Slow Erosion with Rough Geological Layers

Wen Shen,
Department of Mathematics, Penn State University, University Park, PA 16802, U.S.A.
E-mail: shen_w@math.psu.edu

Abstract

In this paper we study slow erosion profile with rough geological layers. The mathematical model is a scalar conservation law which takes the form of an integro-differential equation with discontinuous flux functions. It is has been shown that, for a class of erosion functions, vertical jumps in the profile can occur in finite time even with smooth initial data. Three types of singularities can form in the solution, representing kinks, hyper-kinks and jump discontinuities in the profile. The mathematical model studied in this paper is formulated in a transformed coordinate, where vertical jumps in profile becomes an interval where the unknown is zero after applying a pointwise constraint. Front tracking approximate solutions are designed, for both cases with or without jump discontinuities. Solutions to Riemann problems with discontinuous flux functions are derived, and suitable functionals that measure strengths of various wave types are introduced. Through the establishment of various a-priori estimates, we achieve desired compactness which yields the existence of entropy weak solutions. Finally, a Kruzhkov type entropy inequality is proved, leading to stability and uniqueness of the solutions.

1 Introduction and Preliminaries

We consider the slow erosion model with geological layer

$$z_t - \left[g(z,u) \cdot \exp \int_u^{+\infty} g(z(t,v),v)\, dv \right]_u = 0\,, \qquad z \geq 0\,, \tag{1.1}$$

associated with the initial data

$$z(0,u) = \bar{z}(u)\,. \tag{1.2}$$

In the case where $g = g(z)$, i.e. with homogeneous material for the standing layer, the model was derived in Colombo, Guerra & Shen [6] to model slow erosion of granular matter. The derivation of the model (1.1) goes through some coordinate changes. To enhance the readability of this paper, we now provide a brief derivation of the model (1.1) in the case $g = g(z)$.

Let $u(t,x)$ be the height of a standing profile, where x is the space variable and t is the time variable. The time variable t denotes the total mass of avalanche passed through the profile. We assume that the slope does not change sign, say $u_x > 0$, otherwise this model is not valid. Let $f(u_x)$ be the erosion function, denoting the rate of erosion (for positive values) or rate of deposit (for negative values) per unit amount of mass passing through per unit distance covered in x. There is a *critical slope*, called *the angle of repose*, where no mass exchange happens between the moving and standing layer. In a normalized model, we can make the critical slope to be 1, i.e., $f(1) = 0$, and $f(u_x) > 0$ for $u_x > 1$ and $f(u_x) < 0$ for $u_x < 1$. Then $u(t,x)$ satisfies the following integro-differential equation

$$u_t(t,x) - \left(\exp \int_x^\infty f(u_x(t,y))\, dy\right)_x = 0\,. \tag{1.3}$$

See Shen & Zhang [13] for a detailed derivation of (1.3). Writing $w = u_x$, and differentiating (1.3) in x, one obtains a conservation law for $w(t,x)$

$$w_t(t,x) + \left(f(w) \cdot \exp \int_x^\infty f(w(t,y))\, dy\right)_x = 0\,. \tag{1.4}$$

This model (1.4) was proven by Amadori & Shen [1] as the slow erosion limit of a 2×2 system of balance laws describing dynamics of small avalanches of granular flow, proposed originally by Hadelar & Kuttler [10]. Under suitable assumptions on f, solutions $w(t,x)$ remain strictly positive and uniformly bounded in t. Existence and uniqueness of solutions are proved by Amadori & Shen [2, 3].

For a class of erosion functions f where we allow more erosion for large slope w, the slope w can blow up to infinite in finite time, and the profile $u(t,x)$ will have vertical drops. In this case, model (1.4) is not suitable, and one must use model (1.3). Shen & Zhang in [13] established the global existence of large BV solutions, through a specially designed front tracking approximate solutions that are piecewise polygonal lines with possible vertical jumps. Note that if u has jumps, then u_x contains point masses, and the integral term in (1.3) integrates over $f(u_x)$ where f is non-linear. This caused many technical difficulties.

Under the assumption that $u_x > 0$ for all t, the profile $u(t,x)$ has a well-defined inverse function $X(t,u)$, where $X_u \geq 0$. Treating (t,u) as the independent variables, this coordinate change gives the following equation for the inverse function $X(t,u)$

$$X_t(t,u) - \left(\exp \int_u^\infty g(X_u(t,v))\, dv\right)_u = 0, \qquad X_u \geq 0\,. \tag{1.5}$$

Here g is the erosion function in the new coordinates (t,u), denoting the rate of erosion per distance (in u) covered per unit mass passing through. The erosion function g is related to f by

$$g(z) = z f(1/z)\,,$$

with the following basic properties

$$g(1) = 0, \qquad g(z) > 0 \quad (z < 1), \qquad g(z) < 0 \quad (z > 1)\,.$$

Denoting $z(t,u) \doteq X_u(t,u)$ the slope of the inverse function $X(t,u)$, and differentiating (1.5) in u, we arrive at a conservation law for $z(t,u)$

$$z_t(t,u) - \left(g(z) \cdot \exp \int_u^\infty g(z(t,v))\, dv\right)_u = 0, \qquad z \geq 0\,. \tag{1.6}$$

Note that the equations (1.5) and (1.6) come with a pointwise constraint $X_u \geq 0$, because the coordinate change will have no physical meaning for $X_u < 0$. Under the assumption $g(0) = 0$, the solution of (1.6) remains strictly positive for positive initial data $z(0, u) > 0$, see [2, 6], and no constraint is needed. However, if $g(0) > 0$, then z becomes negative in finite time even with strictly positive initial data $z(0, u)$, thus the constraint must be applied. Global existence of a Lipschitz semigroup solution for (1.6) is proved by Colombo, Guerra & Shen [6].

More recently, existence and local stability of traveling waves for these slow erosion models are achieved by Guerra & Shen [9]. Finally, the uniqueness of entropy weak solutions and the equivalence all these models is proved by Bressan & Shen [4], using backward Euler step combined with a projection operator to accommodate the pointwise constraint.

In this paper we consider (1.1), where the erosion function $g(z, u)$ depends also on the space variable u. Note that u is the height of the profile in the physical coordinate (t, x). In particular, we consider g as a possibly discontinuous function of u. This implies that the standing profile $u(t, x)$ has rough horizontal layers with different material properties. The model (1.1) represents a mountain profile made of horizontal geological layers, with possibly rough transition of material properties between layers.

In the previous models with uniform material, we considered both erosion and deposit phenomena. We assumed that the material of the rolling layer possesses the same properties as the standing layer, even after they were deposited and became part of the standing layer. In this paper, with the standing layer consisting of layers of different materials, it is not natural to assume that the rolling layer will adopt the same properties of the particular standing layer on which they happen to be deposited. Therefore, we consider only the erosion phenomenon, where the slope of the standing profile is always bigger than the critical slope. This means that the initial value for $z(0, u)$ will be smaller than the critical values for z at u.

As mentioned before in the case of homogeneous material with $g = g(z)$, if $g(0) = 0$ and initially we have $z(0, t) > 0$, then $z(t, u) > 0$ for all t. But if $g(0) > 0$, then $z(t, u)$ will reach 0 in finite time, even for positive initial data $z(0, u)$. These properties are expected to hold for the non-homogeneous material case as well. These two cases will be discussed separately, since they require different treatments.

In the case when $g(0, u) = 0$ for all u, the constraint will never be needed. We construct approximate solutions using a special front tracking algorithm, with piecewise constant approximate solutions. Functionals which are used to measure the strength of various types of waves are introduced. A-priori estimates will be derived, in particular a key estimate on the bound of the total wave strengths. These estimates provide sufficient compactness which leads to convergence of approximate solutions, yielding the existence of entropy weak solutions. Furthermore, a Kruzhkov type entropy condition will allow us to achieve uniqueness of solution and continuous dependence on the initial data.

On the other hand, if $z(t, u)$ becomes negative, the constraint $z \geq 0$ must be applied. A projection operator π was introduced by Bressan & Shen in [4], which projects possibly negative-valued functions into the cone of positive functions. The projection operator preserves the conservation laws for $z(t, u)$ as well as for $X(t, u)$. Adopting this additional constraint, we design another front tracking algorithm which treats the profile jump discontinuities in a special way.

At every point u where $g(z,u)$ is discontinuous, we solve our non-local conservation law with discontinuous flux function. Scalar conservation laws with discontinuous flux function is studied in [7, 8], (among many other authors), where a criterion of "minimum-jump" provides the unique entropy weak solutions which is the limit of the vanishing viscosity. The Riemann problems in our front tracking algorithm can be uniquely solved using this criterion.

It is well-known that conservation laws with discontinuous flux function could develop unbounded total variation in its conserved variable, see Temple [11]. Using a different entropy condition, the generalized Lax Condition, Temple [14] introduces a functional to measure wave strength, and showed that the total wave strength is non-increasing at interactions. See also for example its applications in [5] and references therein. This yields a bound on the total variation of the flux function, which is usually the key estimate among the a-priori bounds. Under the assumption that the graphs of the flux functions do not intersect, Temple's function can be used to yield bounded variation of the flux function, for solutions using "minimum-jump" condition. However, in our model the graphs of the erosion functions will intersect in various ways. To handle this new situation, we introduce a new functional to measure the strengths of various types of waves. A bound of these wave strength will yield a bound on the total variation of the flux.

The rest of paper is organized as follows. In Section 2 we give precise definition of the model, and some basic analysis. In Section 3 we discuss solutions of Riemann problems with discontinuous erosion functions. In Section 4 we study the case where the constraint $z \geq 0$ is not applied, and construct approximate solution through a modified front tracking algorithm. Through suitable a-priori estimates, we prove the existence and uniqueness of the entropy weak solutions for the case where $g(0,u) \equiv 0$. Finally, in Section 5 we consider the case $g(0,u) \geq 0$. We combine the constraint into the front tracking algorithm, proving again the existence and uniqueness of the entropy weak solutions for this case.

2 Preliminary and Some Basic Analysis

Since z can be negative without applying the constraint, it is necessary that we extend the definition of $g(z,u)$ onto negative values of z. The extended mapping $z \mapsto g(z,u)$ must be continuous and convex on $z \in [-\infty, +\infty]$. There are many ways of making this extension. In this paper, we let

$$g(z,u) = g(0,u) + \gamma z, \qquad z \leq 0\,, \tag{2.1}$$

where γ is a constant that satisfied

$$\gamma \;\geq\; \max\left\{1, \max_u \{g_z(0,u)\}\right\}\,. \tag{2.2}$$

Our basic assumptions on g include the followings:

(A1) For fixed u, the mapping $z \mapsto g(z,u)$ is $\mathcal{C}^2$ for $z > 0$ and $z < 0$, continuous at $z = 0$, and strictly concave, and

$$g(0,u) \geq 0, \qquad g_z(0,u) < \infty, \qquad g_{zz}(z,u) \leq 0 \quad (z > 0)\,, \tag{2.3}$$

and

$$g_{zz}(z_m, u) \le -c_g < 0, \qquad \text{when} \quad g_z(z_m, u) = 0 \qquad \text{and} \quad z_m > 0\,. \tag{2.4}$$

(A2) The mapping $u \mapsto g(z,u)$ is piecewise continuous with finitely many points of discontinuity, which we denote as $\mathcal{V} = \{V_i\}_{i=1}^{N_v}$. We assume that the set $\mathcal{V}$ is independent of z.

(A3) There exists a piecewise continuous function $A(u)$, with the set of discontinuity points in $\mathcal{V}$, such that

$$A(u) > 0, \qquad g(A(u), u) = 0, \qquad \text{for all} \quad u\,. \tag{2.5}$$

(A4) There exists a piecewise continuous function $B(u)$, with the set of discontinuity points in $\mathcal{V}$, such that

$$B(u) \le 0, \qquad g(B(u), u) = 0, \qquad \text{for all} \quad u\,. \tag{2.6}$$

(A5) The geological layers have bounded variation. We define the distance function between any two erosion functions $g_1(z)$ and $g_2(z)$ as

$$D(g_1, g_2) \doteq \max_{(z_1,z_2)} \{|z_1 - z_2|\} + \max_{(z_1,z_2)} \left\{\left|g_1'(z_1) - g_2'(z_2)\right|\right\}\,, \tag{2.7}$$

where the maximum is taking over the set

$$\left\{(z_1, z_2);\; g_1(z_1) = g_2(z_2) \ge 0,\; \text{and } g_1'(z_1) \cdot g_2'(z_2) \ge 0\right\}\,.$$

We also define the total variation of the geological layers, i.e., total variation of $g(z,u)$ as

$$\|g(\cdot,\cdot)\|_{\text{TV}} \;\doteq\; \sup \left\{\sum_{i=1}^{N} D(g(\cdot, u_i), g(\cdot, u_{i+1}))\right\}\,, \tag{2.8}$$

where the supremum is taken over all $n \ge 1$ and all $(N+1)$-tuples of point u_i such that $u_1 < u_2 < \cdots < u_{N+1}$.

We assume now

$$\|g(\cdot,\cdot)\|_{\text{TV}} \le C\,. \tag{2.9}$$

Here and in the rest of the paper, C denote a generic bounded constant not depending on the critical parameters. We also denote TV$\{\cdot\}$ the total variation of a function. For notation simplicity, we will also denote the integral term as

$$G(u; z) = \exp \int_u^{+\infty} g(z(t,v), v)\, dv\,. \tag{2.10}$$

Thanks to the assumptions **(A1)-(A5)**, the equilibrium profiles $A(u)$ and $B(u)$ have bounded variations.

Lemma 2.1. *Assume that the assumptions* **(A1)-(A5)** *holds. Then the total variations of $A(u)$ and $B(u)$ are bounded, i.e.,*

$$TV\{A(\cdot)\} \leq C, \qquad TV\{B(\cdot)\} \leq C\,. \tag{2.11}$$

Proof. Let $\{u_i\}$ be any set of ordered points such that $u_i < u_{i+1}$. Since

$$g(A(u_{i+1}), u_{i+1}) = g(A(u_i), u_i) = 0\,,$$

by definition (2.7) we have

$$|A(u_{i+1}) - A(u_i)| \leq D(g(A(\cdot), u_{i+1}), g(A(\cdot), u_i))\,.$$

Then, by (2.8), we have that

$$\begin{aligned} \mathrm{TV}\{A(\cdot)\} &= \sup\left\{\sum_{i=1}^{N} |A(u_{i+1}) - A(u_i)|\right\} \\ &\leq \sup\left\{\sum_{i=1}^{N} D(g(A(\cdot), u_{i+1}), g(A(\cdot), u_i))\right\} \;\leq\; \|g\|_{\mathrm{tv}}\,. \end{aligned}$$

The proof for the BV bound for $B(u)$ is completely similar. $\square$

Equation along characteristics. For smooth solutions, formally (1.1) can be rewritten as

$$z_t - g_z G z_u = -g^2 G + g_u G\,. \tag{2.12}$$

This gives us the equations along the characteristics $t \mapsto u(t)$

$$\dot{u} = \frac{d}{dt}u(t) = -\,g_z G\,, \tag{2.13}$$

$$\dot{z} = \frac{d}{dt}z(t, u(t)) = -\,g^2 G + g_u G\,. \tag{2.14}$$

Due to the nonlinearity of the map $z \mapsto g$, singularities will form in finite time, i..e, z will become discontinuous even with smooth initial data $\bar{z}$. These are referred to as kinks. The integral term G remains continuous at such jumps. The propagation speed of the singularity satisfies the Rankine-Hugoniot equation. Let z^-, z^+ be the left and right states of the jump located at u, the kink speed is

$$s = -G(u; z) \cdot \frac{g(z^-, u-) - g(z^+, u+)}{z^- - z^+}\,. \tag{2.15}$$

3 Riemann problem with discontinuous coefficients

In this section we consider the Riemann problem

$$z_t - \left(\tilde{g}(z, u) \cdot \exp \int_u^{+\infty} \tilde{g}(z(v), v)\, dv\right)_u = 0\,, \tag{3.1}$$

with

$$\tilde{g}(z,u) = \begin{cases} g^-(z), & u<0\,, \\ g^+(z), & u>0\,. \end{cases} \tag{3.2}$$

associated with the initial data

$$z(0,u) = \begin{cases} z^-, & u<0\,, \\ z^+, & u>0\,. \end{cases} \tag{3.3}$$

The Cauchy problem is considered locally around a small neighborhood of $u = 0$ for a very short period of time, since piecewise constant function $z(0, u)$ will quickly evolve into non-constant functions on each side of $u = 0$.

