ISENTROPIC FLUID DYNAMICS IN A CURVED PIPE

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Abstract. In this paper we study isentropic flow in a curved pipe. We focus on the consequences of the geometry of the pipe on the dynamics of the flow. More precisely, we present the solution of the general Cauchy problem for isentropic fluid flow in an arbitrarily curved, piecewise smooth pipe. We consider initial data in the subsonic regime, with small total variation about a stationary solution. The proof relies on the front-tracking method and is based on [1].

1. Introduction

Consider a pipe filled with a compressible fluid. The pipe section is far smaller than its length. The pipe is not assumed to be rectilinear. We propose below a modification to the usual isentropic Euler equations that takes into account the pipe's geometry.

First, consider the case of a horizontal pipe with a single elbow. Following [11], along the pipe we use the classical isentropic p-system in Eulerian coordinates

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = 0, \end{cases} \tag{1.1}$$

where t is time, x is the abscissa along the pipe, ρ is the mass density, q is the linear momentum density, i.e., $q = \rho v$ where v is the velocity, and p is the pressure. At the kink, located at, say, $x = 0$, the following conditions on the traces of q and of the dynamic pressure $P = q^2/\rho + p(\rho)$ are imposed:

$$q(t, 0-) = q(t, 0+) \quad \text{and} \quad P(t, 0-) = P(t, 0+) - f\,\kappa \left(2\left|\sin(\vartheta/2)\right| \right) q(t, 0+)\,, \tag{1.2}$$

where the positive parameter f accounts for inhomogeneities in the pipe's walls at the kink and κ depends on the pipe's angle ϑ, see Figure 1. Equivalently, (1.1)–

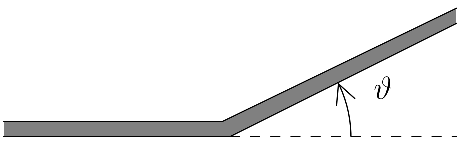

Figure 1. A pipe curved by an angle ϑ at $x = 0$, as considered in (1.1)–(1.2) or (1.3) and in Proposition 2.1.

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(1.2) can be rephrased as a single balance law with a Dirac delta source term in the second equation:

$$
\left\{
\begin{array}{l}
\partial_t \rho + \partial_x q = 0, \\
\partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = - f \, \kappa \left(2 \left| \sin(\vartheta/2) \right| \right) \, q \, \delta_{x=0}.
\end{array}
\right.
\tag{1.3}
$$

Next, we consider a smoothly curved pipe described by the equation $\Gamma = \Gamma(x)$ where x is arc–length. It is reasonable to assume that the dynamics of the fluid is governed by the equations

$$
\left\{
\begin{array}{l}
\partial_t \rho + \partial_x q = 0, \\
\partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = - f(x) \, \kappa \left(\left\| \Gamma''(x) \right\| \right) \, q \, ,
\end{array}
\right.
\tag{1.4}
$$

where $\left\| \Gamma''(x) \right\|$ equals the curvature of the pipe at the location $\Gamma(x)$. We have $\kappa(0) = 0$, and $f(x)$ is an empirical factor that depends on the location along the pipe. A brief derivation of the model can be found in [11].

More generally, we consider an arbitrary piecewise smooth pipe. Call $\bar{x}_0, \ldots, \bar{x}_m$ its corner points, or kinks, and denote by ϑ_i the angle of the pipe at $\bar{x}_i$, see Figure 1. To avoid unphysical behavior we assume that the pipe is horizontal and rectilinear outside a compact set. We are thus led to consider the system:

$$
\left\{
\begin{array}{l}
\partial_t \rho + \partial_x q = 0, \\
\partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = - f(x) \, \kappa \left(\left\| \Gamma''(x) \right\| \right) \, q - \rho \, g \, \sin \alpha(x) \\
\qquad\qquad\qquad\qquad\qquad - \displaystyle\sum_{i=0}^{m} f(\bar{x}_i) \, \kappa \left(2 \left| \sin(\vartheta_i/2) \right| \right) \, q(t, \bar{x}_i +) \, \delta_{x = \bar{x}_i} \, ,
\end{array}
\right.
\tag{1.5}
$$

where $\alpha = \alpha(x)$ describes the inclination of the pipe with respect to the horizontal plane at x and g is gravity. Both κ and α vanish outside a compact set.

