $2 \leq p < \infty$ and suppose $u^0 \in L^p(\Omega, \mathcal{F}_0, P; L^p(\mathbb{R}^d, \phi))$. If $u \in L^p([0, T] \times \Omega, L^p(\mathbb{R}^d, \phi))$ is an entropy solution of (1.1), then*

$$\text{ess sup}_{0 \leq t \leq T} \left\{ E \left[\|u(t)\|_{p, \phi}^p \right] \right\} < \infty.$$

Proof. Let

$$\varphi_\delta(t, x) = \left(1 - \int_0^t J_\delta(\sigma - \tau) d\sigma \right) \phi(x).$$

Let

$$S_R(u) = \begin{cases} R^p + pR^{p-1}(u - R) & \text{for } u \geq R, \\ |u|^p & \text{for } -R < u < R, \\ R^p - pR^{p-1}(u + R) & \text{for } u \leq -R. \end{cases}$$

Strictly speaking, S_R is not in \mathcal{E} , but this can be amended by a simple mollification step (which we ignore). Note that $S_R \rightarrow |\cdot|^p$ pointwise. Furthermore

$$|S'_R(u)| \leq p|u|^{p-1} \quad \text{and} \quad |S''_R(u)| \leq p(p-1)|u|^{p-2}.$$

Under these assumptions it is straightforward to supply estimates of the type (2.3) and (2.4). Hence we may apply the dominated convergence theorem to compute the limit $\lim_{\delta \downarrow 0} \mathbf{Ent}[(S_R, Q_R), \varphi_\delta, 0](u) \geq 0$. This yields

$$\begin{aligned} E \left[\int_{\mathbb{R}^d} S_R(u(\tau)) \phi(x) dx \right] &\leq E \left[\int_{\mathbb{R}^d} S_R(u^0(x)) \phi(x) dx \right] \\ &\quad + E \left[\int_0^\tau \int_{\mathbb{R}^d} Q_R(u, 0) \cdot \nabla \phi dx dt \right] \\ &\quad + \frac{1}{2} E \left[\int_0^\tau \int_{\mathbb{R}^d} \int_Z S''_R(u) \sigma(x, u, z)^2 \phi d\mu(z) dx dt \right], \end{aligned}$$

for almost all $\tau \in [0, T]$. Again, by the dominated convergence theorem, we may send $R \rightarrow \infty$. The result follows. \square

It is enough to consider smooth random variables in Definition 2.1, i.e., random variables of the form

$$V = f(W(h_1), \dots, W(h_n))$$

where $f \in C_c^\infty(\mathbb{R}^n)$, W is the isonormal Gaussian process defined by (1.3), and h_1, \dots, h_n are in $H = L^2([0, T] \times Z)$, see [27, p. 25]. We denote the space of smooth random variables by \mathcal{S} .

Lemma 2.2. *Suppose (\mathcal{A}_f) and (\mathcal{A}_σ) are satisfied. Fix $u \in L^2([0, T] \times \Omega; L^2(\mathbb{R}^d))$, an entropy/entropy-flux pair $(S, Q) \in \mathcal{E}$, and $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$. Then*

$$V \mapsto \mathbf{Ent}[(S, Q), \varphi, V](u)$$

is continuous on $\mathbb{D}^{1,2}$ (in the strong topology).

Remark 2.3. It is not necessary that S'' is compactly supported in the upcoming proof (boundedness and continuity are enough).

Proof. Suppose that $V_n \rightarrow V$ in $\mathbb{D}^{1,2}$ as $n \rightarrow \infty$, and write

$$\begin{aligned} & \mathbf{Ent}[(S, Q), \varphi, V](u) - \mathbf{Ent}[(S, Q), \varphi, V_n](u) \\ &= E \left[\int_{\mathbb{R}^d} (S(u^0(x) - V) - S(u^0(x) - V_n)) \varphi(0, x) dx \right] \\ &+ E \left[\iint_{\Pi_T} (S(u - V) - S(u - V_n)) \partial_t \varphi dx dt \right] \\ &+ E \left[\iint_{\Pi_T} (Q(u, V) - Q(u, V_n)) \cdot \nabla \varphi dx dt \right] \\ &+ E \left[\iint_{\Pi_T} \int_Z (S''(u - V_n) D_{t,z} V_n - S''(u - V) D_{t,z} V) \sigma(x, u, z) \varphi d\mu(z) dx dt \right] \\ &+ \frac{1}{2} E \left[\iint_{\Pi_T} \int_Z (S''(u - V) - S''(u - V_n)) \sigma(x, u, z)^2 \varphi d\mu(z) dx dt \right] \\ &=: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5. \end{aligned}$$

We need to show that $\lim_{n \rightarrow \infty} \mathcal{T}_i(n) = 0$ for $1 \leq i \leq 5$.

First, note that $V_n \rightarrow V$ in $L^2(\Omega)$. Next,

$$|\mathcal{T}_1| \leq \|S\|_{\text{Lip}} E[|V - V_n|] \|\varphi(0)\|_{L^1(\mathbb{R})}.$$

Similarly,

$$|\mathcal{T}_2| \leq \|S\|_{\text{Lip}} E[|V - V_n|] \|\partial_t \varphi\|_{L^1(\Pi_T)}.$$

It follows as $V_n \rightarrow V$ in $L^2(\Omega)$ that $\mathcal{T}_1, \mathcal{T}_2 \rightarrow 0$ as $n \rightarrow \infty$.

Concerning \mathcal{T}_3 , we first observe that for any $\zeta, \xi, \theta \in \mathbb{R}$,

$$\begin{aligned} |Q(\zeta, \xi) - Q(\zeta, \theta)| &= \left| \int_\xi^\zeta S'(z - \xi) \partial f(z) dz - \int_\theta^\zeta S'(z - \theta) \partial f(z) dz \right| \\ &\leq \left| \int_\xi^\zeta (S'(z - \xi) - S'(z - \theta)) \partial f(z) dz \right| + \left| \int_\xi^\theta S'(z - \theta) \partial f(z) dz \right|. \end{aligned}$$

Hence,

$$\begin{aligned} |\mathcal{T}_3| &\leq E \left[\iint_{\Pi_T} \left| \int_V^u (S'(z - V) - S'(z - V_n)) \partial f(z) dz \right| |\nabla \varphi| dx dt \right] \\ &+ E \left[\iint_{\Pi_T} \left| \int_V^{V_n} S'(z - V_n) \partial f(z) dz \right| |\nabla \varphi| dx dt \right] \\ &=: \mathcal{T}_3^1 + \mathcal{T}_3^2. \end{aligned}$$

We want to handle \mathcal{T}_3^1 using the dominated convergence theorem. The uniform bound is obtained as in (2.3). Since

$$|S'(z - \xi) - S'(z - \theta)| \leq \|S''\|_\infty |\xi - \theta| \text{ for all } z \in \mathbb{R},$$

it follows that

$$\psi(\theta) := \int_\xi^\zeta (S'(z - \xi) - S'(z - \theta)) \partial f(z) dz$$

is continuous at ξ and satisfies $\psi(\xi) = 0$. Suppose that $\{V_{n(j)}\}$ is a subsequence converging a.e. to V . By the dominated convergence theorem,

$$\begin{aligned} & \lim_{j \rightarrow \infty} \mathcal{T}_3^1(n(j)) \\ &= E \left[\iint_{\Pi_T} \lim_{j \rightarrow \infty} \left| \int_V^u (S'(z - V) - S'(z - V_{n(j)})) \partial f(z) dz \right| |\nabla \varphi| dx dt \right] = 0. \end{aligned}$$

By the above, any subsequence of $\{\mathcal{T}_3^1(n)\}_{n \geq 1}$ possesses a subsequence converging to 0, and so $\lim_{n \rightarrow \infty} \mathcal{T}_3^1(n) = 0$. Consider \mathcal{T}_3^2 . Since

$$\mathcal{T}_3^2 \leq \|S\|_{\text{Lip}} \|f\|_{\text{Lip}} E [\|V_n - V\| \|\nabla \varphi\|_{L^1(\Pi_T)}],$$

it follows that $\lim_{n \rightarrow \infty} \mathcal{T}_3^2 = 0$.

Concerning the \mathcal{T}_4 -term, we first split it as follows:

$$\begin{aligned} \mathcal{T}_4 &= E \left[\iint_{\Pi_T} \int_Z S''(u - V_n) (D_{t,z} V_n - D_{t,z} V) \sigma(x, u, z) \varphi d\mu(z) dx dt \right] \\ &+ E \left[\iint_{\Pi_T} \int_Z (S''(u - V_n) - S''(u - V)) D_{t,z} V \sigma(x, u, z) \varphi d\mu(z) dx dt \right] \\ &= \mathcal{T}_4^1 + \mathcal{T}_4^2. \end{aligned}$$

By (2.5), $\lim_{n \rightarrow \infty} \mathcal{T}_4^1 = 0$. Owing to (2.5), the dominated convergence theorem implies $\lim_{n \rightarrow \infty} \mathcal{T}_4^2 = 0$. Finally, by (2.4) and the dominated convergence theorem, also $\lim_{n \rightarrow \infty} \mathcal{T}_5 = 0$. \square

For the proof of existence of solutions, it is convenient to introduce a weaker notion of entropy solution based on Young measures (see, e.g., [3, 15, 16, 28]). Instead of representing the solution as an element in $\mathcal{Y}(\Pi_T \times \Omega; \mathbb{R})$ we use the notion of entropy process proposed in [16] or equivalently the strong measure-valued solution proposed in [28]. Any probability measure ν on the real line may be represented by a measurable function $u : [0, 1] \rightarrow \mathbb{R} \cup \{\infty\}$ such that ν is the image of the Lebesgue measure \mathcal{L} on $[0, 1]$ by u . In fact we may take (See [37, § 2.2.2])

$$(2.6) \quad u(\alpha) = \inf \{ \xi \in \mathbb{R} : \nu((-\infty, \xi]) > \alpha \}.$$

A Young measure $\nu \in \mathcal{Y}(\Pi_T \times \Omega; \mathbb{R})$ is thus represented by a (higher dimensional) function $u : \Pi_T \times [0, 1] \times \Omega \rightarrow \mathbb{R}$ where u satisfies $\nu_{t,x,\omega}(B) = \mathcal{L}(u(t, x, \cdot, \omega)^{-1}(B))$ for any measurable $B \subset \mathbb{R}$. The extension to Young measure-valued solutions is thus obtained through the embedding defined by

$$(2.7) \quad \Phi(u)(t, x, \alpha, \omega) = u(t, x, \omega).$$

Given a functional F we define the extension

$$\mathcal{Y}(F)(u) = \int_0^1 F(u(\alpha)) d\alpha,$$

so that $\mathcal{Y}(F) \circ \Phi = F$. For $1 \leq p < \infty$ we let

$$\|u\|_{p, \phi \otimes 1} = \left(\int_0^1 \int_{\mathbb{R}^d} |u(x, \alpha)|^p \phi(x) dx d\alpha \right)^{1/p}.$$

The associated space is denoted by $L^p(\mathbb{R}^d \times [0, 1], \phi)$.

Definition 2.2 (Young measure-valued entropy solution). A Young measure-valued entropy solution $u = u(t, x, \alpha; \omega)$ of (1.1), with initial condition u^0 belonging to $L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$, is a function satisfying:

- (i) u is a predictable process in $L^2([0, T] \times \Omega; L^2(\mathbb{R}^d \times [0, 1], \phi))$.
- (ii) For any random variable $V \in \mathbb{D}^{1,2}$, any entropy/entropy-flux pair (S, Q) in \mathcal{E} , and all nonnegative test functions $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$,

$$(2.8) \quad \mathcal{Y}(\mathbf{Ent}[(S, Q), \varphi, V])(u) \geq 0.$$

The next result is concerned with the essential continuity of the solutions at $t = 0$. A similar argument can be found in [7].

Lemma 2.3 (Initial condition). *Suppose (\mathcal{A}_f) and (\mathcal{A}_σ) are satisfied, and that u^0 belongs to $L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$. Let u be a Young measure-valued entropy solution of (1.1) in the sense of Definition 2.2. Let $S : \mathbb{R} \rightarrow [0, \infty)$ be Lipschitz continuous and satisfy $S(0) = 0$. For any $\psi \in C_c^\infty(\mathbb{R}^d)$,*

$$\mathcal{T}_{r_0} := E \left[\iint_{\Pi_T} \int_{[0,1]} S(u(t, x, \alpha) - u^0(x)) \psi(x) J_{r_0}^+(t) d\alpha dx dt \right] \rightarrow 0 \text{ as } r_0 \downarrow 0.$$

Remark 2.4. The proof does not depend on the differentiability of $J_{r_0}^+$. Hence the above limit may be replaced by

$$\lim_{\tau \downarrow 0} E \left[\frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}^d} \int_{[0,1]} S(u(t, x, \alpha) - u^0(x)) \psi(x) d\alpha dx dt \right] = 0.$$

Proof. Let $S \in C^\infty(\mathbb{R})$ with bounded derivatives. Take

$$\varphi(t, x, y) = \xi_{r_0}(t) \psi(x) J_r(x - y) \text{ where } \xi_{r_0}(t) = 1 - \int_0^t J_{r_0}^+(s) ds.$$

Then let $V = u^0(y)$ in (2.8) and integrate in y . This implies

$$(2.9) \quad \begin{aligned} I &:= E \left[\int_{\mathbb{R}^d} \iint_{\Pi_T} \int_{[0,1]} S(u(t, x, \alpha) - u^0(y)) \psi(x) J_r(x - y) J_{r_0}^+(t) d\alpha dx dt dy \right] \\ &\leq E \left[\int_{\mathbb{R}^d} \iint_{\Pi_T} \int_{[0,1]} Q(u(t, x, \alpha), u^0(y)) \cdot \nabla_x \varphi d\alpha dx dt dy \right] \\ &\quad + E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S(u^0(x) - u^0(y)) \varphi(0, x, y) dx dy \right] \\ &\quad + \frac{1}{2} E \left[\int_{\mathbb{R}^d} \iint_{\Pi_T} \int_{[0,1]} \int_Z S''(u(t, x, \alpha) - u^0(y)) \sigma(x, u, z)^2 \varphi d\mu(z) d\alpha dx dt dy \right] \\ &=: \mathcal{T}^1 + \mathcal{T}^2 + \mathcal{T}^3. \end{aligned}$$

Let us first observe that

$$\begin{aligned} I &= \mathcal{T}_{r_0} + E \left[\int_{\mathbb{R}^d} \iint_{\Pi_T} \int_{[0,1]} (S(u(t, x, \alpha) - u^0(y)) - S(u(t, x, \alpha) - u^0(x))) \right. \\ &\quad \left. \times \psi(x) J_r(x - y) J_{r_0}^+(t) d\alpha dx dt dy \right] =: \mathcal{T}_{r_0} + I^1. \end{aligned}$$

We want to take the limit $r_0 \downarrow 0$. First observe that we have the bound

$$|I^1| \leq \|S\|_{\text{Lip}} E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u^0(x) - u^0(y)| \psi(x) J_r(x - y) dx dy \right] =: R,$$

which is independent of r_0 . Similarly, $|\mathcal{T}^2| \leq R$. Note that $\xi_{r_0} \rightarrow 0$ a.e. as $r_0 \downarrow 0$, so due to assumptions (\mathcal{A}_f) and (\mathcal{A}_σ) , one may conclude by the dominated convergence theorem and estimates similar to those in (2.3) and (2.4) that

$$\lim_{r_0 \downarrow 0} \mathcal{T}^1 = \lim_{r_0 \downarrow 0} \mathcal{T}^3 = 0.$$

Thus, it follows by (2.9) that

$$\lim_{r_0 \downarrow 0} \mathcal{T}_{r_0} \leq 2R.$$

Since $r > 0$ was arbitrary, and $\lim_{r \downarrow 0} R = 0$, we have arrived at $\lim_{r_0 \downarrow 0} \mathcal{T}_{r_0} \leq 0$. The desired result follows, since we can approximate any Lipschitz function uniformly by smooth functions with bounded derivatives. \square

3. THE VISCOUS APPROXIMATION

For each fixed $\varepsilon > 0$, we denote by u^ε the solution of the regularized problem

$$(3.1) \quad \begin{cases} du^\varepsilon + \nabla \cdot f(u^\varepsilon) dt = \int_Z \sigma(x, u^\varepsilon, z) W(dt, dz) + \varepsilon \Delta u^\varepsilon dt, & (t, x) \in \Pi_T, \\ u^\varepsilon(0, x) = u^0(x), & x \in \mathbb{R}^d. \end{cases}$$

As in the deterministic case, the idea is to let $\varepsilon \rightarrow 0$ and obtain a solution to the stochastic conservation law (1.1). The entropy condition is meant to single out this limit as the only proper (weak) solution; the entropy solution. To show that this limit exists, a type of compactness argument is needed [3, 17, 9].

The existence of a unique solution to (3.1) may be found several places [3, 17]. In particular, the semi-group approach presented in [30, ch. 9] may be applied. The functional setting of [30] is that of a Hilbert space, and so the natural choice here is $L^2(\mathbb{R}^d, \phi)$ where $\phi \in \mathfrak{N}$. Due to the new functional setting, we have chosen to include proofs for some of the results relating to (3.1).

3.1. A priori estimates and well-posedness. Let S_ε be the semi-group generated by the heat kernel. That is $S_\varepsilon(t)u = \Phi_\varepsilon(t) \star u$ where

$$\Phi_\varepsilon(t, x) := \frac{1}{(4\varepsilon\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4\varepsilon t}\right).$$

Let $F(u) = \nabla \cdot f(u)$ and G be defined by (1.2). In this setting the key conditions [30, p. 142] for well posedness of (3.1) is given by:

- (F) $D(F)$ is dense in $L^2(\mathbb{R}^d, \phi)$ and there is a function $a : (0, \infty) \rightarrow (0, \infty)$ satisfying $\int_0^T a(t) dt < \infty$ for all $T < \infty$ such that, for all $t > 0$ and $u, v \in D(F)$,

$$\begin{aligned} \|S_\varepsilon(t)F(u)\|_{L^2(\mathbb{R}^d, \phi)} &\leq a(t) \left(1 + \|u\|_{L^2(\mathbb{R}^d, \phi)}\right), \\ \|S_\varepsilon(t)(F(u) - F(v))\|_{L^2(\mathbb{R}^d, \phi)} &\leq a(t) \|u - v\|_{L^2(\mathbb{R}^d, \phi)}. \end{aligned}$$

- (G) $D(G)$ is dense in $L^2(\mathbb{R}^d, \phi)$ and there is a function $b : (0, \infty) \rightarrow (0, \infty)$ satisfying $\int_0^T b^2(t) dt < \infty$ for all $T < \infty$ such that, for all $t > 0$ and $u, v \in D(G)$,

$$\begin{aligned} \|S_\varepsilon(t)G(u)\|_{\mathcal{L}_2(L^2(Z); L^2(\mathbb{R}^d, \phi))} &\leq b(t) \left(1 + \|u\|_{L^2(\mathbb{R}^d, \phi)}\right), \\ \|S_\varepsilon(t)(G(u) - G(v))\|_{\mathcal{L}_2(L^2(Z); L^2(\mathbb{R}^d, \phi))} &\leq b(t) \|u - v\|_{L^2(\mathbb{R}^d, \phi)}. \end{aligned}$$

Suppose $u^0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$. Under assumptions (F) and (G) we may conclude by [30, Theorem 9.15, Theorem 9.29] that there exists a unique predictable process $u^\varepsilon : [0, T] \times \Omega \rightarrow L^2(\mathbb{R}^d, \phi)$ such that

(i)

$$(3.2) \quad \sup_{0 \leq t \leq T} E \left[\|u^\varepsilon(t)\|_{2,\phi}^2 \right] < \infty.$$

(ii) For all $0 \leq t \leq T$

(3.3)

$$\begin{aligned} u^\varepsilon(t, x) &= \int_{\mathbb{R}^d} \Phi_\varepsilon(t, x - y) u^0(y) dy \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \nabla_x \Phi_\varepsilon(t - s, x - y) \cdot f(u^\varepsilon(s, y)) dy ds \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(t - s, x - y) \sigma(y, u^\varepsilon(s, y), z) dy W(ds, dz). \end{aligned}$$

(iii) u^ε is a weak solution of (3.1), i.e., for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any pair of times t_0, t with $0 \leq t_0 \leq t \leq T$,

$$(3.4) \quad \begin{aligned} \int_{\mathbb{R}^d} u^\varepsilon(t) \varphi dx &= \int_{\mathbb{R}^d} u^\varepsilon(t_0) \varphi dx - \int_{t_0}^t \int_{\mathbb{R}^d} f(u^\varepsilon(s)) \cdot \nabla \varphi dx ds \\ &\quad + \int_{t_0}^t \int_{\mathbb{R}^d} \int_Z \sigma(x, u^\varepsilon(s, x), z) \varphi W(ds, dz) dx + \varepsilon \int_{t_0}^t \int_{\mathbb{R}^d} u^\varepsilon \Delta \varphi dx ds, \end{aligned}$$

 dP -almost surley.