The functions $z \mapsto g^-$ and $z \mapsto g^+$ satisfy the assumptions **(A1)-(A5)**, and we denote the zeros of g^-, g^+ as B^-, A^- and B^+, A^+, such that

$$g^-(B^-) = g^-(A^-) = 0, \qquad B^- \le 0 < A^-\,, \tag{3.4}$$

$$g^+(B^+) = g^+(A^+) = 0, \qquad B^+ \le 0 < A^+\,. \tag{3.5}$$

The initial data satisfy

$$B^- \le z^- \le A^-, \qquad B^+ \le z^+ \le A^+. \tag{3.6}$$

Next Lemma provides the existence and uniqueness of the solution to this Riemann problem.

Lemma 3.1. *Under the above setting, there exists a unique solution $z(t, u)$ to the Riemann problem for $t \in [0, \varepsilon]$ where $\varepsilon > 0$ is sufficiently small, on a bounded interval around $u = 0$.*

Proof. Since the integral term is continuous at $u = 0$, the behavior of the solution of the Riemann problem is solely determined by the local part of the flux, i.e., $-\tilde{g}$, which has a jump at $u = 0$. Such a proof exists for a more complicated continuous flux functions, see for example [7]. Here we provide a simpler proof for this simpler case. The analysis here offers motivation for the functional used to control the total variation. The proof takes several steps.

Step 1. Let z_m^-, z_m^+ be the unique values such that

$$\frac{d}{dz} g^-(z_m^-) = 0\,, \qquad \frac{d}{dz} g^+(z_m^+) = 0\,. \tag{3.7}$$

So g^-, g^+ reach their maximum values at z_m^-, z_m^+, respectively.

Let $R[z^-, z^+]$ denote the solution of a Riemann problem with z^-, z^+ as the left and right state. Let $K^-(z^-)$ denote the set of z values where $R[z^-, z]$ consists of waves of non-positive speed. Then, we have

$$K^-(z^-) = \begin{cases} [B^-, z_m^-], & z^- \le z_m^-, \\ [B^-, \bar{z}^-] \cup \{z^-\}, & z^- > z_m^-\,, \end{cases} \tag{3.8}$$

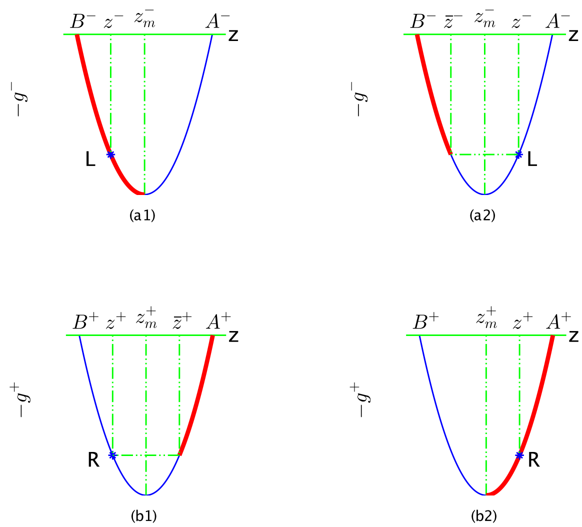

Figure 1: Plot of the set that can be connected with a stationary shock. The set $K^-(z^-)$ is shown by thick red curves in plots (a1) and (a2), while the set $K^+(z^+)$ is shown by thick red curves in plots (b1) and (b2).

where $\bar{z}^-$ is the unique point such that

$$g^-(\bar{z}^-) = g^-(z^-), \qquad \bar{z}^- < z_m^- < z^- .$$

See Figure 1 plots (a1) and (a2) for an illustration. We observe that g^- is a strictly increasing function on the set $K^-(z^-)$, if we exclude the isolated point z^-.

Step 2. Similarly, let $K^+(z^+)$ denote the set of z values where $R[z, z^+]$ consists of waves of non-negative speed. Then, we have (see Figure 1 plots (b1) and (b2))

$$K^+(z^+) = \begin{cases} [\bar{z}^+, A^+] \cup \{z^+\}, & z^+ < z_m^- , \\ [z_m^+, A^+], & z^- \geq z_m^- . \end{cases} \tag{3.9}$$

Here $\bar{z}^+$ is the unique point such that

$$g^+(\bar{z}^+) = g^+(z^+), \qquad \bar{z}^+ > Z^+ > z^+ .$$

We observe also that g^+ is a strictly decreasing function on the set $K^+(z^+)$, if we exclude the isolated point z^+.

Step 3. Combining the results in Step 1 and 2, we conclude that there exist a unique horizontal line connecting the graphs of $g^-(z)$ and $g^+(z)$ with shortest path length, if we do not consider the isolated points. Finally, if the isolated points (one or both) z^-, z^+ shall lie on the horizontal line with the shortest path, then we will select the path with most number of isolated points. This provides the existence and uniqueness of the path for the stationary wave, which in term gives unique solution to the Riemann problem. $\square$

The solution of the Riemann problem consists of two types of wave: (i) The wave lies either on the left or on the right of the jump in $\tilde{g}$. These waves are solutions of the scalar conservation law (3.1) with $\tilde{g} = g^-$ or $\tilde{g} = g^+$. We refer to these waves as z-waves. (ii) The wave is stationary at $u = 0$. It connects the discontinuous flux function $\tilde{g}$ with the condition $g^-(z(0-)) = g^+(z(0+))$. We refer to these as g-waves.

Next Corollary provides the invariant region for the Riemann problem.

Corollary 3.2. *In the setting of Lemma 3.1, we furthermore have*

$$B^- \le z(t,u) \le A^-, \qquad u < 0\,, \tag{3.10}$$

$$B^+ \le z(t,u) \le A^+, \qquad u > 0\,. \tag{3.11}$$

Proof. To show the invariance region for the Riemann solution, it suffices to observe that the $\tilde{g}$ value crossing g-wave is non-negative. This is obvious from the proof of Lemma 3.1. □

The actual solution for the Riemann problem is constructed following the proof of Lemma 3.1. These solutions depend on how the graphs of the flux functions $-g^-(z)$ and $-g^+(z)$ relate to each other. In Figure 2 we illustrate two typical cases, one with the graphs not intersecting, and another one with an intersection point. The graphs shall be self explanatory, with "L" and "R" marking the left and right state, and "M" (or M_1, M_2) as the intermediate state(s). All other cases are constructed in a totally similar way, and we omit the details.

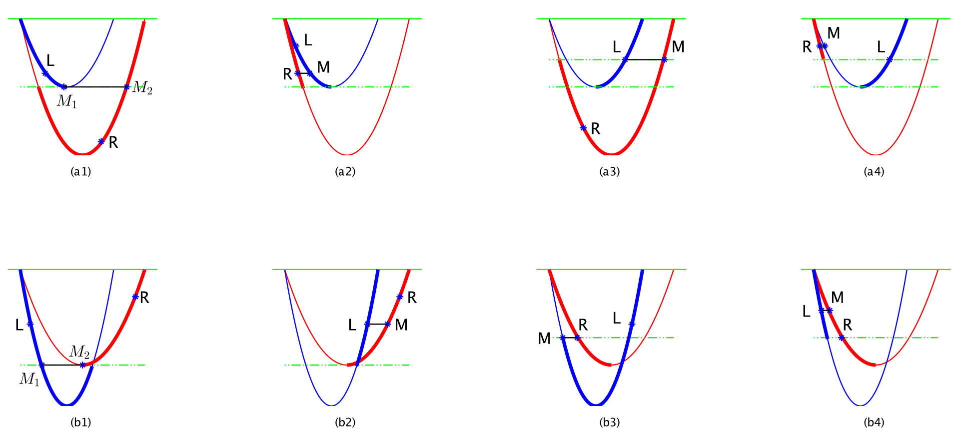

Figure 2: Typical Riemann problem and their solutions. Here blue curve is for $g^-(z)$ and red curve is for $g^+(z)$. The thick curve indicate the case where the left or right state could be taken. The path for the Riemann problem would be L-M-R or L-M_1-M_2-R, depending on cases.

4 An integro-differential equation without constraint

In this section we consider the Cauchy problem for the scalar integro-differential equation

$$(4.1)\qquad z_t - \left[g(z,u) \cdot \exp \int_u^{+\infty} g(z(t,v),v)\, dv \right]_u = 0\,, \qquad z(0,u) = \bar z(u)\,,$$

where we do not apply the constraint $z \geq 0$. The function $g(z,u)$ satisfies the properties **(A1)-(A5)**.

We now define a function $\phi(z;g)$ as

$$(4.2)\qquad \phi(z;g) \doteq \operatorname{sign}(z - z_m)\left[g(z_m) - g(z) \right]\,,$$

where z_m is the unique value such that $g_z(z_m) = 0$ and $g(\cdot)$ attains its maximum value at $g(z_m)$. The existence and uniqueness of such a value follows trivially from the assumptions **(A1)-(A5)**. The function ϕ will be used to measure the strength of z-waves.

The initial data $\bar z(u)$ satisfies the following assumptions:

$$(4.3)\qquad \|\bar z(\cdot) - A(\cdot)\|_{\mathbf{L}^1} < \infty, \qquad 0 \leq \bar z(u) \leq A(u), \qquad \mathrm{TV}\{\phi(\bar z(\cdot); g(\bar z(\cdot), \cdot))\} \leq C\,.$$

We now define the entropy weak solutions for (4.1).

Definition 4.1. *Let $T > 0$, and let $z = z(t,u) \geq 0$ be a bounded, measurable function. We call $z(t,u)$ an entropy weak solution of (1.1) if the following conditions are satisfied:*

(C1) *The map $t \mapsto z(t,\cdot)$ is continuous from $[0,T]$ into $L^1_{loc}(I\!R)$, and $B(u) \leq z(t,u) \leq A(u)$ for every (t,u).*

(C2) *$\|z(t,\cdot) - A(\cdot)\|_{\mathbf{L}^1} \leq \|\bar z(\cdot) - A(\cdot)\|_{\mathbf{L}^1}$ for every $t \in [0,T]$, and $z(t,\cdot) \to \bar z$ in $L^1(I\!R)$ as $t \to 0+$.*

(C3) *Total variation of the map $u \mapsto \phi(z(t,u); g(z(t,u),u))$ is bounded for all $t \in [0,T]$.*

(C4) *The following Kruzhkov inequality holds for all constants c and all non-negative test functions φ,*

$$\begin{aligned}
&\int_0^T \int_R \left[-|z-c|\varphi_t + \operatorname{sign}(z-c)\cdot (g(z,u) - g(c,u)) G(u;z)\varphi_u \right] du\, dt \\
&\qquad \leq \; - \int_0^T \int_R \operatorname{sign}(z-c)\cdot g(c,u) g(z,c) G(u;z)\varphi\, du\, dt \\
(4.4)\qquad &\qquad\quad + \int_0^T \sum_{i=1}^{N_v} \left| g(c,V_i-) - g(c,V_i+) \right| G(V_i;z)\varphi(t,V_i)\, dt\,,
\end{aligned}$$

where $\{V_i\}$ are the points where $u \mapsto g$ is discontinuous.

Here is our first main Theorem.

Theorem 4.2. *Assume that g satisfies the assumption **(A1)-(A5)**, and the initial data $\bar{z}$ satisfies* (4.3)*. Then, there exist a unique entropy weak solution $z(t,u)$ for the Cauchy problem of* (1.1)*, that satisfies the Definition 4.1. Furthermore, let w be the entropy weak solution of* (1.1) *with initial data $\bar{w}$, then it holds*

$$\|z(t,\cdot)-w(t,\cdot)\|_{\mathbf{L}^1} \le e^{Ct}\,\|\bar{z}-\bar{w}\|_{\mathbf{L}^1}\,. \tag{4.5}$$

Note that in Theorem 4.2 we allow the z values to be negative, since we do not apply the constraint. Such solutions might not have physical meaning. We notice that, under further assumption that $B(u)\equiv 0$, the results in Theorem 4.2 still holds. We immediate have the following Theorem.

Theorem 4.3. *In the setting of Theorem 4.2, if $B(u)\equiv 0$, the same results hold, and the solution satisfies $z(t,u)\ge 0$ for all $u\in I\!R$ and $t\in[0,T]$. Therefore, no constraint operator is needed.*

The rest of this section is devoted to the proof of Theorem 4.2. We construct a front tracking approximate solution for (4.1), which is described in detail in Section 4.1. In Section 4.2 we define the functionals used to measure the wave strengths of different types of waves. Interaction estimates are derived in Section 4.3, and special treatment around the g-waves is analyzed in Section 4.4. In Section 4.5 we establish the necessary a-priori estimates for the approximate solutions, which provide compactness and allow us to obtain the convergence of the approximate solutions to the entropy weak solutions in Section 4.6. Finally, uniqueness and stability is achieved in Section 4.7 through a standard variable-doubling technique with the Kruzhkov entropy inequality.

4.1 Front tracking approximation algorithm

Front tracking approximate solutions for (1.1) with homogeneous material $g=g(z)$ were studied in several papers [13, 2, 6], in various coordinates for different classes of erosion functions. These algorithms are different from those for standard conservation laws due to the presence of the integral term in the flux. This integral term causes the constant state of the piecewise constant approximation to vary in t. In the end, the algorithm results in two coupled sets of ODEs, one governing the propagation of the fronts, and the other governing the evolution of the constant values for z between two neighboring fronts.

In this paper, the material is not homogenous, and possibly discontinuous. Jumps in the material are treated as stationary shocks in the solution. Furthermore, since the left and right constant states of these jumps evolve in time, special care must be taken.

Let $\tilde{\varepsilon}>0$ and $\varepsilon>0$ be given, and we now construct a $(\tilde{\varepsilon},\varepsilon)$-front tracking approximate solution. Since $u\mapsto g(z,u)$ has bounded variation, we can approximate it by a function $g^{\tilde{\varepsilon}}(z,u)$ that is piecewise constant in u, with the piecewise constant equilibrium functions $A^{\tilde{\varepsilon}}(u), B^{\tilde{\varepsilon}}(u)$, such that the followings hold

$$\left\|g^{\tilde{\varepsilon}}(\cdot,\cdot)\right\|_{\mathrm{TV}} \le \|g(\cdot,\cdot)\|_{\mathrm{TV}}\,, \qquad \left\|g^{\tilde{\varepsilon}}(\cdot,\cdot)-g(\cdot,\cdot)\right\|_{\mathbf{L}^1} \le C\tilde{\varepsilon}\,, \tag{4.6}$$

$$\mathrm{TV}\{A^{\tilde{\varepsilon}}(\cdot)\} \le \mathrm{TV}\{A(\cdot)\}, \qquad \left\|A^{\tilde{\varepsilon}}(\cdot)-A(\cdot)\right\|_{\mathbf{L}^1} \le C\tilde{\varepsilon}\,, \tag{4.7}$$

$$\mathrm{TV}\{B^{\tilde{\varepsilon}}(\cdot)\} \le \mathrm{TV}\{B(\cdot)\}, \qquad \left\|B^{\tilde{\varepsilon}}(\cdot)-B(\cdot)\right\|_{\mathbf{L}^1} \le C\tilde{\varepsilon}\,. \tag{4.8}$$

Here C is a bounded constant that depends on the total variation of the erosion function g.