The main result of the present paper is that (1.5) generates a Lipschitz continuous semigroup defined globally in time on all initial data that are small perturbations of stationary solutions. The results in [1] also ensure the uniqueness of this semigroup.

The analytic techniques employed here are rooted in the idea of approximating the piecewise smooth pipe with a polygonal one. Indeed, the case of a polygonal pipe can be obtained by *gluing together* systems of the type (1.2), where the source is a sequence of linear combinations of Dirac delta masses, which correspond to stationary discontinuities. At this point, the front-tracking method for systems of hyperbolic conservation laws [3, 12] proves to be a very effective tool. First, front-tracking approximations are defined through the available solutions to Riemann problems, including those at the Dirac masses. Second, front-tracking approximations are extremely accurate in capturing the essential features of the exact solutions to conservation laws. Third, analytic techniques are available that allow to prove the convergence of these approximations. We refer to [3, 12] for further details on the front-tracking method.

2. Main Result

Throughout this paper, $\mathbb{R}^+ = (0, +\infty)$ and $\overline{\mathbb{R}}^+ = [0, +\infty)$. Moreover, we denote the state of the fluid by u, where $u \equiv (\rho, q)$, with $q = \rho \, v$.

We assume that the fluid can be described through the pressure law p satisfying

(p): $p \in \mathbf{C^2}(\mathbb{R}^+; \mathbb{R}^+)$, $p'(\rho) \geq 0$ and $p''(\rho) \geq 0$ for all $\rho > 0$.

A typical example is a polytropic gas with the γ-pressure law $p(\rho) = \rho^\gamma$ for $\gamma \geq 1$.

With reference to the p-system (1.1) recall the following quantities

$$
\begin{aligned}
E(\rho, q) &= \frac{q^2}{2\rho} + \rho \int_{\bar\rho}^{\rho} \frac{p(r)}{r}\, \mathrm{d}r\,, && \text{mathematical entropy,}\\
F(\rho, q) &= \frac{q}{\rho}\left(E(\rho, q) + p(\rho)\right), && \text{entropy flow,}\\
P(\rho, q) &= \frac{q^2}{\rho} + p(\rho), && \text{dynamic pressure.}
\end{aligned}
\tag{2.1}
$$

2.1. **Stationary Solutions.** Assume the pipe is horizontal. Then, both systems (1.3) and (1.4) admit the stationary solution

$$q = 0 \qquad \text{and} \qquad \rho = \text{constant}.$$

In the case of a single kink (1.3), further stationary solutions are given by

$$\rho = \begin{cases} \rho^\ell, & x < 0, \\ \rho^r, & x > 0, \end{cases} \qquad q = \text{constant, where } P(\rho^\ell, q) - P(\rho^r, q) = -f\kappa\left(2\left|\sin(\vartheta/2)\right|\right) q\,.$$

Stationary solutions in the case of a polygonal pipe are obtained by gluing together solutions of the type above, i.e., q is constant while ρ satisfies the jump condition at every kink.

In a smooth pipe with gravity, stationary solutions satisfy

$$\partial_x P\left(\rho(x), q\right) = -f(x)\,\kappa\left(\left\|\Gamma''(x)\right\|\right) q - \rho\, g\, \sin\alpha(x) \qquad \text{and} \qquad q = \text{constant}.$$

Gluing together stationary solutions of the types above yields stationary solutions in the case of a piecewise smooth pipe.

Throughout this paper, by $\bar u = \bar u(x)$ we denote any of the stationary solutions constructed above.

2.2. **The Case of a Single Kink.** We now briefly consider the Riemann Problem for (1.3), referring to [11] for more details.

