ANALYSIS OF A SPLITTING METHOD FOR STOCHASTIC BALANCE LAWS

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ABSTRACT. We analyze a semi-discrete splitting method for conservation laws driven by a semilinear noise term. Making use of fractional BV estimates, we show that the splitting method generates approximate solutions converging to the exact solution, as the time step $\Delta t \to 0$. Under the assumption of a homogenous noise function, and thus the availability of BV estimates, we provide an L^1 error estimate. Bringing into play a generalization of Kružkov's entropy condition, permitting the "Kružkov constants" to be Malliavin differentiable random variables, we establish an L^1 convergence rate of order $\frac{1}{3}$ in Δt.

CONTENTS

1. INTRODCUTION

Recently there have been many works studying the effect of stochastic forcing on scalar conservation laws [3, 4, 6, 8, 9, 10, 16, 20, 19, 12, 36, 35], with emphasis on existence, uniqueness, and stability questions. Deterministic conservation laws exhibit shocks (discontinuous solutions), and a weak formulation coupled with an appropriate entropy condition is required to ensure the well-posedness [23]. The question of uniqueness gets somewhat more difficult by adding a stochastic source term, due to the interaction between noise and nonlinearity. A pathwise theory for conservation laws with stochastic fluxes have been developed in [14, 15, 25, 26].

In this paper we are interested in the convergence of approximate solutions to conservation laws driven by a multiplicative Wiener noise term, i.e., stochastic balance laws of the form

$$du + \operatorname{div} f(u)\, dt = \sigma(x, u)\, dB, \qquad (t, x) \in \Pi_T, \tag{1.1}$$

with initial data:

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{1.2}$$

We denote by ∇ and div $= \nabla\cdot$ the spatial gradient and divergence, respectively. Moreover, $\Pi_T = \mathbb{R}^d \times (0,T)$ for some fixed final time $T > 0$, and $u(x,t)$ is the scalar unknown function that is sought. The random force in (1.1) is driven by a Wiener process $B = B(t) = B(t,\omega)$, $\omega \in \Omega$, over a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, P)$, where P is a probability measure, $\mathscr{F}$ is a σ-algebra, and $\{\mathscr{F}_t\}_{t \geq 0}$ is a right-continuous filtration on $(\Omega, \mathscr{F})$ such that $\mathscr{F}_0$ contains all the P–negligible subsets.

The convection flux $f : \mathbb{R} \to \mathbb{R}^d$ satisfies

$$f \text{ is (globally) Lipschitz continuous on } \mathbb{R}. \qquad (\mathcal{A}_f)$$

Furthermore, we will sometimes make use of the assumption

$$f'' \text{ is uniformly bounded on } \mathbb{R}. \qquad (\mathcal{A}_{f,1})$$

The noise coefficient $\sigma : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy

$$\|\sigma\|_{\text{Lip}} = \sup_{x \in \mathbb{R}^d} \sup_{u \neq v} \left\{ \frac{|\sigma(x,u) - \sigma(x,v)|}{|u-v|} \right\} < \infty, \quad |\sigma(\cdot,0)| \in L^\infty(\mathbb{R}^d). \qquad (\mathcal{A}_\sigma)$$

These assumptions imply

$$|\sigma(x,u) - \sigma(x,v)| \leq \|\sigma\|_{\text{Lip}} |u-v|,$$
$$|\sigma(x,u)| \leq \max\left\{ \|\sigma\|_{\text{Lip}}, \|\sigma(\cdot,0)\|_{L^\infty(\mathbb{R}^d)} \right\} (1+|u|).$$

Furthermore, we often assume the existence of constants M_σ and κ_σ such that

$$|\sigma(x,u) - \sigma(y,u)| \leq M_\sigma |x-y|^{\kappa_\sigma + 1/2} (1+|u|), \qquad \kappa_\sigma \in (0, 1/2]. \qquad (\mathcal{A}_{\sigma,1})$$