To see that conditions (F) and (G) are satisfied we prove the following estimate:

Lemma 3.1. *Let $\phi \in \mathfrak{N}$ and $1 \leq p < \infty$. Suppose $C_\phi \sqrt{4\varepsilon t} \leq 1$. Let $v \in W^{1,p}(\mathbb{R}^d, \phi; \mathbb{R}^d)$ and $u \in L^p(\mathbb{R}^d, \phi)$. Whenever $C_\phi \sqrt{4\varepsilon t} \leq 1$,*

$$(i) \quad \|\Phi_\varepsilon(t) \star u\|_{p,\phi} \leq \kappa_{1,d} \|u\|_{p,\phi},$$

$$(ii) \quad \|\Phi_\varepsilon(t) \star \nabla \cdot v\|_{p,\phi} \leq \frac{\kappa_{2,d}}{\sqrt{\varepsilon t}} \|v\|_{p,\phi},$$

where $\kappa_{1,d} = c_{d-1} \frac{d\alpha(d)}{\pi^{d/2}}$, $\kappa_{2,d} = c_d \frac{d\alpha(d)}{\pi^{d/2}}$, and

$$c_d = \int_0^\infty \zeta^d (1 + \zeta)^2 \exp(\zeta - \zeta^2) d\zeta.$$

The volume of the unit ball in \mathbb{R}^d is denoted by $\alpha(d)$.Before we give a proof let us see why (F) and (G) follows. Recall that we may assume $f(0) = 0$ without any loss of generality. By Lemma 3.1 and (\mathcal{A}_f)

$$\|S_\varepsilon(t)F(u)\|_{L^2(\mathbb{R}^d, \phi)} = \|\Phi_\varepsilon(t) \star \nabla \cdot f(u)\|_{L^2(\mathbb{R}^d, \phi)} \leq \underbrace{\frac{\kappa_{2,d}}{\sqrt{\varepsilon t}} \|f\|_{\text{Lip}}}_{a(t)} \|u\|_{2,\phi}.$$

It remains to observe that $\int_0^T \frac{1}{\sqrt{t}} dt = 2\sqrt{T} < \infty$. The second part of (F) follows similarly. Let us consider (G). First observe that

$$S_\varepsilon(t)G(u)h(x) = \int_Z \left(\int_{\mathbb{R}^d} \Phi_\varepsilon(t, x - y) \sigma(y, u(y), z) dy \right) h(z) d\mu(z).$$

Recall that $HS = \mathcal{L}_2(L^2(Z); L^2(\mathbb{R}^d, \phi))$. By Lemma 3.1 and (\mathcal{A}_σ)

$$\begin{aligned} \|S_\varepsilon(t)G(u)\|_{HS}^2 &= \int_Z \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \Phi_\varepsilon(t, x - y) \sigma(y, u(y), z) dy \right)^2 \phi(x) dx d\mu(z) \\ &= \int_Z \|\Phi_\varepsilon(t) \star \sigma(\cdot, u, z)\|_{2,\phi}^2 d\mu(z) \\ &\leq \kappa_{1,d}^2 \|M\|_{L^2(Z)}^2 (\|\phi\|_{L^1(\mathbb{R}^d)} + \|u\|_{2,\phi})^2. \end{aligned}$$

This yields the first part of condition (G). The second part follows similarly, in view of the Lipschitz assumption on σ .

Proof of Lemma 3.1. Consider (i). By Proposition 6.4,

$$\|\Phi_\varepsilon(t) \star u\|_{p,\phi} \leq \underbrace{\left(\int_{\mathbb{R}^d} |\Phi_\varepsilon(t, x)| (1 + w_{p,\phi}(|x|)) dx \right)}_{\|\Phi(t)\|} \|u\|_{p,\phi}.$$

where

$$w_{p,\phi}(r) = \frac{C_\phi r}{p} \left(1 + \frac{C_\phi r}{p} \right) \exp\left(\frac{C_\phi r}{p}\right).$$

We apply polar coordinates to compute $\|\Phi(t)\|$. This yields

$$\begin{aligned} \|\Phi(t)\| &= \int_0^\infty \int_{\partial B(0,r)} |\Phi_\varepsilon(t)| (r) (1 + w_{p,\phi}(r)) dS(r) dr \\ &= \frac{d\alpha(d)}{(4\varepsilon\pi t)^{d/2}} \int_0^\infty r^{d-1} \exp\left(\frac{C_\phi r}{p} - \frac{r^2}{4\varepsilon t}\right) \\ &\quad \times \left(\exp\left(-\frac{C_\phi r}{p}\right) + \frac{C_\phi r}{p} \left(1 + \frac{C_\phi r}{p} \right) \right) dr. \end{aligned}$$

To simplify, we note that

$$\exp\left(-\frac{C_\phi r}{p}\right) + \frac{C_\phi r}{p} \left(1 + \frac{C_\phi r}{p} \right) \leq \left(1 + \left(\frac{C_\phi r}{p}\right) \right)^2$$

Let $\zeta = r/\sqrt{4\varepsilon t}$. Provided $C_\phi\sqrt{4\varepsilon t} \leq 1$, it follows that

$$\frac{C_\phi r}{p} = \frac{C_\phi}{p} \sqrt{4\varepsilon t} \zeta \leq \zeta.$$

Inserting this we obtain

$$\|\Phi(t)\| \leq \frac{d\alpha(d)}{\pi^{d/2}} \int_0^\infty \zeta^{d-1} (1 + \zeta)^2 \exp(\zeta - \zeta^2) d\zeta$$

Estimate (ii) follows along the same lines. Integration by parts yields

$$\int_{\mathbb{R}^d} \Phi_\varepsilon(t, x - y) \nabla \cdot v(y) dy = \int_{\mathbb{R}^d} \nabla_x \Phi_\varepsilon(t, x - y) \cdot v(y) dy.$$

Hence,

$$|\Phi_\varepsilon(t) \star \nabla \cdot v| \leq |\nabla \Phi(t)| \star |v|.$$

By Proposition 6.4,

$$\|\nabla_x \Phi_\varepsilon(t) \star v\|_{L^p(\mathbb{R}^d)} \leq \underbrace{\left(\int_{\mathbb{R}^d} |\nabla \Phi_\varepsilon(t, x)| (1 + w_{p,\phi}(|x|)) dx \right)}_{\|\nabla \Phi(t)\|} \|v\|_{L^p(\mathbb{R}^d, \phi; \mathbb{R}^d)}.$$

Let $r = |x|$. Then

$$\|\nabla \Phi(t)\| = \int_0^\infty \underbrace{\int_{\partial B(0,r)} |\nabla \Phi_\varepsilon(t)| (r) (1 + w_{p,\phi}(r)) dS(r)}_{\Psi(r)} dr.$$

Now,

$$\nabla \Phi_\varepsilon(t, x) = -\frac{2\pi x}{(4\pi\varepsilon t)^{d/2+1}} \exp\left(-\frac{|x|^2}{4\varepsilon t}\right),$$

and so

$$\Psi(r) = \frac{d\alpha(d)}{2\varepsilon t \pi^{d/2}} \left(\frac{r}{\sqrt{4\varepsilon t}}\right)^d \exp\left(\frac{C_\phi r}{p} - \frac{r^2}{4\varepsilon t}\right) \left(\exp\left(-\frac{C_\phi r}{p}\right) + \frac{C_\phi r}{p} \left(1 + \frac{C_\phi r}{p} \right) \right).$$

Let $\zeta(r) = r/\sqrt{4\varepsilon t}$ and suppose $C_\phi\sqrt{4\varepsilon t} \leq 1$. Then

$$\begin{aligned} \int_0^\infty \left(\frac{r}{\sqrt{4\varepsilon t}}\right)^d \exp\left(\frac{C_\phi}{p}r - \frac{r^2}{4\varepsilon t}\right) \left(1 + \left(\frac{C_\phi}{p}r\right)\right)^2 dr \\ \leq \sqrt{4\varepsilon t} \int_0^\infty \zeta^d (1 + \zeta)^2 \exp(\zeta - \zeta^2) d\zeta. \end{aligned}$$

This concludes the proof of the lemma. \square

The following two estimates constitute the reason why Lemma 3.1 is the key to the well-posedness of (3.1). As we will see, the relevant properties of u^ε follow rather easily with these estimates at hand.

Lemma 3.2. *Let $1 \leq p \leq \infty$ and $\phi \in \mathfrak{N}$. Suppose $v \in C([0, T]; W^{1,p}(\mathbb{R}^d, \phi; \mathbb{R}^d))$. Set*

$$\mathcal{T}[v](t, x) = \int_0^t \int_{\mathbb{R}^d} \Phi_\varepsilon(t-s, x-y) (\nabla \cdot v(s, y)) dy ds.$$

Then, for any $1 \leq q < \infty$,

$$\|\mathcal{T}[v](t)\|_{L^p(\mathbb{R}^d, \phi)}^q \leq \kappa_{2,d}^q \left(2\sqrt{\frac{t}{\varepsilon}}\right)^{q-1} \int_0^t \frac{1}{\sqrt{\varepsilon(t-s)}} \|v(s)\|_{L^p(\mathbb{R}^d, \phi)}^q ds,$$

where $\kappa_{2,d} = c_d \frac{d\alpha(d)}{\pi^{d/2}}$ and c_d is defined in Lemma 3.1.

Proof. By Minkowski's integral inequality [31, p.271] and Lemma 3.1,

$$\begin{aligned} \|\mathcal{T}[v](t)\|_{p,\phi}^q &\leq \left(\int_0^t \|\Phi_\varepsilon(t-s) \star (\nabla \cdot v(s, y))\|_{p,\phi} ds \right)^q \\ &\leq \left(\int_0^t \frac{\kappa_{2,d}}{\sqrt{\varepsilon(t-s)}} \|v(s)\|_{p,\phi} ds \right)^q. \end{aligned}$$

If $q = 1$ we are done, so we may assume $1 < q < \infty$. Let r satisfy $1 = r^{-1} + q^{-1}$ and take

$$h(s) := \left(\frac{1}{\sqrt{\varepsilon(t-s)}}\right)^{1/r} \quad \text{and} \quad g(s) := \left(\frac{1}{\sqrt{\varepsilon(t-s)}}\right)^{1-1/r} \|v(s)\|_{p,\phi}.$$

By Hölder's inequality, $\|hg\|_{L^1([0,t])}^q \leq \|g\|_{L^q([0,t])}^q \|h\|_{L^r([0,t])}^q$, and so

$$\|\mathcal{T}[v](t)\|_{L^p(\mathbb{R}^d)}^q \leq (\kappa_{2,d} \|h\|_{L^r([0,t])})^q \int_0^t \frac{1}{\sqrt{\varepsilon(t-s)}} \|v(s)\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)}^q ds.$$

A simple computation yields

$$\|h\|_{L^r([0,t])}^q = \left(\int_0^t \frac{1}{\sqrt{\varepsilon(t-s)}} ds \right)^{q/r} = \left(2\sqrt{\frac{t}{\varepsilon}}\right)^{q-1}.$$

The result follows. \square

Lemma 3.3. *Let $2 \leq p < \infty$ and $\phi \in \mathfrak{N}$. Suppose $v : \Omega \times [0, T] \times Z \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a predictable process satisfying*

$$|v(s, x, z)| \leq K(s, x)M(z),$$

for $M \in L^2(Z)$ and a process $K \in L^2([0, T]; L^p(\Omega; L^p(\mathbb{R}^d, \phi)))$. Define

$$\mathcal{T}[v](t, x) = \int_0^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(t-s, x-y) v(s, y, z) dy W(dz, ds).$$

Then

$$E \left[\|\mathcal{T}[v](t)\|_{p,\phi}^p \right]^{1/p} \leq c_p^{1/p} \kappa_{1,d} \|M\|_{L^2(Z)} \left(\int_0^t E \left[\|K(s)\|_{p,\phi}^p \right]^{2/p} ds \right)^{1/2},$$

where c_p is the constant appearing in the Burkholder-Davis-Gundy inequality and $\kappa_{1,d} = c_{d-1} \frac{d\alpha(d)}{\pi^{d/2}}$, with c_d defined in Lemma 3.1.

Remark 3.1. To prove this result we use the Burkholder-Davis-Gundy inequality for \mathbb{R} valued processes. Using Banach space valued versions [12, 33], one can derive more general estimates.

Proof. First note that

$$M(t, x) = \int_0^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(\tau - s, x - y) v(s, y, z) dy W(dz, ds)$$

is a martingale on $[0, \tau]$, and so by the Burkholder-Davis-Gundy inequality [22],

$$E \left[\|\mathcal{T}[v](t, x)\|^p \right] \leq c_p E \left[\left(\int_0^t \int_Z |\Phi_\varepsilon(t - s) \star v(s, \cdot, z)(x)|^2 d\mu(z) ds \right)^{p/2} \right].$$

Upon integrating in space and applying Minkowski's inequality, it follows that

$$\begin{aligned} \|\mathcal{T}[v](t)\|_{p,\phi}^2 &\leq c_p^{2/p} E \left[\int_{\mathbb{R}^d} \left(\int_0^t \int_Z |\Phi_\varepsilon(t - s) \star v(s, \cdot, z)(x)|^2 d\mu(z) ds \right)^{p/2} \phi(x) dx \right]^{2/p} \\ &\leq c_p^{2/p} \int_0^t \int_Z \left(E \left[\int_{\mathbb{R}^d} |\Phi_\varepsilon(t - s) \star v(s, \cdot, z)(x)|^p \phi(x) dx \right] \right)^{2/p} d\mu(z) ds. \end{aligned}$$

By Lemma 3.1,

$$\|\mathcal{T}[v](t)\|_{p,\phi}^2 \leq c_p^{2/p} \kappa_{1,d}^2 \int_0^t \int_Z E \left[\|v(s, \cdot, z)\|_{p,\phi}^p \right]^{2/p} d\mu(z) ds.$$

By assumption,

$$\int_0^t \int_Z E \left[\|v(s, \cdot, z)\|_{p,\phi}^p \right]^{2/p} d\mu(z) ds \leq \|M\|_{L^2(Z)}^2 \int_0^t E \left[\|K(s)\|_{p,\phi}^p \right]^{2/p} ds.$$

□

For a Banach space E we denote by $\mathcal{X}_{\beta,q,E}$ the space of pathwise continuous predictable processes $u : [0, T] \times \Omega \rightarrow E$ normed by

$$(3.5) \quad \|u\|_{\beta,q,E} := \left(\sup_{t \in [0, T]} e^{-\beta t} E \left[\|u(t)\|_E^q \right] \right)^{1/q}.$$

The existence of a solution to (3.1) is obtained by the Banach fixed-point theorem, applied to the operator

$$\begin{aligned} \mathcal{S}(u)(t, x) &:= \int_{\mathbb{R}^d} \Phi_\varepsilon(t, x - y) u^0(y) dy \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \nabla_x \Phi_\varepsilon(t - s, x - y) \cdot f(u(s, y)) dy ds \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(t - s, x - y) \sigma(y, u(s, y), z) dy W(ds, dz) \end{aligned}$$

on the space $\mathcal{X}_{\beta,2,L^2(\mathbb{R}^d,\phi)}$, with $\beta \in \mathbb{R}$ sufficiently large. It follows that the sequence $\{u_n\}_{n \geq 1}$ defined inductively by $u_0 = 0$ and $u_{n+1} = \mathcal{S}(u_n)$ converges to u^ε in $\mathcal{X}_{\beta,2,L^2(\mathbb{R}^d,\phi)}$ as $n \rightarrow \infty$. By Lemmas 3.2 and 3.3 we are free to use the space $\mathcal{X}_{\beta,p,L^p(\mathbb{R}^d,\phi)}$ for any $2 \leq p < \infty$ in the fixed-point argument [17].

We can use Lemmas 3.2 and 3.3 to deduce a continuous dependence result. To do this, we need a measure of the distance between the coefficients. For the flux function f , the Lipschitz norm is a reasonable choice. Concerning the noise function σ , we introduce the norm $\|\sigma\|_{\text{Lip}} = \|M_\sigma\|_{L^2(Z)}$, where

$$M_\sigma(z) = \sup_{x \in \mathbb{R}^d} \left\{ \sup_{u \in \mathbb{R}} \frac{|\sigma(x, u, z)|}{1 + |u|} \right\} + \sup_{x \in \mathbb{R}^d} \left\{ \sup_{u \neq v} \frac{|\sigma(x, u, z) - \sigma(x, v, z)|}{|u - v|} \right\}.$$

Note that for any σ satisfying (\mathcal{A}_σ) , we have $\|\sigma\|_{\text{Lip}} < \infty$.

Proposition 3.4 (Continuous dependence). *Let $2 \leq p < \infty$ and $\phi \in \mathfrak{N}$. Let f_1, f_2 satisfy (\mathcal{A}_f) and σ_1, σ_2 satisfy (\mathcal{A}_σ) . Suppose $u_1^0, u_2^0 \in L^p(\Omega, \mathcal{F}_0, P; L^p(\mathbb{R}^d, \phi))$. Let u_1^ε and u_2^ε denote the weak solutions of the corresponding problems (3.1) with $f = f_i, \sigma = \sigma_i$, and $u^0 = u_i^0$, for $i = 1, 2$. Then, for $\beta > 0$ sufficiently large there exists a constant $C = C(\beta, \varepsilon, T, f_1, \sigma_1)$ such that*

$$\begin{aligned} \|u_1^\varepsilon - u_2^\varepsilon\|_{\beta, p, L^p(\mathbb{R}^d, \phi)} &\leq C \left(E \left[\|u_1^0 - u_2^0\|_{p, \phi}^p \right] + \|f_1 - f_2\|_{\text{Lip}} \|u_1^\varepsilon\|_{\beta, p, L^p(\mathbb{R}^d, \phi)} \right. \\ &\quad \left. + \|\sigma_1 - \sigma_2\|_{\text{Lip}} \left(\|\phi\|_{L^1(\mathbb{R}^d)} + \|u_1^\varepsilon\|_{\beta, p, L^p(\mathbb{R}^d, \phi)} \right) \right), \end{aligned}$$

where the norm $\|\cdot\|_{\beta, p, L^p(\mathbb{R}^d, \phi)}$ is defined in (3.5).

Proof. By (3.3),

$$\begin{aligned} u_1^\varepsilon(t, x) - u_2^\varepsilon(t, x) &= \int_{\mathbb{R}^d} \Phi_\varepsilon(t, x - y)(u_1^0(y) - u_2^0(y)) dy \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \nabla_x \Phi_\varepsilon(t - s, x - y) \cdot (f_1(u_1^\varepsilon(s, y)) - f_2(u_2^\varepsilon(s, y))) dy ds \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(t - s, x - y) (\sigma_1(y, u_1^\varepsilon(s, y), z) - \sigma_2(y, u_2^\varepsilon(s, y), z)) dy W(ds, dz) \\ &= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3. \end{aligned}$$

By Lemma 3.1,

$$E \left[\|\mathcal{T}_1(t)\|_{p, \phi}^p \right] \leq \kappa_{1, d}^p E \left[\|u_1^0 - u_2^0\|_{p, \phi}^p \right].$$

where $\kappa_{1, d}$ is defined in Lemma 3.3. Hence

$$(3.6) \quad \|\mathcal{T}_1\|_{\beta, p, L^p(\mathbb{R}^d, \phi)} \leq \kappa_{1, d} E \left[\|u_1^0 - u_2^0\|_{p, \phi}^p \right]^{1/p}.$$

Consider \mathcal{T}_2 . Note that

$$\begin{aligned} E \left[\|f_1(u_1^\varepsilon(s)) - f_2(u_2^\varepsilon(s))\|_{p, \phi}^p \right]^{1/p} &\leq \|f_1 - f_2\|_{\text{Lip}} E \left[\|u_1^\varepsilon(s)\|_{p, \phi}^p \right]^{1/p} \\ &\quad + \|f_2\|_{\text{Lip}} E \left[\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\|_{p, \phi}^p \right]^{1/p}. \end{aligned}$$

By Lemma 3.2,

$$\begin{aligned} E \left[\|\mathcal{T}_2(t)\|_{L^p(\mathbb{R}^d)}^p \right] &\leq \kappa_{2, d}^p \left(4\sqrt{\frac{t}{\varepsilon}} \right)^{p-1} \|f_1 - f_2\|_{\text{Lip}}^p \int_0^t \frac{1}{\sqrt{\varepsilon(t-s)}} E \left[\|u_1^\varepsilon(s)\|_{p, \phi}^p \right] ds \\ &\quad + \kappa_{2, d}^p \left(4\sqrt{\frac{t}{\varepsilon}} \right)^{p-1} \|f_2\|_{\text{Lip}}^p \int_0^t \frac{1}{\sqrt{\varepsilon(t-s)}} E \left[\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\|_{p, \phi}^p \right] ds. \end{aligned}$$

Multiplying by $e^{-\beta t}$ and taking the supremum yields

$$(3.7) \quad \|\mathcal{T}_2\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} \leq \delta_{\beta,1} \|f_1 - f_2\|_{\text{Lip}} \|u_1^\varepsilon\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} \\ + \delta_{\beta,1} \|f_2\|_{\text{Lip}} \|u_1^\varepsilon - u_2^\varepsilon\|_{\beta,p,L^p(\mathbb{R}^d,\phi)},$$

where

$$\delta_{\beta,1} = \kappa_{2,d} \sup_{t \in [0,T]} \left(4\sqrt{\frac{t}{\varepsilon}} \right)^{1-1/p} \left(\int_0^t \frac{e^{-\beta(t-s)}}{\sqrt{\varepsilon(t-s)}} ds \right)^{1/p}.$$

Consider \mathcal{T}_3 . First, observe that

$$|\sigma_1(y, u_1^\varepsilon, z) - \sigma_2(y, u_2^\varepsilon, z)| \leq M_{\sigma_1}(z) |u_1^\varepsilon - u_2^\varepsilon| + M_{\sigma_1 - \sigma_2}(z)(1 + |u_1^\varepsilon|).$$

Due to a simple extension of Lemma 3.3,

$$E \left[\|\mathcal{T}_3(t)\|_{L^p(\mathbb{R}^d)}^p \right]^{1/p} \leq c_p^{1/p} \kappa_{1,d} \|\sigma_1\|_{\text{Lip}} \left(\int_0^t E \left[\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\|_{p,\phi}^p \right]^{2/p} ds \right)^{1/2} \\ + c_p^{1/p} \kappa_{1,d} \|\sigma_1 - \sigma_2\|_{\text{Lip}} \left(\int_0^t E \left[\|1 + |u_1^\varepsilon(s)|\|_{p,\phi}^p \right]^{2/p} ds \right)^{1/2}.$$

Multiplication by $e^{-\beta t/p}$ and then taking the supremum yields

$$(3.8) \quad \|\mathcal{T}_3\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} \leq \delta_{\beta,2} \|\sigma_1\|_{\text{Lip}} \|u_1^\varepsilon - u_2^\varepsilon\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} \\ + \delta_{\beta,2} \|\sigma_1 - \sigma_2\|_{\text{Lip}} (\|\phi\|_{L^1(\mathbb{R}^d)} + \|u_1^\varepsilon\|_{\beta,p,L^p(\mathbb{R}^d,\phi)}),$$

where

$$\delta_{\beta,2} = c_p^{1/p} \kappa_{1,d} \sup_{t \in [0,T]} \left(\int_0^t e^{-\beta 2(t-s)/p} ds \right)^{1/2} \leq c_p^{1/p} \sqrt{\frac{p}{2\beta}}.$$

Here we used that $\|1\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} = \|\phi\|_{L^1(\mathbb{R}^d)}$. Combine (3.6), (3.7), and (3.8), and note that $\delta_{\beta,i} \rightarrow 0$ as $\beta \rightarrow \infty$ for $i = 1, 2$. This concludes the proof. \square

In order to apply Itô's formula to the process $t \mapsto u^\varepsilon(t, x)$ we need to know that the weak (mild) solution u^ε of (3.1) is in fact a strong solution. The following result provides the existence of weak derivatives.