The pipe consists now of two rectilinear tubes connected through a kink at an angle $\vartheta \in (-\pi, \pi)$ located at, say, $x = 0$, so that

$$\Gamma(x) = \begin{cases} (1, 0)\, x, & x < 0, \\ (\cos\vartheta, \sin\vartheta)\, x, & x > 0, \end{cases}$$

see Figure 1. Then, the Riemann Problem for the model (1.1)–(1.2) or (1.3) introduced in [11] reads

$$
\begin{cases}
\partial_t \rho + \partial_x q = 0, \\
\partial_t q + \partial_x P(\rho, q) = 0, \\
[\![q]\!]\,(t, 0) = 0, \\
[\![P]\!]\,(t, 0) = f\,\kappa\left(2\left|\sin(\vartheta/2)\right|\right)\, q(t, 0+), \\
(\rho, q)(0, x) = \begin{cases} (\rho^l, q^l), & x < 0, \\ (\rho^r, q^r), & x > 0, \end{cases}
\end{cases}
\tag{2.2}
$$

where, as usual, we denote[1] $[\![F]\!]\,(t, x) = F(t, x+) - F(t, x-)$ for any function F of the pair (ρ, q). The function κ is assumed to satisfy

($\boldsymbol{\kappa}$): $\kappa \in \mathbf{C^1}(\mathbb{R}; \overline{\mathbb{R}}^+)$, with $\kappa(0) = 0$ and κ is even.

[1]Here $F(x\pm) = \lim_{h\downarrow 0} F(x \pm h)$ for any function F.

We also introduce the *subsonic region*

$$\Omega = \left\{ (\rho, q) \in \mathbb{R}^+ \times \mathbb{R} \colon \left|q/\rho\right| < \sqrt{p'(\rho)} \right\} , \tag{2.3}$$

where the velocity $v = q/\rho$ of the fluid is smaller than the sound speed $\sqrt{p'(\rho)}$. Due to its relevance in the applications, we restrict our attention below to initial data and solutions attaining values in the subsonic region.

Proposition 2.1. *Let* ***(p)*** *and* ***(κ)*** *hold. Fix $f > 0$ and a subsonic stationary solution $\bar{u}$ to* (2.2)*. Then, there exists a $\delta > 0$ such that for all states $u^\ell, u^r \in \Omega$ satisfying*

$$\|u_o - \bar{u}\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^+\times\mathbb{R})} < \delta \qquad \textit{where} \qquad u_o(x) = \begin{cases} u^\ell, & x < 0\,, \\ u^r, & x > 0\,, \end{cases}$$

the Riemann Problem (2.2) *admits a unique self-similar weak entropy solution attaining values in Ω, consisting of a 1-wave supported in $x < 0$, a jump along $x = 0$ and a 2-wave supported in $x > 0$.*

The Riemann Problem (2.2) was analyzed for arbitrary initial states in [11, Section 2] in the isothermal case where the pressure $p(\rho) = \rho$. The well known properties of the p-system allow us to apply [5, Theorem 3.2], so that the Cauchy problem for (2.2) is well posed in $\mathbf{L^1}$. The proof of Proposition 2.1 directly follows from the cited references.

2.3. **The Case of a Piecewise Smooth Pipe.** We now consider a piecewise smooth pipe with finite curvature, see Figure 2. More precisely, we make the following assumptions:

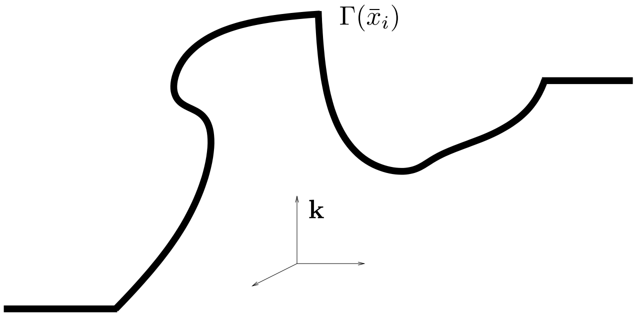

Figure 2. A piecewise smooth pipe.

(Γ): $\Gamma \in \mathbf{C^0}(\mathbb{R};\mathbb{R}^3)$ is such that:
1. Γ is piecewise smooth: there exist $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m$ with $x_{i-1} < x_i$ for all i such that $\Gamma|_{(-\infty,\bar{x}_0]} \in \mathbf{C^2}((-\infty,\bar{x}_0];\mathbb{R}^3)$, $\Gamma|_{[\bar{x}_{i-1},\bar{x}_i]} \in \mathbf{C^2}([\bar{x}_{i-1},\bar{x}_i],\mathbb{R}^3)$ and $\Gamma|_{[\bar{x}_m,+\infty)} \in \mathbf{C^2}([\bar{x}_m,+\infty);\mathbb{R}^3)$;
2. Γ is parametrized by arc–length: $\left\|\Gamma'(x)\right\| = 1$ for all $x \in \mathbb{R}\backslash\{\bar{x}_0,\ldots,\bar{x}_m\}$;
3. Γ has finite curvature: Γ'' vanishes outside a compact set;
4. Γ is horizontal outside a compact set: $\Gamma'(x) \cdot \mathbf{k}$ vanishes outside a compact, where $\mathbf{k}$ denotes the unit vertical vector.