Let $\mathcal{J} = \{U_j\}_{j=0}^N$ denote the (finite) set of points of discontinuities for $g^{\tilde{\varepsilon}}(z,u)$, with $U_0 = -\infty$ and $U_N = +\infty$, and denote the interval

$$I_j = [U_j, U_{j+1})\,, \qquad 0 \le j \le N-1. \tag{4.9}$$

Note that on the sets I_0 and I_{N-1} we must have $g^{\tilde{\varepsilon}}(z,u) = 0$. Note also that the set $\mathcal{J}$ does NOT evolve in time, neither are the intervals I_j. We will include all points V_i where $u \mapsto g$ is discontinuous in the set $\mathcal{J}$, i.e., $\mathcal{V} \subset \mathcal{J}$.

We denote the discrete values of $g^{\tilde{\varepsilon}}(z,u)$, $A^{\tilde{\varepsilon}}(u)$ and $B^{\tilde{\varepsilon}}(u)$ on each interval I_j as

$$g^{\tilde{\varepsilon}}(z,u) = g^j(z), \qquad A^{\tilde{\varepsilon}}(u) = A^j, \qquad B^{\tilde{\varepsilon}}(u) = B^j, \qquad u \in I_j\,. \tag{4.10}$$

Discretization of the initial data. The initial data $\bar{z}(u)$ is approximated by piecewise constant function $\bar{z}^\varepsilon(u)$, satisfying

$$\|\bar{z}^\varepsilon(\cdot) - \bar{z}(\cdot)\|_{\mathbf{L}^1} \le C\varepsilon, \quad 0 \le \bar{z}^\varepsilon(u) \le A^{\tilde{\varepsilon}}(u), \quad \mathrm{TV}\left\{\phi(\bar{z}^\varepsilon, u; g^{\tilde{\varepsilon}})\right\} \le \mathrm{TV}\left\{\phi(\bar{z}, u; g)\right\}\,.$$

This implies

$$\left| \left\|\bar{z}^\varepsilon(\cdot) - A^{\tilde{\varepsilon}}(\cdot)\right\|_{\mathbf{L}^1} - \|\bar{z}(\cdot) - A(\cdot)\|_{\mathbf{L}^1} \right| \le C\varepsilon\,. \tag{4.11}$$

Let $\mathcal{I} = \{u_i\}$ denote the (finite) set of points of discontinuity for $z^\varepsilon(0,t)$, such that

$$\bar{z}^\varepsilon(u) = z_i, \qquad u \in [u_i, u_{i+1})\,. \tag{4.12}$$

In addition, we include all the points U_j in the set $\mathcal{I}$, i.e. $\mathcal{J} \subset \mathcal{I}$. The approximation here depends on both parameters ε and $\tilde{\varepsilon}$. However, for notation simplicity, we will only denote $z^\varepsilon, G^\varepsilon$ etc, without including the specification of the dependence on $\tilde{\varepsilon}$.

Furthermore, we also require an accuracy condition for the integral term. Denote

$$\zeta_i(t) \doteq \left|g^{\tilde{\varepsilon}}(z_i)\,(u_{i+1} - u_i)\right|, \qquad \zeta(t) \doteq \max_i \zeta_i(t)\,. \tag{4.13}$$

We assume

$$\zeta(0) \le \varepsilon\,. \tag{4.14}$$

At any time $t \ge 0$, the integral term is computed as

$$G^\varepsilon(u; z^\varepsilon) = \exp \int_u^\infty g^{\tilde{\varepsilon}}(z^\varepsilon(t,v), v)\, dv\,. \tag{4.15}$$

Thus, the mapping $u \mapsto G^\varepsilon$ is continuous. We denote its value at the grid point u_i by

$$G_i \doteq G^\varepsilon(u_i; z^\varepsilon)\,. \tag{4.16}$$

At $t = 0$, we solve a local Riemann problem at every $u_i \in \mathcal{I}$. If the solution consists of some rarefaction z-wave, they are approximated by finitely many small upward jumps of size ε, and we insert more points in the set $\mathcal{I}$ and rearrange the indices. Thanks to the properties of the initial data $\bar{z}$ in (4.3), total number of fronts is finite at $t = 0$. Let $\eta(t)$ denote the maximum size of an upward z jump in the rarefaction fronts, we have

$$\eta(0) \leq \varepsilon \,. \tag{4.17}$$

Wave speeds. If u_i is a z-wave, then it travels with Rankine-Hugoniot speed, regardless of the fact that it is a shock front or a small rarefaction front. Let $u_i \in (U_j, U_{j+1})$ for some j. We have

$$\dot{u}_i(t) = -\frac{g^j(z_i) - g^j(z_{i-1})}{z_i - z_{i-1}} G_i \,. \tag{4.18}$$

If $u_i \in \mathcal{J}$ is a g-wave, then $\dot{u}_i = 0$, and special treatment will be described below.

The evolution of the constant value z_i is governed by the following ODEs (see [6])

$$\dot{z}_i(t) = \frac{G_{i+1} - G_i}{u_{i+1} - u_i} g^j(z_i) \,. \tag{4.19}$$

Here j is the index such that $[u_i, u_{i+1}] \in [U_j, U_{j+1}]$, and we should use the erosion function $g^j(z)$.

We now have two sets of ODEs, governing the evolution of the front positions u_i and the constant states z_i. These are solved until the first interaction point, where a new Riemann problem is solved. The interaction estimates are discussed in detail in the following Section 4.3. The algorithm then continues. In addition, special cares are needed for the g-waves, which we discuss below.

Treatments around g-waves. Let (z^-, z^+) be the left and right states of a g-wave located at $u_i \in \mathcal{J}$ at time t. As time evolves, the values $z^-(t), z^+(t)$ changes in time according to (4.19). This implies that the evolution of the flux values g_i, g_{i-1} are different. After a while, we will no longer have $g^{\tilde{\varepsilon}}(z^+, u_i+) = g^{\tilde{\varepsilon}}(z^-, u_i-)$ at the g-wave.

We discretize time into intervals of length ε, and let $t_k = k\varepsilon$. Consider a g-wave located at u_i. At $t = 0$, a Riemann problem with discontinuous coefficient is solved at u_i. During the interval $t \in (0, \varepsilon)$, we allow the flux $g^{\tilde{\varepsilon}}$ to differ at the left and right states of g-wave. If in case a z-wave interacts with the g-wave during this time interval, then a new Riemann problem will be solved at this interaction. Finally, at $t = \varepsilon$ we solve a new Riemann problem at u_i, such that $g^{\tilde{\varepsilon}}(z^+, u_i+) = g^{\tilde{\varepsilon}}(z^-, u_i-)$ at $t = \varepsilon+$. The process is then iterated until $t = T$.

Accuracy of the integral term. We also ensure the accuracy of the integral term around the g-wave. Let u_i be a g-wave and u_{i+1} and u_{i-1} are two neighboring z-wave that are moving away from the g-wave, then at a later time it could occur that

$$\zeta_{i-1}(t) \geq 2\varepsilon, \qquad \text{and/or} \qquad \zeta_i(t) \geq 2\varepsilon \,.$$

In this case we will insert new fronts on the interval $[u_{i-1}, u_i]$ and/or $[u_i, u_{i+1}]$, such that the ζ_i values on these new intervals are bounded by ε. Since $g^{\tilde{\varepsilon}}(z^\varepsilon, u) \geq 0$, the integral term is decreasing in u, therefore $\dot{z}_i < \dot{z}_{i+1}$ if u_i is a newly inserted front. Thus, the front will evolve into a rarefaction wave.

Wave front changes type. It is possible that some z-shock front would shrink to size zero in finite time, and then evolve into a rarefaction fan. In this case we will simply keep the front and let it become a rarefaction front. However, a rarefaction front will never change into a shock, because if $z^- = z^+$ as the left and right states, the $\dot{z}^- \leq \dot{z}^+$ because $G^- \geq G^+$ and $\dot{z} = -(g)^2 G$.

4.2 Wave strength

Wave strength for z-waves. Recall the function $\phi(z; g)$ defined in (4.2). Wave strength for z-waves is defined through this function. Let z^-, z^+ be the left and right states of a z-wave located at u, and let g be the flux function at u. We define the strength $F(z)$ of this z-wave as

$$F(z) \doteq \left|\phi(z^-; g) - \phi(z^+; g)\right| . \tag{4.20}$$

Wave strength for g-waves. Definition of wave strength for g-waves is a bit more involved. In the literature, using the "minimum-jump" entropy condition, under the assumption that the graphs of the left and right flux functions do not intersect, such wave strength is well-studied, see Temple [14]. However, in our case the graphs can intersect, and the Temple functional can not be use.

In this paper we propose a new definition of the g-wave strength. Such a wave strength depends on how the maxima of g^-, g^+ are related to each other, as well as on the way the graphs of g^- and g^+ intersect with each other. We denote z_m^- and z_m^+ the points where g^-, g^+ reach their max value, respectively, and denote their max values as $g_m^- = g^-(z_m^-)$ and $g_m^+ = g^+(z_m^+)$.

First, we locate the unique point $(\hat{z}, \hat{g})$ as follows.

- If the graphs of g^- and g^+ has an intersection point z_0 satisfying

  $$(g^-)'(z_0) \geq 0, \qquad \text{and} \qquad (g^+)'(z_0) \leq 0\,, \tag{4.21}$$

  then we let

  $$\hat{z} = z_0, \qquad \hat{g} = g^-(z_0) = g^+(z_0)\,. \tag{4.22}$$

- Otherwise, $\hat{z}$ will be the point where g^- or g^+ attains its maximum value, whichever has the smaller max value. To be precise, we let

  $$\hat{z} = z_m^-, \qquad \hat{g} = g_m^-, \qquad \text{if} \quad g_m^- \leq g_m^+\,, \tag{4.23}$$

  $$\hat{z} = z_m^+, \qquad \hat{g} = g_m^+, \qquad \text{if} \quad g_m^- > g_m^+\,. \tag{4.24}$$

  Note that this includes the cases where the two graphs do not intersect with each other, or they intersect but the intersection points do not satisfy (4.21).

By the assumptions **(A1)-(A5)** on g^-, g^+, the existence and uniqueness of the values $(\hat{z}, \hat{g})$ is obvious. Now, we let

$$M^- = g_m^- - \hat{g}, \qquad M^+ = g_m^+ - \hat{g}\,. \tag{4.25}$$

Both these quantities are non-negative. See Figure 3 for an illustration.

We consider now $t = t_k+$, where a Riemann problem is just solved at all the g-waves, such that the flux g is continuous at all the g-waves. We have the following Lemma.

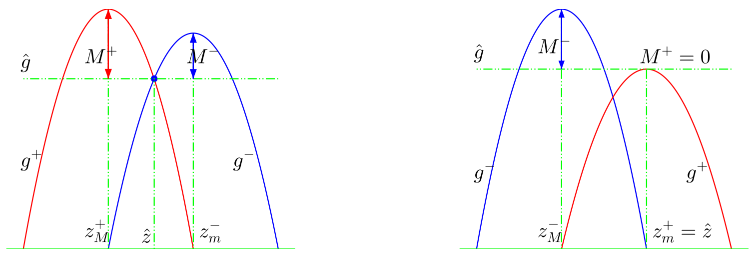

Figure 3: Definitions of M^-, M^+ for various cases.

Lemma 4.4. *Let z^-, z^+ be the left and right states of a g-wave. Then, if $(g^-)'(z^-) = 0$, we have $M^- = 0$. Similarly, if $(g^+)'(z^+) = 0$, we have $M^+ = 0$.*

Proof. We first consider the case $(g^-)'(z^-) = 0$. This could happen in two situations. (i) The graphs of g^- and g^+ have an intersection point z_0 with $(g^-)'(z_0) = 0$ and $-(g^+)'(z_0) \geq 0$ so that (4.21) holds. Then $\hat{z} = z_0$, and $z_m^- = z_0$, therefore $M^- = 0$; (ii) The graphs of g^- and g^+ don't have an intersection point satisfying (4.21), and $g_m^- \leq g_m^+$ with $z^- = \hat{z} = z_m^-$, therefore $M^- = 0$.

The proof for the case $(g^+)'(z^+) = 0$ is entirely similar, and we omit the details. $\square$

We are now ready to define the wave strength $\tilde{F}(g)$ for a g-wave at $t = t_k+$ where a Riemann problem is just solved. Let z^-, z^+ be the left and right states of this g-wave. The wave strength depends on the signs of the derivatives $(g^-)'(z^-)$ and $(g^+)'(z^+)$. We define

$$\tilde{F}(g) \doteq \begin{cases} 2M^- + 2M^+, & \text{if } -(g^-)'(z^-) \leq 0, \quad -(g^+)'(z^+) \geq 0\,, \\ 2M^- + 4M^+, & \text{if } -(g^-)'(z^-) \leq 0, \quad -(g^+)'(z^+) \leq 0\,, \\ 4M^- + 2M^+, & \text{if } -(g^-)'(z^-) > 0, \quad -(g^+)'(z^+) \geq 0\,, \\ \text{(unstable)}, & \text{if } -(g^-)'(z^-) > 0, \quad -(g^+)'(z^+) \leq 0\,. \end{cases} \tag{4.26}$$

Thanks to Lemma 4.4, the function $\tilde{F}(g)$ is continuous across the cases $(g^-)'(z^-) = 0$ and $(g^+)'(z^+) = 0$. Note also the last case with $-(g^-)'(z^-) > 0, \ -(g^+)'(z^+)$, the g-wave could not be part of the path in a Riemann solution. It can only be a stand-alone g-front which is the entire solution of a Riemann problem, therefore it can only occur at $t = 0$. Such a front is highly unstable. With any small perturbation on the data (z^-, z^+), such g-fronts will no longer exist. Therefore, if such a front shall occur, we make a small perturbation to avoid it.

Note that in the case where the two graphs do not intersect, the definition (4.26) reduces to those by Temple [14].

Finally, when $t > t_k$, the flux might differ slightly on each side of the g-front. Denote the new flux values as $\tilde{g}^-, \tilde{g}^+$ for the left and right states, respectively. We define the wave strength for a g-wave as

$$F(g) \doteq \tilde{F}(g) + \left|\tilde{g}^-(z^-(t)) - \tilde{g}^+(z^+(t))\right| \,. \tag{4.27}$$

Note that in the definition of $\tilde{F}$, the flux value g is at t_k, right after a Riemann problem was solved, while the values $\tilde{g}^-, \tilde{g}^+$ are for $t > t_k$.

4.3 Interaction estimates

Interaction between z-waves. When two z-waves interact, since the flux $(-g)$ is strictly convex, the solution behaves like that for a standard scalar conservation law with convex flux. Thus, two z-rarefaction fronts will never intersect. If two z-shocks interact, they merge into one single larger z-shock. And if a z-shock interacts with a z-rarefaction front, one of them will be canceled. The out-going wave is a single z-wave, either shock or a really small rarefaction front, depending on the sizes of the incoming waves. The total wave strength, measured in the function ϕ, is clearly non-increasing at such interactions.