A prevailing difficulty affecting convergence/error analysis is related to the time discretization and the interplay between noise and nonlinearity. Up to now there are only a few studies investigating this problem. Holden and Risebro [17] study a one-dimensional equation with bounded initial data and a compactly supported, homogeneous noise function $\sigma = \sigma(u)$, ensuring L^∞-bounds on the solution. An operator splitting method is used to construct approximate solutions, and it is shown that a subsequence of these approximations converges to a (possible non-unique) weak solution. Recently this work was generalized to stochastic entropy solutions and extended to the multi-dimensional case by Bauzet [1]. Kröker and Rohde [21] analyze semi-discrete (time continuous) finite volume methods. They use the compensated compactness method to prove convergence to a stochastic entropy solution for one-dimensional equations, with non-homogeneous noise function $\sigma = \sigma(x,u)$. Bauzet, Charrier, and Galloüet [2] analyze fully discrete finite volume methods for multi-dimensional equations, with homogeneous noise function $\sigma = \sigma(u)$. Their proof relies on weak BV (energy) estimates and a uniqueness result for measure-valued stochastic entropy solutions.

In this paper, as in [17, 1], we will investigate the semi-discrete splitting method for calculating approximations to stochastic entropy solutions of (1.1). Roughly speaking, this method is based on "splitting off" the effect of the stochastic source $\sigma(x,u)\,dB$. This Godunov-type operator splitting can be used to extend sophisticated numerical methods for deterministic conservation laws to stochastic balance laws. Generally speaking, the tag "operator splitting" refers to the well-known idea of constructing numerical methods for complicated partial differential equations by reducing them to a progression of simpler equations, each of which can be solved by some tailor-made numerical method. The operator splitting approach is described in a large number of articles and books. We do not survey the literature here, referring the reader instead to the bibliography in [18]. The main focus of the book [18] is on convergence results, within classes of discontinuous functions, for general splitting algorithms for deterministic nonlinear partial differential equations.

Compared to the earlier results of Holden-Risebro and Bauzet [1, 17], the main contributions of the present paper are twofold. First, we establish convergence of the splitting approximations to a stochastic entropy solution in the case of non-homogeneous noise functions $\sigma = \sigma(x, u)$. Whenever σ has a dependency on the spatial position x, BV estimates are no longer available and the approach resorted to in [17, 1] does not apply. Following an idea laid out in [6], and independently in [8], we derive a fractional BV_x estimate, which, via an interpolation argument *à la* Kružkov, is turned into a temporal equicontinuity estimate. These a priori estimates, along with Young measures and an earlier uniqueness result, are used to show that splitting approximations converge to a stochastic entropy solution.

Let us make a few comments about the convergence proof. In the deterministic case, the spatial and temporal estimates would imply strong (L^1) compactness of the splitting approximations. In the stochastic setting, we have the randomness variable ω for which there is no compactness; as a matter of fact, possible "oscillations" in ω may prevent strong compactness. In the literature, the standard way of dealing with this issue is to look for tightness (weak compactness) of the probability laws of the approximations. Then an application of the Skorokhod representation theorem provides a new probability space and new random variables, with the same laws as the original variables, that do converge strongly (almost surely) in ω to some limit. Equipped with almost sure convergence, it is not difficult to show that the limit variable is a so-called martingale solution, i.e., the limit is probabilistic weak in the sense that the stochastic basis is now viewed as part of the solution. One can pass (*à la* Yamada & Watanabe) from martingale to pathwise solutions provided there is a strong uniqueness result. In the present paper we will *not* follow this "traditional" approach. Instead we will utilize Young measures, parametrized over (t, x, ω), to represent weak limits of nonlinear functions, thereby obtaining weak convergence of the splitting approximations towards a so-called Young measure-valued stochastic entropy solution. We use the spatial and temporal translation estimates to conclude that the limit is a solution in this sense. Weak convergence is then upgraded to strong convergence in (t, x, ω) *a posteriori*, thanks to the fact that these measure-valued solutions are L^1 stable (unique). After the works of Tartar, DiPerna, and others, weak compactness arguments of this type (propagation of compactness) are frequently used in the nonlinear PDE literature, cf., e.g., [11, 27, 30, 33], and recently in the context of stochastic equations [1, 2, 3, 4, 19, 36].