On the friction term f, we require the following condition:

***(f)*:** $f \in (\mathbf{C^0} \cap \mathbf{L}^\infty)(\mathbb{R};\mathbb{R})$ and $f \geq 0$.

Lemma 2.2. *Let Γ satisfy ($\boldsymbol{\Gamma}$). Then, $\Gamma' \in \mathbf{BV}(\mathbb{R};\mathbb{R}^3)$ and its weak derivative is the measure*

$$\mu = \Gamma'' \, \mathrm{d}\mathcal{L} + \sum_{i=0}^{m} \left(\Gamma'(x_i+) - \Gamma'(x_i-)\right) \delta_{x=x_i} \,.$$

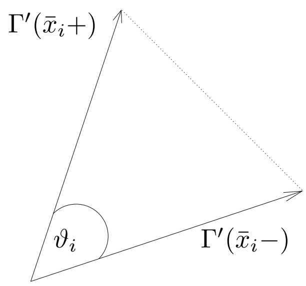

FIGURE 3. Justification of (2.4): here, $\left\|\Gamma'(\bar{x}_i+)\right\| = \left\|\Gamma'(\bar{x}_i+)\right\| = 1$.

Remark that the above expression of μ admits a geometric interpretation. For $i = 0, \ldots, m$, call ϑ_i the angle at $\bar{x}_i$ such that $\cos\vartheta_i = \Gamma'(\bar{x}_i-)\cdot\Gamma'(\bar{x}_i+)$. Elementary geometric considerations, see Figure 3, show that

$$\left\|\Gamma'(\bar{x}_i+) - \Gamma'(\bar{x}_i-)\right\| = \sqrt{2(1-\cos\vartheta_i)} = 2\left|\sin(\vartheta_i/2)\right|, \tag{2.4}$$

as used in [11].

Definition 2.3. *Let $T > 0$ and fix a stationary state $\bar{u} \in \mathbb{R}^+ \times \mathbb{R}$. By a* weak solution *to* (1.4) *we mean a map*

$$u = (\rho, q) \in \mathbf{C^0}\left([0,T]; \bar{u} + (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^+ \times \mathbb{R})\right)$$

such that $u_o = u_{|t=0}$ and for any function $\varphi \in \mathbf{C_c^1}((0,T)\times\mathbb{R};\mathbb{R})$, we have

$$\int_{\mathbb{R}} \int_0^T (\rho\,\partial_t\varphi + q\,\partial_x\varphi)\,\mathrm{d}t\ \mathrm{d}x = 0,$$

$$\begin{aligned}\int_{\mathbb{R}} \int_0^T \left(q\,\partial_t\varphi + P(\rho,q)\partial_x\varphi\right)\mathrm{d}t\ \mathrm{d}x = {}& \int_{\mathbb{R}} \int_0^T f(x)\,\kappa\left(\left\|\Gamma''(x)\right\|\right)\, q(t,x)\,\varphi(t,x)\,\mathrm{d}t\ \mathrm{d}x \\ & + \sum_{i=0}^{m} \int_0^T f(\bar{x}_i)\,\kappa\left(2\sin(\vartheta_i/2)\right)\, q(t,\bar{x}_i)\,\varphi(t,\bar{x}_i)\,\mathrm{d}t \\ & + \int_{\mathbb{R}} \int_0^T \rho(t,x)\,g\,\sin\alpha(x)\,\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x\,.\end{aligned}$$