Proposition 3.5. *Fix $\phi \in \mathfrak{N}$ and a multiindex α . Make the following assumptions:*

- (i) *The flux-function f belongs to $C^{|\alpha|}(\mathbb{R}; \mathbb{R}^d)$ with all bounded derivatives.*
- (ii) *For each fixed $z \in Z$, $(x, u) \mapsto \sigma(x, u, z)$ belongs to $C^{|\alpha|}(\mathbb{R}^d \times \mathbb{R})$ and for each $0 < \zeta \leq \alpha$ and $0 \leq n \leq |\alpha|$ there exists $M_{\zeta,n} \in L^2(Z)$ such that*

$$\begin{cases} \partial_1^\zeta \partial_2^n \sigma(x, u, z) \leq M_{\zeta,n}(z), & 1 \leq n \leq |\alpha|, \\ \partial_1^\zeta \sigma(x, u, z) \leq M_{\zeta,0}(z)(1 + |u|). \end{cases}$$

- (iii) *The initial function u^0 satisfies for all $\zeta \leq \alpha$,*

$$E \left[\|\partial^\zeta u^0\|_{p,\phi}^p \right] < \infty \quad (2 \leq p < \infty).$$

Let u^ε be the weak solution of (3.1). For any $\zeta \leq \alpha$, there exists a predictable process

$$(t, x, \omega) \mapsto \partial_x^\zeta u^\varepsilon(t, x, \omega) \text{ in } L^p([0, T] \times \Omega; L^p(\mathbb{R}^d, \phi))$$

such that for all $\varphi \in C_c^\infty(\Pi_T)$,

$$\iint_{\Pi_T} \partial_x^\zeta u^\varepsilon \varphi \, dx dt = (-1)^{|\zeta|} \iint_{\Pi_T} u^\varepsilon \partial_x^\zeta \varphi \, dx dt, \quad dP\text{-almost surely.}$$

To prove Proposition 3.5 we apply

Lemma 3.6. *Let $\sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R})$ and suppose $u \in C^\infty(\mathbb{R}^d)$. For any multiindex α , let $\partial_x^\alpha := \prod_k \partial_{x_k}^{\alpha_k}$. Then*

$$\partial_x^\alpha \sigma(x, u(x)) = \sum_{\zeta \leq \alpha} \sum_{\gamma \in \pi(\zeta)} C_{\gamma, \alpha} \partial_1^{\alpha - \zeta} \partial_2^{|\gamma|} \sigma(x, u(x)) \prod_{i=1}^{|\gamma|} \partial_x^{\gamma^i} u(x).$$

Here $\pi(\zeta)$ denote all partitions of ζ , i.e., all multiindices $\gamma = \{\gamma^i\}_{i \geq 1}$ such that $\sum \gamma^i = \zeta$. Furthermore, $|\gamma|$ denotes the number of terms in the partition γ .

Remark 3.2. Whenever $\zeta \neq 0$ we assume that the terms γ^i in the partition γ satisfies $\gamma^i \neq 0$. If $\zeta = 0$ we let $\gamma = \gamma^1 = 0$ and by convention let $|\gamma| = 0$.

Proof. One may prove by induction and the chain-rule that

$$\partial_x^\alpha \sigma(x, u(x)) = [(\partial_z + \partial_y)^\alpha \sigma(y, u(z))]_{y=x, z=x}.$$

By the binomial theorem,

$$(\partial_z + \partial_y)^\alpha \sigma(y, u(z)) = \sum_{\zeta \leq \alpha} \binom{\alpha}{\zeta} \partial_y^{\alpha - \zeta} \partial_z^\zeta \sigma(y, u(z)).$$

Thanks to [19, Propositions 1 and 2], it follows that

$$\partial_z^\zeta \sigma(y, u(z)) = \sum_{\gamma \in \pi(\zeta)} M_\gamma \partial_u^{|\gamma|} \sigma(y, u(z)) \prod_i \partial_z^{\gamma^i} u(z),$$

where M_γ is a constant. The result follows by combining the above identities. \square

Proof of Proposition 3.5. We divide the proof into two steps.

Step 1 (uniform estimates on $\{u^n\}_{n \geq 1}$). For all $\zeta < \alpha$, suppose

$$(3.9) \quad \sup_{0 \leq s \leq T} E \left[\|\partial^\zeta u^n(s)\|_{p, \phi}^p \right] \leq C_{\zeta, p} \quad (2 \leq p < \infty).$$

We claim that there exists a constant $C \geq 0$, independent of β and n , and a number $\delta_\beta \geq 0$ such that

$$(3.10) \quad \|\partial^\alpha u^{n+1}\|_{\beta, p, L^p(\mathbb{R}^d, \phi)} \leq C + \delta_\beta \|\partial^\alpha u^n\|_{\beta, p, L^p(\mathbb{R}^d, \phi)},$$

where $\delta_\beta < 1$ for some $\beta > 0$, and $\|\cdot\|_{\beta, p, L^p(\mathbb{R}^d, \phi)}$ is defined in (3.5). Given (3.10), it follows that

$$\|\partial_x^\alpha u^n\|_{\beta, p, L^p(\mathbb{R}^d, \phi)} \leq C \sum_{k=0}^{n-1} \delta_\beta^k \leq \frac{C}{1 - \delta_\beta},$$

and we are done.

To establish (3.10), observe that the weak derivative satisfies

$$\begin{aligned} \partial_x^\alpha u^{n+1}(t, x) &= \int_{\mathbb{R}^d} \Phi_\varepsilon(t, x - y) \partial_y^\alpha u^0(y) dy \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \nabla_x \Phi_\varepsilon(t - s, x - y) \cdot \partial_y^\alpha f(u^n(s, y)) dy ds \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(t - s, x - y) \partial_y^\alpha \sigma(y, u^n(s, y), z) dy W(ds, dz) \\ &=: \mathcal{T}_1(t, x) + \mathcal{T}_2(t, x) + \mathcal{T}_3(t, x). \end{aligned}$$

To justify this, multiply by a test function and apply the Fubini theorem [39, p.297]. By the triangle inequality we may estimate each term separately.

Consider \mathcal{T}_1 . By Lemma 3.1,

$$E \left[\|\mathcal{T}_1(t)\|_{p, \phi}^p \right] \leq \kappa_{1, d}^p E \left[\|\partial^\alpha u^0\|_{p, \phi}^p \right],$$

where $\kappa_{1,d}$ is defined in Lemma 3.3. By assumption (iii) it follows that there exists a constant C such that

$$(3.11) \quad \|\mathcal{T}_1\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} \leq C.$$

Consider \mathcal{T}_2 . By Lemma 3.2,

$$\begin{aligned} \|\mathcal{T}_2\|_{\beta,p,L^p(\mathbb{R}^d,\phi)}^p &= \sup_{t \in [0,T]} e^{-\beta t} E \left[\|\mathcal{T}_2(t)\|_{p,\phi}^p \right] \\ &\leq \kappa_{2,d}^p \left(2\sqrt{\frac{T}{\varepsilon}} \right)^{p-1} \sup_{t \in [0,T]} \int_0^t \frac{e^{-\beta(t-s)}}{\sqrt{\varepsilon(t-s)}} e^{-\beta s} E \left[\|\partial^\alpha f(u^n(s))\|_{p,\phi}^p \right] ds \\ &\leq \kappa_{2,d}^p \left(2\sqrt{\frac{T}{\varepsilon}} \right)^{p-1} \left(\int_0^T \frac{e^{-\beta(t-s)}}{\sqrt{\varepsilon(t-s)}} ds \right) \|\partial^\alpha f(u^n(s))\|_{\beta,p,L^p(\mathbb{R}^d,\phi)}^p. \end{aligned}$$

By Lemma 3.6, the triangle inequality, and the generalized Hölder's inequality,

$$\begin{aligned} E \left[\|\partial^\alpha f(u^n(s))\|_{p,\phi}^p \right]^{1/p} &\leq \sum_{\gamma \in \pi(\alpha)} C_{\gamma,\alpha} \|\partial^{|\gamma|} f\|_\infty E \left[\left\| \prod_{i=1}^{|\gamma|} \partial^{\gamma^i} u^n(s) \right\|_{p,\phi}^p \right]^{1/p} \\ &\leq \sum_{\gamma \in \pi(\alpha)} C_{\gamma,\alpha} \|\partial^{|\gamma|} f\|_\infty \prod_{i=1}^{|\gamma|} E \left[\|\partial^{\gamma^i} u^n(s)\|_{q_i,\phi}^{q_i} \right]^{1/q_i}, \end{aligned}$$

where $q_i = |\alpha|p/|\gamma_i|$. By assumption there exists a constant C , independent of n , such that

$$E \left[\|\partial^{\gamma^i} u^n(s)\|_{q_i,\phi}^{q_i} \right] \leq C,$$

for all terms where $\gamma^i < \alpha$. Since $C_{\alpha,\alpha} = 1$, there is another constant C such that

$$E \left[\|\partial^\alpha f(u^n(s))\|_{p,\phi}^p \right]^{1/p} \leq C + \|f'\|_\infty E \left[\|\partial^\alpha u^n(s)\|_{p,\phi}^p \right]^{1/p},$$

for all $n \geq 1$. Multiply by $e^{-\beta t/p}$ and take the supremum to obtain

$$\|\partial^\alpha f(u^n)\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} \leq C + \|f'\|_\infty \|\partial^\alpha u^n\|_{\beta,p,L^p(\mathbb{R}^d,\phi)}.$$

It follows that

$$(3.12) \quad \begin{aligned} \|\mathcal{T}_2\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} &\leq c_d \left(2\sqrt{\frac{T}{\varepsilon}} \right)^{1-1/p} \left(\int_0^T \frac{e^{-\beta(t-s)}}{\sqrt{\varepsilon(t-s)}} ds \right)^{1/p} \\ &\quad \times \left(C + \|f'\|_\infty \|\partial^\alpha u^n\|_{\beta,p,L^p(\mathbb{R}^d,\phi)} \right). \end{aligned}$$

Consider \mathcal{T}_3 . By Lemma 3.6

$$\begin{aligned} \mathcal{T}_3(t,x) &= \sum_{\zeta \leq \alpha} \sum_{\gamma \in \pi(\zeta)} C_{\gamma,\alpha} \int_0^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(t-s, x-y) \\ &\quad \times \partial_1^{\alpha-\zeta} \partial_2^{|\gamma|} \sigma(y, u^n(s,y), z) \left(\prod_{i=1}^{|\gamma|} \partial_y^{\gamma^i} u^n(s,y) \right) dy W(ds, dz) \\ &= \mathcal{T}_3^0(t,x) + \mathcal{T}_3^1(t,x), \end{aligned}$$

where \mathcal{T}_3^0 contains the term with $\zeta = 0$. By Lemma 3.3, assumption (ii), and the generalised Hölder inequality,

$$E \left[\left\| \mathcal{T}_3^1(t) \right\|_{L^p(\mathbb{R}^d)}^p \right]^{1/p} \leq \sum_{0 < \zeta \leq \alpha} \sum_{\gamma \in \pi(\zeta)} C_{\gamma, \alpha} c_p^{1/p} \kappa_{1,d} \|M_{\alpha - \zeta, |\gamma|}\|_{L^2(Z)} \\ \times \left(\int_0^t \prod_{i=1}^{|\gamma|} E \left[\left\| \partial^{\gamma^i} u^n(s) \right\|_{L^{q_i}(\mathbb{R}^d)}^{q_i} \right]^{2/q_i} ds \right)^{1/2},$$

where $q_i = |\zeta|p/|\gamma_i|$. The term \mathcal{T}_3^0 is estimated similarly by applying the second case of assumption (ii). It follows from (3.9) that there exists a constant C such that

$$E \left[\left\| \mathcal{T}_3(t) \right\|_{p, \phi}^p \right]^{1/p} \leq C + c_p^{1/p} \kappa_{1,d} \|M_{0,1}\|_{L^2(Z)} \left(\int_0^t E \left[\left\| \partial^\alpha u^n(s) \right\|_{p, \phi}^p \right]^{2/p} ds \right)^{1/2}.$$

Multiplying by $(e^{-\beta t})^{1/p}$ and taking the supremum yields

(3.13)

$$\| \mathcal{T}_3 \|_{\beta, p, L^p(\mathbb{R}^d, \phi)} \leq C + c_p^{1/p} \kappa_{1,d} \|M_{0,1}\|_{L^2(Z)} \\ \times \sup_{t \in [0, T]} \left(\int_0^t e^{-2\beta(t-s)/p} \left(e^{-\beta s} E \left[\left\| \partial^\alpha u^n(s) \right\|_{p, \phi}^p \right] \right)^{2/p} ds \right)^{1/2} \\ \leq C + c_p^{1/p} \kappa_{1,d} \|M_{0,1}\|_{L^2(Z)} \sqrt{\frac{p}{2\beta}} \| \partial^\alpha u^n \|_{\beta, p, L^p(\mathbb{R}^d, \phi)}.$$

Combining (3.11), (3.12), and (3.13) we obtain inequality (3.10), where

$$\delta_\beta = \kappa_{2,d} \left(2\sqrt{\frac{T}{\varepsilon}} \right)^{1-1/p} \left(\int_0^T \frac{e^{-\beta(t-s)}}{\sqrt{\varepsilon(t-s)}} ds \right)^{1/p} \|f'\|_\infty \\ + c_p^{1/p} \kappa_{1,d} \|M_{0,1}\|_{L^2(Z)} \sqrt{\frac{p}{2\beta}}.$$

It is clear that $\delta_\beta \rightarrow 0$ as $\beta \rightarrow \infty$ and so (3.10) follows. By induction, estimate (3.9) holds for all $\zeta \leq \alpha$.

Step 2 (convergence of u^n). Fix $\zeta \leq \alpha$. We apply Theorem 6.9 to the family $\{\partial^\zeta u^n\}_{n \geq 1}$ on the space

$$(X, \mathcal{A}, \mu) = (\Omega \times \Pi_T, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d), dP \otimes dt \otimes \phi(x) dx).$$

By means of (3.9),

$$\sup_{n \geq 1} \left\{ E \left[\iint_{\Pi_T} |\partial^\zeta u^n(t, x)|^2 \phi(x) dx dt \right] \right\} < \infty.$$

Hence, $\{\partial^\zeta u^n\}_{n \geq 1}$ has a Young measure limit $\nu^\zeta \in \mathcal{Y}(\Omega \times \Pi_T)$. Next, define $\partial^\zeta u^\varepsilon(t, x, \omega) := \int_{\mathbb{R}} d\nu_{t,x,\omega}^\zeta$. By definition, the limit has a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable version. Furthermore, $\partial^\zeta u^\varepsilon \in L^p(\Omega \times [0, T]; L^p(\mathbb{R}^d, \phi))$, cf. proof of Theorem 4.1 and Lemma 6.8. Let us show that $\partial^\zeta u^\varepsilon$ is the weak derivative of u^ε . To this end, observe that

$$\iint_{\Pi_T} u^n(t, x) \partial^\zeta \varphi dx dt = (-1)^{|\zeta|} \iint_{\Pi_T} \partial^\zeta u^n(t, x) \varphi dx dt,$$

for any $\varphi \in C_c^\infty(\mathbb{R}^d)$. By Lemma 6.11(ii) and Theorem 6.10, there is a subsequence $\{n(j)\}_{j \geq 1}$ such that for any $A \in \mathcal{F}$,

$$\begin{aligned} \lim_{j \rightarrow \infty} E \left[\mathbb{1}_A \iint_{\Pi_T} \partial^\zeta u^{n(j)} \varphi \, dx dt \right] &= E \left[\iint_{\Pi_T} \int_{\mathbb{R}} \varphi(t, x) \mathbb{1}_A(\omega) \, d\nu_{t,x,\omega}^\zeta(\xi) \, dx dt \right] \\ &= E \left[\mathbb{1}_A \iint_{\Pi_T} \partial^\zeta u^\varepsilon \varphi \, dx dt \right]. \end{aligned}$$

As $u^n \rightarrow u^\varepsilon$ in $\mathcal{X}_{\beta,2,L^2(\mathbb{R}^d,\phi)}$, it follows that

$$\lim_{n \rightarrow \infty} E \left[\mathbb{1}_A \iint_{\Pi_T} u^n \partial^\zeta \varphi \, dx dt \right] = E \left[\mathbb{1}_A \iint_{\Pi_T} u^\varepsilon \partial^\zeta \varphi \, dx dt \right].$$

The conclude the proof. \square

3.2. Malliavin differentiability. In this subsection we will establish the Malliavin differentiability of the viscous approximation. Furthermore, we observe that the Malliavin derivative satisfies a linear parabolic equation. This equation is then applied to show that $D_{r,z} u^\varepsilon(t, x) \rightarrow \sigma(x, u^\varepsilon(r, x), z)$ as $t \downarrow r$ in a weak sense (Lemma 3.8); a property which is crucial in the proof of uniqueness.

Proposition 3.7 (Malliavin derivative of viscous approximation). *Suppose (\mathcal{A}_f) and (\mathcal{A}_σ) are satisfied. Fix $\phi \in \mathfrak{N}$ and $u^0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$. Let u^ε be the solution of (3.1). Then u^ε belongs to $\mathbb{D}^{1,2}(L^2([0, T]; L^2(\mathbb{R}^d, \phi)))$ and*

$$(3.14) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{\mathbb{D}^{1,2}(L^2(\mathbb{R}^d, \phi))} < \infty.$$

Furthermore, for $dr \otimes d\mu$ -a.a. (r, z) , the $L^2(\mathbb{R}^d, \phi)$ -valued process $\{D_{r,z} u^\varepsilon(t)\}_{t > r}$ is a predictable weak solution of

$$(3.15) \quad \begin{cases} dw + \nabla \cdot (f'(u^\varepsilon)w) \, dt = \int_Z \partial_2 \sigma(x, u^\varepsilon, z') w \, W(dt, dz') + \varepsilon \Delta w \, dt, & t \in [r, T], \\ w(r, x, z) = \sigma(x, u^\varepsilon(r, x), z), \end{cases}$$

while $D_{r,z} u^\varepsilon(t) = 0$ if $r > t$. Furthermore

$$(3.16) \quad \operatorname{ess\,sup}_{r \in [0, T]} \left\{ \sup_{t \in [0, T]} E \|D_r u^\varepsilon(t)\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 \right\} < \infty.$$

Remark 3.3. Let $w_{r,z}(t, x) = D_{r,z} u^\varepsilon(t, x)$, $t > r$. Estimate (3.16) may be seen as a consequence of the Grönwall-type estimate

$$E \left[\|w_{r,z}(t)\|_{L^2(\mathbb{R}^d, \phi)}^2 \right] \leq \left(1 + C e^{C(t-r)} \right) E \left[\|w_{r,z}(r)\|_{L^2(\mathbb{R}^d, \phi)}^2 \right],$$

for $t \geq r$. From the perspective of a uniqueness result (see Lemma 3.8), it is of interest to know whether one can derive such estimates independent of ε .

Proof. We divide the proof into two steps.

Step 1 (uniform bounds). Consider the Picard approximation $\{u^n\}_{n \geq 1}$ of u^ε . We want to prove that

$$(3.17) \quad \sup_{0 \leq t \leq T} \|u^n(t)\|_{\mathbb{D}^{1,2}(L^2(\mathbb{R}^d, \phi))} \leq C, \quad \text{for all } n \geq 1.$$

By [27, Proposition 1.2.4, Proposition 1.3.8, Proposition 1.2.8],

$$\begin{aligned}
(3.18) \quad & D_{r,z}u^{n+1}(t,x) \\
&= \int_{\mathbb{R}^d} \Phi_\varepsilon(t-r, x-y)\sigma(y, u^n(r,y), z) dy \\
&\quad - \int_r^t \int_{\mathbb{R}^d} \nabla_x \Phi_\varepsilon(t-s, x-y) \cdot f'(u^n(s,y)) D_{r,z}u^n(s,y) dy ds \\
&\quad + \int_r^t \int_Z \int_{\mathbb{R}^d} \Phi_\varepsilon(t-s, x-y) \partial_2 \sigma(y, u^n(s,y), z') D_{r,z}u^n(s,y) dy W(ds, dz') \\
&=: \mathcal{T}_1^n + \mathcal{T}_2^n + \mathcal{T}_3^n,
\end{aligned}$$

for all $r \in (0, t]$. Whenever $r > t$, $D_r u^{n+1}(t) = 0$ since u^{n+1} is adapted, see [27, Corollary 1.2.1]. Hence, given (3.17), we conclude that $\{D_{r,z}u^\varepsilon\}_{t \geq r}$ is a mild solution of (3.15) for almost all $(r, z) \in (0, t] \times Z$. Next, we apply [30, Theorem 9.15] to infer that the mild solution is also a weak solution.

Let us prove (3.17). Recall that

$$\|u(t)\|_{\mathbb{D}^{1,2}(L^2(\mathbb{R}^d, \phi))}^2 = E \left[\|u(t)\|_{2, \phi}^2 \right] + E \left[\|Du(t)\|_{H \otimes L^2(\mathbb{R}^d, \phi)}^2 \right],$$

where H denotes the space $L^2([0, T] \times Z)$. To obtain a uniform bound we want to establish the existence of a constant C such that

$$(3.19) \quad \|Du^{n+1}\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)} \leq C + \delta_\beta \|Du^n\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)} \quad \text{where } \delta_\beta < 1,$$

for some $\beta > 0$. Under these assumptions it follows that

$$\|Du^n\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)} \leq C \sum_{k=0}^{n-1} \delta_\beta^k \leq \frac{C}{1 - \delta_\beta},$$

and (3.17) follows. To establish (3.19) we proceed by (3.18) to estimate each term separately.