Our second main contribution is an L^1 error estimate of the form $O(\Delta t^{\frac{1}{3}})$, for homogeneous noise functions $\sigma = \sigma(u)$. Except for the expected convergence rate for the vanishing viscosity method [6], this appears to be the first error estimate derived for approximate solutions to stochastic conservation laws. The rate $\frac{1}{3}$ should be compared to the first order convergence rate available for conservation laws with deterministic source [24]. Our proof relies on BV estimates and a generalization of the Kružkov entropy condition, allowing the "Kružkov constants" to be Malliavin differentiable random variables, which was put forward in the recent work [19].

The remaining part of this paper is organized as follows: Section 2 collects some preliminarily material along with the relevant notion of (stochastic entropy) solution. The operator splitting method is defined precisely in Section 3. A series of a priori estimates are derived in Section 4, which are subsequently used in Section 5 to prove convergence towards a stochastic entropy solution. Section 6 is devoted to the proof of the error estimate. Section 7 is an appendix collecting some definitions and useful results used elsewhere in the paper.

2. Preliminaries

In this article, as in [19], we apply certain weighted L^p spaces. Since we do *not* assume $\sigma(x,0) \equiv 0$, weighted spaces on $\mathbb{R}^d$ provide a convenient alternative to working on the torus as in [8, 10]. The weights used herein turns out to be suitable also for the fractional BV_x estimates, cf. Proposition 4.4.

Let $\mathfrak{N}$ be the set of all nonzero $\phi \in C^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for which there exists a constant C such that $|\nabla \phi| \leq C\phi$. An example is $\phi(x) = e^{-\sqrt{1+|x|^2}}$. Set

$$C_\phi = \inf \left\{ C \mid |\nabla \phi| \leq C\phi \right\}.$$

For $\phi \in \mathfrak{N}$, we use the weighted L^p-norm $\|\cdot\|_{p,\phi}$ defined by

$$\|u\|_{p,\phi} := \left(\int_{\mathbb{R}^d} |u(x)|^p \phi(x)\, dx \right)^{1/p}.$$

The corresponding weighted L^p-space is denoted by $L^p(\mathbb{R}^d, \phi)$. Similarly, we define

$$\|u\|_{\infty,\phi^{-1}} := \sup_{x \in \mathbb{R}^d} \left\{ \frac{|u(x)|}{\phi(x)} \right\}, \quad u \in C(\mathbb{R}^d). \tag{2.1}$$

Some useful results regarding functions in $\mathfrak{N}$ are collected in Section 7.2.

We denote by $\mathscr{E}$ the set of non-negative convex functions in $C^2(\mathbb{R})$ such that S' is bounded and S'' compactly supported. A pair of functions (S, Q) is called an entropy/entropy-flux pair if $S : \mathbb{R} \to \mathbb{R}$ is C^2 and $Q = (Q_1, \ldots, Q_d) : \mathbb{R} \mapsto \mathbb{R}^d$ satisfies $Q' = S'f'$. An entropy/entropy-flux pair (S, Q) is said to belong to $\mathscr{E}$ if S belongs to $\mathscr{E}$.

Let $\mathscr{P}$ denote the predictable σ-algebra on $[0, T] \times \Omega$ with respect to $\{\mathscr{F}_t\}$, see, e.g., [7, § 2.2]. In general we are working with eqiuvalence classes of functions with respect to the measure $dt \otimes dP$. The equivalence class u is said to be *predictable* if it has a version $\tilde{u}$ that is $\mathscr{P}$-measurable. Equivalently, we could ask for any representative to be $\mathscr{P}^*$ measurable, where $\mathscr{P}^*$ is the completion of $\mathscr{P}$ with respect to $dt \otimes dP$. Note that any (jointly) measurable and adapted process is $\mathscr{P}^*$-measurable, cf., e.g., [7, Theorem 3.7].