The weak solution (ρ,q) is a weak entropy solution *if for any function $\varphi \in \mathbf{C_c^1}((0,T)\times\mathbb{R};\mathbb{R}^+)$, we have*

$$\begin{aligned}&\int_{\mathbb{R}} \int_0^T \left(E(\rho,q)\,\partial_t\varphi + F(\rho,q)\,\partial_x\varphi\right)\mathrm{d}t\,\mathrm{d}x \\ &+ \int_{\mathbb{R}} \int_0^T \partial_q E(\rho,q)\left(f(x)\,\kappa\left(\left\|\Gamma''(x)\right\|\right)\, q(t,x) + \rho(t,x)\,g\,\sin\alpha(x)\right)\varphi\,\mathrm{d}t\,\mathrm{d}x \geq 0\,.\end{aligned}$$

Theorem 2.4. *Let ($\boldsymbol{p}$), ($\boldsymbol{\Gamma}$), ($\boldsymbol{f}$), and ($\boldsymbol{\kappa}$) hold. Fix a subsonic stationary solution $\bar{u}$. Then, there exist $\hat{\delta}, \check{\delta}$, and $L \in \mathbb{R}^+$ such that* (1.4) *generates a semigroup*

$$S\colon \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D}$$

with the properties:

(1) *The domain $\mathcal{D}$ is non-trivial and its elements have uniformly bounded total variation:*

$$\left\{u \in \bar{u} + \mathbf{L^1}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R}) \colon \operatorname{TV}(u) \leq \check{\delta}\right\} \subseteq \mathcal{D},$$
$$\left\{u \in \bar{u} + \mathbf{L^1}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R}) \colon \operatorname{TV}(u) \leq \hat{\delta}\right\} \supseteq \mathcal{D}.$$

(2) *For all $u_o \in \mathcal{D}$, the map $t \mapsto S_t u_o$ is a weak entropy solution to* (1.5) *in the sense of Definition* 2.3.

(3) *S is Lipschitz continuous with respect to the $\mathbf{L^1}$ norm, i.e., for $u, u' \in \mathcal{D}$*

$$\left\|S_{t'} u' - S_t u\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^+\times\mathbb{R})} \leq L \left(\left\|u' - u\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^+\times\mathbb{R})} + \left|t' - t\right|\right).$$

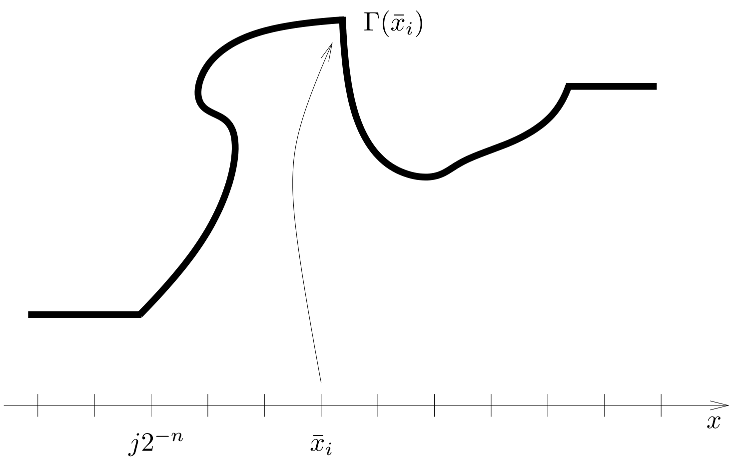

FIGURE 4. Discretization of (1.5) leading to (2.5).

Proof. We follow the construction in [1]. In the discretization of the pipe, we assume for simplicity that all kinks are at a dyadic abscissa. In other words, without any loss of generality, we assume that for all $i = 0, \ldots, m$, we have $\bar{x}_i = j_i 2^{-n_i}$ for suitable $n_i \in \mathbb{N}$ and $j_i \in \{-2^{2n_i}, \ldots, 2^{2n_i}\}$, see Figure 4.

