ON THE LARGE TIME BEHAVIOR OF PERIODIC ENTROPY SOLUTIONS TO SCALAR CONSERVATION LAWS

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Abstract. We prove that a periodic entropy solution to a one-dimensional scalar conservation law converges as time $t \to +\infty$ to a traveling wave. Moreover, the flux function is shown to be affine on the segment $[\alpha, \beta]$ containing the essential range of the traveling wave profile, and the speed of the traveling wave coincides with the slope of the line $v = \varphi(u)$, $u \in [\alpha, \beta]$.

Key words. conservation laws; entropy solutions; compensated compactness; decay property; traveling waves

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1. Introduction. We consider a scalar conservation law

$$u_t + \varphi(u)_x = 0, \quad (t, x) \in \Pi = (0, +\infty) \times \mathbb{R}. \tag{1.1}$$

The flux function is supposed to be only continuous: $\varphi(u) \in C(\mathbb{R})$. We recall the notion of an entropy solution to the Cauchy problem for equation (1.1) with initial data

$$u(0, x) = u_0(x) \in L^\infty(\mathbb{R}) \tag{1.2}$$

in the sense of Kruzhkov [6].

Definition 1.1. *A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1.1), (1.2) if for all $k \in \mathbb{R}$*

$$\frac{\partial}{\partial t}|u - k| + \frac{\partial}{\partial x}[\operatorname{sign}(u - k)(\varphi(u) - \varphi(k))] \le 0 \tag{1.3}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$), and

$$\operatorname*{ess\,lim}_{t \to 0+} u(t, \cdot) = u_0 \quad \text{in } L^1_{loc}(\mathbb{R}). \tag{1.4}$$

Here $\operatorname{sign} u = \begin{cases} 1 & , \quad u > 0, \\ -1 & , \quad u \le 0 \end{cases}$ and relation (1.3) means that for each test function $h = h(t, x) \in C^1_0(\Pi)$, $h \ge 0$,

$$\int_\Pi [|u - k| h_t + \operatorname{sign}(u - k)(\varphi(u) - \varphi(k)) h_x] dt dx \ge 0.$$

Taking in (1.3) $k = \pm R$, $R \ge \|u\|_\infty$, we derive that $u_t + \varphi(u)_x = 0$ in $\mathcal{D}'(\Pi)$, i.e., an e.s. $u = u(t, x)$ is a weak solution of (1.1). As was shown in [12] (see also [7, 8] for more details), for every $u_0(x) \in L^\infty(\mathbb{R})$ there exists a unique e.s. to problem (1.1), (1.2). We underline that $\varphi(u)$ is assumed to be only continuous, and it is essential that we have only one space variable (in the multidimensional case some conditions on the continuity modulus of flux functions are necessary for the uniqueness, cf. [7, 8]).

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Let $u_n(t,x) \in L^\infty(\Pi)$ be a sequence of e.s. of equation (1.1), which converges weakly-$*$ in $L^\infty(\Pi)$ to a function $u(t,x)$. The following result was established in [16] (in the general case of multidimensional conservation laws).

THEOREM 1.2. *Assume that for almost each $(t,x) \in \Pi$ the function $\varphi(u)$ is not affine in any vicinity of the point $u(t,x)$. Then $u_n(t,x) \to u(t,x)$ as $n \to \infty$ in $L^1_{loc}(\Pi)$ (strongly).*

Multidimensional version of Theorem 1.2 is proved with the help of a rather complicated variant of H-measures, see details in [16]. But in the one-dimensional situation the statement of Theorem 1.2 can be established by more common compensated compactness method, in the similar way as in [17, 1] (see also book [3]). For the sake of completeness we provide the proof of Theorem 1.2 in the next section 2.