Next we collect some basic material related to Malliavin calculus. We refer to [29] for an introduction to the topic. The Malliavin calculus is developed with respect to the isonormal Gaussian process $W : L^2([0, T]) \to \mathcal{H}^1$, defined by $W(h) := \int_0^T h\, dB$. Here $\mathcal{H}^1$ is the subspace of $L^2(\Omega, \mathscr{F}, P)$ consisting of zero-mean Gaussian random variables. We denote by $\mathcal{S}$ the class of smooth random variables of the form

$$V = f(W(h_1), \ldots, W(h_n)),$$

where $f \in C_c^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n \in L^2([0, T])$ and $n \geq 1$. For such random variables, the Malliavin derivative is defined by

$$DV = \sum_{i=1}^n \partial_i f(W(h_1), \ldots, W(h_n)) h_i,$$

where ∂_i denotes the derivative with respect to the i-th variable. The space $\mathcal{S}$ is dense in $L^2(\Omega, \mathscr{F}, P)$. Furthermore, the operator D is closable as a map from $L^2(\Omega)$ to $L^2(\Omega; L^2([0, T]))$ [29, Proposition 1.2.1]. The domain of D in $L^2(\Omega)$ is denoted by $\mathbb{D}^{1,2}$. That is, $\mathbb{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|V\|_{\mathbb{D}^{1,2}} = \left\{ E\left[|V|^2 \right] + E\left[\|DV\|_{L^2([0,T])}^2 \right] \right\}^{1/2}.$$

For the generalization of the above notations and results to Hilbert space-valued random variables, see [29, Remark 2, p.31].

We use the notion of stochastic entropy solution introduced in [19], which is a refinement of the notion introduced by Feng and Nualart [12].

Definition 2.1. Fix $\phi \in \mathfrak{N}$. A stochastic entropy solution u of (1.1)–(1.2) with $u_0 \in L^2(\Omega, \mathscr{F}_0, P; L^2(\mathbb{R}^d, \phi))$, is a stochastic process

$$u = \{u(t) = u(t, x) = u(t, x; \omega)\}_{t \in [0, T]}$$

satisfying the following conditions:

(i) u is a predictable process in $L^2([0, T] \times \Omega; L^2(\mathbb{R}^d, \phi))$.

(ii) For any random variable $V \in \mathbb{D}^{1,2}$ and any entropy, entropy-flux pair $(S, Q) \in \mathscr{E}$,

$$E\left[\iint_{\Pi_T} S(u - V)\partial_t \varphi + Q(u, V) \cdot \nabla \varphi \, dxdt + \int_{\mathbb{R}^d} S(u_0(x) - V)\varphi(0, x) \, dx\right]$$

$$- E\left[\iint_{\Pi_T} S''(u - V)\sigma(x, u)D_t V \varphi \, dxdt\right]$$

$$+ \frac{1}{2}E\left[\iint_{\Pi_T} S''(u - V)\sigma(x, u)^2 \varphi \, dxdt\right] \geq 0,$$

for all non-negative $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^d)$.

Here $L^2([0, T] \times \Omega; L^2(\mathbb{R}^d, \phi))$ denotes the Lebesgue-Bochner space and $D_t V$ denotes the Malliavin derivative of V evaluated at time t. By [19, Lemma 2.2] it suffices to consider $V \in \mathcal{S}$ in (ii). In [19], the existence and uniqueness of entropy solutions in the sense of Definition 2.1 is established under assumptions $(\mathcal{A}_f)$, $(\mathcal{A}_\sigma)$, and $(\mathcal{A}_{\sigma,1})$. We also mention that whenever $u^0 \in L^p(\Omega; L^p(\mathbb{R}^d, \phi))$ with $2 \leq p < \infty$,

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \left\{ E\left[\|u(t)\|_{p,\phi}^p\right] \right\} < \infty.$$