Interaction between g-waves. Since g-waves are stationary, they will never interact with each other.

Interaction between a g-wave and a z-wave. We now consider the case when a z-wave interact with a stationary g-wave. We first consider the case where $g^-(z^-) = g^+(z^+)$, where (g^-, g^+) and (z^-, z^+) are the left and right erosion functions and z values, respectively. We show in next Lemma that the total wave strength is non-increasing after this interaction.

Lemma 4.5. *At the interaction of a z-wave and a g-wave, the total wave strength is non-increasing.*

Proof. We denote (g_{in}, z_{in}) and (g_{out}, z_{out}) as the incoming and the out-going waves, respectively We will prove the lemma for various cases.

Case 1. We assume that the incoming and out-going g-waves g_{in} and g_{out} have the same signs for $(g^-)'$ and $(g^+)'$ at the left and right states. We show that the total wave strength remains unchanged after the interaction.

We consider the case where a z-wave approaches a g-wave on the left. (The case where a z-wave approaches a g-wave on the right is completely similar, and we omit the details.) We let z^-, z^M, z^+ denote the left, middle and right states of the out-coming waves, where (z^-, z^M) is a g-wave and (z^M, z^+) is a z-wave. See Fig. 4 plot (a1) for an illustration. We use the same notations such as z_m^-, z_m^+ and $\hat{z}$ as in the proof of Lemma 3.1. Then, we have

$$\begin{aligned} \left|\phi^-(z^-) - \phi(z^m)\right| &= \left|g^-(z^-) - g(z^m)\right| = \left|g^+(z^M) - g^+(z^+)\right| \\ &= \left|\phi^+(z^M) - \phi^+(z^+)\right| , \end{aligned} \tag{4.28}$$

therefore wave strengths of the out-coming z-wave and g-wave remain the same as those of the incoming waves.

Case 2. We assume now that the incoming and out-going g-waves g_{in} and g_{out} does not have the same signs for $(g^-)'$ and $(g^+)'$ at the left and right state. These are the cases where the path of the g-wave "flipped" around the point $(\hat{z}, \hat{g})$. We will show that the total wave strength is non-increasing after the interaction.

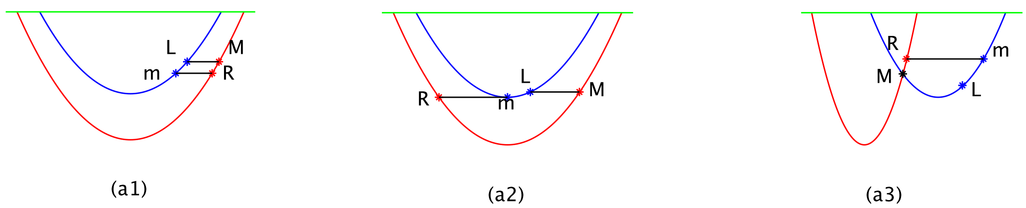

Figure 4: Interactions between g-wave and z-wave. The blue curve is the graph of g^- while the red one for g^+. Here the incoming states before intersection follow the path L-m-R, while the out-going states follow the path L-M-R.

To fix the idea, we consider the case where a z-wave approaches a g-wave on the left, while the other case where a z-wave approaches a g-wave on the right is completely similar. We let z^-, z^M, z^+ denote the left, middle and right states of the out-coming waves, where (z^-, z^M) is a g-wave and (z^M, z^+) is a z-wave. Then, this could only happen when $-(g^+)'$ changes from negative sign at incoming g-wave to positive at out-going g-wave, because at the middle state of a Riemann solution we will never have $-(g^+)' < 0$. In Fig. 4 plots (a2) and (a3) we illustrate two possible cases.

Consider now plot (a2). We have a z-shock approaching a g-wave on the left, resulting in an out-going z-shock traveling with positive speed. It holds

$$F(z_{in}) = \left|\phi^-(z^-) - \phi^-(z^m)\right|, \qquad F(z_{out}) = \left|\phi^+(z^+) - \phi^+(z^M)\right| ,$$
$$F(g_{out}) = 2M^+, \qquad F(g_{in}) = 4M^+ ,$$

so

$$\begin{aligned} F(z_{in}) - F(z_{out}) &= \left|\phi^-(z^-) - \phi^-(z^m)\right| - \left|\phi^+(z^+) - \phi^+(z^M)\right| \\ &= -2M^+ = F(g_{out}) - F(g_{in}) , \end{aligned} \tag{4.29}$$

which gives

$$\sum F_{in} = F(z_{in}) + F_(g_{in}) = F(z_{out}) + F(g_{out}) = \sum F_{out} , \tag{4.30}$$

indicating that the wave strength is unchanged.

For plot (a3), an incoming z-rarefaction approaches a g-wave on the left, resulting in two outgoing z-waves and a g-wave sandwiched in between. These two z-waves are: a left-going shock L-M and a right-going z-rarefaction M-R. The g-wave path is located at the intersection point M. The total strength of incoming waves is

$$\sum F_{in} = \left(4M^- + 2M^+\right) + \left|\phi^-(z^-) - \phi^-(z^m)\right| .$$

The total strength for the three out-going waves is

$$\begin{aligned} \sum F_{out} &= \left|\phi^-(z^-) - \phi^-(\hat{z})\right| + 2M^- + \left|\phi^+(z^+) - \phi^+(\hat{z})\right| \\ &\leq 2M^- + 2M^- + \left|\phi^-(z^-) - \phi^-(z^m)\right| = \sum F_{in} . \end{aligned}$$

In general, the increase in the wave strength at the interaction for the z-wave is bounded by $2M^+$, while the decrease in strength of the g-wave is exactly $2M^+$. Overall, the total wave strength is non-increasing at such an interaction. $\square$

4.4 Treatment of the evolution of g at a g-wave

Next Lemma shows that the total wave strength at time $t = t_k$, where new Riemann problems are solved at all g-waves, is non-increasing.

Lemma 4.6. *Let u_i be the location of a g-wave. When a Riemann problem is solved at time $t = t_k$ at u_i, the total wave strength is non-increasing.*

Proof. We discuss all possible cases, which are illustrated in Figure 5.
Case 1. If

$$-(g^-)'(z^-(t)) < 0, \qquad -(g^+)'(z^+(t)) > 0\,,$$

then the g-wave must be at the intersection point of the graphs of g^- and g^+. Then

$$-\frac{d}{dt}g^-(z^-(t)) > 0, \qquad -\frac{d}{dt}g^+(z^+(t)) < 0\,.$$

After time step ε, the solution of the new Riemann problem with left and right states $(z^-(t+\varepsilon), z^+(t+\varepsilon))$ will yield a z-rarefaction waves going left and a z-shock going right, each of size $\mathcal{O}(\varepsilon)$, with the g-wave in the middle whose position is unchanged.

Before the Riemann problem, the strength of the g-wave is

$$F(g) = \tilde{F}(g) + \left(g^+(z^+) - g^-(z^-)\right)\,.$$

In the solution of the Riemann problem, the total strength of the three waves are

$$\left(g^+(z^+) - g^m(z^m)\right) + \tilde{F}(g) + \left(g^m(z^m) - g^-(z^-)\right)\,,$$

which exactly equals to $F(g)$.
Case 2. If

$$-(g^-)'(z^-(t)) = 0, \qquad -(g^+)'(z^+(t)) > 0\,,$$

then we must have $z^- = \hat{z} = z_m^-$. The evolution is the same as Case 1. The total wave strength is unchanged after solving the Riemann problem.
Case 3. If

$$-(g^-)'(z^-(t)) \le 0, \qquad -(g^+)'(z^+(t)) \le 0\,,$$

then

$$-\frac{d}{dt}g^-(z^-(t)) > 0, \qquad -\frac{d}{dt}g^+(z^+(t)) > 0\,.$$

After a small time, the new Riemann problem will be solved by left-going z-wave and a g-wave connecting the right state $z^+(t+\varepsilon)$. Again, the total wave strength is unchanged after solving the Riemann problem.
Case 4. If

$$-(g^-)'(z^-(t)) > 0, \qquad -(g^+)'(z^+(t)) > 0\,,$$

then

$$-\frac{d}{dt}g^-(z^-(t)) < 0, \qquad -\frac{d}{dt}g^+(z^+(t)) < 0\,.$$

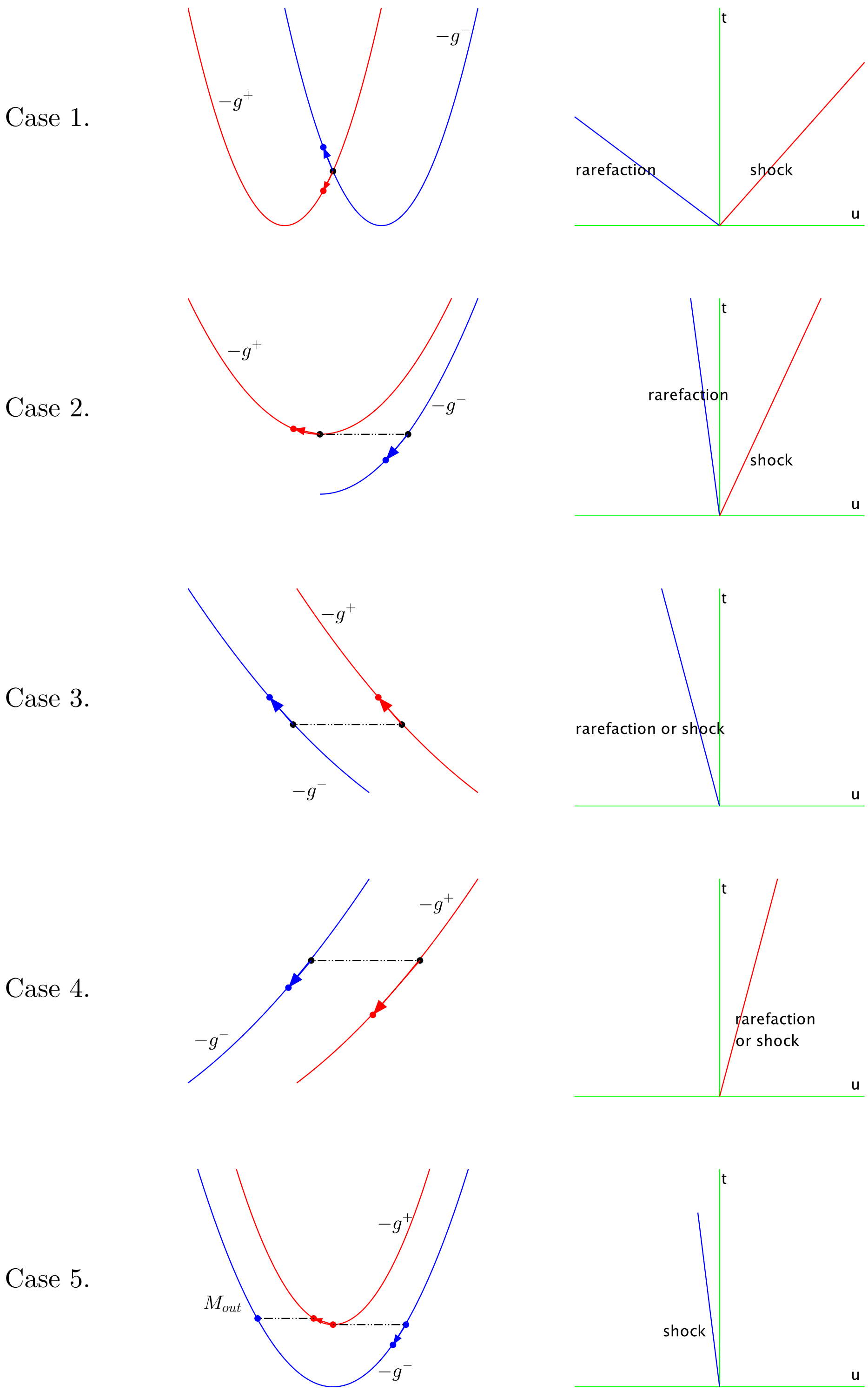

Figure 5: All 5 Cases.

The new Riemann problem will have the g-wave connected to the left $z^-(t+\varepsilon)$. Still, the total wave strength is unchanged after solving the Riemann problem.

Case 5. This is a so-called *flipping cases.* We consider

$$-(g^-)'(z^-(t)) < 0, \qquad -(g^+)'(z^+(t)) = 0\,,$$

then we must have $z^+ = \hat{z} = z_m^+$. So

$$-\frac{d}{dt}g^-(z^-(t)) < 0, \qquad -\frac{d}{dt}g^+(z^+(t)) > 0\,.$$

The new path of the g-wave is connected to the new right state $z^+(t+\varepsilon)$.

Before the Riemann problem, the wave strength is

$$F_-(g) = g^-(z^-) - g^+(z^+) + 4(g_m^- - g_m^+)\,.$$

After solving the Riemann problem, we have sum of the strength for z-shock and g-wave is

$$\big(2g_m^- g^-(z^-) - g^+(z^+)\big) + 2(g_m^- - g_m^+) = F_-(g)\,.$$

Total wave strength is unchanged after solving the Riemann problem. □

Finally, we re-consider the wave interaction between a z-wave and a g-wave, when $g^-(z^-) = g^+(z^+) + \mathcal{O}(\varepsilon)$. In this case, the wave strength is still non-increasing.

Lemma 4.7. *Consider an interaction between an incoming z-wave and g-wave, where the left and right flux values around the g-wave might not be the same. Then, the total wave strength is non-increasing after interaction.*

Proof. We can treat this interaction as a two-step interaction. Consider a z-wave approaching a g-wave on the left, and denote z^-, z^m, z^+ as the three states between the two incoming waves. In step 1 we solve the Riemann problem with (z^m, z^+) as left and right states. Thanks to Lemma 4.6, the wave strength is non-increasing. In step 2, we treat the interaction between the z-wave and the resulting g-wave from the Riemann problem in step 1. Thanks to Lemma 4.5, the wave strength is again non-increasing.

For the case when a z-wave approaching a g-wave on the right, with z^-, z^m, z^+ as the three states of the incoming waves, then in step 1 we solve the Riemann problem with (z^-, z^m), and in step 2 the resulting g-wave would interact with the z-wave coming from the right. The same result holds. □

4.5 A-priori estimate for the front tracking

In this section we establish various a-priori estimates for the $(\tilde{\varepsilon}, \varepsilon)$-front tracking approximate solutions.

4.5.1 Upper and lower bound.

Lemma 4.8. *Let $z^\varepsilon(t,u)$ be the piecewise constant approximate solution generated by the $(\tilde{\varepsilon}, \varepsilon)$-front tracking algorithm. Then, as long as the algorithm holds, we have*

$$B^{\tilde{\varepsilon}}(u) \le z^\varepsilon(t,u) \le A^{\tilde{\varepsilon}}(u), \qquad \textit{for all } \; u, t \ge 0\,. \tag{4.31}$$

Proof. There are three situations to consider.

1. Consider a constant state z_i and let j be the index such that $[u_i, u_{i+1}] \in [U_j, U_{j+1}]$. When it is not in any interaction, the evolution is governed by (4.19). If $z_i = A^j$ or $z_i = B^j$, then $g_i = 0$ so $G_i = G_{i+1}$, and $\dot{z}_i = 0$. Thus, the values B^j and A^j are critical points for this ODE. If the initial data lies between them, so will the solution for all $t > 0$.