Consider \mathcal{T}_1^n . By Lemma 3.1 and assumption (\mathcal{A}_σ) ,

$$\begin{aligned}
\|\mathcal{T}_1^n(r, t)\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 &= \int_Z \|\Phi_\varepsilon(t-r) \star \sigma(\cdot, u^n(r), z)\|_{2, \phi}^2 d\mu(z) \\
&\leq \kappa_{1,d}^2 \|M\|_{L^2(Z)}^2 (\|\phi\|_{L^1(\mathbb{R}^d)} + \|u^n(r)\|_{2, \phi})^2,
\end{aligned}$$

for each $0 \leq r < t$. It follows that

$$(3.20)$$

$$\begin{aligned}
(3.21) \quad & \|\mathcal{T}_1^n\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)} \\
&\leq \kappa_{1,d} \|M\|_{L^2(Z)} \left(\sup_{0 \leq t \leq T} e^{-\beta t} E \left[\int_0^t (\|\phi\|_{L^1(\mathbb{R}^d)} + \|u^n(r)\|_{2, \phi})^2 dr \right] \right)^{1/2} \leq C.
\end{aligned}$$

Let us consider \mathcal{T}_2^n . By Lemma 3.2,

$$\begin{aligned}
\|\mathcal{T}_2^n(r, t)\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 &= \int_Z \left\| \int_r^t \nabla \Phi_\varepsilon(t-s) \star f'(u^n(s)) D_{r,z}u^n(s) ds \right\|_{2, \phi}^2 d\mu(z) \\
&\leq 2\kappa_{2,d}^2 \|f'\|_\infty^2 \int_r^t \frac{1}{\sqrt{\varepsilon(t-s)}} \int_Z \|D_{r,z}u^n(s)\|_{2, \phi}^2 d\mu(z) ds.
\end{aligned}$$

Multiplication by $e^{-\beta t}$ and integration in r yields

$$\begin{aligned} & e^{-\beta t} E \left[\|\mathcal{T}_2^n(t)\|_{H \otimes L^2(\mathbb{R}^d, \phi)}^2 \right] \\ & \leq 2\kappa_{2,d}^2 \|f'\|_\infty^2 \sqrt{\frac{t}{\varepsilon}} \int_0^t \frac{e^{-\beta(t-s)}}{\sqrt{\varepsilon(t-s)}} e^{-\beta s} E \left[\|Du^n(s)\|_{H \otimes L^2(\mathbb{R}^d, \phi)}^2 \right] ds. \end{aligned}$$

It follows that

$$(3.22) \quad \|\mathcal{T}_2^n\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)} \leq \kappa_{2,d} \|f'\|_\infty \left(2\sqrt{\frac{T}{\varepsilon}} \int_0^T \frac{e^{-\beta(T-s)}}{\sqrt{\varepsilon(T-s)}} ds \right)^{1/2} \times \|Du^n(s)\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)}.$$

Consider \mathcal{T}_3^n . Due to (\mathcal{A}_σ) ,

$$|\partial_2 \sigma(x, u, z') D_{r,z} u^n(s, y)| \leq M(z') |D_{r,z} u^n(s, y)|.$$

By Lemma 3.3,

$$\begin{aligned} & E \left[\|\mathcal{T}_3^n(r, t)\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 \right] \\ & \leq c_2 \kappa_{1,d}^2 \|M\|_{L^2(Z)}^2 \int_r^t E \left[\|D_r u^n(s)\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 \right] ds. \end{aligned}$$

Integrate in r and multiply by $e^{-\beta t}$ to obtain

$$\begin{aligned} & e^{-\beta t} E \left[\|\mathcal{T}_3^n(t)\|_{H \otimes L^2(\mathbb{R}^d, \phi)}^2 \right] \\ & \leq c_2 \kappa_{1,d}^2 \|M\|_{L^2(Z)}^2 \int_0^t e^{-\beta(t-s)} e^{-\beta s} E \left[\|Du^n(s)\|_{H \otimes L^2(\mathbb{R}^d, \phi)}^2 \right] ds. \end{aligned}$$

Hence

$$(3.23) \quad \|\mathcal{T}_3^n\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)} \leq c_2^{1/2} \kappa_{1,d} \|M\|_{L^2(Z)} \frac{1}{\sqrt{\beta}} \|Du^n\|_{\beta, 2, H \otimes L^2(\mathbb{R}^d, \phi)}.$$

Combining (3.20), (3.22), and (3.23) yields (3.19) with

$$\delta_\beta = \kappa_{2,d} \|f'\|_\infty \left(2\sqrt{\frac{T}{\varepsilon}} \int_0^T \frac{e^{-\beta(T-s)}}{\sqrt{\varepsilon(T-s)}} ds \right)^{1/2} + c_2^{1/2} \kappa_{1,d} \|M\|_{L^2(Z)} \frac{1}{\sqrt{\beta}},$$

where $\kappa_{2,d}$ is the constant from Lemma 3.2, while c_2 and $\kappa_{1,d}$ are the constants from Lemma 3.3. Note that $\delta_\beta \downarrow 0$ as $\beta \rightarrow \infty$. Leaving out the integration in r throughout Step 1, we deduce the estimate

$$(3.24) \quad \|D_r u^n\|_{\beta, 2, L^2(Z) \otimes L^2(\mathbb{R}^d, \phi)} \leq \frac{C}{1 - \delta_\beta}.$$

Step 2 (convergence). Let E denote the space $L^2([0, T]; L^2(\mathbb{R}^d, \phi))$ and recall that $H = L^2(Z \times [0, T])$. Consider $\{u_n\}_{n \geq 1}$ as a sequence in $\mathbb{D}^{1,2}(E)$. By (3.17) and the Hilbert space valued version of [27, Lemma 1.2.3] (see [8, Lemma 5.2]), it follows that u^ε belongs to $\mathbb{D}^{1,2}(E)$ and that $Du^n \rightharpoonup Du^\varepsilon$ in $L^2(\Omega; H \otimes E)$, i.e., for any $h \in H, \varphi \in E$, and $V \in L^2(\Omega)$,

$$E [\langle Du^n, h \otimes \varphi \rangle_{H \otimes E} V] \rightarrow E [\langle Du^\varepsilon, h \otimes \varphi \rangle_{H \otimes E} V].$$

It follows that the map

$$(t, \omega) \mapsto Du^\varepsilon(t, \omega) \in L^2(H \otimes L^2(\mathbb{R}^d, \phi))$$

is \mathcal{P}^* -measurable. Note that Lemma 6.8 extends to cover this case so that $(t, \omega) \mapsto D_{r,z} u^\varepsilon(t, \omega)$ is \mathcal{P}^* -measurable $dr \otimes d\mu$ almost all $(r, z) \in [0, T] \times Z$.

For each fixed $t \in [0, T]$, we can conclude by (3.17) as above that $u^\varepsilon(t) \in \mathbb{D}^{1,2}(L^2(\mathbb{R}^d, \phi))$, where $Du^n(t) \rightharpoonup Du^\varepsilon(t)$ along some subsequence. Besides, this

limit agrees dt -almost everywhere with the evaluation of the limit taken in $\mathbb{D}^{1,2}(E)$. This follows by definition for smooth Hilbert space valued random variables and may be extended to the general case by approximation. The weak lower semi-continuity of the norm yields (3.14). Similarly, we may apply (3.24) and the Banach-Alaoglu theorem to extract a weakly convergent subsequence in the space $L^\infty([0, T]; \mathcal{X}_{\beta, 2, L^2(Z) \otimes L^2(\mathbb{R}^d, \phi)})$. This yields the bound (3.16).

As above, $u^\varepsilon(t, x) \in \mathbb{D}^{1,2}$ for $dt \otimes dx$ -almost all (t, x) , and for such (t, x) we have $D_{r,z}u^\varepsilon(t, x) = Du^\varepsilon(t, x, r, z)$ for $d\mu \otimes dr$ almost all (r, z) where $Du^\varepsilon(t, x, r, z)$ denotes the evaluation of the limit taken in $\mathbb{D}^{1,2}(E)$. Taking the Malliavin derivative of (3.3) (as above on u^{n+1}) it follows that $t \mapsto D_{r,z}u^\varepsilon(t)$ is a mild solution of (3.15) for $dr \otimes d\mu$ almost all (r, z) . To conclude that it is a weak solution we can easily verify conditions (F) and (G), with $F(w, t) = \nabla \cdot (f'(u^\varepsilon(t))w)$ and

$$G(w, t)h(x) = \int_Z \partial_2 \sigma(x, u^\varepsilon(t, x), z)w(x)h(z) d\mu(z).$$

□

The next result concerns the limit of $D_{r,z}u^\varepsilon(t, x)$ as $t \downarrow r$. In view of Lemma 3.7, this is a question about the satisfaction of the initial condition for (3.15).

Lemma 3.8. *Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be non-negative. In the setting of Proposition 3.7, for $\Psi \in L^2(\Omega \times Z; L^2(\mathbb{R}^d, \phi))$, set*

$$\mathcal{T}_{r_0}(\Psi) := E \left[\iiint_{Z \times \Pi_T} (D_{r,z}u^\varepsilon(t, x) - \sigma(x, u^\varepsilon(r, x), z)) J_{r_0}^+(t-r) \Psi \phi dt dx d\mu(z) \right].$$

Then there exists a constant C independent of r_0 such that

$$(3.25) \quad |\mathcal{T}_{r_0}(\Psi)| \leq CE \left[\|\Psi\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 \right]^{1/2}.$$

and $\lim_{r_0 \downarrow 0} \mathcal{T}_{r_0}(\Psi) = 0$ for dr -almost all $r \in [0, T]$.

Proof. Note that $\mathcal{T}_{r_0} = \mathcal{T}_{r_0}^1 - \mathcal{T}_{r_0}^2$, and consider each term separately. By Hölder's inequality,

$$\begin{aligned} |\mathcal{T}_{r_0}^1| &= \left| \int_0^T E \left[\int_Z \int_{\mathbb{R}^d} \Psi(x, z) D_{r,z}u^\varepsilon(t, x) \phi(x) d\mu(z) dx \right] J_{r_0}^+(t-r) dt \right| \\ &\leq \operatorname{ess\,sup}_{t \in [0, T]} E \left[\int_Z \int_{\mathbb{R}^d} \Psi(x, z) D_{r,z}u^\varepsilon(t, x) \phi(x) d\mu(z) dx \right] \\ &\leq E \left[\|\Psi\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 \right]^{1/2} \operatorname{ess\,sup}_{t \in [0, T]} \left\{ E \left[\|D_r u^\varepsilon(t)\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 \right]^{1/2} \right\}. \end{aligned}$$

Furthermore, due to (\mathcal{A}_σ) ,

$$\begin{aligned} \mathcal{T}_{r_0}^2 &= E \left[\int_Z \int_{\mathbb{R}^d} \Psi(x, z) \sigma(x, u^\varepsilon(r, x), z) \phi(x) dx d\mu(z) \right] \\ &\leq \|M\|_{L^2(Z)} E \left[\|\Psi\|_{L^2(Z; L^2(\mathbb{R}^d, \phi))}^2 \right]^{1/2} E \left[\|1 + |u^\varepsilon(r)|\|_{L^2(\mathbb{R}^d, \phi)}^2 \right]^{1/2}. \end{aligned}$$

The uniform bound (3.25) follows by (3.16) and (3.2). Note that $\Psi \mapsto \mathcal{T}_{r_0}(\Psi)$ is a linear functional on $L^2(\Omega \times Z; L^2(\mathbb{R}^d, \phi))$, for each $r_0 > 0$. By (3.25) the family $\{\mathcal{T}_{r_0}\}_{r_0 > 0}$ is uniformly continuous. Hence, by approximation, it suffices to prove the lemma for Ψ smooth in x with bounded derivatives. Let

$$\varphi(t, x, z) = \Psi(x, z) \phi(x) \xi_{r_0, r}(t), \quad \xi_{r_0, r}(t) = 1 - \int_0^t J_{r_0}^+(\sigma - r) d\sigma.$$

By Proposition 3.7,

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} \sigma(x, u^\varepsilon(r, x), z) \varphi(r, x, z) dx \\
&+ \int_r^T \int_{\mathbb{R}^d} D_{r,z} u^\varepsilon(t, x) \partial_t \varphi(t, x, z) dx dt \\
&+ \int_r^T \int_{\mathbb{R}^d} f'(u^\varepsilon(t, x)) D_{r,z} u^\varepsilon(t, x) \cdot \nabla \varphi(t, x, z) dx dt \\
&+ \varepsilon \int_r^T \int_{\mathbb{R}^d} D_{r,z} u^\varepsilon(t, x) \Delta \varphi(t, x, z) dx dt \\
&+ \int_r^T \int_Z \int_{\mathbb{R}^d} \partial_2 \sigma(x, u^\varepsilon(t, x), z') D_{r,z} u^\varepsilon(t, x) \varphi(t, x, z) dx W(dz', dt),
\end{aligned}$$

$dr \otimes d\mu \otimes dP$ -almost all (r, z, ω) . Note that

$$\partial_t \varphi(t, x, z) = -\Psi(x, z) \phi(x) J_{r_0}^+(t - r).$$

Taking expectations and integrating in z we obtain

$$\begin{aligned}
\mathcal{T}_{r_0}(\Psi) &= E \left[\int_Z \int_r^T \int_{\mathbb{R}^d} f'(u^\varepsilon(t, x)) D_{r,z} u^\varepsilon(t, x) \cdot \nabla(\Psi \phi) \xi_{r_0}(t, r) dx dt d\mu(z) \right] \\
&+ \varepsilon E \left[\int_Z \int_r^T \int_{\mathbb{R}^d} D_{r,z} u^\varepsilon(t, x) \Delta(\Psi \phi) \xi_{r_0}(t, r) dx dt d\mu(z) \right].
\end{aligned}$$

As $\lim_{r_0 \downarrow 0} \xi_{r_0, r}(t) = 0$ for all $t > r$, it follows by the dominated convergence theorem that $\lim_{r_0 \downarrow 0} \mathcal{T}_{r_0}(\Psi) = 0$. \square

4. EXISTENCE OF ENTROPY SOLUTIONS

We will now prove the existence entropy solutions, as defined in Section 2.

Theorem 4.1. *Fix $\phi \in \mathfrak{N}$ and $2 \leq p < \infty$. Suppose $u^0 \in L^p(\Omega, \mathcal{F}_0, P; L^p(\mathbb{R}^d, \phi))$, and (\mathcal{A}_f) and (\mathcal{A}_σ) hold. There exists a Young measure-valued entropy solution u of (1.1) in the sense of Definition 2.2. Moreover, $u \in L^p(\Omega \times [0, T]; L^p(\mathbb{R}^d \times [0, 1], \phi))$.*

The bounds in Proposition 3.5 blow up as $\varepsilon \downarrow 0$. Below we establish bounds that are independent of the regularization parameter $\varepsilon > 0$.

Lemma 4.2 (Uniform bounds). *Suppose (\mathcal{A}_f) and (\mathcal{A}_σ) hold, and u^0 belongs to $L^p(\Omega, \mathcal{F}_0, P; L^p(\mathbb{R}^d, \phi))$ for some even number $p \geq 2$ and $\phi \in \mathfrak{N}$. Then there exists a constant C , depending on u^0, f, σ, p, T and ϕ but not on ε , such that*

$$(4.1) \quad E \left[\|u^\varepsilon(t)\|_{p, \phi}^p \right] \leq C, \quad t \in [0, T].$$

Proof. Assume that u^0, f, σ satisfy the assumptions of Proposition 3.5 for $|\alpha| \leq 2$. The general result follows by approximation, in view of Proposition 3.4. Let us for the moment also assume $\phi \in \mathfrak{N} \cap C^\infty$, so that there is a constant $C_{\phi, 2}$ satisfying $|\Delta \phi| \leq C_{\phi, 2} \phi$. By Proposition 3.5, u^ε is a strong solution of (3.1). Hence we may apply Itô's formula to the function $S(u) = |u|^p$, cf. Step 2 in the upcoming proof

of Theorem 4.1. After multiplying by ϕ and integrating the result in x ,

$$\begin{aligned} \|u^\varepsilon(t)\|_{p,\phi}^p &= \|u^0\|_{p,\phi}^p \\ &- p \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^{p-1} \operatorname{sign}(u^\varepsilon(s,x)) (\nabla \cdot f(u^\varepsilon(s,x)) - \varepsilon \Delta u^\varepsilon(s,x)) \phi(x) dx ds \\ &+ p \int_0^t \int_Z \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^{p-1} \operatorname{sign}(u^\varepsilon(s,x)) \sigma(x, u^\varepsilon(s,x), z) \phi(x) dx W(ds, dz) \\ &+ \frac{1}{2} p(p-1) \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^{p-2} \int_Z \sigma^2(x, u^\varepsilon(s,x), z) \phi(x) d\mu(z) dx ds. \end{aligned}$$

Let $q(u) = p \int_0^u |z|^{p-1} \operatorname{sign}(z) \partial f(z) dz$ and note that by (\mathcal{A}_f) ,

$$(4.2) \quad |q(u)| = \left| \int_0^u p |z|^{p-1} \operatorname{sign}(z) \partial f(z) dz \right| \leq \|f\|_{\text{Lip}} |u|^p.$$

It follows that $q(u^\varepsilon(t))\phi \in L^1(\Omega; L^1(\mathbb{R}^d; \mathbb{R}^d))$ for $0 \leq t \leq T$.

By the chain rule and integration by parts,

$$\begin{aligned} \mathcal{T}_1 &:= \int_0^t \int_{\mathbb{R}^d} p |u^\varepsilon(s,x)|^{p-1} \operatorname{sign}(u^\varepsilon(s,x)) \nabla \cdot f(u^\varepsilon(s,x)) \phi(x) dx ds \\ &= \int_0^t \int_{\mathbb{R}^d} \partial q(u^\varepsilon(s,x)) \cdot \nabla u^\varepsilon(s,x) \phi(x) dx ds \\ &= - \int_0^t \int_{\mathbb{R}^d} q(u^\varepsilon(s,x)) \cdot \nabla \phi(x) dx ds. \end{aligned}$$

By (4.2) and the fact that $\phi \in \mathfrak{N}$,

$$|\mathcal{T}_1| \leq C_\phi \|f\|_{\text{Lip}} \int_0^t \|u^\varepsilon(s)\|_{p,\phi}^p ds.$$

Again by the chain rule and integration by parts,

$$\begin{aligned} \mathcal{T}_2 &:= \varepsilon \int_0^t \int_{\mathbb{R}^d} p |u^\varepsilon(s,x)|^{p-1} \operatorname{sign}(u^\varepsilon(s,x)) \Delta u^\varepsilon(s,x) \phi(x) dx ds \\ &= -\varepsilon p(p-1) \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^{p-2} |\nabla u^\varepsilon(s,x)|^2 \phi(x) dx ds \\ &\quad - \varepsilon \int_0^t \int_{\mathbb{R}^d} p |u^\varepsilon(s,x)|^{p-1} \operatorname{sign}(u^\varepsilon(s,x)) \nabla u^\varepsilon(s,x) \cdot \nabla \phi(x) dx ds \\ &= -\varepsilon p(p-1) \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^{p-2} |\nabla u^\varepsilon(s,x)|^2 \phi(x) dx ds \\ &\quad + \varepsilon \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^p \Delta \phi(x) dx ds. \end{aligned}$$

Using the assumption on ϕ ,

$$|\mathcal{T}_2| \leq C_{\phi,2\varepsilon} \int_0^t \|u^\varepsilon(s)\|_{p,\phi}^p ds.$$

Finally, by assumption (\mathcal{A}_σ) ,

$$\begin{aligned} \mathcal{T}_3 &:= \frac{1}{2} p(p-1) \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^{p-2} \int_Z \sigma^2(x, u^\varepsilon(s,x), z) \phi(x) d\mu(z) dx ds \\ &\leq \frac{1}{2} p(p-1) \|M\|_{L^2(Z)}^2 \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon(s,x)|^{p-2} (1 + |u^\varepsilon(s,x)|)^2 \phi(x) dx ds \\ &\leq p(p-1) \|M\|_{L^2(Z)}^2 \left(\int_0^t \|u^\varepsilon(s)\|_{p-2,\phi}^{p-2} ds + \int_0^t \|u^\varepsilon(s)\|_{p,\phi}^p ds \right). \end{aligned}$$

After taking expectations and summarizing our findings, we arrive at

$$\begin{aligned} E \left[\|u^\varepsilon(t)\|_{p,\phi}^p \right] &\leq E \left[\underbrace{\|u^0\|_{p,\phi}^p + p(p-1) \|M\|_{L^2(Z)}^2}_{C_2} \int_0^t E \left[\|u^\varepsilon(s)\|_{p-2,\phi}^{p-2} \right] ds \right. \\ &\quad \left. + \underbrace{\left(C_\phi \|f\|_{\text{Lip}} + \varepsilon C_{\phi,2} + p(p-1) \|M\|_{L^2(Z)}^2 \right)}_{C_1} \int_0^t E \left[\|u^\varepsilon(s)\|_{p,\phi}^p \right] ds, \right. \end{aligned}$$

and hence, appealing to Grönwall's inequality,

$$E \left[\|u^\varepsilon(t)\|_{p,\phi}^p \right] \leq C_2 (1 + C_1 t e^{C_1 t}).$$

Observe that the result holds for p if it also holds for $p-2$, as long as u^0 belongs to $L^p(\Omega; L^p(\mathbb{R}^d, \phi))$. For this reason, (4.1) follows by induction. In view of Lemma 6.5, it takes an easy approximation argument to extend to the case $\phi \in \mathfrak{N}$. \square

Remark 4.1. In the forgoing proof, it is certainly possible to apply the Burkholder-Davis-Gundy inequality, resulting in the improvement

$$E \left[\sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^p(\mathbb{R}^d, \phi)}^p \right] \leq C,$$

for some constant C independent of ε .

Proof of Theorem 4.1. We divide the proof into two main steps.