Suppose that the initial data is periodic. Without loss of generality, we may assume that the period equals 1, i.e., $u_0(x+1) = u_0(x)$ almost everywhere in $\mathbb{R}$. Then the unique e.s. $u(t,x)$ of problem (1.1), (1.2) is space periodic: $u(t,x+1) = u(t,x)$ a.e. in Π and satisfies the following conservation property: for a.e. $t > 0$

$$\int_0^1 u(t,x)dx = I \doteq \int_0^1 u_0(x), \tag{1.5}$$

see the proof in [14]. It is also known (see, for example, [9, Corollary 3.3]) that for two e.s. $u_1(t,x)$, $u_2(t,x)$ of (1.1), (1.2) with initial functions $u_{01}(x)$, $u_{02}(x)$, respectively, the following L^1-contraction property holds

$$\int_0^1 |u_1(t,x) - u_2(t,x)|dx \le \int_0^1 |u_{01}(x) - u_{02}(x)|dx \quad \text{for a.e. } t > 0. \tag{1.6}$$

Based on Theorem 1.2 and approach developed by G.-Q. Chen and H. Frid in [2], we establish the following decay property of periodic e.s.

THEOREM 1.3. *Assume that the function $\varphi(u)$ is not affine in any vicinity of the point I. Then*

$$\operatorname*{ess\,lim}_{t\to\infty} u(t,\cdot) = I \quad \text{in } L^1([0,1]). \tag{1.7}$$

The statement of Theorem 1.3 follows from the more general result of [16] (in the case $\varphi(u) \in C^2(\mathbb{R})$ one can rely also on [4]). For completeness we reproduce the proof of Theorem 1.3 in section 3 below.

It turns out that in general case of arbitrary continuous flux $\varphi(u)$ the following asymptotic property holds, which is the main our result.

THEOREM 1.4. *There exists a 1-periodic function $v(y) \in L^\infty(\mathbb{R})$ and a constant $c \in \mathbb{R}$ such that*

$$u(t,x) - v(x-ct) \underset{t\to+\infty}{\to} 0 \quad \text{in } L^1([0,1]).$$

Besides, $\int_0^1 v(x)dx = I$ and $\varphi(u) - cu = \mathrm{const}$ on the minimal segment $[\alpha(v), \beta(v)]$ containing values $v(y)$ for a.e. $y \in [0,1]$.

Observe that the statement of Theorem 1.3 follows from Theorem 1.4. Indeed, under assumptions of Theorem 1.3 the profile $v(x) = I$ a.e. on $\mathbb{R}$. Otherwise, by Theorem 1.4

$$\alpha(v) = \operatorname{ess\,inf} v(y) < I < \beta(v) = \operatorname{ess\,sup} v(y),$$

and the flux function $\varphi(u) = cu + \mathrm{const}$ is affine in the vicinity $(\alpha(v), \beta(v))$ of I. But this is impossible in view of the condition of Theorem 1.3.

The proof of Theorem 1.4 is contained in the last section 4.

In the proof of Theorem 1.2 we will use results of the theory of measure valued functions (Young measures). Recall (see [5, 17]) that a measure-valued function on Π is a weakly measurable map $(t, x) \mapsto \nu_{t,x}$ of Π into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of $\nu_{t,x}$ means that for each continuous function $g(\lambda)$ the function $(t, x) \to \langle \nu_{t,x}, g(\lambda) \rangle = \int g(\lambda) d\nu_{t,x}(\lambda)$ is measurable on Π.

We say that a measure-valued function $\nu_{t,x}$ is *bounded* if there exists $R > 0$ such that $\operatorname{supp} \nu_{t,x} \subset [-R, R]$ for almost all $(t, x) \in \Pi$. We shall denote by $\|\nu_{t,x}\|_\infty$ the smallest such R.

Finally, we say that measure-valued functions of the kind $\nu_{t,x}(\lambda) = \delta(\lambda - u(t, x))$, where $u(t, x) \in L^\infty(\Pi)$ and $\delta(\lambda - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are *regular*. We identify these measure-valued functions and the corresponding functions $u(t, x)$, so that there is a natural embedding $L^\infty(\Pi) \subset \mathrm{MV}(\Pi)$, where $\mathrm{MV}(\Pi)$ is the set of bounded measure-valued functions on Π.

Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem of Tartar (see [17]).

Theorem 1.5. *Let $u_n(t, x) \in L^\infty(\Pi)$, $n \in \mathbb{N}$, be a bounded sequence. Then there exist a subsequence $u_r(t, x) = u_{n_r}(t, x)$ and a measure-valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ such that*

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_r) \underset{r\to\infty}{\rightharpoonup} \langle \nu_{t,x}, g(\lambda) \rangle \quad \text{weakly-}* \text{ in } L^\infty(\Pi). \tag{1.8}$$

Besides, $\nu_{t,x}$ is regular, i.e., $\nu_{t,x}(\lambda) = \delta(\lambda - u(t, x))$ if and only if $u_r(t, x) \underset{r\to\infty}{\to} u(t, x)$ in $L^1_{loc}(\Pi)$.