2. Next, consider an interaction time. If the interaction is between to z-waves, then it follows the wave interaction for a scalar conservation law with convex flux, and a maximum principle applies. Thus (4.31) holds. If the interaction is between a z-wave and a g-wave, then the outgoing waves are the solution of the Riemann problem with left and right states satisfying (4.31). Since this is an invariant region for Riemann problems, the solution also satisfies (4.31).

3. Finally, consider a g-front when we solve new Riemann problems with (z^-, z^+) as the left and right states. By Case 1, we know that the Riemann data satisfies (4.31). Again, by the invariant region property so will the solution of the Riemann problem. □

4.5.2 Discrete L^1-norm bound.

Lemma 4.9. *Let $z^\varepsilon(t,u)$ be the piecewise constant approximate solution generated by the $(\tilde{\varepsilon}, \varepsilon)$-front tracking algorithm. Then for any $t \in (0, T]$, it holds*

$$\left\| z^\varepsilon(t,\cdot) - A^{\tilde{\varepsilon}}(\cdot) \right\|_{\mathbf{L}^1} \le \left\| \bar{z}^\varepsilon(\cdot) - A^{\tilde{\varepsilon}}(\cdot) \right\|_{\mathbf{L}^1} + Ct\varepsilon\,, \tag{4.32}$$

where the constant C depends on the total variation of the erosion function $g^{\tilde{\varepsilon}}$, but not on ε or $\tilde{\varepsilon}$.

Proof. We remark that $z(t,u) = A(u)$ is an equilibrium solution for (1.1). Furthermore equation (1.1) can be written as,

$$(z(t,u) - A(u))_t - \left[(g(z,u) - g(A(u),u)) \cdot \exp \int_{-\infty}^{u} (g(z(v),v) - g(A(v),v))\, dv \right]_u = 0\,. \tag{4.33}$$

This indicates that $(z(t,u) - A(u))$ is a conserved quantity. Formally its $\mathbf{L}^1$-norm is non-increasing in t.

For the $(\tilde{\varepsilon}, \varepsilon)$-front tracking approximate solution, we denote

$$I(t) \doteq \left\| z^\varepsilon(t,\cdot) - A^{\tilde{\varepsilon}}(\cdot) \right\|_{\mathbf{L}^1} = \sum_i (A_i - z_i)(u_{i+1} - u_i)\,.$$

We use the fact that $z^\varepsilon(t,u) \le A^{\tilde{\varepsilon}}(u)$ to eliminate the absolute value sign. Here A_i is the discrete $A^{\tilde{\varepsilon}}$ value on the interval $u \in (u_i, u_{i+1}]$.

For notation simplicity, we denote

$$g_i(t) \doteq g^j(z_i(t)), \qquad \text{where } j \text{ is the index such that } [u_i, u_{i+1}] \in [U_j, U_{j+1}]\,. \tag{4.34}$$

By using the evolution equation (4.19) for $\dot z_i$, and summation by parts, we get

$$\begin{aligned} I'(t) &= \sum_i -\dot z_i(u_{i+1}-u_i) + \sum_i (A_i - z_i)(\dot u_{i+1}-\dot u_i) \\ &= \sum_i -g_i(G_{i+1}-G_i) + \sum_i (A_i - z_i)(\dot u_{i+1}-\dot u_i) \\ &= \sum_i \left[(g_i - g_{i-1})G_i - (A_i - A_{i-1})\dot u_i + (z_i - z_{i-1})\dot u_i\right] \doteq \sum_i I_i(t)\,. \end{aligned}$$

There are two cases. First, if $u_i \notin \mathcal{J}$, then

$$A_i = A_{i-1} \qquad \text{and} \qquad \dot u_i = -\frac{g_i - g_{i-1}}{z_i - z_{i-1}} G_i, \qquad \Rightarrow \quad I_i(t) = 0\,.$$

Otherwise, if $u_i = U_j \in \mathcal{J}$ for some j, then $\dot U_j = 0$ and

$$I_j(t) = (g_j(t) - g_{j-1}(t))G_j\,.$$

Since the integral term satisfies the estimate

$$G_j = \exp \int_{U_j} g^{\tilde\varepsilon}(z^\varepsilon(t,v),v)\,dv \le \exp\{M_1 I(t)\}, \qquad M_1 = \|g_z^{\tilde\varepsilon}\|_\infty\,,$$

we now have

$$I'(t) = \sum_{\{j: U_j \in \mathcal{J}\}} (g_j(t) - g_{j-1}(t))G_j \le \exp\{M_1 I(t)\} \cdot \sum_{\{j: U_j\in\mathcal{J}\}} |g_j(t) - g_{j-1}(t)|\,. \tag{4.35}$$

On the interval $t \in (t_k, t_{k+1})$, by Taylor expansions of $g_j(t)$ and $g_{j-1}(t)$, we have

$$\begin{aligned} g_j(t) - g_{j-1}(t) &= g_j(t_k) + \varepsilon g_j'(t_k) - g_{j-1}(t_k) - \varepsilon g_{j-1}(t_k) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon\Big[g_j'(t_k) - g_{j-1}'(t_k)\Big] + \mathcal{O}(\varepsilon^2)\,. \end{aligned} \tag{4.36}$$

Here we used that fact that $g_j(t_k) = g_{j-1}(t_k)$ at a g-front U_j. The notation $\mathcal{O}(\varepsilon^2)$ denotes a quantity whose absolute value is bounded by $C\varepsilon^2$ for some constant C not depending on ε. Then, (4.36) imples

$$\sum_{j: U_j \in \mathcal{J}} |g_j(t) - g_{j-1}(t)| \le \varepsilon M_2, \qquad M_2 = \left\|g^{\tilde\varepsilon}\right\|_{\text{TV}}\,. \tag{4.37}$$

Then we have, for $t \in [t_k, t_{k+1}]$

$$\dot I(t) \le \varepsilon \exp\{M_1 I(t)\} \cdot \sum_{u_i\in\mathcal{J}} \left|g_i'(t_k) - g_{i-1}'(t_k)\right| \le \varepsilon M_2 \exp\{M_1 I(t)\}\,.$$

Consider now the ODE on interval $t \in [t_k, t_{k+1}]$,

$$\dot J(t) = \varepsilon M_2 \exp\{M_1 J(t)\}, \qquad J(t_k) = I(t_k)\,. \tag{4.38}$$

It is an separable equation, whose solution can be computed as

$$\exp\{-M_1 J(t)\} - \exp\{-M_1 J(t_k)\} = -\varepsilon M_1 M_2 (t - t_k)\,. \tag{4.39}$$

Since $J(t)$ is increasing, we have $J(t) > J(t_k) > 0$ for $t > t_k$. Straight computation gives

$$M_1 J(t) = M_1 J(t_k) - \ln\left(1 - \varepsilon M_1 M_2 (t - t_k) \exp\{-M_1 J(t_k)\}\right)\,.$$

Using $\exp\{-M_1 J(t_k)\} \leq 1$, and $-\ln(1-\varepsilon) \leq 2\varepsilon$ for ε sufficiently small, we have the following bound

$$J(t) \leq J(t_k) + 2\varepsilon M_2 (t - t_k)\,. \tag{4.40}$$

By comparison principle, it holds $I(t) \leq J(t)$ for $t \in [t_k, t_{k+1}]$,

$$I(t_{k+1}) \leq I(t_k) + M_2 \varepsilon^2\,.$$

Summing up over time, we get the estimate for $t \in [0, T]$

$$I(t) \leq I(0) + \varepsilon M_2 t\,.$$

□

4.5.3 Bound on the integral term in the flux

Lemma 4.10. *The integral term $G^\varepsilon(u; z^\varepsilon(t,u))$ satisfies*

$$\frac{1}{C} \leq G^\varepsilon \leq C, \qquad C = \exp\left\{\|g_z^{\tilde{\varepsilon}}\|_\infty \cdot \left\|z^\varepsilon(t,\cdot) - A^{\tilde{\varepsilon}}(\cdot)\right\|_{\mathbf{L}^1}\right\}\,. \tag{4.41}$$

Proof. Since G^ε is computed as (4.15), and we have

$$\left|\int_u^\infty g^{\tilde{\varepsilon}}(z^\varepsilon(t,v), v)\, dv\right| \leq \|g_z^{\tilde{\varepsilon}}\|_\infty \cdot \left\|z^\varepsilon(t,\cdot) - A^{\tilde{\varepsilon}}(\cdot)\right\|_{\mathbf{L}^1}\,,$$

and the result follows. □

Lemma 4.11. *For every $t \in [0,T]$, the mapping $u \mapsto G^\varepsilon$ is non-increasing, and therefore has bounded variation.*

Proof. Since $g(z,u) \geq 0$ for $B^{\tilde{\varepsilon}}(u) \leq z^\varepsilon(t,u) \leq A^{\tilde{\varepsilon}}(u)$, the mapping $u \mapsto G^\varepsilon$ is non-increasing. Since G^ε is uniformly bounded by Lemma 4.10, it has bounded variation. □

4.5.4 Bound on the accuracy of the approximation

Lemma 4.12. *Let z^ε be an $(\tilde{\varepsilon}, \varepsilon)$-approximate solution that satisfies* (4.17) *and* (4.14)*. Then, for any $t \in [0,T]$, it holds*

$$\eta(t) \leq C\varepsilon, \qquad \zeta(t) \leq C\varepsilon\,, \tag{4.42}$$

for some constant C independent of ε.

Proof. Let u_i be a z-rarefaction front, with z_{i-1} and z_i as the left and right value. Denote

$$\eta_i(t) = z_i(t) - z_{i-1}(t) > 0\,.$$

Since this is a rarefaction front, we must have

$$[u_{i-1}, u_{i+1}] \in [U_j, U_{j+1}]\,,$$

for some index j (which depends on i), and the erosion functions on the left and right of the front will be the same, i.e, $g^j(z)$. By the Intermediate Value Theorem, (4.19) can be written as

$$\dot{z}_i(t) = g^j(z_i) G^\varepsilon_z(\tilde{u}_i; z^\varepsilon) = -\left(g^j(z_i)\right)^2 G^\varepsilon(\tilde{u}_i; z^\varepsilon)\,, \tag{4.43}$$

for some $\tilde{u}_i$ that lies between u_i and u_{i+1}. Furthermore, we have

$$|G^\varepsilon(\tilde{u}_i; z^\varepsilon) - G^\varepsilon(\tilde{u}_{i-1}; z^\varepsilon)| \le C\zeta\,. \tag{4.44}$$

The evolution of η_i satisfies

$$\begin{aligned}
\dot{\eta}_i(t) &= \dot{z}_i(t) - \dot{z}_{i-1}(t) \;=\; -\left(g^j(z_i)\right)^2 G^\varepsilon(\tilde{u}_i; z^\varepsilon) + \left(g^j(z_{i-1})\right)^2 G^\varepsilon(\tilde{u}_{i-1}; z^\varepsilon) \\
&= \left[\left(g^j(z_{i-1})\right)^2 - \left(g^j(z_i)\right)^2\right] G^\varepsilon(\tilde{u}_i; z^\varepsilon) + \left(g^j(z_{i-1})\right)^2 \left[G^\varepsilon(\tilde{u}_{i-1}; z^\varepsilon) - G^\varepsilon(\tilde{u}_i; z^\varepsilon)\right] \\
&\le C\eta + C\zeta\,. \qquad (4.45)
\end{aligned}$$

For the evolution of ζ_i, we only need to consider the case where u_i and u_{i+1} are two neighboring z-waves, since the interval next to g-waves are specially treated already. Assume that

$$[u_{i-1}, u_{i+2}] \in [U_j, U_{j+1}]$$

for some index j (which depends on i), so the erosion function over the interval $[u_{i-1}, u_{i+2}]$ is g^j. Then we have

$$\begin{aligned}
\dot{\zeta}_i(t) &= (\dot{u}_{i+1} - \dot{u}_i) g^j(z_i) + (u_{i+1} - u_i)(g^j)'(z_i)\dot{z}_i \\
&= (\dot{u}_{i+1} - \dot{u}_i) g^j(z_i) + (g^j)'(z_i)(G_{i+1} - G_i) g^j(z_i) \\
&= g^j(z_i)\left[\left(\dot{u}_{i+1} + (g^j)'(z_i) G_{i+1}\right) + \left(-(g^j)'(z_i) G_i - \dot{u}_i\right)\right]\,.
\end{aligned}$$

If the front u_i and/or u_{i+1} are/is a shock, then the term(s) involving $\dot{u}_i$ and/or $\dot{u}_{i+1}$ will be negative. If one of them is a rarefaction front, say u_i, then that term is positive, and its size depends on the size of the rarefaction front

$$-(g^j)'(z_i) G_i - \dot{u}_i \le C\eta\,.$$

This yields

$$\dot{\zeta}_i \le C\eta\,. \tag{4.46}$$

Taking max over i in (4.45) and (4.46), we get a system of ordinary differential inequalities

$$\dot{\eta} \le C\eta + C\zeta, \qquad \dot{\zeta}_i \le C\eta\,. \tag{4.47}$$

By a standard comparison argument, we conclude (4.42). $\square$

Next Lemma provides the accuracy of the flux at g-waves.

Lemma 4.13. *Let U_j be a g-wave. Then*

$$\left|g^j(z^\varepsilon(U_j-))G^\varepsilon(U_j;z^\varepsilon) - g^{j-1}(z^\varepsilon(U_j+))G^\varepsilon(U_j;z^\varepsilon)\right| \le C\varepsilon\,. \tag{4.48}$$

Proof. Thanks to (4.36), we have

$$\left|g^j(z^\varepsilon(U_j-)) - g^{j-1}(z^\varepsilon(U_j+))\right| \le C\varepsilon\,.$$

Combine with Lemma 4.10, this yields the result. □

4.5.5 Bound on the total wave strength

We denote the total wave strength at time t to be the sum over all z-waves and g-waves strength. We denote

$$Q(t) \doteq \sum_{u_i \notin \mathcal{J}} F_z^i + \sum_{u_i \in \mathcal{J}} F_g^i \doteq Q_z(t) + Q_g(t)\,. \tag{4.49}$$

Thanks to Lemma 4.5 and Lemma 4.6, the total wave strength is non-increasing at interaction, as well as at the time when new Riemann problems are solved at g-waves. Therefore, we only need to bound the growth of the total wave strength outside these situations.

We have the following Lemma.