Step 1 (convergence). We apply Theorem 6.9 to the viscous approximation $\{u^\varepsilon\}_{\varepsilon>0}$ on the measure space

$$(X, \mathcal{A}, \mu) = (\Omega \times \Pi_T, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d), dP \otimes dt \otimes \phi(x)dx).$$

By Lemma 4.2,

$$\sup_{\varepsilon>0} \left\{ E \left[\iint_{\Pi_T} |u^\varepsilon(t, x)|^2 \phi(x) dx dt \right] \right\} < \infty,$$

so we may take $\zeta(\xi) = \xi^2$. It follows that there exists a Young measure $\nu = \nu_{t,x,\omega}$ such that for any Carathéodory function $\psi = \psi(u, t, x, \omega)$ satisfying

$$\psi(u^\varepsilon(\cdot), \cdot) \rightharpoonup \bar{\psi}(\cdot) \text{ in } L^1(\Omega \times \Pi_T),$$

we have

$$\bar{\psi}(t, x, \omega) = \int_{\mathbb{R}} \psi(\xi, t, x, \omega) d\nu_{t,x,\omega}(\xi) = \int_0^1 \psi(u(t, x, \alpha, \omega), t, x, \omega) d\alpha.$$

Here $u(t, x, \cdot, \omega)$ is defined through (2.6), i.e.,

$$u(t, x, \alpha, \omega) = \inf \{ \xi \in \mathbb{R} : \nu_{t,x,\omega}((-\infty, \xi]) > \alpha \}.$$

We want to show that the limit u is measurable, i.e. that it has a version \tilde{u} such that for any $\beta \in \mathbb{R}$,

$$B_\beta = \{(t, x, \alpha, \omega) : \tilde{u}(t, x, \alpha, \omega) \geq \beta\} \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}([0, 1]).$$

Note that

$$\tilde{u}(t, x, \alpha, \omega) \geq \beta \Leftrightarrow \inf_{\xi \in \mathbb{R}} \{ \nu_{t,x,\omega}((-\infty, \xi]) > \alpha \} \geq \beta \Leftrightarrow \nu_{t,x,\omega}((-\infty, \beta]) \leq \alpha.$$

By definition of the Young measure we pick a version (not relabeled) such that, the mapping $(t, x, \omega) \mapsto \nu_{t,x,\omega}((-\infty, \beta])$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. Furthermore, if it was finitely valued it would be clear that B_β is in the product topology, i.e., $B_\beta \in \mathcal{F} \otimes \mathcal{B}(\Pi_T) \otimes \mathcal{B}([0, 1])$. Hence, the result follows upon approximation by simple functions [11, Example 5.3.1].

Let us show that $u \in L^p([0, T] \times \Omega; L^p(\mathbb{R}^d \times [0, 1], \phi))$. That, is

$$(4.3) \quad E \left[\iint_{\Pi_T} \int_0^1 |u(t, x, \alpha)|^p \phi(x) d\alpha dx dt \right] < \infty.$$

Let $\psi \in C_c^\infty(\mathbb{R})$ be supported in $(-1, 1)$ and satisfy $0 \leq \psi \leq 1, \psi(0) = 1$. Take $\psi_R(u) = \psi(u/R)$. It follows that $\lim_{R \rightarrow \pm\infty} \psi_R(u) = 1$ for each $u \in \mathbb{R}$. Moreover, take $S_R(u) = |u|^p \psi_R(u)$, and note that $S_R(u) \uparrow |u|^p$ for all $u \in \mathbb{R}$. Since S_R is compactly supported it follows that $\{S_R(u^\varepsilon)\}_{\varepsilon > 0}$ is uniformly integrable on X . By Theorem 6.10, there is a subsequence $\varepsilon_j \downarrow 0$ such that

$$\begin{aligned} & E \left[\iint_{\Pi_T} \int_0^1 S_R(u(t, x, \alpha)) \phi(x) d\alpha dx dt \right] \\ &= \lim_{j \rightarrow \infty} E \left[\iint_{\Pi_T} S_R(u^{\varepsilon_j}(t, x)) \phi(x) dx dt \right] \\ &\leq \limsup_{\varepsilon \downarrow 0} E \left[\iint_{\Pi_T} |u^\varepsilon(t, x)|^p \phi(x) dx dt \right] \leq C \quad (\text{by Lemma 4.2}), \end{aligned}$$

for some constant C independent of R . The claim (4.3) follows upon letting $R \rightarrow \infty$, applying the monotone convergence theorem.

Step 2 (entropy condition). Let us for the moment assume that f, σ, u^0 satisfy the assumptions of Proposition 3.5 for all multiindices $|\alpha| \leq 2$. Fix an entropy/entropy-flux pair (S, Q) in \mathcal{E} , a nonnegative test function $\varphi \in C_c^\infty([0, T] \times \mathbb{R})$, and a random variable $V \in \mathcal{S}$. We want to apply the anticipating Itô formula (Theorem 6.7), for fixed $x \in \mathbb{R}^d$, to $X_t = u^\varepsilon(t, x)$ and $F(X, V, t) = S(X - V)\varphi(t, x)$.

By Proposition 3.5 u^ε is a strong solution of (3.1).

$$\begin{aligned} u^\varepsilon(t, x) &= u^0(x) + \int_0^t \varepsilon \Delta u^\varepsilon(s, x) - \nabla \cdot f(u^\varepsilon(s, x)) ds \\ &\quad + \int_0^t \int_Z \sigma(x, u^\varepsilon(s, x), z) W(ds, dz), \quad dx \otimes dP\text{-almost surely.} \end{aligned}$$

To this end, consider the weak form (3.4), integrate by parts (cf. Proposition 3.5), and use a (separating) countable subset $\{\varphi_n\}_{n \geq 1} \subset C_c^\infty(\mathbb{R}^d)$ of test functions. The anticipating Itô formula yields, after taking expectations and integrating in x ,

$$(4.4) \quad \begin{aligned} 0 &= E \left[\int_{\mathbb{R}^d} S(u^0 - V) \varphi(0) dx \right] \\ &\quad + E \left[\int_0^T \int_{\mathbb{R}^d} S(u^\varepsilon(t) - V) \partial_t \varphi(t) dx dt \right] \\ &\quad - E \left[\int_0^T \int_{\mathbb{R}^d} \nabla \cdot f(u^\varepsilon(t)) S'(u^\varepsilon(t) - V) \varphi(t) dx dt \right] \\ &\quad + E \left[\varepsilon \int_0^T \int_{\mathbb{R}^d} \Delta u^\varepsilon(t) (S'(u^\varepsilon(t) - V) \varphi(t)) dx dt \right] \\ &\quad - E \left[\int_0^T \int_{\mathbb{R}^d} \int_Z S''(u^\varepsilon(t) - V) \varphi(t) \sigma(x, u^\varepsilon(t), z) D_{t,z} V d\mu(z) dx dt \right] \\ &\quad + \frac{1}{2} E \left[\int_0^T \int_{\mathbb{R}^d} \int_Z S''(u^\varepsilon(t) - V) \varphi(t) \sigma(x, u^\varepsilon(t), z)^2 d\mu(z) dx dt \right], \end{aligned}$$

where $D_{t,z}V$ is the Malliavin derivative of V at (t, z) . By the chain rule and integration by parts, it follows from (4.4) that

$$\begin{aligned}
& E \left[\int_0^T \int_{\mathbb{R}^d} \underbrace{S(u^\varepsilon(t, x) - V) \partial_t \varphi(t, x)}_{\psi_1(u^\varepsilon, \cdot)} + \underbrace{Q(u^\varepsilon(t, x), V) \cdot \nabla \varphi(t, x)}_{\psi_2(u^\varepsilon, \cdot)} dx dt \right] \\
& + E \left[\int_{\mathbb{R}^d} S(u^0(x) - V) \varphi(0, x) dx \right] \\
(4.5) \quad & \geq E \left[\int_0^T \int_{\mathbb{R}^d} \underbrace{S''(u^\varepsilon(t, x) - V) \int_Z \sigma(x, u^\varepsilon(t, x), z) D_{t,z} V d\mu(z) \varphi(t, x)}_{\psi_3(u^\varepsilon, \cdot)} dx dt \right] \\
& - \frac{1}{2} E \left[\int_0^T \int_{\mathbb{R}^d} \underbrace{S''(u^\varepsilon(t, x) - V) \int_Z \sigma^2(x, u^\varepsilon(t, x), z) d\mu(z) \varphi(t, x)}_{\psi_4(u^\varepsilon, \cdot)} dx dt \right] \\
& - \varepsilon E \left[\int_0^T \int_{\mathbb{R}^d} S(u^\varepsilon(t, x) - V) \Delta \varphi(t, x) dx dt \right].
\end{aligned}$$

At this point we may apply Proposition 3.4 to relax the assumptions on f, σ, u^0 to the ones listed in Theorem 4.1, leaving the details to the reader.

Obviously,

$$\lim_{\varepsilon \downarrow 0} E \left[\varepsilon \int_0^T \int_{\mathbb{R}^d} S(u^\varepsilon(t, x) - V) \Delta \varphi(t, x) dx dt \right] = 0.$$

For the remaining terms, it suffices by Step 1 and the upcoming Theorem 6.10 to show that $\{\psi_i(u^\varepsilon, \cdot) \phi^{-1}\}_{\varepsilon > 0}$ is uniformly integrable ($i = 1, 2, 3, 4$).

Regarding ψ_1 , by Lemma 6.11(ii), we must show that

$$(4.6) \quad \sup_{\varepsilon > 0} E \left[\iint_{\Pi_T} |\psi_1(u^\varepsilon(t, x), t, x) \phi^{-1}(x)|^2 \phi(x) dx dt \right] < \infty.$$

As S is in \mathcal{E} and $\varphi \in C_c^\infty(\Pi_T)$,

$$\begin{aligned}
|\psi_1(u^\varepsilon(t, x), t, x) \phi^{-1}(x)| &= |S(u^\varepsilon(t, x) - V) \partial_t \varphi(t, x) \phi^{-1}(x)|^2 \\
&\leq 2 \|S\|_{\text{Lip}}^2 \|\partial_t \varphi(t)\|_{\infty, \phi^{-1}} (|u^\varepsilon(t, x)|^2 + |V|^2).
\end{aligned}$$

So (4.6) follows by Lemma 4.2. The term involving ψ_2 follows along the same lines.

Consider the term involving the Malliavin derivative, namely ψ_3 . As above, $\{\psi_3(u^\varepsilon, \cdot) \phi^{-1}\}_{\varepsilon > 0}$ is uniformly integrable provided

$$\sup_{\varepsilon > 0} E \left[\iint_{\Pi_T} |\psi_3(u^\varepsilon(t, x), t, x) \phi^{-1}(x)|^2 \phi(x) dx dt \right] < \infty.$$

By (\mathcal{A}_σ) ,

$$\left| \int_Z \sigma(x, u^\varepsilon(t, x), z) D_{t,z} V d\mu(z) \right|^2 \leq \|M\|_{L^2(Z)}^2 \|D_t V\|_{L^2(Z)}^2 (1 + |u^\varepsilon(t, x)|)^2.$$

Recall that V is uniformly bounded and also that $\text{supp}(S'') \subset (-R, R)$ for some $R < \infty$. Hence,

$$S''(u^\varepsilon - V)(1 + |u^\varepsilon|) \leq \|S''\|_\infty (1 + R + \|V\|_\infty).$$

Consequently,

$$\begin{aligned} & E \left[\iint_{\Pi_T} |\psi_3(u^\varepsilon(t, x), t, x) \phi^{-1}(x)|^2 \phi(x) dx dt \right] \\ & \leq \|S''\|_\infty^2 \|M\|_{L^2(Z)}^2 (1+R+\|V\|_\infty)^2 \|\varphi\|_{\infty, \phi^{-1}}^2 E \left[\int_0^T \|D_t V\|_{L^2(Z)}^2 dt \right] \|\phi\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Consider the ψ_4 -term. By (\mathcal{A}_σ)

$$S''(u^\varepsilon - V) \int_Z \sigma^2(x, u^\varepsilon, z) d\mu(z) \leq \|S''\|_\infty \|M\|_{L^2(Z)}^2 (1 + R + \|V\|_\infty)^2.$$

Hence

$$\begin{aligned} & E \left[\iint_{\Pi_T} |\psi_4(u^\varepsilon(t, x), t, x) \phi^{-1}(x)|^2 \phi(x) dx dt \right] \\ & \leq \|S''\|_\infty^2 \|M\|_{L^2(Z)}^4 (1 + R + \|V\|_\infty)^4 \|\varphi\|_{\infty, \phi^{-1}}^2 \iint_{\Pi_T} \phi(x) dx dt. \end{aligned}$$

Summarizing, upon sending $\varepsilon \downarrow 0$ along a subsequence, it follows that

$$\mathcal{Y}(\mathbf{Ent}[(S, Q), \varphi, V])(u) \geq 0,$$

where u is the process defined in Step 1. Finally, the result follows for general $V \in \mathbb{D}^{1,2}$ by the density of $\mathcal{S} \subset \mathbb{D}^{1,2}$ and Lemma 2.2. \square

5. UNIQUENESS OF ENTROPY SOLUTIONS

To prove the uniqueness of Young measure-valued entropy solutions, we need an additional assumption on σ : there exists $M \in L^2(Z)$ and $\kappa > 0$ such that

$$(\mathcal{A}_{\sigma,1}) \quad |\sigma(x, u, z) - \sigma(y, u, z)| \leq M(z) |x - y|^{\kappa+1/2} (1 + |u|),$$

for $x, y \in \mathbb{R}^d$ and $u \in \mathbb{R}$. Actually, it suffices that the criterion is satisfied locally, i.e., for each compact $K \subset \mathbb{R}^d \times \mathbb{R}^d$ there exists $M = M_K$ such that $(\mathcal{A}_{\sigma,1})$ is satisfied for all $(x, y) \in K$.

Theorem 5.1. *Fix $\phi \in \mathfrak{N}$, and suppose $u^0 \in L^2(\Omega, P, \mathcal{F}_0; L^2(\mathbb{R}^d, \phi))$. Under assumptions (\mathcal{A}_f) , (\mathcal{A}_σ) , and $(\mathcal{A}_{\sigma,1})$ there exists but one Young measure-valued entropy solution to (1.1) in the sense of Definition 2.2. Moreover, this solution is an entropy solution in the sense of Definition 2.1.*

The proof is found at the end of this section. As discussed in the introduction, due to the lack of Malliavin differentiability at the hyperbolic level, the uniqueness argument will invoke the viscous approximations and their limit taken in the weak sense of Young measures.

Retracing the proof of Theorem 5.1, making some small modifications, we obtain the following spatial regularity result:

Proposition 5.2 (Spatial regularity). *Fix $\phi \in \mathfrak{N}$, and suppose u^0 belongs to $L^2(\Omega, P, \mathcal{F}_0; L^2(\mathbb{R}^d, \phi))$. Under assumptions (\mathcal{A}_f) , (\mathcal{A}_σ) , and $(\mathcal{A}_{\sigma,1})$ the entropy solution u to (1.1) satisfies*

$$\begin{aligned} & E \left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u(t, x+z) - u(t, x-z)| \phi(x) J_r(z) dx dz \right] \\ & \leq CE \left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u^0(x+z) - u^0(x-z)| \phi(x) J_r(z) dx dz \right] + \mathcal{O}(r^\kappa), \end{aligned}$$

where the constant C depends only on $C_\phi, \|f\|_{\text{Lip}}, T$, and κ is the exponent from assumption $(\mathcal{A}_{\sigma,1})$.

See [14, 9] for similar results, and how to turn this result into a fractional BV estimate. The proof of Proposition 5.2 is found at the very end of the section.

The next lemma contains the ‘‘entropy condition’’ at the parabolic level, which will be utilized later in the uniqueness proof.

Lemma 5.3. *For each fixed $\varepsilon > 0$, let u^ε be the solution of (3.1). Suppose $V \in L^2(\Omega)$ is \mathcal{F}_s -measurable for some $s \in (0, T)$, and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ with $\text{supp}(\varphi) \subset (s, T) \times \mathbb{R}^d$. Then*

$$\begin{aligned} & E \left[\iint_{\Pi_T} S(u^\varepsilon - V) \partial_t \varphi + Q(u^\varepsilon, V) \cdot \nabla \varphi \, dx dt d\alpha \right] \\ & \geq -\frac{1}{2} E \left[\iint_{\Pi_T} \int_Z S''(u^\varepsilon - V) \sigma(x, u^\varepsilon, z)^2 \varphi(t, x) \, d\mu(z) \, dx dt d\alpha \right] \\ & \quad - \varepsilon E \left[\iint_{\Pi_T} S(u^\varepsilon(t) - V) \Delta \varphi(t) \, dx dt \right], \end{aligned}$$

for any entropy/entropy-flux pair (S, Q) in \mathcal{E} .

Proof. Consider (4.5). Note that for any $V \in \mathcal{S}_b$ that is \mathcal{F}_s -measurable,

$$E \left[\iint_{\Pi_T} \int_Z S''(u^\varepsilon(t) - V) \sigma(x, u^\varepsilon(t), z) D_{t,z} V \varphi(t) \, d\mu(z) \, dx dt \right] = 0,$$

thanks to [27, Proposition 1.2.8]. The general result follows by approximation as in Lemma 2.2. \square

The following ‘‘doubling of variables’’ lemma is at the heart of the matter. To some extent it may be instructive to compare its proof with the rather involved computations in [17, Lemma 3.2] and [3, Section 4.1].

Lemma 5.4. *Suppose $(\mathcal{A}_f), (\mathcal{A}_\sigma)$ hold. Fix $\phi \in \mathfrak{N}$, and let $\{u^\varepsilon\}_{\varepsilon>0}$ be a sequence of viscous approximations with initial condition $u^0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$. Let v be a Young measure-valued entropy solution in the sense of Definition 2.1 with initial condition $v^0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$.*

For any $0 < \gamma < \frac{1}{2}T$ take $t_0 \in [0, T - 2\gamma]$ and define

$$\xi_{\gamma, t_0}(t) := 1 - \int_0^t J_\gamma^+(s - t_0) \, ds.$$

Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be non-negative and define

$$\varphi(t, x, s, y) = \frac{1}{2^d} \psi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right) \xi_{\gamma, t_0}(t) J_{r_0}^+(t-s).$$

Let S_δ be a function satisfying

$$S'_\delta(\sigma) = 2 \int_0^\sigma J_\delta(z) \, dz, \quad S_\delta(0) = 0.$$

Furthermore, define

$$Q_\delta(u, c) = \int_c^u S'_\delta(z-c) f'(z) \, dz,$$

and note that the pair (S_δ, Q_δ) belongs to \mathcal{E} .

Then

$$(5.1) \quad L \geq R + F + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3,$$

where

$$\begin{aligned}
L &= E \left[\iint_{\Pi_T} \int_{\mathbb{R}^d} S_\delta(v^0(y) - u^\varepsilon(t, x)) \varphi(t, x, 0, y) dy dx dt \right], \\
R &= -E \left[\iiint_{\Pi_L^2} \int_{[0,1]} S_\delta(v - u^\varepsilon) (\partial_s + \partial_t) \varphi d\beta dX \right], \\
F &= -E \left[\iiint_{\Pi_L^2} \int_{[0,1]} Q_\delta(u^\varepsilon, v) \cdot \nabla_x \varphi + Q_\delta(v, u^\varepsilon) \cdot \nabla_y \varphi d\beta dX \right], \\
\mathcal{T}_1 &= -\frac{1}{2} E \left[\iiint_{\Pi_L^2} \int_{[0,1]} \int_Z S_\delta''(v - u^\varepsilon) (\sigma(y, v, z) - \sigma(x, u^\varepsilon, z))^2 \varphi d\mu(z) d\beta dX \right], \\
\mathcal{T}_2 &= E \left[\iiint_{\Pi_L^2} \int_{[0,1]} \int_Z S_\delta''(v - u^\varepsilon) (D_{s,z} u^\varepsilon - \sigma(x, u^\varepsilon, z)) \sigma(y, v, z) \varphi d\mu(z) d\beta dX \right], \\
\mathcal{T}_3 &= -\varepsilon E \left[\iiint_{\Pi_L^2} \int_{[0,1]} S_\delta(u^\varepsilon - v) \Delta_x \varphi d\beta dX \right],
\end{aligned}$$

where $dX = dx dt dy ds$.

Remark 5.1. In [17, Section 4.6] the authors prove existence of a *strong* entropy solution. The additional condition attached to the notion of strong solution stems from the difficulties in sending $\varepsilon \downarrow 0$ before $r_0 \downarrow 0$. In our setting, the existence of a strong entropy solution amounts to showing that we can send $r_0 \downarrow 0$ and $\varepsilon \downarrow 0$ simultaneously in such a way that $\lim_{(\varepsilon, t) \downarrow (0, s)} \mathcal{T}_2 = 0$. This requires a careful study of how the continuity properties of (3.15) depends on ε , cf. Lemma 3.8. We do not proceed along this path in this paper, instead we let $r_0 \downarrow 0$ before $\varepsilon \downarrow 0$ as in [3].