2. Proof of Theorem 1.2. We use the notation $\operatorname{Co} A$ for the convex hull of a set $A \subset \mathbb{R}$. Evidently, if A is compact then $\operatorname{Co} A = [\min A, \max A]$ is the minimal segment containing A.

The following technical lemma is borrowed from [13, Lemma 3.3].

Lemma 2.1. *Assume that ν is a compactly supported finite nonnegative Borel measure on $\mathbb{R}$, $[a, b] = \operatorname{Co} \operatorname{supp} \nu$; $H(u) \in C(\mathbb{R})$, and for each $k \in (a, b)$*

$$\int (H(\lambda) - H(k)) \operatorname{sign}^+(\lambda - k) d\nu(\lambda) = 0, \tag{2.1}$$

where $\operatorname{sign}^+(\lambda) = (1 + \operatorname{sign} \lambda)/2$ is the Heaviside function. Then $H(u) \equiv \mathrm{const}$ on $[a, b]$.

For the sake of completeness we reproduce below the proof.

Proof. First, observe that by continuity of $H(\lambda)$ equality (2.1) holds for each $k \in [a, b]$. Since $H(u)$ is continuous, there exist such $k_1, k_2 \in [a, b]$ that

$$H(k_1) = H_- \doteq \min_{u\in[a,b]} H(u), \quad H(k_2) = H_+ \doteq \max_{u\in[a,b]} H(u).$$

Then it follows from (2.1) with $k = k_1, k_2$ that $H(b) = H_- = H_+$. Indeed, assuming that $H(b) \neq H_-$, we claim that $k_1 < b$ and the nonnegative function $(H(\lambda) - H(k_1)) \operatorname{sign}^+(\lambda - k_1)$ is strictly positive in some neighborhood $(b - \delta, b]$ of the point b. Since $\nu((b - \delta, b]) > 0$ we conclude that the integral in equality (2.1) with

$k = k_1$ is positive, which contradicts to this equality. Hence, $H(b) = H_-$. By the similar reasons, using (2.1) with $k = k_2$, we claim that $H(b) = H_+$. Evidently, the equality $H_- = H_+$ can hold only if $H(u) \equiv \text{const} = H(b)$ on the segment $[a, b]$. The proof is complete. $\square$

We denote

$$\eta_k^+(u) = (u-k)^+ = \max(u-k, 0), \quad \psi_k^+(u) = (\varphi(u) - \varphi(k))\operatorname{sign}^+(u-k);$$
$$\eta_k^-(u) = (u-k)^- = \max(k-u, 0), \quad \psi_k^-(u) = (\varphi(u) - \varphi(k))\operatorname{sign}^-(u-k),$$

where the functions $\operatorname{sign}^\pm(u) = (\operatorname{sign} u \pm 1)/2$. If $u = u(t, x)$ is an e.s. of (1.1) then it satisfies (1.1) in $\mathcal{D}'(\mathbb{R})$, and this yields

$$\frac{\partial}{\partial t}(u-k) + \frac{\partial}{\partial x}(\varphi(u) - \varphi(k)) = 0 \quad \text{in } \mathcal{D}'(\Pi) \tag{2.2}$$

for all $k \in \mathbb{R}$. Putting (1.3) together with (2.2) multiplied by ± 1, we derive that for each $k \in \mathbb{R}$

$$l_k^\pm \doteq \frac{\partial}{\partial t}\eta_k^\pm(u) + \frac{\partial}{\partial x}\psi_k^\pm(u) \le 0 \quad \text{in } \mathcal{D}'(\Pi). \tag{2.3}$$