Introduce the set $\mathcal{K}_n$ of indices that correspond to kinks, namely

$$\mathcal{K}_n = \left\{j \in \{-2^{2n}, \ldots, 2^{2n}\} \colon \exists i \in \{0, \ldots, m\} \text{ such that } \bar{x}_i = j\, 2^{-n}\right\}.$$

The procedure in [1, Theorem 3], by means of front-tracking approximate solutions to (1.5), constructs an *exact* solution u^n to the following approximation of (1.5):
(2.5)

$$\left\{\begin{aligned}
&\partial_t \rho + \partial_x q = 0,\\
&\partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho)\right) = -\sum_{j \notin \mathcal{K}_n} f(j2^{-n})\, \kappa \left(\left\|\Gamma''(j2^{-n})\right\|\right) q(t, j2^{-n})\, \delta_{x=j2^{-n}}\\
&\qquad\qquad\qquad\qquad\qquad - \sum_{j=-2^{2n}}^{2^{2n}} \rho(t, j2^{-n})\, g\, \sin\alpha(j2^{-n})\, \delta_{x=j2^{-n}}\\
&\qquad\qquad\qquad\qquad\qquad - \sum_{i=0}^{m} f(\bar{x}_i)\, \kappa \left(2\left|\sin(\vartheta_i/2)\right|\right) q(t, \bar{x}_i+)\, \delta_{x=\bar{x}_i}.
\end{aligned}\right.$$

An application of [1, Theorem 6] yields for any $n \in \mathbb{N}$ the existence of a semigroup $S^n \colon \mathbb{R}^+ \times \mathcal{D}^n \to \mathcal{D}^n$ satisfying (1) with $\mathcal{D}$ replaced by $\mathcal{D}^n$, (2) with (1.5) replaced by (2.5), and (3) for suitable $\hat{\delta}, \check{\delta}$ and L independent of n.

We now let $n \to +\infty$ and follow the procedure in [1, Theorem 8]. Remark that (2.5) differs from the equation considered in [1] by the last term

$$-\sum_{i=0}^{m} f(\bar{x}_i)\, \kappa \left(2\left|\sin(\vartheta_i/2)\right|\right)\, q(t,\bar{x}_i+)\, \delta_{x=\bar{x}_i}$$

on the right-hand side of the second equation. However, this term is independent of n and does not prevent the application of techniques used in [1, Theorem 8], see also [10]. □

3. Other Applications

The p-system (1.1) is of use in a variety of situations and the procedure presented above may well be applied to them.

3.1. **Water Flowing in a Pipe.** A different scenario that admits the same treatment presented in Section 2 is that of water flowing in a pipe. Neglecting friction along the walls, in a horizontal pipe the Saint-Venant equations [17] read

$$\begin{cases} \partial_t a + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{a} + p(a)\right) = 0\,. \end{cases} \tag{3.1}$$

Here, as usual, t is time, x the coordinate along the tube, $a = a(t,x)$ is the area of the wet cross-section, $q = q(t,x)$ is the water flow, so that $q = a\,v$, where $v = v(t,x)$ is the averaged speed of water at time t and position x. The hydrostatic term $p = p(a)$ is defined as in [2, Section 3.2], namely

$$p(a) = g \int_0^a \big(h(a) - h(\alpha)\big)\, \mathrm{d}\alpha \tag{3.2}$$

where $h = h(a)$ is the height of water corresponding to a, see Figure 5. Here g is the acceleration due to gravity. In the case of water pipes, the function h is often

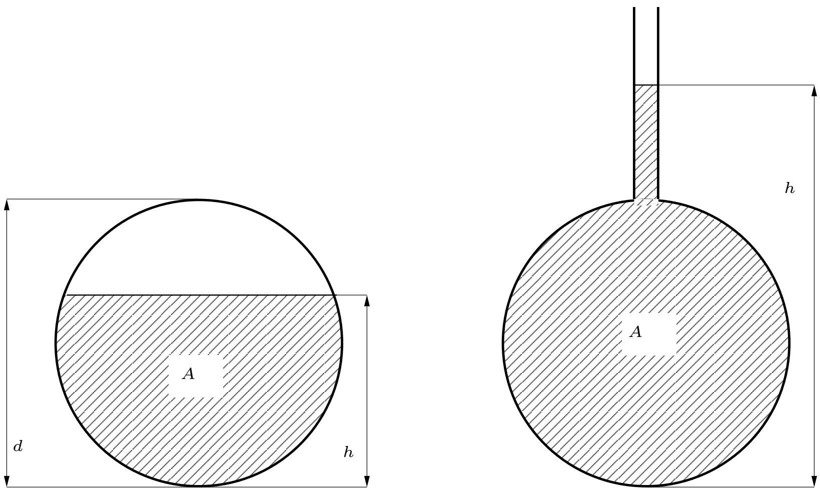

Figure 5. Notation used in (3.1) and (3.2). Left: the cross section of a standard pipe used in the modeling of free surface flows. Right: a pipe with the fictitious Preissmann slot used to describe pressurized flows.