Proof. Recall that $\text{supp}(J_{r_0}^+) \subset (0, 2r_0)$, so $J_{r_0}^+(t - s)$ is zero whenever $s \geq t$. Applying Lemma 5.3 with $V = v(s, y, \beta)$ and integrating in y, s, β , we obtain

$$\begin{aligned}
& E \left[\iiint_{\Pi_T^2} \int_{[0,1]} S_\delta(u^\varepsilon - v) \partial_t \varphi + Q_\delta(u^\varepsilon, v) \cdot \nabla \varphi d\beta dX \right] \\
(5.2) \quad & \geq -\frac{1}{2} E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z S_\delta''(u^\varepsilon - v) \sigma(x, u^\varepsilon, z)^2 \varphi d\mu(z) d\beta dX \right] \\
& \quad - \varepsilon E \left[\iiint_{\Pi_T^2} \int_{[0,1]} S_\delta(u^\varepsilon - v) \Delta_x \varphi d\beta dX \right].
\end{aligned}$$

Similarly, in the entropy inequality for $v = v(s, y, \beta)$ we take $V = u^\varepsilon(t, x)$ and integrate in t, x , resulting in

$$\begin{aligned}
& E \left[\iiint_{\Pi_T^2} \int_{[0,1]} S_\delta(v - u^\varepsilon) \partial_s \varphi + Q_\delta(v, u^\varepsilon) \cdot \nabla_y \varphi d\beta dX \right] \\
& + E \left[\iint_{\Pi_T} \int_{\mathbb{R}^d} S_\delta(v^0(y) - u^\varepsilon(t, x)) \varphi(t, x, 0, y) dy dx dt \right] \\
(5.3) \quad & \geq E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z S_\delta''(v - u^\varepsilon) D_{s,z} u^\varepsilon \sigma(y, v, z) \varphi d\mu(z) d\beta dX \right] \\
& \quad - \frac{1}{2} E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z S_\delta''(v - u^\varepsilon) \sigma(y, v, z)^2 \varphi d\mu(z) d\beta dX \right].
\end{aligned}$$

The result follows by adding (5.2) and (5.3). \square

Proposition 5.5 (Kato inequality). *Fix $\phi \in \mathfrak{N}$. Suppose (\mathcal{A}_f) , (\mathcal{A}_σ) , and $(\mathcal{A}_{\sigma,1})$ hold. Let u be the Young measure-valued limit of the viscous approximations $\{u^\varepsilon\}_{\varepsilon>0}$ with initial condition $u^0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$, constructed in Theorem 4.1. Let v be a Young measure-valued entropy solution in the sense of Definition 2.1 with initial condition $v^0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathbb{R}^d, \phi))$. Then, for almost all $t_0 \in (0, T)$ and any non-negative $\psi \in C_c^\infty(\mathbb{R}^d)$,*

$$(5.4) \quad \begin{aligned} & E \left[\int_{\mathbb{R}^d} \iint_{[0,1]^2} |u(t_0, x, \alpha) - v(t_0, x, \beta)| \psi(x) d\alpha d\beta dx \right] \\ & \leq E \left[\int_{\mathbb{R}^d} |u^0(x) - v^0(x)| \psi(x) dx \right] \\ & \quad + E \left[\int_0^{t_0} \int_{\mathbb{R}^d} \iint_{[0,1]^2} \text{sign}(u(t, x, \alpha) - v(t, x, \beta)) \right. \\ & \quad \quad \left. \times (f(u(t, x, \alpha)) - f(v(t, x, \beta))) \cdot \nabla \psi(x) d\beta d\alpha dx dt \right]. \end{aligned}$$

Proof. Starting off from (5.1), we send r_0 and ε to zero (in that order). Next, we send (δ, r) to $(0, 0)$ simultaneously. In view of Limits 3 and 4, we let $\delta(r) = r^{1+\eta}$ with $0 < \eta < 2\kappa - 1$. Finally, we send $\gamma \downarrow 0$. We arrive at the Kato inequality (5.4) thanks to the upcoming Limits 1–6. \square

Remark 5.2. Later we will make repeated use of two elementary identities. Set

$$\xi_r(x) := \frac{1}{2^d} \int_{\mathbb{R}^d} \psi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right) dy.$$

Then note that $\xi_r = \psi \star J_r$. Indeed, making the change of variable $z = (x+y)/2$, it follows that $(x-y)/2 = x-z$ and $dy = 2^d dz$. Next, consider the change of variables

$$\Phi(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2} \right) = (\tilde{x}, z).$$

By the change of variables formula

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} g(\tilde{x}, z) d\tilde{x} dz = \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(\Phi(x, y)) |\det(\partial\Phi(x, y))| dx dy,$$

for any measurable function $g(\cdot, \cdot)$. A computation yields $|\det(\partial\Phi(x, y))| = 1/2^d$. It follows that

$$\begin{aligned} & \frac{1}{2^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \underbrace{h(x, y) \psi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right)}_{g(\Phi(x, y))} dx dy \\ & = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \underbrace{h(\tilde{x} + z, \tilde{x} - z) \psi(\tilde{x}) J_r(z)}_{g(\tilde{x}, z)} d\tilde{x} dz, \end{aligned}$$

for any measurable function $h(\cdot, \cdot)$. Most of the time we drop the tilde and write x instead of \tilde{x} .

Limit 1. With L defined in Lemma 5.4,

$$\lim_{r_0 \downarrow 0} L = E \left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(v^0(x-z) - u^0(x+z)) \psi(x) J_r(z) dx dz \right].$$

If $\delta = \delta(r)$ is a nondecreasing function satisfying $\delta(r) \downarrow 0$ as $r \downarrow 0$, then

$$\lim_{\gamma, (\delta, r), \varepsilon, r_0 \downarrow 0} L = E \left[\|v^0 - u^0\|_{1, \psi} \right].$$

Proof. Note that

$$(5.5) \quad |S_\delta(b) - S_\delta(a)| = \left| \int_a^b S'_\delta(z) dz \right| \leq |b - a|.$$

Hence, due to Remark 5.2,

$$\begin{aligned} & \left| L - E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_\delta(v^0(y) - u^0(x)) \frac{1}{2^d} \psi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right) dx dy \right] \right| \\ & \leq E \left[\iint_{\Pi_T} |u^0(x) - u^\varepsilon(t, x)| J_{r_0}^+(t) (\psi \star J_r)(x) dx dt \right]. \end{aligned}$$

Arguing as in Lemma 2.3 for the viscous approximation, it follows that

$$\begin{aligned} \lim_{r_0 \downarrow 0} L &= E \left[\frac{1}{2^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S_\delta(v^0(y) - u^0(x)) \psi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right) dx dy \right] \\ &= E \left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(v^0(x-z) - u^0(x+z)) \psi(x) J_r(z) dx dz \right]. \end{aligned}$$

This proves the first limit. The second limit follows by the dominated convergence theorem and Lemma 6.1. \square

Remark 5.3. To establish Limits 2 and 3 we need to send $\varepsilon \downarrow 0$ in terms of the form

$$E \left[\iiint_{\Pi_T \times \mathbb{R}^d \times [0,1]} \Psi(u^\varepsilon(t, x, \omega), t, x, y, \beta, \omega) \underbrace{\frac{1}{2^d} \phi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right)}_{d\eta_{\phi,r}} d\beta dy dx dt \right],$$

where Ψ is continuous in the first variable. Essentially we proceed as in the proof of Theorem 4.1, but now the underlying measure space is $\Pi_T \times \mathbb{R}^d \times [0, 1] \times \Omega$ instead of $\Pi_T \times \Omega$. By Lemma 4.2 and Remark 5.2,

$$\begin{aligned} & \sup_{\varepsilon > 0} \left\{ E \left[\iiint_{\Pi_T \times \mathbb{R}^d \times [0,1]} |u^\varepsilon(t, x)|^2 d\eta_{\phi,r} \right] \right\} \\ & = \sup_{\varepsilon > 0} \left\{ E \left[\int_0^T \|u^\varepsilon(t)\|_{2, \phi \star J_r}^2 dt \right] \right\} < \infty. \end{aligned}$$

By Theorem 6.9, there exists $\nu \in \mathcal{Y}(\Pi_T \times \mathbb{R}^d \times [0, 1] \times \Omega)$ such that whenever $\Psi(u^\varepsilon, \cdot) \rightarrow \bar{\Psi}$ along some subsequence in $L^1(\Pi_T \times \mathbb{R}^d \times [0, 1] \times \Omega, d\eta_{\phi,r} \otimes dP)$,

$$\bar{\Psi} = \int_{\mathbb{R}} \Psi(\xi, t, x, y, \beta, \omega) d\nu_{t,x,\omega}(\xi) = \int_0^1 \Psi(u(t, x, \alpha, \omega), t, x, y, \beta, \omega) d\alpha,$$

where u is defined through (2.6). The fact that $\nu_{t,x,y,\beta,\omega} = \nu_{t,x,\omega}$ comes out since the limit is independent of y, β when Ψ is independent of y, β . For measurability considerations, see Step 1 in proof of Theorem 4.1.

Limit 2. With R defined in Lemma 5.4,

$$\lim_{\gamma, \varepsilon, r_0 \downarrow 0} R = E \left[\iiint_{\mathbb{R}^d \times \mathbb{R}^d \times [0,1]^2} S_\delta(v(t_0, x-z, \beta) - u(t_0, x+z, \alpha)) \psi(x) J_r(z) d\alpha d\beta dx dz \right],$$

for dt -a.a. $t_0 \in [0, T]$. If $\delta = \delta(r)$ is a nondecreasing function satisfying $\delta(r) \downarrow 0$ as $r \downarrow 0$, then

$$\lim_{\gamma, (\delta, r), \varepsilon, r_0 \downarrow 0} R = E \left[\int_{\mathbb{R}^d} \iint_{[0,1]^2} |v(t_0, x, \beta) - u(t_0, x, \alpha)| \psi(x) d\alpha d\beta dx \right],$$

for dt -a.a. $t_0 \in [0, T]$.

Proof. Since

$$(\partial_s + \partial_t)\varphi(t, x, s, y) = -\frac{1}{2^d} \psi\left(\frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) J_\gamma^+(t-t_0) J_{r_0}^+(t-s),$$

it follows that

$$R = E \left[\frac{1}{2^d} \iiint_{\Pi_T^2} \int_{[0,1]} S_\delta(v - u^\varepsilon) \psi\left(\frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) \times J_\gamma^+(t-t_0) J_{r_0}^+(t-s) d\beta dX \right].$$

Thanks to

$$|S_\delta(v - u^\varepsilon)| \leq |v| + |u^\varepsilon|,$$

we can apply the dominated convergence theorem and Lemma 6.2, resulting in

$$\lim_{r_0 \downarrow 0} R = E \left[\iiint_{\Pi_T \times \mathbb{R}^d \times [0,1]} \underbrace{S_\delta(v(t, y, \beta) - u^\varepsilon(t, x)) J_\gamma^+(t-t_0) (\psi \phi^{-1})\left(\frac{x+y}{2}\right)}_{\Psi(u^\varepsilon, \cdot)} \times \frac{1}{2^d} \phi\left(\frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) d\beta dy dx dt \right].$$

By Lemma 6.11(ii), $\{\Psi_\varepsilon(u^\varepsilon, \cdot)\}$ is uniformly integrable, and so, cf. Theorem 6.10, we can extract a weakly convergent subsequence. By Remark 5.3 and 5.2

$$\begin{aligned} \lim_{\varepsilon, r_0 \downarrow 0} R &= E \left[\iint_{\Pi_T} \int_{\mathbb{R}^d} \iint_{[0,1]^2} S_\delta(v(t, y, \beta) - u(t, x, \alpha)) J_\gamma^+(t-t_0) \right. \\ &\quad \left. \times \frac{1}{2^d} \psi\left(\frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) d\alpha d\beta dy dx dt \right] \\ &= E \left[\iint_{\Pi_T} \int_{\mathbb{R}^d} \iint_{[0,1]^2} S_\delta(v(t, x-z, \beta) - u(t, x+z, \alpha)) \right. \\ &\quad \left. \times J_\gamma^+(t-t_0) \psi(x) J_r(z) d\alpha d\beta dz dx dt \right] \end{aligned}$$

Note that

$$|S_\delta(a-b) - S_\delta(c-d)| \leq |b-d| + |a-c|, \quad a, b, c, d \in \mathbb{R}.$$

Applying this inequality and Lemma 6.2, we can send $\gamma \downarrow 0$ to obtain the first inequality. To send $(\delta, r) \downarrow (0, 0)$ we apply the dominated convergence theorem and Lemma 6.1, yielding

$$\lim_{r, \varepsilon, r_0 \downarrow 0} R = E \left[\iint_{\Pi_T} \iint_{[0,1]^2} |v(t, x, \beta) - u(t, x, \alpha)| \psi(x) J_\gamma^+(t-t_0) d\alpha d\beta dx dt \right].$$

To send $\gamma \downarrow 0$ we apply Lemma 6.2. This provides the second limit. \square

Limit 3. With F defined in Lemma 5.4,

$$(5.6) \quad \lim_{\gamma, \varepsilon, r_0 \downarrow 0} F_1 = E \left[\int_0^{t_0} \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times [0,1]^2} S'_\delta(u(t, x+z, \alpha) - v(t, x-z, \beta)) \right. \\ \left. \times (f(u(t, x+z, \alpha)) - f(v(t, x-z, \beta))) \cdot \nabla \psi(x) J_r(z) d\alpha d\beta dx dz dt \right] \\ + \mathcal{O} \left(\delta + \frac{\delta}{r} \right).$$

If $\delta : [0, \infty) \rightarrow [0, \infty)$ satisfy $\lim_{r \rightarrow 0} \frac{\delta(r)}{r} = 0$, then

$$(5.7) \quad \lim_{\gamma, r, \varepsilon, r_0 \downarrow 0} F = -E \left[\int_0^{t_0} \int_{\mathbb{R}^d} \int_{[0,1]^2} \text{sign}(u(t, x, \alpha) - v(t, x, \beta)) \right. \\ \left. \times (f(u(t, x, \alpha)) - f(v(t, x, \beta))) \cdot \nabla \psi(x) d\alpha d\beta dx dt \right].$$

Proof. Using integration by parts,

$$Q_\delta(u^\varepsilon, v) = S'_\delta(u^\varepsilon - v)(f(u^\varepsilon) - f(v)) - \int_v^{u^\varepsilon} S''_\delta(z - v)(f(z) - f(v)) dz$$

and

$$Q_\delta(v, u^\varepsilon) = S'_\delta(v - u^\varepsilon)(f(v) - f(u^\varepsilon)) - \int_{u^\varepsilon}^v S''_\delta(z - u^\varepsilon)(f(z) - f(u^\varepsilon)) dz.$$

Due to the symmetry of S_δ ,

$$F = -E \left[\iiint_{\Pi_T^2} \int_{[0,1]} Q_\delta(u^\varepsilon, v) \cdot \nabla_x \varphi + Q_\delta(v, u^\varepsilon) \cdot \nabla_y \varphi d\beta dX \right] \\ = -E \left[\iiint_{\Pi_T^2} \int_{[0,1]} S'_\delta(u^\varepsilon - v)(f(u^\varepsilon) - f(v)) \cdot (\nabla_x + \nabla_y) \varphi d\beta dX \right] \\ + E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \left(\int_v^{u^\varepsilon} S''_\delta(z - v)(f(z) - f(v)) dz \right) \cdot \nabla_x \varphi d\beta dX \right] \\ + E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \left(\int_{u^\varepsilon}^v S''_\delta(z - u^\varepsilon)(f(z) - f(u^\varepsilon)) dz \right) \cdot \nabla_y \varphi d\beta dX \right] \\ = -F_1 + F_2 + F_3,$$

where $dX = dx dt dy ds$ as in Lemma 5.4. Note that

$$(5.8) \quad \left| \int_v^u S''_\delta(z - v)(f(z) - f(v)) dz \right| \leq \|f\|_{\text{Lip}} \delta, \quad u, v \in \mathbb{R}.$$

To see this, recall that $S''_\delta(\sigma) = 2J_\delta(\sigma)$. By (\mathcal{A}_f) ,

$$\left| \int_v^u S''_\delta(z - v)(f(z) - f(v)) dz \right| \leq 2 \|f\|_{\text{Lip}} \text{sign}(u - v) \int_v^u J_\delta(z - v) |z - v| dz,$$

and letting $\xi = |z - v|/\delta$,

$$\text{sign}(u - v) \int_v^u J_\delta(z - v) |z - v| dz = \delta \int_0^{\delta^{-1}|u-v|} J(\xi) \xi d\xi \leq \frac{\delta}{2}.$$

In view of (5.8), it is clear that

$$F_2 \leq \|f\|_{\text{Lip}} \delta \iiint_{\Pi_T^2} |\nabla_x \varphi| dX.$$

A computation shows $\|\nabla\varphi\|_{L^1(\Pi_T^2)} \leq C(1+r^{-1})$, for some constant C depending only on J, T, ψ . Consequently,

$$F_2 \leq C \|f\|_{\text{Lip}} \delta \left(1 + \frac{1}{r}\right).$$

The same type of estimate applies to F_3 .

Let us consider F_1 . Observe that

$$(\nabla_x + \nabla_y)\varphi(t, x, s, y) = \frac{1}{2^d} \nabla\psi \left(\frac{x+y}{2}\right) J_r \left(\frac{x-y}{2}\right) \xi_{\gamma, t_0}(t) J_{r_0}^+(t-s).$$

For $\delta > 0$, define

$$\mathcal{F}_\delta(a, b) := S'_\delta(a-b)(f(a) - f(b)), \quad a, b \in \mathbb{R},$$

and note that $(t, b) \mapsto \mathcal{F}_\delta(u^\varepsilon(t, x), b)$ obeys the hypotheses of Lemma 6.2. By the dominated convergence theorem and Lemma 6.2,

$$\begin{aligned} \lim_{r_0 \downarrow 0} F_1 = E \left[\iint_{\Pi_T} \int_{\mathbb{R}^d} \int_{[0,1]} \underbrace{\mathcal{F}_\delta(u^\varepsilon(t, x), v(t, y, \beta)) \cdot \zeta \left(\frac{x+y}{2}\right) \xi_{\gamma, t_0}(t)}_{\Psi(u^\varepsilon, \cdot)} \right. \\ \left. \times \frac{1}{2^d} \phi \left(\frac{x+y}{2}\right) J_r \left(\frac{x-y}{2}\right) d\beta dy dx dt \right], \end{aligned}$$

where $\zeta(x) = \phi^{-1}(x)\nabla\psi(x)$. The uniform integrability of $\{\Psi(u^\varepsilon, \cdot)\}_{\varepsilon>0}$ follows thanks to Lemma 6.11(ii). Indeed, $|\zeta| \leq C_\phi$ and $|\mathcal{F}_\delta(u^\varepsilon, v)| \leq \|f\|_{\text{Lip}} |u^\varepsilon - v|$, so

$$|\Psi(u^\varepsilon, \cdot)|^2 \leq 2C_\phi^2 \|f\|_{\text{Lip}}^2 (|u^\varepsilon|^2 + |v|^2).$$

By Theorem 6.10 and Remark 5.3

$$\begin{aligned} \lim_{\varepsilon, r_0 \downarrow 0} F_1 = E \left[\iint_{\Pi_T} \int_{\mathbb{R}^d} \iint_{[0,1]^2} \mathcal{F}_\delta(u(t, x, \alpha), v(t, y, \beta)) \right. \\ \left. \cdot \frac{1}{2^d} \nabla\psi \left(\frac{x+y}{2}\right) J_r \left(\frac{x-y}{2}\right) \xi_{\gamma, t_0}(t) d\alpha d\beta dy dx dt \right], \end{aligned}$$

along a subsequence. Sending $\gamma \downarrow 0$, applying Remark 5.2, yields (5.6).

Next we want to prove (5.7). To send $r \downarrow 0$ we apply Lemma 6.1. It is easily verified that condition (i) and (iii) are satisfied with $F_\delta = \mathcal{F}_\delta$. Consider condition (ii). Since $S'_\delta(-\sigma) = -S'_\delta(\sigma)$ for all $\sigma \in \mathbb{R}$, it follows that

$$\begin{aligned} \mathcal{F}_\delta(a, b) - \mathcal{F}_\delta(a, c) &= S'_\delta(b-a)(f(b) - f(a)) - S'_\delta(c-a)(f(c) - f(a)) \\ &= \int_b^c \partial_z(S'_\delta(z-a)(f(z) - f(a))) dz \\ &= \int_b^c S''_\delta(z-a)(f(z) - f(a)) dz + \int_b^c S'_\delta(z-a)f'(z) dz, \end{aligned}$$

for $a, b, c \in \mathbb{R}$. By (5.8),

$$|\mathcal{F}_\delta(a, b) - \mathcal{F}_\delta(a, c)| \leq \underbrace{\left| \int_b^c S''_\delta(z-a)(f(z) - f(a)) dz \right|}_{\leq \|f\|_{\text{Lip}} 2\delta} + \underbrace{\left| \int_b^c S'_\delta(z-a)f'(z) dz \right|}_{\leq \|f\|_{\text{Lip}} |b-c|}.$$

This and the symmetry of \mathcal{F}_δ , i.e., $\mathcal{F}_\delta(a, b) = \mathcal{F}_\delta(b, a)$ for $a, b \in \mathbb{R}$, yields condition (ii). Hence, by Lemma 6.1,

$$\lim_{(\delta, r), \varepsilon, r_0 \downarrow 0} F_1 = E \left[\iint_{\Pi_T} \iint_{[0,1]^2} \text{sign}(u(t, x, \alpha) - v(t, x, \beta)) \right. \\ \left. \times (f(u(t, x, \alpha)) - f(v(t, x, \beta))) \cdot \nabla \psi(x) \xi_{\gamma, t_0}(t) d\beta d\alpha dx dt \right].$$

At long last, Limit (5.7) follows by sending $\gamma \downarrow 0$. \square

Limit 4. Suppose assumptions $(\mathcal{A}_{\sigma,1})$ and (\mathcal{A}_σ) hold. With \mathcal{T}_1 defined in Lemma 5.4,

$$\mathcal{T}_1 = \mathcal{O} \left(\frac{r^{2\kappa+1}}{\delta} + \delta \right).$$

Proof. By assumption $(\mathcal{A}_{\sigma,1})$ and (\mathcal{A}_σ) ,

$$|\sigma(y, v, z) - \sigma(x, u^\varepsilon, z)| \leq M(z) |y - x|^\kappa (1 + |u^\varepsilon|) + M(z) |v - u^\varepsilon|.$$

and thus

$$|\mathcal{T}_1| = \frac{1}{2} E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z S_\delta''(v - u^\varepsilon) (\sigma(y, v, z) - \sigma(x, u^\varepsilon, z))^2 \varphi d\mu(z) d\beta dX \right] \\ \leq \|M\|_{L^2(Z)}^2 E \left[\iiint_{\Pi_T^2} \int_{[0,1]} S_\delta''(v - u^\varepsilon) |x - y|^{2\kappa+1} (1 + |u^\varepsilon|)^2 \varphi d\beta dX \right] \\ + \|M\|_{L^2(Z)}^2 E \left[\iiint_{\Pi_T^2} \int_{[0,1]} S_\delta''(v - u^\varepsilon) |v - u^\varepsilon|^2 \varphi d\beta dX \right] \\ =: \mathcal{T}_1^1 + \mathcal{T}_1^2.$$

Since $J_r(\frac{x-y}{2}) = 0$ whenever $|x - y| \geq 2r$,

$$\mathcal{T}_1^1 \leq 4 \|M_K\|_{L^2(Z)}^2 \|J\|_\infty \frac{r^{2\kappa+1}}{\delta} E \left[\iiint_{\Pi_T^2} (1 + |u^\varepsilon|)^2 \varphi dX \right].$$

Moreover, as

$$E \left[\iiint_{\Pi_T^2} (1 + |u^\varepsilon|)^2 \varphi dX \right] \leq \int_0^T E \left[\|1 + u^\varepsilon(t)\|_{2, \psi \star J_r}^2 \right] dt,$$

there is a constant $C > 0$, independent of $r_0, \varepsilon, \delta, \gamma, r$, such that $\mathcal{T}_1^1 \leq Cr^{2\kappa+1}\delta^{-1}$. Regarding the second term \mathcal{T}_1^2 , observe that

$$S_\delta''(v - u^\varepsilon) |v - u^\varepsilon|^2 = J_\delta(v - u^\varepsilon) |v - u^\varepsilon|^2 \leq 2 \|J\|_\infty \delta.$$

Hence, $\mathcal{T}_1^2 \leq \delta 2 \|J\|_\infty \|M\|_{L^2(Z)}^2 \|\varphi\|_{L^1(\Pi_T^2)}$. \square

Let us consider the term involving the Malliavin derivative.