According to the Schwartz representation theorem, the distributions $l_k^\pm = -\mu_k^\pm$, where $\mu_k^\pm$ are locally finite nonnegative Borel measures on Π. Notice that $l_k^+ - l_k^-$ coincides with zero distribution (2.2) and in particular $\mu_k^+ = \mu_k^- = \mu_k$. As is easy to see, $\mu_k = 0$ whenever $|k| > M \ge \|u\|_\infty$. Let $|k| \le M$ and $K \subset \Pi$ be a compact set. We choose a nonnegative function $h_K = h_K(t, x) \in C_0^1(\Pi)$ such that $h_K = 1$ on the set K. Then

$$\mu_k(K) \le \int_\Pi h_K(t,x) d\mu_k(t,x) = -\langle l_k^+, h_K\rangle =$$
$$\int_\Pi [\eta_k^+(u)(h_K)_t + \psi_k^+(u)(h_K)_x] dtdx \le$$
$$C(K, M) \doteq 2(M + \max_{|u|\le M} |\varphi(u)|) \int_\Pi (|(h_K)_t| + |(h_K)_x|) dtdx. \tag{2.4}$$

Now we consider a bounded sequence $u_n = u_n(t, x)$ of e.s. of (1.1) and assume that $u_n \rightharpoonup u$ as $n \to \infty$ weakly-$*$ in $L^\infty(\Pi)$. Passing to a subsequence $u_r = u_{n_r}(t, x)$, we can suppose that this subsequence converges to a bounded measure valued function $\nu_{t,x} \in \mathrm{MV}(\Pi)$ in the sense of relation (1.8).

Proposition 2.2. *For almost every $(t, x) \in \Pi$ the flux function is affine on $\operatorname{Co}\operatorname{supp}\nu_{t,x}$.*

Proof. Obviously, $\|\nu_{t,x}\|_\infty \le M = \sup_{n\in\mathbb{N}} \|u_n\|_\infty$. Since $u_r(t, x)$ is an e.s. of (1.1), then for all $k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\eta_k^\pm(u_r) + \frac{\partial}{\partial x}\psi_k^\pm(u_r) = -\mu_{kr}, \tag{2.5}$$

where μ_{kr}, $r \in \mathbb{N}$, is a sequence in the space $\mathrm{M}_{loc}(\Pi)$ of locally finite Borel measures on Π.

In view of (2.4) for each compact set $K \subset \Pi$ $0 \le \mu_{kr}(K) \le C(K, M)$, that is, the sequences μ_{kr}, $r \in \mathbb{N}$, are bounded in $\mathrm{M}_{loc}(\Pi)$. By Murat interpolation lemma [11] (also see [17, Lemma 28]), the sequences (2.5) are pre-compact in the Sobolev space $W_{2,loc}^{-1}(\Pi)$ for each $k \in \mathbb{R}$. Recall that $W_{2,loc}^{-1}(\Pi)$ is a locally convex space of distributions $l = l(t,x)$ such that lf belongs to the Sobolev space $W_2^{-1}(\mathbb{R}^2)$ for all $f = f(t,x) \in C_0^\infty(\Pi)$. The topology in $W_{2,loc}^{-1}(\Pi)$ is generated by the family of semi-norms $l \to \|lf\|_{W_2^{-1}}$, $f(t,x) \in C_0^\infty(\Pi)$.

By Tartar–Murat compensated compactness [10, 17] the quadratic functional $q(\bar\lambda) = (\lambda_1\lambda_4 - \lambda_2\lambda_3)$, $\bar\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, is weakly continuous on the sequences $(\eta_k^+(u_r), \psi_k^+(u_r), \eta_l^-(u_r), \psi_l^-(u_r))$ for all $k,l \in \mathbb{R}$. By (1.8) this can be written as the following commutation relation: for a.e. $(t,x) \in \Pi$

$$\begin{aligned} &\langle \nu_{t,x}(\lambda), \eta_k^+(\lambda)\psi_l^-(\lambda) - \eta_l^-(\lambda)\psi_k^+(\lambda)\rangle = \\ &\langle \nu_{t,x}(\lambda), \eta_k^+(\lambda)\rangle\langle \nu_{t,x}(\lambda), \psi_l^-(\lambda)\rangle - \langle \nu_{t,x}(\lambda), \eta_l^-(\lambda)\rangle\langle \nu_{t,x}(\lambda), \psi_k^+(\lambda)\rangle. \end{aligned} \tag{2.6}$$