chosen introducing the so-called Preissmann slot. It is an artificial modification of the cross section of a tube, see Figure 5, right, to merge free surface flow and pressurized flow in a combined model. In the case of free surface flow the physical geometry is used. In the case of pressurized flow a narrow slot is added to the model, so that the *height* of water is extended beyond the tube diameter d. This widely used technique, see, e.g., [7, 14, 16], allows us to consider both regimes in a single model.

With suitable choices of the term f, it is natural to consider the following extension of (3.1) to describe the dynamics of water in a curved pipe:

$$\begin{cases} \partial_t a + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{a} + p(a)\right) = -f(x)\,\kappa(x)\,q\,, \end{cases} \tag{3.3}$$

with p defined by the pressure law (3.2) satisfies **(p)**. Referring to the case depicted in Figure 5 and to [2, Section 3.2], calling r the radius of the pipe and d the width of the Preissmann slot, we have

$$h(a) = \begin{cases} \sqrt{\frac{2}{\pi}\,a}, & a \in \left[0, \frac{\pi}{2}\,r^2\right]\,, \\ 2r - \sqrt{2r^2 - \frac{2}{\pi}\,a}, & a \in \left(\frac{\pi}{2}\,r^2, \pi\,r^2 - \frac{1}{2\pi}\,d^2\right]\,, \\ \frac{a}{d} - \frac{1}{2\pi}\,d + 2r - \pi\,\frac{r^2}{d}, & a \in \left(\pi r^2 - \frac{1}{2\pi}\,d^2, +\infty\right)\,. \end{cases} \tag{3.4}$$

Straightforward computations show that the pressure law (3.2) with h defined as in (3.4) satisfies **(p)**, so that the results in Section 2 can be applied also to (3.3).

3.2. **A Pipe with a Varying Section.** The dynamics of a fluid in a pipe with a slowly varying section $a = a(x)$ is described by the well known equations

$$\begin{cases} \partial_t (a\,\rho) + \partial_x (a\,q) = 0, \\ \partial_t (a\,q) + \partial_x \left(a\left(\frac{q^2}{\rho} + p(\rho)\right)\right) = 0, \end{cases} \quad \text{or} \quad \begin{cases} \partial_t \rho + \partial_x q = -\frac{q}{a}\,\partial_x a, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho)\right) = -\frac{q^2}{a\,\rho}\,\partial_x a, \end{cases} \tag{3.5}$$

where $p = p(r)$ is the pressure law and, as in the previous section, $\rho = \rho(t,x)$ is the fluid density and $q = q(t,x)$ is its linear momentum density. The equivalence between the two systems (3.5) is proved in [6, Lemma 2.6]. This problem has been widely considered in the literature, see, for instance, [4, 6, 8, 9, 13, 15].

The system on the right in (3.5) clearly shows that a sudden change in the pipe section, i.e., a discontinuity in the function a, yields a Dirac delta function as source term in both equations. Similarly to what was done in Section 2, it is then natural

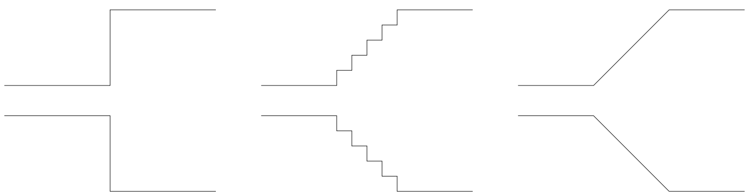

FIGURE 6. Left: a single junction between two pipes. Middle: a sequence of junctions. Right: a pipe with a smoothly varying section.

to select a class of solutions to (3.5) in the case of a single junction as in Figure 6, left,

$$a(x) = \begin{cases} a^-, & x < 0\,, \\ a^+, & x > 0\,, \end{cases}$$

pass to the case of a piecewise constant section $a = a(x)$ as in Figure 6, center, and, in the limit, re-obtain equations (3.5). We refer to [6] for the details.

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