Limit 5. With \mathcal{T}_2 defined in Lemma 5.4,

$$\lim_{r_0 \downarrow 0} \mathcal{T}_2 = 0.$$

Proof. Let us split \mathcal{T}_2 as follows:

$$\begin{aligned}
\mathcal{T}_2 &= E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z S''_\delta(v - u^\varepsilon(s, x)) \left(D_{s,z} u^\varepsilon(t, x) - \sigma(x, u^\varepsilon(s, x), z) \right) \right. \\
&\quad \left. \times \sigma(y, v, z) \varphi d\mu(z) d\beta dX \right] \\
&+ E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z \left(S''_\delta(v - u^\varepsilon(t, x)) - S''_\delta(v - u^\varepsilon(s, x)) \right) \right. \\
&\quad \left. \times D_{s,z} u^\varepsilon(t, x) \sigma(y, v, z) \varphi d\mu(z) d\beta dX \right] \\
&+ E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z S''_\delta(v - u^\varepsilon(s, x)) \left(\sigma(x, u^\varepsilon(s, x), z) - \sigma(x, u^\varepsilon(t, x), z) \right) \right. \\
&\quad \left. \times \sigma(y, v, z) \varphi d\mu(z) d\beta dX \right] \\
&+ E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z \left(S''_\delta(v - u^\varepsilon(s, x)) - S''_\delta(v - u^\varepsilon(t, x)) \right) \right. \\
&\quad \left. \times \sigma(x, u^\varepsilon, z) \sigma(y, v, z) \varphi d\mu(z) d\beta dX \right] \\
&=: \mathcal{T}_2^1 + \mathcal{T}_2^2 + \mathcal{T}_2^3 + \mathcal{T}_2^4.
\end{aligned}$$

Consider \mathcal{T}_2^1 . We want to apply Lemma 3.8 for fixed (s, y, β) with

$$\begin{aligned}
\Psi_{s,y,\beta}(x, z) &= S''_\delta(v - u^\varepsilon(s, x)) \sigma(y, v, z), \\
\phi_y(x) &= \frac{1}{2^d} \psi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right).
\end{aligned}$$

Then

$$\mathcal{T}_2^1 = \iint_{\Pi_T} \int_0^1 \mathcal{T}_{r_0}(\Psi_{s,y,\beta}) d\beta ds dy.$$

By means of Lemma 3.8, $\lim_{r_0 \downarrow 0} \mathcal{T}_{r_0}(\Psi_{s,y,\beta}) = 0 ds dy d\beta$ -a.e., and so $\lim_{r_0 \downarrow 0} \mathcal{T}_2^1 = 0$ by the dominated convergence theorem. To this end, in view of (3.25), there exists a constant C , not depending on r_0 , such that

$$\begin{aligned}
|\mathcal{T}_{r_0}(\Psi_{s,y,\beta})|^2 &\leq C^2 E \left[\iint_{Z \times \mathbb{R}^d} |\Psi_{s,y,\beta}(x, z)|^2 \phi_y(x) dx d\mu(z) \right] \\
&\leq C^2 \|S''_\delta\|_\infty^2 E \left[\int_Z |\sigma(y, v, z)|^2 (\psi \star J_r)(y) d\mu(z) \right] \\
&\leq C^2 \|S''_\delta\|_\infty^2 \|M\|_{L^2(Z)}^2 E [(1 + |v|)^2 (\psi \star J_r)(y)].
\end{aligned}$$

Due to the compact support of $\psi \star J_r$, we see that $|\mathcal{T}_{r_0}(\Psi_{s,y,\beta})|$ is dominated by an integrable function.

Let us consider \mathcal{T}_2^2 . Note that

$$\begin{aligned}
&|S''_\delta(v - u^\varepsilon(t, x)) - S''_\delta(v - u^\varepsilon(s, x))| \\
&\leq \underbrace{\max \left\{ 2 \|S''_\delta\|_\infty, \|S''_\delta\|_{\text{Lip}} |u^\varepsilon(t, x) - u^\varepsilon(s, x)| \right\}}_{\Psi}.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} \mathcal{T}_2^2 &\leq E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z \Psi^2(s, t, x) |\sigma(y, v, z)|^2 \varphi d\mu(z) d\beta dX \right]^{1/2} \\ &\quad \times E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z |D_{s,z} u^\varepsilon(t, x)|^2 \varphi d\mu(z) d\beta dX \right]^{1/2} \\ &=: F_1 \times F_2. \end{aligned}$$

By the uniform boundedness of Ψ we can apply the dominated convergence theorem and Lemma 6.2, to conclude that $\lim_{r_0 \downarrow 0} F_1 = 0$. It remains to show that $|F_2| \leq C$, with C independent of $r_0 > 0$. We deduce easily

$$\begin{aligned} F_2^2 &= \int_0^T \int_0^T E \left[\|D_s u^\varepsilon(t)\|_{L^2(Z; L^2(\mathbb{R}^d, \psi \star J_r))}^2 \right] J_{r_0}^+(t-s) \xi_{\gamma, t_0}(t) ds dt \\ &\leq \int_0^T \sup_{t \in [0, T]} \left\{ E \left[\|D_s u^\varepsilon(t)\|_{L^2(Z; L^2(\mathbb{R}^d, \psi \star J_r))}^2 \right] \right\} ds, \end{aligned}$$

and so $|F_2|$ is uniformly bounded by (3.16).

Consider \mathcal{T}_2^3 . By Hölder's inequality and (\mathcal{A}_σ) ,

$$\begin{aligned} |\mathcal{T}_2^3| &\leq \|S_\delta''\|_\infty \|M\|_{L^2(Z)} E \left[\iiint_{\Pi_T^2} \int_{[0,1]} \int_Z |\sigma(y, v, z)|^2 \varphi d\mu(z) d\beta dX \right]^{1/2} \\ &\quad \times E \left[\iiint_{\mathbb{R}^d \times [0, T]^2} |u^\varepsilon(s, x) - u^\varepsilon(t, x)|^2 (\psi \star J_r)(x) J_{r_0}^+(t-s) dx dt ds \right]^{1/2} \end{aligned}$$

By the dominated convergence theorem and Lemma 6.2, $\lim_{r_0 \downarrow 0} \mathcal{T}_2^3 = 0$.

The term \mathcal{T}_2^4 is treated in the same manner as \mathcal{T}_2^2 , resulting in $\lim_{r_0 \downarrow 0} \mathcal{T}_2^4 = 0$. \square

Limit 6. With \mathcal{T}_3 defined in Lemma 5.4,

$$\mathcal{T}_3 = \mathcal{O}(\varepsilon).$$

Proof. Note that

$$|S_\delta(u^\varepsilon - v) \Delta_x \varphi| \leq (|u^\varepsilon| + |v|) |\Delta_x \varphi|.$$

Using this inequality, it follows from Lemma 4.2 that

$$E \left[\iiint_{\Pi_T^2} S_\delta(u^\varepsilon - v) \Delta_x \varphi d\beta dX \right] \leq C,$$

for some constant $C > 0$ independent of ε and r_0 . \square

Having established Proposition 5.5, the proof of Theorem 5.1 follows easily.

Proof of Theorem 5.1. In the setting of Proposition 5.5, suppose $u^0 = v^0$. Let $\{\phi_R\}_{R>1}$ be as in Lemma 6.6 and take $\psi = \phi_R$ in (5.4). Exploiting that ϕ belongs to \mathfrak{N} , sending $R \rightarrow \infty$ yields

$$\eta(t_0) \leq C_\phi \|f\|_{\text{Lip}} \int_0^{t_0} \eta(t) dt,$$

where

$$\eta(t) = E \left[\iiint_{\mathbb{R}^d \times [0, 1]^2} |u(t, x, \alpha) - v(t, x, \beta)| \phi(x) d\beta d\alpha dx \right].$$

An application of Grönwall's inequality gives $\eta(t_0) = 0$ for dt -a.a. $t_0 \in [0, T]$. Hence, any Young measure-valued entropy solution of (1.1) (in the sense of Definition 2.2)

with initial value u^0 , coincides with the one obtained in Theorem 4.1. Furthermore, $u(t, x, \alpha) = \hat{u}(t, x) := \int u(t, x, \alpha) d\alpha$ for $dtdxd\alpha dP$ -a.a. (t, x, α, ω) . It follows that \hat{u} is an entropy solution (in the sense of definition 2.1). \square

Proof of Proposition 5.2. Let $\{\phi_R\}_{R>1}$ be as in Lemma 6.6, and start off from Lemma 5.4 with $\psi = \phi_R$ and $v^0 = u^0$. We then compute the limits $r_0 \downarrow 0$, $\varepsilon \downarrow 0$, and $\gamma \downarrow 0$ (in that order). Recall that by Theorem 5.1, $v = u$ with $u = \lim_{\varepsilon \downarrow 0} u^\varepsilon$. Furthermore, u is a solution according to Definition 2.1. Due to Limits 1–6 we arrive at the inequality

$$(5.9) \quad \begin{aligned} & E \left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(u^0(x-z) - u^0(x+z)) \phi_R(x) J_r(z) dx dz \right] \\ & \geq E \left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(u(t_0, x-z) - u(t_0, x+z)) \phi_R(x) J_r(z) dx dz \right] \\ & \quad + E \left[\int_0^{t_0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} S'_\delta(u(t, x+z) - u(t, x-z)) \right. \\ & \quad \quad \quad \left. \times (f(u(t, x+z)) - f(u(t, x-z))) \cdot \nabla \phi_R(x) J_r(z) dx dz dt \right] \\ & \quad + \mathcal{O} \left(\delta + \frac{\delta}{r} + \frac{r^{2\kappa+1}}{\delta} \right), \end{aligned}$$

where $\mathcal{O}(\cdot)$ is independent of R , cf. Limits 3 and 4 and Lemmas 6.5 and 6.6.

Note that

$$|S_\delta(\sigma) - |\sigma|| \leq \delta, \quad \forall \sigma \in \mathbb{R},$$

and $|\nabla \phi| \leq C_\phi \phi$. With the help of Lemma 6.6, we can now send $R \rightarrow \infty$ in (5.9), obtaining

$$\eta(t_0) \leq \eta(0) + C_\phi \|f\|_{\text{Lip}} \int_0^{t_0} \eta(t) dt + \mathcal{O} \left(\delta + \frac{\delta}{r} + \frac{r^{2\kappa+1}}{\delta} \right),$$

where

$$\eta(t) = E \left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u(t, x-z) - u(t, x+z)| \phi(x) J_r(z) dx dz \right].$$

By Grönwall's inequality,

$$\eta(t) \leq \left(1 + C_\phi \|f\|_{\text{Lip}} t e^{C_\phi \|f\|_{\text{Lip}} t} \right) \left(\eta(0) + \mathcal{O} \left(\delta + \frac{\delta}{r} + \frac{r^{2\kappa+1}}{\delta} \right) \right).$$

Prescribing $\delta = r^{\kappa+1}$ concludes the proof. \square

6. APPENDIX

6.1. Some “doubling of variables” tools.

Lemma 6.1. *Suppose $u, v \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\{F_\delta\}_{\delta>0}$ satisfy:*

- (i) *There is $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $F_\delta \rightarrow F$ pointwise as $\delta \downarrow 0$.*
- (ii) *There exists a constant $C > 0$ such that*

$$|F_\delta(a, b) - F_\delta(c, d)| \leq C(|a - c| + |b - d| + \delta),$$

for all $a, b, c, d \in \mathbb{R}$ and all $\delta > 0$.

(iii) *There is a constant $C > 0$ such that*

$$|F_\delta(a, a)| \leq C(1 + |a|) \text{ for all } \delta > 0.$$

Fix $\psi \in C_c(\mathbb{R}^d)$. Suppose that $\delta : [0, \infty) \rightarrow [0, \infty)$ satisfies $\delta(r) \downarrow 0$ as $r \downarrow 0$. Set

$$\begin{aligned} \mathcal{T}_r &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_{\delta(r)}(u(x), v(y)) \frac{1}{2^d} \psi\left(\frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) dy dx \\ &\quad - \int_{\mathbb{R}^d} F(u(x), v(x)) \psi(x) dx. \end{aligned}$$

Then $\mathcal{T}_r \rightarrow 0$ as $r \downarrow 0$.

Proof. Due to Remark 5.2,

$$\begin{aligned} \mathcal{T}_r &= \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} F_{\delta(r)}(u(x+z), v(x-z)) \psi(x) dx}_{g_\delta(z)} J_r(z) dz \\ &\quad - \underbrace{\int_{\mathbb{R}^d} F(u(x), v(x)) \psi(x) dx}_{g(0)}. \end{aligned}$$

Suppose for the moment that given a number $\varepsilon > 0$, there exists two numbers $\eta = \eta(\varepsilon) > 0$ and $\delta = \delta_0(\varepsilon) > 0$ such that

$$(6.1) \quad |g_\delta(z) - g(0)| \leq \varepsilon, \text{ whenever } |z| \leq \eta \text{ and } \delta < \delta_0.$$

The change of variables $z = r\zeta$ yields

$$|\mathcal{T}_r| \leq \int_{\mathbb{R}^d} |g_\delta(z) - g(0)| J_r(z) dz = \int_{\mathbb{R}^d} |g_\delta(r\zeta) - g(0)| J(\zeta) d\zeta.$$

Fix $\varepsilon > 0$, and pick η, δ_0 as dictated by (6.1). Let $r_0 > 0$ satisfy $r_0 \leq \eta$ and $\delta(r_0) \leq \delta_0$. It follows by (6.1) that $|\mathcal{T}_{r_0}| \leq \varepsilon$. Hence, $\mathcal{T}_r \downarrow 0$ as $r \downarrow 0$.

Let us now prove (6.1). By assumption (ii),

$$\begin{aligned} |g_\delta(z) - g(0)| &\leq C \int_{\mathbb{R}^d} |u(x+z) - u(x)| \psi(x) dx + C \int_{\mathbb{R}^d} |v(x-z) - v(x)| \psi(x) dx \\ &\quad + \int_{\mathbb{R}^d} |F_\delta(u(x), v(x)) - F(u(x), v(x))| \psi(x) dx + C\delta \|\psi\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Because of assumptions (i) and (iii), we can apply the dominated convergence theorem to conclude that

$$\lim_{\delta \downarrow 0} \int_{\mathbb{R}^d} |F_\delta(u(x), v(x)) - F(u(x), v(x))| \psi(x) dx = 0.$$

It remains to show that

$$(6.2) \quad \lim_{z \rightarrow 0} \int_{\mathbb{R}^d} |u(x+z) - u(x)| \psi(x) dx = 0.$$

The term involving v follows by the same argument. Pick a compact $K \subset \mathbb{R}^d$ such that $\bigcup_{|z| \leq 1} \text{supp}(\psi(\cdot + z)) \subset K$. Fix $\varepsilon > 0$. By the density of continuous functions in $L^1(K)$, we can find $w \in C(K)$ such that $\|w - u\|_{L^1(K)} \leq \varepsilon$. Then

$$\int_{\mathbb{R}^d} |u(x+z) - u(x)| \psi(x) dx \leq 2 \|\psi\|_\infty \varepsilon + \int_{\mathbb{R}^d} |w(x+z) - w(x)| \psi(x) dx,$$

for any $|z| \leq 1$. Next we send $z \rightarrow 0$. The claim (6.2) follows by the dominated convergence theorem and the arbitrariness of $\varepsilon > 0$. \square

Lemma 6.2. *Let $v \in L^p([0, T])$, $1 \leq p < \infty$. Moreover, Let $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable in the first variable and Lipschitz in the second variable,*

$$|F(s, a) - F(s, b)| \leq C |a - b|, \quad \forall a, b \in \mathbb{R}, \forall s \in [0, T],$$

for some constant $C > 0$. Set

$$\mathcal{T}_{r_0}(s) = \left(\int_0^T |F(s, v(t)) - F(s, v(s))|^p J_{r_0}^+(t-s) dt \right)^{1/p}.$$

Then $\mathcal{T}_{r_0}(s) \rightarrow 0$ ds -a.e. as $r_0 \downarrow 0$.

Proof. We can write $v = v_1^n + v_2^n$ with v_1^n continuous and $\|v_2^n\|_{L^p([0, T])} \leq 1/n$. This is possible since the continuous functions are dense in $L^p([0, T])$. Assuming $s \in [0, T - 2r_0]$, an application of the triangle inequality gives

$$\begin{aligned} |\mathcal{T}_{r_0}(s)| &\leq C \left(\int_0^T |v(t) - v(s)|^p J_{r_0}^+(t-s) dt \right)^{1/p} \\ &\leq C \left(\int_0^T |v_1^n(t) - v_1^n(s)|^p J_{r_0}^+(t-s) dt \right)^{1/p} \\ &\quad + (|v_2^n|^p \star J_{r_0}(s))^{1/p} + |v_2^n(s)|. \end{aligned}$$

Sending $r_0 \downarrow 0$, it follows that $\lim_{r_0 \downarrow 0} |\mathcal{T}_{r_0}(s)| \leq 2|v_2^n(s)|$ for ds -a.a. $s \in [0, T]$. Since $v_2^n \rightarrow 0$ in $L^p([0, T])$, it has a subsequence that converges ds -a.e., and this concludes the proof. \square

6.2. Weighted L^p spaces. First we make some elementary observations regarding functions in \mathfrak{N} (see Section 2 for the definition of \mathfrak{N}).

Lemma 6.3. *Suppose $\phi \in \mathfrak{N}$ and $0 < p < \infty$. Then, for $x, z \in \mathbb{R}^d$,*

$$\left| \phi^{1/p}(x+z) - \phi^{1/p}(x) \right| \leq w_{p,\phi}(|z|) \phi^{1/p}(x),$$

where

$$w_{p,\phi}(r) = \frac{C_\phi}{p} r \left(1 + \frac{C_\phi}{p} r e^{C_\phi r/p} \right),$$

which is defined for all $r \geq 0$. As a consequence it follows that if $\phi(x_0) = 0$ for some $x_0 \in \mathbb{R}^d$, then $\phi \equiv 0$ (and by definition $\phi \notin \mathfrak{N}$).

Proof. Set $g(\lambda) = \phi^{1/p}(x + \lambda z)$. Then

$$g'(\lambda) = \frac{1}{p} \phi^{1/p-1}(x + \lambda z) (\nabla \phi(x + \lambda z) \cdot z).$$

Since $\phi \in \mathfrak{N}$, it follows that $|g'(\lambda)| \leq \frac{C_\phi}{p} g(\lambda) |z|$. Hence

$$g(\lambda) \leq g(0) + \frac{C_\phi}{p} |z| \int_0^\lambda g(\xi) d\xi.$$

By Grönwall's inequality,

$$g(\lambda) \leq g(0) \left(1 + \frac{C_\phi}{p} |z| \lambda e^{C_\phi |z| \lambda/p} \right).$$

Hence,

$$|g(1) - g(0)| \leq \frac{C_\phi}{p} |z| g(0) \left(1 + \frac{C_\phi}{p} |z| e^{C_\phi |z|/p} \right).$$

This concludes the proof. \square

Next, we consider an adaption of Young's inequality for convolutions.

Proposition 6.4. Fix $\phi \in \mathfrak{N}$. Suppose $f \in C_c(\mathbb{R}^d)$, and $g \in L^p(\mathbb{R}^d, \phi)$ for some finite $p \geq 1$. Then

$$\|f \star g\|_{L^p(\mathbb{R}^d, \phi)} \leq \left(\int_{\mathbb{R}^d} |f(x)| (1 + w_{p, \phi}(|x|)) dx \right) \|g\|_{L^p(\mathbb{R}^d, \phi)}$$

where $w_{p, \phi}$ is defined in Lemma 6.3.