Let $\{J_\delta\}_{\delta > 0}$ be a sequence of symmetric mollifiers on $\mathbb{R}^d$, i.e.,

$$J_\delta(x) = \frac{1}{\delta^d} J\left(\frac{x}{\delta}\right), \tag{2.2}$$

where $J \geq 0$ is a smooth, symmetric function satisfying $\operatorname{supp}(J) \subset B(0, 1)$ and $\int J = 1$. For $d = 1$, we set $J^+(x) = J(x - 1)$, so that $\operatorname{supp}(J^+) \subset (0, 2)$.

Under the additional assumption $(\mathcal{A}_{\sigma,1})$, [19, Proposition 5.2] asserts that the entropy solution u satisfies

$$E\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u(t, x + z) - u(t, x - z)| J_r(z)\phi(x) \, dx\right]$$

$$\leq e^{C_\phi \|f\|_{\mathrm{Lip}} t} E\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_0(x + z) - u_0(x - z)| J_r(z)\phi(x) \, dx\right] + \mathcal{O}(r^{\kappa_\sigma}), \tag{2.3}$$

where κ_σ is given in $(\mathcal{A}_{\sigma,1})$. Whenever $\sigma(x, u) = \sigma(u)$, the last term on the right-hand side vanishes, i.e., $\mathcal{O}(\ldots) = 0$.

3. Operator splitting

We will now describe the basic operator splitting method for (1.1). Let $\mathcal{S}_{\mathrm{CL}}(t)$ be the solution operator that maps an initial function $v_0(x)$ to the unique entropy solution of the deterministic conservation law

$$\partial_t v + \operatorname{div} f(v) = 0, \qquad v(0, x) = v_0(x), \tag{3.1}$$

that is, if $v(t) := \mathcal{S}_{\mathrm{CL}}(t)v_0$, then v is the unique entropy solution of (3.1). More precisely, for each $\tau \in [0, T]$,

$$\int_{\mathbb{R}^d} |v^0(x) - c| \, \varphi(0, x) \, dx - \int_{\mathbb{R}^d} |v(\tau) - c| \, \varphi(\tau, x) \, dx$$

$$+ \int_0^\tau \int_{\mathbb{R}^d} |v - c| \, \partial_t \varphi + \operatorname{sign}(v - c)\left(f(v) - f(c)\right) \cdot \nabla \varphi \, dxdt \geq 0,$$

for all $c \in \mathbb{R}$ and all non-negative $\varphi \in C_c^\infty([0,T] \times \mathbb{R})$. Note that the integrals are well defined due to the global Lipschitz assumption $(\mathcal{A}_f)$. Recall that the entropy solution has a version that belongs to $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$ [5]. As we frequently need to consider the evaluation $v(t)$ it is convenient for us to assume that v has this property. Let $u, v \in L^1(\mathbb{R}^d, \phi)$ where $\phi \in \mathfrak{N}$. Then, for any $t \in [0,T]$,

$$\|\mathcal{S}_{\text{CL}}(t)v - \mathcal{S}_{\text{CL}}(t)u\|_{1,\phi} \le e^{C_\phi \|f\|_{\text{Lip}} t} \|u - v\|_{1,\phi}.$$

Suppose $u \in L^1(\Omega, \mathscr{F}_s, P; L^1(\mathbb{R}^d, \phi))$ for some $s \in [0,T]$. Let $s \le t \le T$. By considering the composition $\Omega \ni \omega \mapsto u(\omega) \mapsto \mathcal{S}_{\text{CL}}(t-s)u(\omega)$, it follows that $\mathcal{S}_{\text{CL}}(t-s)u$ is $\mathscr{F}_s$-measurable as an element in $L^1(\mathbb{R}^d, \phi)$, cf. [28, § 3.3].