It is clear that (2.6) holds for $(t,x) \in P$, where P is a set of common Lebesgue point of the functions $(t,x) \to \langle \nu_{t,x}(\lambda), p(\lambda)\rangle$, $p(\lambda) \in C(\mathbb{R})$. Since the space $C(\mathbb{R})$ is separable, we see that $P \subset \Pi$ is a set of full measure. We fix $(t,x) \in P$, $\nu = \nu_{t,x}$, and assume that the segment $[a,b] = \operatorname{Co}\operatorname{supp}\nu$ is not trivial, i.e., $a < b$. Then it follows from (2.6) that for each $k,l \in (a,b)$ such that $l < k$

$$\langle \nu(\lambda), \eta_k^+(\lambda)\rangle\langle \nu(\lambda), \psi_l^-(\lambda)\rangle - \langle \nu(\lambda), \eta_l^-(\lambda)\rangle\langle \nu(\lambda), \psi_k^+(\lambda)\rangle = 0 \tag{2.7}$$

because, evidently, $\eta_k^+(\lambda)\psi_l^-(\lambda) = \eta_l^-(\lambda)\psi_k^+(\lambda) \equiv 0$. Since $[a,b]$ is the minimal segment containing $\operatorname{supp}\nu$ then

$$\langle \nu(\lambda), \eta_k^+(\lambda)\rangle = \int (\lambda - k)^+ d\nu(\lambda) > 0, \quad \langle \nu(\lambda), \eta_l^-(\lambda)\rangle = \int (\lambda - l)^- d\nu(\lambda) > 0$$

and (2.7) implies that for each $l,k \in (a,b)$, $l < k$

$$I_-(l) \doteq \frac{\langle \nu(\lambda), \psi_l^-(\lambda)\rangle}{\langle \nu(\lambda), \eta_l^-(\lambda)\rangle} = I_+(k) = \frac{\langle \nu(\lambda), \psi_k^+(\lambda)\rangle}{\langle \nu(\lambda), \eta_k^+(\lambda)\rangle}.$$

Clearly, this can hold only if $I_-(l) = I_+(k) = C$, where $C = \text{const}$. In particular, $I_+(k) = C$, which implies that

$$\int \operatorname{sign}^+(\lambda - k)(\varphi(\lambda) - C\lambda - (\varphi(k) - Ck))d\nu(\lambda) = \langle \nu(\lambda), \psi_k^+(\lambda)\rangle - C\langle \nu(\lambda), \eta_k^+(\lambda)\rangle = 0$$

for all $k \in (a,b)$. By Lemma 2.1, applied to the function $H(\lambda) = \varphi(\lambda) - C\lambda$, we conclude that $\varphi(\lambda) - C\lambda = \text{const}$ on $[a,b]$, that is, $\varphi(\lambda)$ is affine on $[a,b] = \operatorname{Co}\operatorname{supp}\nu_{t,x}$. In the case $a = b$ this statement is trivially fulfilled. To conclude the proof, it only remains to see that $(t,x) \in P$ is arbitrary. □

Now we are ready to conclude the proof of Theorem 1.2. Let, as above, u_r be a subsequence of u_n, convergent to a measure valued function $\nu_{t,x}$ in the sense of relation (1.8), and let $P \subset \Pi$ be the set of full measure defined in the proof of Proposition 2.2. Suppose $(t,x) \in P$, $[a,b] = \operatorname{Co}\operatorname{supp}\nu_{t,x}$. If $a < b$ then $u(t,x) = \int \lambda d\nu_{t,x}(\lambda) \in (a,b)$ and by Proposition 2.2 the flux $\varphi(u)$ is affine on (a,b). According to the assumption of Theorem 1.2 this may happen only for a set of (t,x) of null Lebesgue measure. Hence, $b = a = u(t,x)$ for a.e. $(t,x) \in \Pi$, that is, $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$. This means that the measure valued function $\nu_{t,x}$ is regular. By Theorem 1.5 the subsequence $u_r \to u$ as $r \to \infty$ in $L^1_{loc}(\Pi)$. To conclude the proof it only remains to notice that the limit function $u(t,x)$ does not depend on the choice of subsequence. Therefore, the original sequence u_n converges as $n \to \infty$ to the same limit $u = u(t,x)$ in $L^1_{loc}(\Pi)$.