Lemma 4.14. *The total wave strength for the front tracking approximate solution remain bounded for finite time, i.e.,*

$$Q(t) \le e^{Ct}(Q(0)+1) - 1, \qquad 0 \le t \le T\,. \tag{4.50}$$

Proof. We recall the definition of ϕ in (4.2). Let $t \in [0,T]$. We denote the discrete values of ϕ as

$$\phi^j(z) \doteq \phi(z,u;g^\varepsilon), \qquad u \in (U_j, U_{j+1}]\,. \tag{4.51}$$

and let

$$\phi_i(t) \doteq \phi^{j_i}(z_i(t)), \qquad [u_i,u_{i+1}] \in [U_{j_i}, U_{j_i+1}]\,, \tag{4.52}$$

where the j_i is a j index that depends on i. We denote the discrete function as

$$\phi^\varepsilon(t,u) \doteq \phi(z^\varepsilon(t,u),u;g^\varepsilon(z^\varepsilon(t,u),u))\,. \tag{4.53}$$

Then, the total wave strength for z-waves is

$$Q_z(t) = \sum_{u_i \notin \mathcal{J}} |\phi_i(t) - \phi_{i-1}(t)|\,, \qquad \dot{Q}_z = \sum_{u_i \notin \mathcal{J}} \left|\dot{\phi}_i(t) - \dot{\phi}_{i-1}(t)\right|\,.$$

For Q_g, it consists of the part with $\tilde{F}$ which does not change during the interval $t \in (t_k, t_{k+1}]$, and another part $|\phi_i(t) - \phi_{i-1}(t)|$ which changes in t. So

$$\dot{Q}_g = \sum_{u_i \in \mathcal{J}} \left|\dot{\phi}_i(t) - \dot{\phi}_{i-1}(t)\right|\,.$$

Therefore,

$$\dot{Q}(t) = \sum_i \left|\dot{\phi}_i(t) - \dot{\phi}_{i-1}(t)\right| . \tag{4.54}$$

By definition (4.2), we have

$$\left(\phi^{j_i}\right)'(z) = \left|\left(g^{j_i}\right)'(z)\right|, \qquad \dot{\phi}_i(t) = \left(\phi^{j_i}\right)'(z_i) \cdot \dot{z}_i . \tag{4.55}$$

We now have

$$\begin{aligned}
\dot{Q} &= \sum_i \left|\left(\phi^{j_i}\right)'(z_i) \cdot \dot{z}_i - \left(\phi^{j_{i-1}}\right)'(z_{i-1}) \cdot \dot{z}_{i-1}\right| \\
&= \sum_i \left|\left(\phi^{j_i}\right)'(z_i) \left(g^{j_i}(z_i)\right)^2 G^\varepsilon(z^\varepsilon;\tilde{u}_i) - \left(\phi^{j_{i-1}}\right)'(z_{i-1}) \left(g^{j_{i-1}}(z_{i-1})\right)^2 G^\varepsilon(z^\varepsilon;\tilde{u}_{i-1})\right| \\
&\le \sum_i \left|\left(\phi^{j_i}\right)'(z_i) \left(g^{j_i}(z_i)\right)^2 - \left(\phi^{j_{i-1}}\right)'(z_{i-1}) \left(g^{j_{i-1}}(z_{i-1})\right)^2\right| G^\varepsilon(z^\varepsilon;\tilde{u}_i) \\
&\quad + \sum_i \left|G^\varepsilon(z^\varepsilon;\tilde{u}_i) - G^\varepsilon(z^\varepsilon;\tilde{u}_{i-1})\right| \left(\phi^{j_{i-1}}\right)'(z_{i-1}) \left(g^{j_{i-1}}(z_{i-1})\right)^2 .
\end{aligned} \tag{4.56}$$

Thanks to the uniform bounds on g, g_z, the second term is bounded by $C\left\|G^\varepsilon\right\|_{\text{TV}}$. Using Lemma 4.11, G^ε is a BV function. Therefore, the second term is bounded by a constant.

We now consider the first term. Assume that u_i and a z-wave which lies on the interval I_j for some index j. By definition (4.2), we have

$$\phi^j(z_m) = 0, \qquad \left(\phi^j\right)'(z_m) = 0, \qquad \left(\phi^j\right)''(z_m^j) \neq 0 , \tag{4.57}$$

where z_m^j is the unique z value such that $\left(g^j\right)'(z_m^j) = 0$. We define the function

$$\psi^j(z) \doteq \left(\phi^j\right)'(z) \cdot \left(g^j(z)\right)^2 . \tag{4.58}$$

Then, we have

$$\frac{d\psi^j}{d\phi^j}(z) = \frac{d\psi^j}{dz}\frac{1}{\left(\phi^j\right)'(z)} = \left(\phi^j\right)''(z)\frac{\left(g^j\right)^2(z)}{\left(\phi^j\right)'(z)} + 2g^j(z)\left(g^j\right)'(z) . \tag{4.59}$$

The second term is bounded. For the first term, we need to verify the limit as $z \to z_m$ where $\left(g^j\right)'(z_m) = 0$. By L'Hôpital's Rule, we have

$$\lim_{z\to z_m^j} \frac{\left(g^j\right)^2(z)}{\left(\phi^j\right)'(z)} = \lim_{z\to z_m^j} \frac{2g^j(z)\left(g^j\right)'(z)}{\left(\phi^j\right)''(z)} = 0 . \tag{4.60}$$

Therefore, we conclude

$$\left|\frac{d\psi^j}{d\phi^j}(z)\right| \le C, \qquad \text{for all } \ z \in [B^j, A^j] . \tag{4.61}$$

Then, for any $z_1, z_2 \in [B^j, A^j]$, we have (by Intermediate Value Theorem)

$$\left|\psi^j(z_1) - \psi^j(z_2)\right| \le \left(\psi^j\right)'(\tilde{z})\left|\phi^j(z_1) - \phi^j(z_2)\right| \le C\left|\phi^j(z_1) - \phi^j(z_2)\right| . \tag{4.62}$$

If u_i is a z-wave, then (4.62) implies

$$\sum_{u_i \notin \mathcal{J}} \left| \left(\phi^{j_i}\right)'(z_i) \left(g^{j_i}(z_i)\right)^2 - \left(\phi^{j_{i-1}}\right)'(z_{i-1}) \left(g^{j_{i-1}}(z_{i-1})\right)^2 \right|$$
$$\leq C \sum_{u_i \notin \mathcal{J}} \left| \phi^{j_i}(z_i) - \phi^{j-1}(z_{i-1}) \right| . \tag{4.63}$$

On the other hand, if u_i is a g-wave, then ϕ^{j_i} and $\phi^{j_{i-1}}$ are in different intervals of I_j, and they are two different functions. Consider an time interval $t \in [t_k, t_{k+1}]$ and expand the functions around $t = t_k$, we have

$$\left| \left(\phi^{j_i}\right)'(z_i) \left(g^{j_i}(z_i)\right)^2 - \left(\phi^{j_{i-1}}\right)'(z_{i-1}) \left(g^{j_{i-1}}(z_{i-1})\right)^2 \right|$$
$$\leq \left| \left(\phi^{j_i}\right)'(z_i(t_k)) - \left(\phi^{j_{i-1}}\right)'(z_{i-1}(t_k)) \right| \left(g^{j_i}(z_i(t_k))\right)^2 + C\varepsilon .$$

Therefore,

$$\sum_{u_i \in \mathcal{J}} \left| \left(\phi^{j_i}\right)'(z_i) \left(g^{j_i}(z_i)\right)^2 - \left(\phi^{j_{i-1}}\right)'(z_{i-1}) \left(g^{j_{i-1}}(z_{i-1})\right)^2 \right|$$
$$\leq C \sum_{u_i \in \mathcal{J}} \left| \phi^{j_i}(z_i) - \phi^{j_{i-1}}(z_{i-1}) \right| + C\varepsilon . \tag{4.64}$$

Combining (4.63) and (4.64), estimate(4.56) becomes

$$\dot{Q} \leq CQ + C . \tag{4.65}$$

A standard comparison argument yields (4.50), completing the proof. □

4.5.6 Continuity in time

Lemma 4.15. *The $(\tilde{\varepsilon}, \varepsilon)$-approximate solution satisfies*

$$\|z^\varepsilon(t,\cdot) - z^\varepsilon(\tau,\cdot)\|_{\mathbf{L}^1} \leq Ce^{Ct} \cdot |t - \tau| , \qquad t, \tau \in [0, T] . \tag{4.66}$$

Proof. By Lemma 4.14, we have

$$\mathrm{TV}\{\phi(z^\varepsilon(t,\cdot), \cdot; g^{\tilde{\varepsilon}})\} \leq Q(t) \leq e^{Ct}[Q(0) + 1] - 1 . \tag{4.67}$$

Recall the definition (4.2). It implies that the mapping $t \to g^{\tilde{\varepsilon}}(z^\varepsilon, u)$ also has bounded variation, i.e.,

$$\mathrm{TV}\{g^{\tilde{\varepsilon}}(z^\varepsilon(t,\cdot), \cdot)\} \leq \mathrm{TV}\{\phi(z^\varepsilon(t,\cdot), \cdot; g^{\tilde{\varepsilon}})\} \leq e^{Ct}[Q(0) + 1] - 1 . \tag{4.68}$$

By the finite propagation speed property, the $\mathbf{L}^1$ continuity in time follows from a rather standard argument, which we present below. Assume now $0 \leq t < \tau \leq T$, and $\xi_n(t)$ be a smooth approximation to the characteristic function $\chi_{[t,\tau]}$, such that $\xi_n \to \chi_{[t,\tau]}$ in $\mathbf{L}^1$ and ξ_n' approaches $\delta_t - \delta_\tau$ where δ denotes the Dirac Delta function. Let $\varphi(x)$

denote a test function with $|\varphi| \leq 1$. Since the front tracking approximate provides weak solutions, we have

$$\int_0^T \int_{I\!R} z^\varepsilon \left(\varphi(x)\xi_n(t)\right)_t - g^{\tilde{\varepsilon}}(z^\varepsilon, u)G^\varepsilon(u; z^\varepsilon)\, du\, dt = 0\,.$$

Taking the limit in n, we get

$$\int_{I\!R} \varphi(x)\left(z^\varepsilon(\tau, u) - z^\varepsilon(t, u)\right)\, du = \int_t^\tau \int_{I\!R} \varphi'(x) g^{\tilde{\varepsilon}}(z^\varepsilon, u)G^\varepsilon(u; z^\varepsilon)\, du\, ds\,.$$

Then it follows, after using (4.68) and Lemma 4.11,

$$\begin{aligned}
\|z^\varepsilon(t,\cdot) - z^\varepsilon(\tau,\cdot)\|_{\mathbf{L}^1} &= \sup_{|\varphi|\leq 1} \int_{I\!R} \varphi(x)\left(z^\varepsilon(\tau, u) - z^\varepsilon(t, u)\right)\, du \\
&= \sup_{|\varphi|\leq 1} \int_t^\tau \int_{I\!R} \varphi'(x) g^{\tilde{\varepsilon}}(z^\varepsilon, u)G^\varepsilon(u; z^\varepsilon)\, du\, ds \\
&\leq \int_t^\tau \mathrm{TV}\left\{g^{\tilde{\varepsilon}}(z^\varepsilon(s,\cdot),\cdot)G^\varepsilon(\cdot; z^\varepsilon)\right\}\, du\, ds \\
&\leq (\tau - t)\left\|g^{\tilde{\varepsilon}}\right\|_\infty \mathrm{TV}\left\{G^\varepsilon(\cdot; z^\varepsilon)\right\} + \mathrm{TV}\left\{g^{\tilde{\varepsilon}}(z^\varepsilon(t,\cdot),\cdot)\right\}\|G^\varepsilon\|_\infty \\
&\leq Ce^{Ct}(\tau - t)\,,
\end{aligned}$$

completing the proof. □

4.5.7 Bound on the total number of fronts

Thanks to Lemma 4.14, the total wave strength is bounded for $t \leq T$. We introduce a threshold δ, say $\delta = \frac{1}{3}\varepsilon$. If two neighboring z-rarefaction fronts are very small such that both wave strengths are less than the threshold δ, we will merge then into one z-rarefaction front. Thanks to the property (2.4) of the function g, we immediately have the following Lemma.

Lemma 4.16. *Given $\tilde{\varepsilon}, \varepsilon > 0$ and $T > 0$, the $(\tilde{\varepsilon}, \varepsilon)$-front tracking approximation algorithm will have finitely many front during $t \in [0, T]$.*

4.6 Convergence of front tracking approximation

Thanks to the a-priori estimates, a standard compactness argument (see for example [2] and [4]) and the argument of Temple [14], we establish the convergence of the $(\tilde{\varepsilon}-\varepsilon)$-front tracking approximate solution as $\varepsilon \to 0+$. We denote this limit as the $\tilde{\varepsilon}$-approximate solution $z^{\tilde{\varepsilon}}$. We have proved the following Lemma.

Lemma 4.17. *Let $z^{\tilde{\varepsilon}}$ be the limit of the $(\tilde{\varepsilon} - \varepsilon)$-front tracking approximate solution as $\varepsilon \to 0+$. Then the followings hold.*

(i) $B^{\tilde{\varepsilon}}(u) \leq z^{\tilde{\varepsilon}}(t, u) \leq A^{\tilde{\varepsilon}}(u)$ *for all* $u \in I\!R$.

(ii) $\left\|z^{\tilde{\varepsilon}}(t,\cdot) - A^{\tilde{\varepsilon}}(\cdot)\right\|_{\mathbf{L}^1} \leq \left\|\bar{z}^{\tilde{\varepsilon}}(\cdot) - A^{\tilde{\varepsilon}}(\cdot)\right\|_{\mathbf{L}^1}.$

(iii) $C^{-1} \leq G^{\tilde{\tilde{\varepsilon}}} \leq C$.

(iv) *The total wave strength is bounded*

$$Q(t) \leq e^{Ct}(Q(0)+1) - 1\,,$$

where $Q(t)$ *is defined in* (4.49)*, and the wave strength for g-waves is defined in* (4.26) *since* $F(g) = \tilde{F}(g)$.

(v) $\left\|z^{\tilde{\tilde{\varepsilon}}}(t,\cdot) - z^{\tilde{\tilde{\varepsilon}}}(\tau,\cdot)\right\|_{\mathbf{L}^1} \leq C \cdot |t-\tau|$.

Note that $z^{\tilde{\tilde{\varepsilon}}}$ is no longer piecewise constant. It provides a weak solution for the integro-differential equation with an erosion function $g^{\tilde{\tilde{\varepsilon}}}(z,u)$ that is piecewise constant in u.

We now show that the $\tilde{\varepsilon}$-approximate solution solutions satisfy the Kruzhkov entropy condition. We have the following Lemma.

Lemma 4.18. *The* $\tilde{\varepsilon}$*-approximate solution* $z^{\tilde{\tilde{\varepsilon}}}$ *satisfies the following Kruzhkov-type entropy inequality for all constants* c *and all non-negative test functions* φ*,*

$$\begin{aligned}
&\int_0^T \int_R \left[-|z^{\tilde{\tilde{\varepsilon}}} - c|\varphi_t + \mathcal{G}(g^{\tilde{\tilde{\varepsilon}}}, z^{\tilde{\tilde{\varepsilon}}}, c, u)\varphi_u\right]\,du\,dt \\
&\qquad \leq \int_0^T \int_R \operatorname{sign}(z^{\tilde{\tilde{\varepsilon}}} - c)\cdot g^{\tilde{\tilde{\varepsilon}}}(c,u)\tilde{G}(u; z^{\tilde{\tilde{\varepsilon}}})_u \varphi\,du\,dt \\
&\qquad\quad + \int_0^T \sum_j \left|g^{\tilde{\tilde{\varepsilon}}}(c,U_j-) - g^{\tilde{\tilde{\varepsilon}}}(c,U_j+)\right| \tilde{G}(U_j; z^{\tilde{\tilde{\varepsilon}}})\varphi(t,U_j)\,dt + C\tilde{\varepsilon}\,,
\end{aligned} \tag{4.69}$$

where $\mathcal{G}$ *is defined as*

$$\mathcal{G}(g,z,c,u) \doteq \operatorname{sign}(z-c)\cdot(g(z,u) - g(c,u))\tilde{G}(u;z)\,. \tag{4.70}$$

and $\tilde{G}(u; z^{\tilde{\tilde{\varepsilon}}})$ *is the linear spline interpolation of the integral term* $G^{\tilde{\tilde{\varepsilon}}}$ *at the knots* $U_j \in \mathcal{J}$.