Proof. First observe that

$$\begin{aligned} \|f \star g\|_{L^p(\mathbb{R}^d, \phi)}^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)g(y) dy \right|^p \phi(x) dx \\ &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)| \left(\frac{\phi(x)}{\phi(y)} \right)^{1/p} |g(y)| \phi^{1/p}(y) dy \right)^p dx. \end{aligned}$$

By Lemma 6.3(iii),

$$\left(\frac{\phi(x)}{\phi(y)} \right)^{1/p} \leq \frac{1}{\phi^{1/p}(y)} \left(\phi^{1/p}(y) + \left| \phi^{1/p}(x) - \phi^{1/p}(y) \right| \right) \leq (1 + w_{p, \phi}(|x-y|)).$$

Set

$$\zeta(x) := |f(x)| (1 + w_{p, \phi}(|x|)), \quad \xi(x) := |g(x)| \phi^{1/p}(x).$$

Then, by Young's inequality for convolutions,

$$\|f \star g\|_{L^p(\mathbb{R}^d, \phi)} \leq \|\zeta \star \xi\|_{L^p(\mathbb{R}^d)} \leq \|\zeta\|_{L^1(\mathbb{R}^d)} \|\xi\|_{L^p(\mathbb{R}^d)}.$$

□

Lemma 6.5. Fix $\phi \in \mathfrak{N}$, and let $w_{p, \phi}$ be defined in Lemma 6.3. Then

- (i) If $\varphi \in L^1(\mathbb{R}^d)$ is nonnegative, then $\phi \star \varphi \in \mathfrak{N}$ with $C_{\phi \star \varphi} = C_\phi$.
- (ii) Let $\varphi \in L^1(\mathbb{R}^d)$ have support in $B(0, 1)$. Take $\varphi_\delta(z) = \frac{1}{\delta^d} \varphi\left(\frac{z}{\delta}\right)$. Then, for any $u \in L^p(\mathbb{R}^d, \phi)$,

$$\left| \|u\|_{p, \phi}^p - \|u\|_{p, \phi \star \varphi_\delta}^p \right| \leq w_{1, \phi}(\delta) \|\varphi\|_{L^1(\mathbb{R}^d)} \|u\|_{p, \phi}^p.$$

- (iii) Let $\varphi \in C_c^2(\mathbb{R}^d)$. Then

$$|\Delta(\phi \star \varphi)(x)| \leq C_\phi \left(\int_{\mathbb{R}^d} |\nabla \varphi(z)| (1 + w_{1, \phi}(|z|)) dz \right) \phi(x).$$

If φ has support in $B(0, 1)$ this yields

$$|\Delta(\phi \star \varphi_\delta)(x)| \leq \frac{1}{\delta} (1 + w_{1, \phi}(|\delta|)) C_\phi \|\nabla \varphi\|_{L^1(\mathbb{R}^d)} \phi(x).$$

Proof. Consider (i). By Young's inequality for convolutions,

$$\|\phi \star \varphi\|_{L^1(\mathbb{R}^d)} \leq \|\phi\|_{L^1(\mathbb{R}^d)} \|\varphi\|_{L^1(\mathbb{R}^d)}.$$

Furthermore, $\phi \star \varphi \in C^1(\mathbb{R}^d)$ and

$$|\nabla(\phi \star \varphi)(x)| = \left| \int_{\mathbb{R}^d} \varphi(y) \nabla \phi(x-y) dy \right| \leq C_\phi (\phi \star \varphi)(x).$$

Consider (ii). By Lemma 6.3,

$$\begin{aligned} \left| \|u\|_{p, \phi}^p - \|u\|_{p, \phi \star \varphi_\delta}^p \right| &= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x)|^p (\phi(x-z) - \phi(x)) \varphi_\delta(z) dz dx \right| \\ &\leq \|u\|_{p, \phi}^p \int_{\mathbb{R}^d} w_{1, \phi}(|z|) \varphi_\delta(z) dz. \end{aligned}$$

This proves (ii). Consider (iii). Integration by parts yields

$$\begin{aligned} |\Delta(\phi \star \varphi)(x)| &= \left| \int_{\mathbb{R}^d} \nabla \varphi(x-y) \cdot \nabla \phi(y) dy \right| \\ &\leq C_\phi \int_{\mathbb{R}^d} |\nabla \varphi(x-y)| \phi(y) dy. \end{aligned}$$

By Lemma 6.3,

$$\int_{\mathbb{R}^d} |\nabla \varphi(x-y)| \phi(y) dy \leq \left(\int_{\mathbb{R}^d} |\nabla \varphi(x-y)| (1 + w_{1,\phi}(|x-y|)) dy \right) \phi(x).$$

□

Lemma 6.6. *Let $\phi \in \mathfrak{N}$. Then there exists $\{\phi_R\}_{R>1} \subset C_c^\infty(\mathbb{R}^d)$ such that*

- (i) $\phi_R \rightarrow \phi$ and $\nabla \phi_R \rightarrow \nabla \phi$ pointwise in \mathbb{R}^d as $R \rightarrow \infty$,
- (ii) \exists a constant C independent of $R > 1$ such that

$$\max \left\{ \|\phi_R\|_{\infty, \phi^{-1}}, \|\nabla \phi_R\|_{\infty, \phi^{-1}} \right\} \leq C.$$

Proof. Modulo a mollification step, we may assume $\phi \in C^\infty$. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ satisfy $0 \leq \zeta \leq 1$, $\zeta(0) = 1$. Let $\phi_R(x) := \phi(x)\zeta(R^{-1}x)$. Then

$$\nabla \phi_R(x) = \nabla \phi(x)\zeta(R^{-1}x) + R^{-1}\phi(x)\nabla \zeta(R^{-1}x).$$

Hence (i) follows. Clearly, $\|\phi_R\|_{\infty, \phi^{-1}} = \sup_x \{|\phi_R(x)|\phi^{-1}(x)\} = \|\zeta\|_\infty$. Furthermore,

$$|\nabla \phi_R(x)| \leq (C_\phi \zeta(R^{-1}x) + R^{-1}|\nabla \zeta(R^{-1}x)|) \phi(x).$$

Hence, $\|\nabla \phi_R\|_{\infty, \phi^{-1}} \leq C_\phi + R^{-1}\|\nabla \zeta\|_\infty$. □

6.3. A version of Itô's formula. Here we establish the particular anticipating Itô formula applied in the proof of Theorem 4.1.

Theorem 6.7. *Let*

$$X(t) = X_0 + \int_0^t \int_Z u(s, z) W(dz, ds) + \int_0^t v(s) ds,$$

where $u : [0, T] \times Z \times \Omega \rightarrow \mathbb{R}$ and $v : [0, T] \times \Omega \rightarrow \mathbb{R}$ are jointly measurable and $\{\mathcal{F}_t\}$ -adapted processes, satisfying

$$(6.3) \quad E \left[\left(\int_0^T \int_Z u^2(s, z) d\mu(z) ds \right)^2 \right] < \infty, \quad E \left[\int_0^T v^2(s) ds \right] < \infty.$$

Let $F : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}$ be twice continuously differentiable. Suppose there exists a constant $C > 0$ such that for all $(\zeta, \lambda, t) \in \mathbb{R}^2 \times [0, T]$,

$$\begin{aligned} |F(\zeta, \lambda, t)|, |\partial_3 F(\zeta, \lambda, t)| &\leq C(1 + |\zeta| + |\lambda|), \\ |\partial_1 F(\zeta, \lambda, t)|, |\partial_{1,2}^2 F(\zeta, \lambda, t)|, |\partial_1^2 F(\zeta, \lambda, t)| &\leq C. \end{aligned}$$

Let $V \in \mathcal{S}$. Then $s \mapsto \partial_1 F(X(s), V, s)u(s)$ is Skorohod integrable, and

$$\begin{aligned}
F(X(t), V, t) &= F(X_0, V, 0) \\
&+ \int_0^t \partial_3 F(X(s), V, s) ds \\
&+ \int_0^t \int_Z \partial_1 F(X(s), V, s)u(s, z) W(dz, ds) \\
&+ \int_0^t \partial_1 F(X(s), V, s)v(s) ds \\
&+ \int_0^t \int_Z \partial_{1,2}^2 F(X(s), V, s)D_{s,z}Vu(s, z) d\mu(z) ds \\
&+ \frac{1}{2} \int_0^t \int_Z \partial_1^2 F(X(s), V, s)u^2(s, z) d\mu(z) ds, \quad dP\text{-almost surely.}
\end{aligned}$$

Proof. The proof follows [27, Theorem 3.2.2 and Proposition 1.2.5]. We give an outline and some details where there are considerable differences. Furthermore, we leave out the t variable F as this is a standard modification.

Set $t_i^n = \frac{it}{2^n}$, $0 \leq i \leq 2^n$. By Taylor's formula,

$$\begin{aligned}
F(X(t), V) &= F(X_0, V) + \underbrace{\sum_{i=0}^{2^n-1} \partial_1 F(X(t_i^n), V)(X(t_{i+1}^n) - X(t_i^n))}_{\mathcal{T}_n^1} \\
&\quad + \underbrace{\frac{1}{2} \sum_{i=0}^{2^n-1} \partial_1^2 F(\bar{X}_i, V)(X(t_{i+1}^n) - X(t_i^n))^2}_{\mathcal{T}_n^2},
\end{aligned}$$

where \bar{X}_i denotes a random intermediate point between $X(t_i^n)$ and $X(t_{i+1}^n)$. As in the proof of [27, Proposition 1.2.5],

$$\mathcal{T}_n^2 \rightarrow \frac{1}{2} \int_0^t \int_Z \partial_1^2 F(X(s), V)u^2(s, z) d\mu(z) ds, \quad \text{in } L^1(\Omega) \text{ as } n \rightarrow \infty.$$

Note that

$$\begin{aligned}
\mathcal{T}_n^1 &= \underbrace{\sum_{i=0}^{2^n-1} \partial_1 F(X(t_i^n), V) \int_{t_i^n}^{t_{i+1}^n} \int_Z u(s, z) W(dz, ds)}_{\mathcal{T}_n^{1,1}} \\
&\quad + \underbrace{\sum_{i=0}^{2^n-1} \partial_1 F(X(t_i^n), V) \int_{t_i^n}^{t_{i+1}^n} v(s) ds}_{\mathcal{T}_n^{1,2}}.
\end{aligned}$$

Clearly,

$$\mathcal{T}_n^{1,2} \rightarrow \int_0^t \partial_1 F(X(s), V)v(s) ds, \quad \text{in } L^1(\Omega) \text{ as } n \rightarrow \infty.$$

Consider $\mathcal{T}_n^{1,1}$. By [27, Proposition 1.3.5], $s \mapsto \partial_1 F(X(t_i^n), V)u(s)$ is Skorohod integrable on $[t_i^n, t_{i+1}^n]$ and

$$\mathcal{T}_n^{1,1} = \underbrace{\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \int_Z \partial_1 F(X(t_i^n), V)u(s, z) W(dz, ds)}_{\mathcal{T}_n^{1,1,1}} + \underbrace{\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \int_Z \partial_{1,2}^2 F(X(t_i^n), V)D_{s,z}Vu(s, z) d\mu(z) ds}_{\mathcal{T}_n^{1,1,2}}.$$

As before

$$\mathcal{T}_n^{1,1,2} \rightarrow \int_0^t \int_Z \partial_{1,2}^2 F(X(s), V)D_{s,z}Vu(s, z) d\mu(z) ds, \quad \text{in } L^1(\Omega) \text{ as } n \rightarrow \infty.$$

Consider $\mathcal{T}_n^{1,1,1}$. Let

$$\zeta_n(s, z) = \sum_{i=0}^{2^n-1} \partial_{1,2}^2 F(X(t_i^n), V) \mathbf{1}_{[t_i^n, t_{i+1}^n)}(s) D_{s,z}Vu(s, z),$$

and note that ζ_n is Skorohod integrable on $[0, t]$. We need to show the following:

- (i) There exists $\zeta \in L^2(\Omega; H)$ such that $\zeta_n \rightarrow \zeta$ in $L^2(\Omega; H)$.
- (ii) There exists a $G \in L^2(\Omega)$ such that for each $U \in \mathcal{S}$

$$E \left[\int_0^t \int_Z \zeta_n(s, z) W(dz, ds) U \right] \rightarrow E[GU].$$

Then we may conclude by [27, Proposition 1.3.6] that ζ is Skorohod integrable and $\int_0^t \zeta(s) dW(s) = G$. The result then follows. Consider (i). Let

$$\zeta(s, z) = \partial_{1,2}^2 F(X(s), V)D_{s,z}Vu(s, z).$$

Then

$$E \left[\int_0^t \int_Z |\zeta_n(s) - \zeta(s)|^2 d\mu(z) ds \right] \leq E \left[H_n \int_0^t \int_Z |D_{s,z}Vu(s, z)|^2 d\mu(z) ds \right],$$

where

$$H_n = \sup_{|t_i^n - s| \leq t2^{-n}} \left\{ \left| \partial_{1,2}^2 F(X(t_i^n), V) - \partial_{1,2}^2 F(X(s), V) \right|^2 \right\}.$$

Hence, (i) follows by the dominated convergence theorem. Consider (ii). The existence of a random variable G follows by the convergence of the other terms. This also yields the weak convergence. It remains to check that $G \in L^2(\Omega)$. This is a consequence of assumptions (6.3). \square

6.4. The Lebesgue-Bochner space. Let (X, \mathcal{A}, μ) be a σ -finite measure space and E a Banach space. In the previous sections $X = [0, T] \times \Omega$, $\mu = dt \otimes dP$, E is typically $L^p(\mathbb{R}^d, \phi)$ for some $1 \leq p < \infty$, and \mathcal{A} is the predictable σ -algebra \mathcal{P} . A function $u : X \rightarrow E$ is *strongly μ -measurable* if there exists a sequence of μ -simple functions $\{u_n\}_{n \geq 1}$ such that $u_n \rightarrow u$ μ -almost everywhere. By a *μ -simple function* $s : X \rightarrow E$ we mean a function of the form

$$s(\zeta) = \sum_{k=1}^N \mathbf{1}_{A_k}(\zeta) x_k, \quad \zeta \in X,$$

where $x_k \in E$ and $A_k \in \mathcal{A}$ satisfy $\mu(A_k) < \infty$ for all $1 \leq k \leq N$. The Lebesgue-Bochner space $L^p(X, \mathcal{A}, \mu; E)$ is the linear space of μ -equivalence classes of strongly measurable functions $u : X \rightarrow E$ satisfying

$$\int_X \|u(\xi)\|_E^p d\mu(\xi) < \infty.$$

A map $u : X \rightarrow E$ is *weakly μ -measurable* if the map $\xi \mapsto \langle u(\xi), \varphi^* \rangle$ has a μ -version which is \mathcal{A} -measurable for each φ^* in the dual space E^* . By the Pettis measurability theorem [32, Theorem 1.11], strong μ -measurability is equivalent to weak μ -measurability, whenever E is separable.

For $u \in L^1(X, \mathcal{A}, \mu; L^1(\mathbb{R}^d, \phi))$, it is convenient to know that $\zeta \mapsto u(\zeta)(x)$ has a μ -version which is \mathcal{A} -measurable for almost all x . In fact this is crucial to the manipulations performed in the previous sections. The following result verifies that this is indeed the case.

Lemma 6.8. *Let (X, \mathcal{A}, μ) be a σ -finite measure space and $\phi \in \mathfrak{N}$. Let*

$$\Psi : L^1(X \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d), d\mu \otimes d\phi) \rightarrow L^1(X, \mathcal{A}, \mu; L^1(\mathbb{R}^d, \phi))$$

be defined by $\Psi(u)(\xi) = u(\xi, \cdot)$. Then Ψ is an isometric isomorphism.

Remark 6.1. The measure space $(X \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d), d\mu \otimes d\phi)$ is not necessarily complete. Strictly speaking we should rather consider its completion. What this ensures is that every representative is measurable with respect to the complete σ -algebra. A remedy is to define $L^1(X \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d), d\mu \otimes d\phi)$ by asking that any element u has a $d\mu \otimes d\phi$ -version \tilde{u} which is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. Now, $\tilde{u}(\cdot, x)$ is \mathcal{A} -measurable, and so for $d\phi$ -almost all x , $u(\cdot, x)$ has a μ -version which is \mathcal{A} -measurable.

Proof. Let us first check that $\Psi(u) \in L^1(X; L^1(\mathbb{R}^d, \phi))$. By the Pettis measurability theorem [32, Theorem 1.11], strong μ -measurability follows due to the separability of $L^1(\mathbb{R}^d, \phi)$ if $\Psi(u)$ is weakly μ -measurable. That is, for any $\varphi \in L^\infty(\mathbb{R}^d, \phi)$, the map

$$\xi \mapsto \int_{\mathbb{R}^d} \varphi(x) u(\xi, x) \phi(x) dx,$$

has a μ -version which is \mathcal{A} -measurable. This is a consequence of Fubini's theorem [11, Proposition 5.2.2]. The fact that Ψ is an isometry is obvious. It remains to prove that Ψ is surjective. Let $v \in L^1(X; L^1(\mathbb{R}^d, \phi))$. By definition there exists a sequence $\{v_n\}_{n \geq 1}$ of simple functions such that $v_n \rightarrow v$ μ -almost everywhere. Set

$$v_n(\xi) = \sum_{k=1}^{N_n} \mathbf{1}_{A_{k,n}}(\xi) f_{k,n}, \quad u_n(\xi, x) = v_n(\xi)(x),$$

where $A_{k,n} \in \mathcal{A}$, $f_{k,n} \in L^1(\mathbb{R}^d, \phi)$. Note that u_n is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, and $\Psi(u_n) = v_n$. By the Lebesgue dominated convergence theorem, $v_n \rightarrow v$ in $L^1(X, L^1(\mathbb{R}^d, \phi))$ [32, Proposition 1.16]. By the isometry property, $\{u_n\}_{n \geq 1}$ is Cauchy, and so by completeness there exists u such that

$$u_n \rightarrow u \text{ in } L^1(X \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d), d\mu \otimes d\phi).$$

Since

$$\begin{aligned} \int_X \|v - \Psi(u)\|_{1,\phi} d\mu &= \lim_{n \rightarrow \infty} \int_X \|v_n - \Psi(u)\|_{1,\phi} d\mu \\ &= \lim_{n \rightarrow \infty} \iint_{X \times \mathbb{R}^d} |u_n(\xi, x) - u(\xi, x)| d\mu \otimes d\phi(\xi, x) = 0, \end{aligned}$$

it follows that $\Psi(u) = v$. □

6.5. Young measures. The purpose of this subsection is to provide a reference for some results concerning Young measures and their application as generalized limits. Let (X, \mathcal{A}, μ) be a σ -finite measure space, and $\mathcal{P}(\mathbb{R})$ denote the set of probability measures on \mathbb{R} . In the previous sections X is typically $\Pi_T \times \Omega$. A *Young measure* from X into \mathbb{R} is a function $\nu : X \rightarrow \mathcal{P}(\mathbb{R})$ such that $x \mapsto \nu_x(B)$ is \mathcal{A} -measurable for every Borel measurable set $B \subset \mathbb{R}$. We denote by $\mathcal{Y}(X, \mathcal{A}, \mu; \mathbb{R})$, or simply $\mathcal{Y}(X; \mathbb{R})$ if the measure space is understood, the set of all Young measures from X into \mathbb{R} . The following theorem is proved in [29, Theorem 6.2] in the case that $X \subset \mathbb{R}^n$ and μ is the Lebesgue measure:

Theorem 6.9. *Let (X, \mathcal{A}, μ) be a σ -finite measure space. Let $\zeta : [0, \infty) \rightarrow [0, \infty]$ be a continuous, nondecreasing function satisfying $\lim_{\xi \rightarrow \infty} \zeta(\xi) = \infty$ and $\{u^n\}_{n \geq 1}$ a sequence of measurable functions such that*

$$\sup_n \int_X \zeta(|u^n|) d\mu(x) < \infty.$$

Then there exist a subsequence $\{u^{n_j}\}_{j \geq 1}$ and $\nu \in \mathcal{Y}(X, \mathcal{A}, \mu; \mathbb{R})$ such that for any Carathéodory function $\psi : \mathbb{R} \times X \rightarrow \mathbb{R}$ with $\psi(u^{n_j}(\cdot), \cdot) \rightarrow \bar{\psi}$ in $L^1(X)$, we have

$$\bar{\psi}(x) = \int_{\mathbb{R}} \psi(\xi, x) d\nu_x(\xi).$$

The proof is based on the embedding of $\mathcal{Y}(X; \mathbb{R})$ into $L_{w*}^\infty(X, \mathcal{M}(\mathbb{R}))$. Here $\mathcal{M}(\mathbb{R})$ denotes the space of Radon measures on \mathbb{R} . The crucial observation is that $(L^1(X, C_0(\mathbb{R})))^*$ is isometrically isomorphic to $L_{w*}^\infty(X, \mathcal{M}(\mathbb{R}))$ also in the case that (X, \mathcal{A}, μ) is an abstract σ -finite measure space. It is relatively straightforward to go through the proof and extend to this more general case [26, Theorem 2.11]. Note however that the use of weighted L^p spaces allows us to stick with the version for finite measure spaces.

6.6. Weak compactness in L^1 . To apply Theorem 6.9 one must first be able to extract from $\{\psi(\cdot, u^n(\cdot))\}_{n \geq 1}$ a weakly convergent subsequence in $L^1(X)$. The key result is the Dunford-Pettis Theorem.

Definition 6.1. Let $\mathcal{K} \subset L^1(X, \mathcal{A}, \mu)$.

- (i) \mathcal{K} is *uniformly integrable* if for any $\varepsilon > 0$ there exists $c_0(\varepsilon)$ such that

$$\sup_{f \in \mathcal{K}} \int_{|f| \geq c} |f| d\mu < \varepsilon, \quad \text{whenever } c > c_0(\varepsilon).$$

- (ii) \mathcal{K} has *uniform tail* if for any $\varepsilon > 0$ there exists $E \in \mathcal{A}$ with $\mu(E) < \infty$ such that

$$\sup_{f \in \mathcal{K}} \int_{X \setminus E} |f| d\mu < \varepsilon.$$

If \mathcal{K} satisfies both (i) and (ii) it is said to be *equiintegrable*.

Remark 6.2. Note that (ii) is void when μ is finite. As a consequence uniform integrability and equiintegrability are equivalent for finite measure spaces.

Theorem 6.10 (Dunford-Pettis). *Let (X, \mathcal{A}, μ) be a σ -finite measure space. A subset \mathcal{K} of $L^1(X)$ is relatively weakly sequentially compact if and only if it is equiintegrable.*

By the Eberlain-Šmulian theorem [40], in the weak topology of a Banach space, relative weak compactness is equivalent with relative sequentially weak compactness. There are a couple of well known reformulations of uniform integrability.

Lemma 6.11. *Suppose $\mathcal{K} \subset L^1(X)$ is bounded. Then \mathcal{K} is uniformly integrable if and only if:*

(i) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\sup_{f \in \mathcal{K}} \int_E |f| d\mu < \varepsilon, \quad \text{whenever } \mu(E) < \delta(\varepsilon).$$

(ii) There is an increasing function $\Psi : [0, \infty) \rightarrow [0, \infty)$ such that $\Psi(\zeta)/\zeta \rightarrow \infty$ as $\zeta \rightarrow \infty$ and

$$\sup_{f \in \mathcal{K}} \int_X \Psi(|f(x)|) d\mu(x) < \infty.$$

Remark 6.3. Suppose there exists $g \in L^1(X)$ such that $|f| \leq g$ for all $f \in \mathcal{K}$. Then

$$\sup_{f \in \mathcal{K}} \int_E |f| d\mu \leq \int_E g d\mu, \quad \forall E \in \mathcal{A}.$$

Since $\{g\} \subset L^1(X)$ is uniformly integrable, it follows by Lemma 6.11(i) that \mathcal{K} is uniformly integrable.

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