3. Proof of Theorem 1.3. To prove the decay property, we use the approach developed in [2], which relies on the precompactness of the rescaled sequence $u_n = u(nt, nx)$, $n \in \mathbb{N}$. Recall that $u(t,x)$ is an e.s. of (1.1), (1.2) with a 1-periodic initial function $u_0(x) \in L^\infty(\mathbb{R})$.

As follows from [15, Corollary 7.1], after possible correction on a set of null measure, an e.s. $u(t,x)$ is continuous on $[0,+\infty)$ as a map $t \mapsto u(t,\cdot)$ of $[0,+\infty)$ into $L^1([0,1])$. Thus, we can and will always assume that $u(t,\cdot) \in C([0,+\infty), L^1([0,1]))$.

In view of (1.5), $\int_0^1 u(t,x)dx = I = \int_0^1 u_0(x)dx$ for all $t > 0$ (already without extraction of a set of t of null measure).

Lemma 3.1. *The sequence*

$$u_n \underset{n\to\infty}{\rightharpoonup} I \quad \textit{weakly-}* \textit{ in } L^\infty(\Pi).$$

Proof. We introduce the sequence $v_n = u_n - I$. First observe that for each integer $m \in \mathbb{Z}$ and $p(t) \in L^1((0,+\infty))$

$$I_n \doteq \int_{(0,+\infty)\times[0,1]} v_n(t,x)p(t)e^{2\pi imx}dtdx = 0 \ \forall n > |m|, \tag{3.1}$$

where $i = \sqrt{-1}$ is the imaginary unit. Indeed, for $m = 0$ this reduces to the equality $\int_0^1 v_n(t,x)dx = 0$ following from (1.5). If $m \neq 0$ then using that $u_n(t, x+1/n) = u_n(t,x)$ for a.e. $(t,x) \in \Pi$, we find

$$I_n = \int_{(0,+\infty)\times[0,1]} v_n(t,x+1/n)p(t)e^{2\pi im(x+1/n)}dtdx =$$
$$e^{2\pi im/n}\int_{(0,+\infty)\times[0,1]} v_n(t,x)p(t)e^{2\pi imx}dtdx = e^{2\pi im/n}I_n.$$

Since $n > |m|$ then $e^{2\pi im/n} \neq 1$ and we conclude that $I_n = 0$. In view of (3.1)

$$\lim_{n\to\infty}\int_{(0,+\infty)\times[0,1]} v_n(t,x)h(t,x)dtdx = 0$$

for all function $h(t,x)$ of the kind $h(t,x) = \sum_{j=1}^l p_j(t)e^{2\pi im_jx}$, $m_j \in \mathbb{Z}$, $p_j \in L^1((0,+\infty))$, $j = 1,\dots,l$. Since functions of such kind are dense in $L^1((0,+\infty)\times[0,1])$ while the sequence v_n is bounded in $L^\infty((0,+\infty)\times[0,1])$, we obtain that $v_n \rightharpoonup 0$ as $n \to \infty$ weakly-$*$ in $L^\infty((0,+\infty)\times[0,1])$ and, in view of spatial periodicity, also in $L^\infty(\Pi)$. The proof is complete. □

It is clear that $u_n(t,x) = u(nt,nx)$ is an e.s. of (1.1), (1.2) with initial data $u_0(nx)$. By Lemma 3.1 the sequence u_n converges weakly to the constant I. By the assumption of Theorem 1.3, the flux function $\varphi(u)$ is not affine in any vicinity of I. Applying Theorem 1.2, we conclude that $u_n \to I$ as $n \to \infty$ in $L^1_{loc}(\Pi)$. This implies that there is a subsequence $u_r = u_{n_r}(t,x)$ such that for a.e. $t > 0$ $u_r(t,x) \underset{r\to\infty}{\to} I$ in $L^1([0,1])$. Making the change of variables $y = n_r x$ and using the space periodicity of u, we find that for a.e. $t > 0$

$$\int_0^1 |u(n_rt,y) - I|dy = \int_0^1 |u(n_rt,n_rx) - I|dx \underset{r\to\infty}{\to} 0. \tag{3.2}$$

We fix such $t = t_0 > 0$. Then for all $t > n_r t_0$

$$\int_0^1 |u(t,y) - I|dy \le \int_0^1 |u(n_r t_0, y) - I|dy, \tag{3.3}$$

by the L^1-contraction property (1.6) (with "initial time" $n_r t_0$). In view of (3.2) it follows from (3.3) that $\lim_{t\to\infty} u(t,x) = I$ in $L^1([0,1])$, and decay property (1.7) holds.