Proof. On the interval $I_j = [U_j, U_{j+1}]$, we have $g = g^j(z)$ which does not depend on u, and the limit of the front tracking approximate solutions $z^{\tilde{\tilde{\varepsilon}}}$ generate the entropy weak solutions (see [2]), and a Kruzhkov-type entropy inequality holds (see [4])

$$\begin{aligned}
&\int_0^T \int_{U_j}^{U_{j+1}} \left[-|z^{\tilde{\tilde{\varepsilon}}} - c|\varphi_t + \mathcal{G}(g^j, z^{\tilde{\tilde{\varepsilon}}}, c, u)\varphi_u\right]\,du\,dt \\
&\qquad \leq \int_0^T \int_{U_j}^{U_{j+1}} \operatorname{sign}(z^{\tilde{\tilde{\varepsilon}}} - c)\cdot g^j(c)\tilde{G}(u; z^{\tilde{\tilde{\varepsilon}}})_u \psi\,du\,dt \\
&\qquad\quad + \int_0^T \mathcal{G}(g^j, z^{\tilde{\tilde{\varepsilon}}}, c, u)\varphi\Big|_{u=U_j}^{u=U_{j+1}}\,dt\,.
\end{aligned} \tag{4.71}$$

Summing over j, and perform summation-by-parts for the last term in (4.71), we get

$$\begin{aligned}
&\int_0^T \int_R \left[-|z^{\tilde{\tilde{\varepsilon}}} - c|\varphi_t + \mathcal{G}(g^{\tilde{\tilde{\varepsilon}}}, z^{\tilde{\tilde{\varepsilon}}}, c, u)\varphi_u\right]\,du\,dt \\
&\leq \int_0^T \int_R \operatorname{sign}(z^{\tilde{\tilde{\varepsilon}}} - c)\cdot g^{\tilde{\tilde{\varepsilon}}}(c,u)\tilde{G}(u; z^{\tilde{\tilde{\varepsilon}}})_u\varphi\,du\,dt + \int_0^T \sum_j \mathcal{G}(g^j, z^{\tilde{\tilde{\varepsilon}}}, c, u)\varphi\Big|_{u=U_j+}^{u=U_j-}\,dt\,.
\end{aligned}$$

Here the last term becomes

$$\begin{aligned}
&\int_0^T \sum_j \mathcal{G}(g^j, z^{\tilde{\varepsilon}}, c, u)\varphi\Big|_{u=U_j+}^{u=U_j-} dt \\
&= \int_0^T \sum_j \operatorname{sign}(z^{\tilde{\varepsilon}} - c) \cdot (g^{\tilde{\varepsilon}}(z^{\tilde{\varepsilon}}, u) - g^{\tilde{\varepsilon}}(c, u))\tilde{G}(u; z^{\tilde{\varepsilon}})\varphi\Big|_{u=U_j+}^{u=U_j-} dt \\
&\leq \int_0^T \sum_j \left[\left|g^{j-1}(z^{\tilde{\varepsilon}}(U_j-), U_j-) - g^j(z^{\tilde{\varepsilon}}(U_j+), U_j+)\right| + \left|g^{j-1}(c) - g^j(c)\right|\right] \\
&\qquad \cdot \tilde{G}(U_j; z^{\tilde{\varepsilon}})\varphi(t, U_j)\, dt \\
&\leq \tilde{\varepsilon}\, T \left\|g^{\tilde{\varepsilon}}\right\|_{\mathrm{tv}} \|G^{\tilde{\varepsilon}}\|_\infty + \int_0^T \sum_j \left|g^{j-1}(c) - g^j(c)\right| \tilde{G}(U_j; z^{\tilde{\varepsilon}})\varphi(t, U_j)\, dt\,.
\end{aligned}$$

Recalling that $g^j(c) = g^{\tilde{\varepsilon}}(c, U_j+)$ and $g^{j-1}(c) = g^{\tilde{\varepsilon}}(c, U_j-)$, we complete the proof. □

We are now ready to prove the existence of entropy weak solutions for (1.1), which is rather standard after establishing the a-priori estimates. Taking the limit $\tilde{\varepsilon} \to 0+$, thanks to the a-priori estimates in Lemma 4.17 and Lemma 4.18, a standard compactness argument yields the convergence of the solutions $z^{\tilde{\varepsilon}} \to z$, which satisfies the Definition 4.1.

4.7 Uniqueness and stability of solutions

We adapt an approach used in [5] (section 4). Let z, w be two entropy weak solutions with initial data $\bar{z}, \bar{w}$. Let φ^n be a family of smooth positive test function such that $\varphi^n = 0$ at every point V_i where $u \mapsto g$ is discontinuous, and $\varphi^n \to \varphi$ in $\mathbf{L}^1(I\!R)$ as $n \to \infty$. A construction of these functions φ^n is given in [5].

Using first φ_n as the test function, by a standard Kruzhkov analysis [12], (see also [4], Section 6), we get

$$\begin{aligned}
&\int_0^T \int_{I\!R} \left[- |z - w|\, \varphi^n_t + \operatorname{sign}(z - w)(g(z, u) - g(w, u))\frac{1}{2}\left(G(u; z) + G(u; w)\right) \varphi^n_u\right] du\, dt \\
&\quad \leq -\int_0^T \int_{I\!R} \operatorname{sign}(z - w) \left[g(z, u)G(u; z) - g(w)G(u; w)\right] g(z)\varphi^n\, du\, dt\,.
\end{aligned}$$

Then, we take the limit $n \to \infty$. By Lebesque dominated convergence theorem, we have

$$\begin{aligned}
&\int_0^T \int_{I\!R} \left[- |z - w|\, \varphi_t + \operatorname{sign}(z - w)(g(z, u) - g(w, u))\frac{1}{2}\left(G(u; z) + G(u; w)\right) \varphi_u\right] du\, dt \\
&\quad \leq -\int_0^T \int_{I\!R} \operatorname{sign}(z - w) \left[g(z, u)G(u; z) - g(w)G(u; w)\right] g(z)\varphi\, du\, dt \\
&+ \int_0^T \sum_j \operatorname{sign}(z - w)(g(z, u) - g(w, u))\Big|_{u=U_j-}^{u=U_j+} \cdot \frac{1}{2}\left(G(U_j; z) + G(U_j; w)\right) \varphi(t, U_j)\, du\, dt\,.
\end{aligned}$$

By the same arguments as in [5][eqn (4.12)], the last term above is less than 0. We now have

$$\begin{aligned}&\int_0^T \int_{I\!R} \left[-|z-w|\,\varphi_t + \mathrm{sign}(z-w)(g(z,u)-g(w,u))\frac{1}{2}\,(G(u;z)+G(u;w))\,\varphi_u \right] du\,dt \\ &\quad\le\ -\int_0^T \int_{I\!R} \mathrm{sign}(z-w)\,[g(z,u)G(u;z)-g(w)G(u;w)]\,g(z)\varphi\,du\,dt\,.\end{aligned}$$

Now, following the analysis in [4] (proof of Theorem 6.1, with $\Theta = 0$), we arrive at, for any $0 \le t_1 \le t_2 \le T$, and some constant C that does not depend on t_1, t_2,

$$\|z(t_2,\cdot,)-w(t_2,\cdot)\|_{\mathbf{L}^1} \le \|z(t_1,\cdot,)-w(t_1,\cdot)\|_{\mathbf{L}^1} + C\int_{t_1}^{t_2} \|z(t,\cdot,)-w(t,\cdot)\|_{\mathbf{L}^1}\,dt\,. \tag{4.72}$$

By Gronwall's lemma, for any $0 \le t \le T$ we finally get

$$\|z(t,\cdot,)-w(t,\cdot)\|_{\mathbf{L}^1} \le e^{Ct}\,\|z(0,\cdot,)-w(0,\cdot)\|_{\mathbf{L}^1}\ , \tag{4.73}$$

implying continuous dependence on initial data, and thus the uniqueness of entropy weak solutions. This completes the proof for Theorem 4.2.

5 Slow erosion model with constraints

5.1 Preliminaries and main results

We now study the case where $B(u)$ is negative on some intervals of u, and the solution $z(t,u)$ could become negative in finite time. The pointwise constraint $z \ge 0$ will be applied. We adopt the projection operator introduced in [4]. For the convenience of the readers, we repeat the definition of the operator and its main properties.

Consider the sets

$$Z \ \doteq\ \left\{ z \in \mathbf{L}^1_{loc}(I\!R)\,;\quad \lim_{|u|\to\infty} z(u) = 1, \quad \|z(\cdot)-1\|_{\mathbf{L}^1} \le M \right\}, \tag{5.1}$$

$$Z^+ \ \doteq\ \{ z \in X\,;\quad z(u) \ge 0\}\,. \tag{5.2}$$

For a given $z \in Z$, define

$$F(u) \doteq \int_0^u \int_0^v z(s)\,ds\,dv\,, \quad F'(u) = \int_0^u z(s)\,ds\,; \quad F''(u) = z(u) \quad \text{(for a.e. } u)\,. \tag{5.3}$$

Let F_* be the lower convex envelope of F, namely

$$F_*(u) \ \doteq\ \min\ \left\{ \theta F(a) + (1-\theta)F(b)\,;\quad \theta \in [0,1]\,,\quad u = \theta a + (1-\theta) b \right\}. \tag{5.4}$$

The projection operator $\pi : Z \mapsto Z^+$ is now defined by setting

$$\pi z(u) \ \doteq\ F_*''(u)\,. \tag{5.5}$$

Since F_* is convex, its second derivative is non-negative. Hence $\pi z \in Z^+$.

The next Lemma, proved in [4], collects the main properties of this operator.

Lemma 5.1. *Let $\pi : Z \mapsto Z^+$ be the operator defined at* (5.5). *Then the following holds.*

(i) $\pi z = z$ *for every* $z \in Z^+$.

(ii) *For any* a, b *where* $\pi z(a) \geq 0, \pi z(b) \geq 0$, *one has*

$$\int_a^b \pi z(u)\, dx = \int_a^b z(u)\, du\,, \qquad \int_a^b \int_a^u \pi z(v)\, dv\, du = \int_a^b \int_a^u z(v)\, dv\, du\,. \tag{5.6}$$

Moreover

$$\int_a^\xi \int_a^u \pi z(v)\, dv\, du \;\leq\; \int_a^\xi \int_a^u z(v)\, dv\, du \qquad \textit{for all} \;\; \xi \in I\!\!R\,. \tag{5.7}$$

(iii) *(monotonicity) If* $z, w \in Z$ *and* $z(u) \leq w(u)$ *for a.e.* u, *then* $\pi z(u) \leq \pi w(u)$ *for a.e.* u.

(iv) *(*$\mathbf{L}^1$*-contractivity) For any* $z, w \in Z$, *we have* $\|\pi z - \pi w\|_{\mathbf{L}^1} \;\leq\; \|z - w\|_{\mathbf{L}^1}$.

(v) *(BV stability) For any* $z \in Z$ *having bounded total variation, one has*

$$TV\{\pi z\} \;\leq\; TV\{z\}\,.$$

(vi) *(Dissipative property) Let* $z \in Z$. *For any constant* $c > 0$ *and any non-negative test function* $\varphi \in \mathcal{C}_c^\infty(I\!\!R)$ *one has*

$$\begin{aligned} \int_{I\!\!R} |\pi z(u) - c|\, \varphi(u)\, du \;\;\leq\;\; & \int_{I\!\!R} |z(u) - c|\, \varphi(u)\, du \\ & - \int_{I\!\!R} \mathrm{sign}(\pi z(u) - c)\Theta^z(u)\, \varphi_u(u)\, du\,, \end{aligned}$$

where

$$\Theta^z(u) \doteq \int_{-\infty}^u [\pi z(v) - z(v)]\, dv\,.$$

We are now ready to state the definition of entropy weak solution for (1.1) where g satisfies the assumptions **(A1)-(A5)**.

Definition 5.2. *Let* $T > 0$, *and let* $z = z(t, u) \geq 0$ *be a bounded, measurable function. We call* $z(t, u)$ *an entropy weak solution of* (1.1) *if the following conditions are satisfied:*

(D1) *The map* $t \mapsto z(t, \cdot)$ *is continuous from* $[0, T]$ *into* $L^1_{loc}(I\!\!R)$, *and* $0 \leq z(t, u) \leq A(u)$ *for every* (t, u).

(D2) $z(t, \cdot) \to \bar{z}$ *in* $L^1(I\!\!R)$ *as* $t \to 0+$.

(D3) *Total variation of* $\phi(z(t, \cdot), \cdot)$ *is bounded for all* $t \in [0, T]$.

(D4) *There exists a measurable function* $\Theta = \Theta(t,u)$ *with compact support in* $[0,T]$ *such that*

$$
\begin{aligned}
z(t,u) \ > \ 0 \qquad &\Longrightarrow \qquad \Theta(t,u) \ = \ 0\,, \\
z(t,a) > 0, \quad z(t,b) > 0 \qquad &\Longrightarrow \qquad \int_a^b \Theta(t,u)\,du = 0\,.
\end{aligned} \tag{5.8}
$$

Moreover, for any constant $c \geq 0$ *and every non-negative test function* $\varphi \in \mathcal{C}_c^\infty(]0,T[\times I\!R)$*, the following entropy inequality holds:*

$$
\begin{aligned}
&\int_0^T \int_R \left[-|z-c|\varphi_t + \mathrm{sign}(z-c)\cdot(g(z,u)-g(c,u))\,G(u;z)\varphi_u\right] du\,dt \\
&\leq \ -\int_0^T \int_R \mathrm{sign}(z-c)\cdot g(c,u)g(z,c)G(u;z)\varphi\,du\,dt \\
&\qquad + \int_0^T \sum_{i=1}^{N_v} |g(c,V_i-)-g(c,V_i+)|\,G(V_i;z)\varphi(t,V_i)\,dt \\
&\qquad - \int_0^T \int_{I\!R} \mathrm{sign}(z-c)\cdot\Theta(t,u)\,\psi_u\,du\,dt\,.
\end{aligned} \tag{5.9}
$$

Here $\{V_i\}$ *are the points where* $u \mapsto g$ *is discontinuous.*

Note that thanks to the time continuity property **(D1)**, one can take test functions φ that vanish on the boundary $t = 0$ and $t = T$.

Now we state the second main Theorem of the paper.

Theorem 5.3. *For the Cauchy problem* (1.1) *where* g *satisfies the assumptions **(A1)-(A5)**, there exists a unique entropy weak solution as in Definition 5.2. Furthermore, let* z, w *be the entropy weak solutions with initial data* $\bar{z}, \bar{w}$*, it holds*

$$
\|z(t,\cdot)-w(t,\cdot)\|_{\mathbf{L}^1} \leq e^{Ct}\,\|\bar{z}-\bar{w}\|_{\mathbf{L}^1}\,, \qquad t \in [0,T]\,. \tag{5.10}
$$

5.2 Front tracking algorithm

In [4], a flux splitting technique is used to treat the integro-differential equation and the constraint separately. Here we use a different approach. We adopt a front tracking algorithm where the constraint operator is instantly applied, an approach similar to [6].

When z approaches 0 on an interval $[u^-, u^+]$, a new wave type is formed. In the physical coordinate, this indicates that the profile $u(t,x)$ has a vertical drop at x, with the drop size $(u^+ - u^-)$. We refer to these waves as u-waves. Such a wave contains two fronts in the (t,u)-coordinates, one on the left and one on the right of the wave, denoted by $u^-(t)$ and $u^+(t)$.