Similarly, for $s \le t \le T$, we let $\mathcal{S}_{\text{SDE}}(t,s)$ denote the two-paramater semigroup defined by $\mathcal{S}_{\text{SDE}}(t,s)w^s = w(t)$, where w is the strong solution of

$$w(t,x) = w^s(x) + \int_s^t \sigma(x, w(r,x)) \, dB(r).$$

Suppose $w^s, v^s \in L^1(\Omega, \mathscr{F}_s, P; L^1(\mathbb{R}^d, \phi))$. Then

$$E\left[\|\mathcal{S}_{\text{SDE}}(t,s)w^s - \mathcal{S}_{\text{SDE}}(t,s)v^s\|_{1,\phi}\right] = E\left[\|w^s - v^s\|_{1,\phi}\right]. \tag{3.2}$$

To see this, let $S_\delta \to |\cdot|$ as $\delta \downarrow 0$ and consider the quantity $S_\delta(w(t,x) - v(t,x))$. Next, apply Itô's formula, multiply by ϕ and let $\delta \downarrow 0$. Due to (3.2),

$$\mathcal{S}_{\text{SDE}}(\cdot, s) : L^1(\Omega, \mathscr{F}_s, P; L^1(\mathbb{R}^d, \phi)) \to L^1([s,T] \times \Omega, \mathscr{P}_{[s,T]}, dt \otimes dP; L^1(\mathbb{R}^d, \phi)),$$

where $\mathscr{P}_{[s,T]}$ denotes the predictable σ-algebra relative to $\{\mathscr{F}_t\}_{s \le t \le T}$ on $[s,T] \times \Omega$.

Fix $N \in \mathbb{N}$, specify $\Delta t = T/N$, and set $t_n = n\Delta t$. The operator splitting for (1.1), with initial condition $u^0 = u^0(x;\omega)$, is the sequence $\{u^n = u^n(x;\omega)\}_{n=0}^N$ defined recursively by

$$u^{n+1}(x;\omega) = [\mathcal{S}_{\text{SDE}}(t_{n+1}, t_n; \omega) \circ \mathcal{S}_{\text{CL}}(\Delta t)] u^n(x;\omega), \tag{3.3}$$

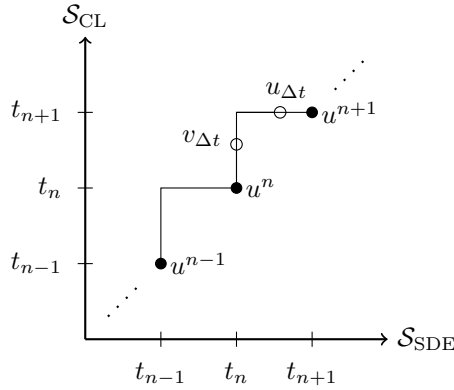for $n = 0, 1, \ldots, N-1$. A graphical representation of the splitting is given in Figure 1.

FIGURE 1. A graphical representation of $\{u^n\}$, $u_{\Delta t}$, $v_{\Delta t}$.

To investigate the convergence of the semi-discrete splitting algorithm (3.3), we need to work with functions that are not only defined for each $t_n = n\Delta t$, but in the entire interval $[0,T]$. To this end, we introduce two different "time-interpolants" $u_{\Delta t}(t) = u_{\Delta t}(t,x;\omega)$ and $v_{\Delta t}(t) = v_{\Delta t}(t,x;\omega)$, defined for $n = 0, \ldots, N-1$ by

$$u_{\Delta t}(t) = \mathcal{S}_{\text{SDE}}(t, t_n) \circ \mathcal{S}_{\text{CL}}(\Delta t)u^n, \qquad t \in (t_n, t_{n+1}], \tag{3.4}$$

and

$$v_{\Delta t}(t) = \mathcal{S}_{\mathrm{CL}}(t - t_n)u^n, \qquad t \in [t_n, t_{n+1}), \tag{3.5}$$

respectively, cf. Figure 1. As $u_{\Delta t}$ is discontinuous at t_n we introduce the right limit $u_{\Delta t}((t_n)+) = \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$. Similarly, let $v_{\Delta t}((t_{n+1})-) = \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$.