4. Proof of Theorem 1.4. If $\varphi(u)$ is not affine in any vicinity of I then it follows from Theorem 1.3 that $v(y) \equiv I$ and $\alpha(v) = \beta(v) = I$. Otherwise, assume that $\varphi(u)$ is affine on some maximal interval (a,b), $-\infty \le a < I < b \le +\infty$: $\varphi(u) - cu = \text{const}$ on (a,b).

Let $b < +\infty$ and $u_+ = u_+(t,x)$ be the e.s. of (1.1), (1.2) with initial data $u_0(x) + b - I > u_0$. By the comparison principle [7, 8] we see that $u_+ \ge u$. Since $\int_0^1 (u_0(x) + b - I)dx = b$ while $\varphi(u)$ is not affine in any vicinity of b, it follows from Theorem 1.3 that $u_+(t,\cdot) \to b$ in $L^1([0,1])$ as $t \to +\infty$. In view of the inequality $u \le u_+$, we find that $(u(t,\cdot) - b)^+ \to 0$ in $L^1([0,1])$ as $t \to +\infty$. In the similar way, if $a > -\infty$ then $u \ge u_-$, where $u_- = u_-(t,x)$ is the e.s. of (1.1), (1.2) with initial data $u_0(x) + a - I < u_0$. By Theorem 1.3 again we see that $u_-(t,\cdot) \to a$ in $L^1([0,1])$ as $t \to +\infty$ because $\int_0^1 (u_0(x) + a - I)dx = a$ while $\varphi(u)$ is not affine in any vicinity of a. This implies that $(a - u(t,\cdot))^+ \underset{t\to+\infty}{\to} 0$ in $L^1([0,1])$. It follows from the above limit relations that

$$u(t,\cdot) - s_{a,b}(u(t,\cdot)) \underset{t\to+\infty}{\to} 0 \quad \text{in } L^1([0,1]), \tag{4.1}$$

where $s_{a,b}(u) = \min(b, \max(a,u))$ is the cut-off function (it is possible that $a = -\infty$ or $b = +\infty$).

We take a strictly increasing sequence $t_k \to +\infty$ such that $ct_k \in \mathbb{Z}$ and define $v_k(x) = s_{a,b}(u(t_k,x))$.

By the construction $\varphi(v_k) = cv_k + \text{const}$, which readily implies that $v_k(x - ct)$ is an e.s. of (1.1), (1.2) with initial data v_k. By the L^1-contraction property (1.6) and the assumption $ct_k \in \mathbb{Z}$, $\forall t > t_k$

$$\int_0^1 |u(t,x) - v_k(x - ct)|dx \le \int_0^1 |u(t_k,x) - v_k(x - ct_k)|dx =$$

$$\delta_k \doteq \int_0^1 |u(t_k,x) - v_k(x)|dx \to 0 \quad \text{as } k \to \infty, \tag{4.2}$$

where the latter limit relation follows from (4.1). Taking in (4.2) $t = t_l$, where $l > k$, we find

$$\int_0^1 |v_l(x) - v_k(x)|dx = \int_0^1 |s_{a,b}(u(t_l,x)) - v_k(x - ct_l)|dx \le$$
$$\int_0^1 |u(t_l,x) - v_k(x - ct_l)|dx \le \delta_k.$$

We see that v_k is a Cauchy sequence in $L^1([0,1])$. Therefore v_k converges as $k \to \infty$ to a 1-periodic function v. It is clear that $a \le v \le b$. In view of (4.2)

$$\int_0^1 |u(t,x) - v(x - ct)|dx \le \int_0^1 |u(t,x) - v_k(x - ct)|dx +$$
$$\int_0^1 |v_k(x) - v(x)|dx \le \delta_k + \int_0^1 |v_k(x) - v(x)|dx \to 0$$

as $t \to +\infty$. Here $k = k(t) = \max\{k : t > t_k\} \to \infty$. We also notice that

$$\int_0^1 v(x)dx = \int_0^1 v(x-ct)dx = \lim_{t\to+\infty} \int_0^1 u(t,x)dx = I.$$

This completes the proof of Theorem 1.4.

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