We now derive the speeds of these fronts, based on the projection operator and its properties in (5.6). Denote the integral term G^-, G^+ at u^-, u^+ respectively, and let $z^- = z(t,u^-(t)-)$ and $z^+ = z(t,u^+(t)+)$. Also, denote g^-, g^+ the erosion function at u^-- and u^++. The two fronts u^-, u^+ travel with speeds

$$
\dot{u}^-(t) = -\frac{g^-(z^-)-g^-(0)}{z^-}G^- - \frac{\mu^-}{z^-}, \qquad \dot{u}^+(t) = -\frac{g^+(z^-)-g^+(0)}{z^+}G^+ + \frac{\mu^+}{z^+}\,, \tag{5.11}
$$

where μ^-, μ^+ satisfy the two conditions in (5.6), namely

$$\begin{aligned}
\mu^- + \mu^+ &= \int_{u^-}^{u^+} g^2(0,u) \cdot G(t,u)\, du\,, \\
\mu^-(u^+ - u^-) &= -\int_{u^-}^{u^+} \int_{u^-}^{u} g^2(0,v) \cdot G(t,v)\, dv\, du\,.
\end{aligned}$$

These two conditions uniquely determine the values of μ^-, μ^+, i.e.,

$$\mu^- = -\frac{1}{u^+ - u^-} \cdot \int_{u^-}^{u^+} \int_{u^-}^{u} g^2(0,v) G(t,v)\, dv\, du\,, \tag{5.12}$$

$$\mu^+ = \int_{u^-}^{u^+} g^2(0,u) \cdot G(t,u)\, du + \frac{1}{u^+ - u^-} \cdot \int_{u^-}^{u^+} \int_{u^-}^{u} g^2(0,v) G(t,v)\, dv\, du\,. \tag{5.13}$$

Since $g(0,u)$ is piecewise constant, these integrals can be easily computed.

We also enforce an admissibility condition for size of upward jump at u^+. We require

$$\dot{u}^+ \geq -g_z(z^+, u^+) \cdot G^+\,. \tag{5.14}$$

A similar admissibility condition was used in [6], to ensure that the characteristics remains at least tangent to the trajectory $u^+(t)$ from the righthand side of the u-shock. Such a condition would be satisfied in the solution of a flux splitting algorithm after one takes the limit $\Delta t \to 0+$.

The $(\tilde{\varepsilon}-\varepsilon)$-front tracking algorithm remains more or less the same as the the previous case, which is described in Section 4.1. The new feature here is the presence of the u-waves. The left and right fronts of the u-shock travels with the speed given in (5.12)-(5.13), which may interact with other fronts. There are four new phenomena.

- If a u-wave interacts with a z-wave, the z-wave will merge into the u-wave.
- If a u-wave approaches a g-wave, the g-wave will also merge into the u-wave.
- It also happens that z-rarefaction fronts would be added to the right of the right front of a u-wave, to ensure the admissibility condition (5.14).
- Finally, as u-wave passes through a g-wave, the g-wave could reappear outside the u-wave.

5.3 New wave strengths and interaction estimates

The definition of wave strength for z-waves remain unchanged, but we need to redefine the strength of the g-waves. Let z^-, z^+ be the left and right states of a g-wave, connecting the flux functions g^-, g^+. Since the projection operator is applied instantly, we have $z^-, z^+ \geq 0$. Recall the definitions in Section 4.2 for $(\hat{z}, \hat{g})$, (M^-, M^+) in (4.25), and $F(g)$ in (4.27). Then, the new definition for g-wave strength is

$$F^N(g) \doteq F(g) + \left| g^-(0) - g^+(0) \right| \,. \tag{5.15}$$

We also need to define the strength for the u-waves. Let $z^\varepsilon = 0$ on the interval $[u^-, u^+]$, with (z^-, z^+), (G^-, G^+) as the values at u^-, u^+, and let g^-, g^+ be the corresponding flux functions at u^-, u^+, respectively. Recall the definition of $\phi(z; g)$ in (4.2). We define

$$D^- \doteq \left|\phi(z^-; g^-) - \phi(0; g^-)\right|, \qquad D^+ \doteq \left|\phi(z^+; g^+) - \phi(0; g^+)\right| . \tag{5.16}$$

If $u^k \in [u^-, u^+]$ is a point where g is discontinuous, and (g^l, g^r) are the left and right flux functions around u^k, then we let

$$D_k \doteq 2M^- + 4M^+ + \left|g^l(0) - g^r(0)\right| . \tag{5.17}$$

If $\{u_k\}_{k=1}^{n_g}$ are the set of points where g is discontinuous on the interval $[u^-, u^+]$, we define the strength of this u-wave as

$$F(u) \doteq D^- + \sum_{k=1}^{n_g} D_k + D^+ . \tag{5.18}$$

We have the following Lemma on the interaction estimates.

Lemma 5.4. *The wave strength is unchanged at interactions between u-wave with z-wave, and u-wave with g-wave.*

Proof. When a u-wave (either left or right front) interacts with a z-wave, the z-wave merges into the u-wave, and only the part D_z will be effected in the strength of the new u-wave. Thanks to the definition of the z-wave (4.20), this is the same as the interaction of two z-waves, so the total strength is non-increasing.

We now discuss the case when a u-wave interacts with a g-wave. Let (g^l, g^r), (ϕ^l, ϕ^r), and (z^l, z^r) denote the erosion functions, ϕ functions, and z values on the left and right of the g-wave, respectively. We consider several cases.

(1). We consider a right u-front that approaches a g-wave on the left. Note that the g-front is merged into the u-wave after interaction, say it is number n_g of the g-fronts in the u-wave. Let g_{in}, u_{in} denote the incoming waves, and u_{out} the out-going u-waves.

The change in the strength of the u-wave is

$$F(u_{out}) - F(u_{in}) \quad = \quad \left[D^+_{out} - D^+_{in}\right] + D_{n_g} .$$

If $g^l(0) \geq g^r(0)$, then let $z^l_0 < 0$ be the unique value such that $g^l(z^l_0) = g^r(0)$. We have

$$D^+_{out} - D^+_{in} \quad = \quad [\phi^r(z^r) - \phi^r(0)] - \left[\phi^l(z^l) - \phi^l(z^l_0)\right] - [g^l(0) - g^r(0)] .$$

Now consider three difference cases for the incoming g-wave.

(i). If $-(g^l)'(z^l) < 0, -(g^r)'(z^r) \leq 0$, then

$$\phi^r(z^r) - \phi^r(0) = \phi^l(z^l) - \phi^l(z^l_0) ,$$

so

$$\begin{aligned} F(u_{out}) - F(u_{in}) &= -[g^l(0) - g^r(0)] + \Big|g^l(0) - g^r(0)\Big| + 2M^- + 4M^+ \\ &= 2M^- + 4M^+ \;=\; F(g_{in})\,. \end{aligned}$$

(ii). If $-(g^l)'(z^l) > 0, -(g^r)'(z^r) > 0$, then

$$[\phi^r(z^r) - \phi^r(0)] - \Big[\phi^l(z^l) - \phi^l(z_0^l)\Big] = 2(g_m^r - g_m^l) = 2(M^+ - M^-)\,,$$

so

$$\begin{aligned} F(u_{out}) - F(u_{in}) &= 2(M^+ - M^-) - [g^l(0) - g^r(0)] + \Big|g^l(0) - g^r(0)\Big| + 2M^- + 4M^+ \\ &= 4M^- + 2M^+ \;=\; F(g_{in})\,. \end{aligned}$$

(iii). If $-(g^l)'(z^l) < 0, -(g^r)'(z^r) > 0$, then

$$[\phi^r(z^r) - \phi^r(0)] - \Big[\phi^l(z^l) - \phi^l(z_0^l)\Big] = 2M^+\,,$$

so

$$\begin{aligned} F(u_{out}) - F(u_{in}) &= 2M^+ - [g^l(0) - g^r(0)] + \Big|g^l(0) - g^r(0)\Big| + 2M^- + 4M^+ \\ &= 4M^- + 4M^+ \;=\; F(g_{in})\,. \end{aligned}$$

In all three cases, the total wave strength is unchanged.

On the other hand, if $g^l(0) \le g^r(0)$, we then let $z_0^r < 0$ be the unique value such that $g^r(z_0^r) = g^l(0)$. Similarly, we have

$$D_{out}^+ - D_{in}^+ = [\phi^r(z^r) - \phi^r(z_0^r)] - \Big[\phi^l(z^l) - \phi^l(0)\Big] - [g^r(0) - g^l(0)]\,.$$

Going through the three cases of the incoming g-wave in a similar way, we conclude

$$F(u_{out}) - F(u_{in}) \;=\; [\phi^r(z^r) - \phi^r(0)] - \Big[\phi^l(z^l) - \phi^l(z_0^l)\Big] + 2M^- + 4M^+ \;=\; F(g_{in})\,,$$

and total wave strength is still unchanged.

(2). If a right u-front approaches a g-wave on the right, it is completely similar to Case (1), and we omit the details.

(3). We consider a left u-front approaches a g-wave on the left. Note that the g-front is contained in the u-wave before interaction, and it is the left most g-fronts, thus with index $k = 1$. This g-wave is "released" from the u-wave after interaction. Let $(g^l, g^r), (\phi^l, \phi^r)$ be the functions g, ϕ at the left and right of the g-front, respectively. We also let (z^l, z^r) be the left and right values for z around the g-wave after the interaction. Denote u_{in} and (u_{out}, g_{out}) as the incoming and out-going waves. We have

$$F(u_{in}) - F(u_{out}) = [D_{out}^- - D_{in}^+] + D_1 = [\phi^r(z^r) - \phi^r(0)] - \Big[\phi^l(z^l) - \phi^l(0)\Big] + D_1\,.$$

A similar discussion as the one for Case (1) for the cases $g^l(0) \ge g^r(0)$ and $g^l(0) \le g^r(0)$ now yields the result

$$F(u_{in}) - F(u_{out}) = F(g_{out})\,,$$

showing that the wave strength is unchanged at such interactions.

(4). The case when left u-front approaches a g-wave on the right is similar to Case (3). We omit the details. □

5.4 Existence and Uniqueness of Entropy Weak Solutions; Proof of Theorem 5.3

The convergence of the $(\tilde\varepsilon, \varepsilon)$-front tracking approximate solutions as $\varepsilon \to 0+$ now follows from a standard compactness argument, similar to that of the case where $z > 0$ in Section 4.6. We denote this limit as the $\tilde\varepsilon$-approximate solution $z^{\tilde\varepsilon}$. Thanks to the estimates in Lemma 5.4, the solution $z^{\tilde\varepsilon}$ satisfies the a-priori estimates in Lemma 4.17, with $B^{\tilde\varepsilon}(u) \equiv 0$ and the wave strength function F replaced by F^N. Combining Lemma 4.18 and the dissipative property (vi) of the projection operator π in Lemma 5.1, we obtain next Lemma, showing that $z^{\tilde\varepsilon}$ satisfies a discrete entropy condition.

Lemma 5.5. *There exists a measurable function $\Theta^{\tilde\varepsilon} = \Theta^{\tilde\varepsilon}(t,u)$ with compact support in $[0,T]$, that satisfies (5.8) with z replaced by $z^{\tilde\varepsilon}$. Furthermore, the $\tilde\varepsilon$-approximate solution $z^{\tilde\varepsilon}$ satisfies the following Kruzhkov-type entropy inequality for all constants c and all non-negative test functions φ,*

$$\begin{aligned}
&\int_0^T \int_R \Big[-|z^{\tilde\varepsilon} - c|\varphi_t + \mathrm{sign}(z^{\tilde\varepsilon} - c) \cdot (g^{\tilde\varepsilon}(z^{\tilde\varepsilon}, u) - g^{\tilde\varepsilon}(c,u))\tilde G(u; z^{\tilde\varepsilon}) \cdot \varphi_u \Big]\, du\, dt \\
&\qquad\le \int_0^T \int_R \mathrm{sign}(z^{\tilde\varepsilon} - c) \cdot g^{\tilde\varepsilon}(c,u) \tilde G(u; z^{\tilde\varepsilon})_u \varphi\, du\, dt \\
&\qquad\quad + \int_0^T \sum_j \left| g^{\tilde\varepsilon}(c, U_j-) - g^{\tilde\varepsilon}(c, U_j+) \right| \tilde G(U_j; z^{\tilde\varepsilon}) \varphi(t, U_j)\, dt + C\tilde\varepsilon \\
&\qquad\quad - \int_0^T \int_{I\!R} \mathrm{sign}(z^{\tilde\varepsilon} - c) \cdot \Theta^{\tilde\varepsilon}(t,u)\, \varphi_u\, du\, dt\,.
\end{aligned} \tag{5.19}$$

where $\tilde G(u; z^{\tilde\varepsilon})$ is the linear spline interpolation of the integral term $G^{\tilde\varepsilon}$ at the knots $U_j \in \mathcal{J}$.

Finally, taking the limit $\tilde\varepsilon \to 0+$, thanks to the a-priori estimates, we obtain the existence of entropy weak solutions.

For the uniqueness, one can apply a standard Kruzhkov analysis, combining the analysis in Section 4.7 and [4] Section 6 (to treat the term with Θ), reaching the result

$$\|\hat z(t,\cdot) - z(t,\cdot)\|_{\mathbf{L}^1} \le e^{Ct} \cdot \|\hat z(0,\cdot) - z(0,\cdot)\|_{\mathbf{L}^1}\ , \tag{5.20}$$

where $z(t,u)$ and $\hat z(t,u)$ are two entropy weak solutions.

References

[1] D. Amadori and W. Shen, The slow erosion limit in a model of granular flow, *Arch. Rational Mech. Anal.*, **199** (2011), 1–31.

[2] D. Amadori and W. Shen, Front tracking approximations for slow erosion, *Discr. Contin. Dyn. Syst.*, **32** (2012), 1481–1502.

[3] D. Amadori and W. Shen, An integro-differential conservation law arising in a model of granular flow, *J. Hyp. Diff. Eq.*, **9** (2012), 105–131.

[4] A. Bressan and W. Shen, A Semigroup Approach to an Integro-Differential Equation Modeling Slow Erosion, *J. Differential Equations* **257** (2014), 2360-2403.

[5] G.M. Coclite and N.H. Risebro, Conservation laws with time dependent discontinuous coefficients. *SIAM J. Math. Anal.* **36** (2005), 1293–1309.

[6] R. M. Colombo, G. Guerra, and W. Shen, Lipschitz semigroup for an integro–differential equation for slow erosion, *Quart. Appl. Math.*, **70** (2012), 539–578.

[7] T. Gimse and N. H. Risebro, Riemann problems with a discontinuous flux function, in *Proceedings of Third Internat. Conf. on Hyperbolic Problems*, Studentlitteratur, Lund, pp. 488-502.

[8] T. Gimse and N. H. Risebro, Solution of the Cauchy problem for a conservation law with a discontinuous flux function, *SIAM J. Math. Anal.*, **23** (1992), pp. 635–648.

[9] G. Guerra and W. Shen, Existence and stability of traveling waves for an integro-differential equation for slow erosion. *J. Differential Equations* **256** (2014), 253–282.

[10] K.P. Hadeler and C. Kuttler, Dynamical models for granular matter, *Granular Matter*, **2** (1999), 9–18.

[11] E. Isaacson and B. Temple, Nonlinear resonance in inhomogeneous systems of conservation laws. Mathematics of nonlinear science (Phoenix, AZ, 1989), *Contemp. Math.*, **108**, Amer. Math. Soc., Providence, RI, (1990), pp 63?77.

[12] S. Kruzhkov, First-order quasilinear equations with several space variables, *Math. USSR Sb.* **10** (1970), 217–273.

[13] W. Shen and T. Y. Zhang, Erosion profile by a global model for granular flow, *Arch. Rational Mech. Anal.* **204** (2012), 837–879.

[14] B. Temple, Global solution of the Cauchy problem for a class of 2x2 non-strictly hyperbolic conservation laws, *Adv. in Appl. Math.*, **3** (1982), pp. 335–375.