4. A priori estimates

To establish the convergence of $\{u_{\Delta t}\}_{\Delta t > 0}, \{v_{\Delta t}\}_{\Delta t > 0}$ we will need a series of a priori estimates. These are also crucial when deriving the error estimate. The following result explains the introduction of the weight functions $\mathfrak{N}$.

Proposition 4.1 (Local L^p estimates). *Suppose $(\mathcal{A}_f)$ and $(\mathcal{A}_\sigma)$ are satisfied, $2 \leq p < \infty$ and $M \geq \|f\|_{\mathrm{Lip}}$. Let $\{u^n\}$ be the splitting solutions defined by (3.3), with initial condition $u^0 \in L^p(\Omega, \mathscr{F}_0, P; L^p_{\mathrm{loc}}(\mathbb{R}^d))$. For $t \in (0, T)$ and $R > 0$, set $\Gamma(t) = \max\{0, R - Mt\}$. Suppose $\phi \in C^1(\mathbb{R})$ is non-negative and satisfies $|\nabla \phi| \leq C_\phi \phi$. Then there exist constants C_1 and C_2 depending only on p, σ, f, C_ϕ such that*

$$E\left[\int_{B(0,\Gamma(t_n))} |u^n(x)|^p \phi(x)\, dx\right] \leq e^{C_1 t_n} E\left[\int_{B(0,R)} |u^0(x)|^p \phi(x)\, dx\right]$$
$$+ C_2 t_n e^{C_1 t_n} \int_{B(0,R)} \phi(x)\, dx. \tag{4.1}$$

If $\sigma(x,0) = 0$, then $C_2 = 0$. Here, $B(0,R)$ denotes the open ball with radius R centered at 0.

Remark 4.2. Suppose $\phi \in \mathfrak{N}$ and $u^0 \in L^p(\Omega; L^p(\mathbb{R}^d, \phi))$. Then $\phi \in L^1(\mathbb{R}^d)$ and the right hand side of (4.1) is bounded independently of $R > 0$. It follows that $u^n \in L^p(\Omega; L^p(\mathbb{R}^d, \phi))$.

Proof. 1. Deterministic step. We want to prove the following: With $1 \leq p < \infty$, let $v^0 \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and $v(t) = \mathcal{S}_{\mathrm{CL}}(t)v^0$. Then, for any $0 < \tau \leq T$,

$$\int_{B(0,\Gamma(\tau))} |v(\tau,x)|^p \phi(x)\, dx \leq e^{\|f\|_{\mathrm{Lip}} C_\phi t} \int_{B(0,R)} |v^0(x)|^p \phi(x)\, dx. \tag{4.2}$$

We might as well assume $\Gamma(\tau) > 0$. As v is an entropy solution of (3.1),

$$\iint_{\Pi_T} \int_{\mathbb{R}^d} S(v(t,x))\partial_t \varphi + Q(v(t,x)) \cdot \nabla_x \varphi\, dx dt + \int_{\mathbb{R}} S(v^0(x))\varphi(0,x)\, dx \geq 0, \tag{4.3}$$

for all nonnegative $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^d)$, for any convex $S \in C^2$ with S' bounded and $Q' = S'f'$. Let $0 < \delta < \min\{\Gamma(\tau), \frac{1}{2}\tau\}$. Take $\varphi(t,x) = \psi_\delta(t)H_\delta(\Gamma(t), |x|)\phi(x)$, where

$$\psi_\delta(t) = 1 - \int_0^t J_\delta^+(\tau - \zeta)\, d\zeta \quad \text{and} \quad H_\delta(L, r) = \int_{-\delta}^L J_\delta(\zeta - r)\, d\zeta.$$

Under the assumption $\phi \in C^\infty(\mathbb{R}^d)$ it follows that φ is a non-negative function in $C_c^\infty([0,T) \times \mathbb{R}^d)$. However, by approximation, it suffices with $\phi \in C^1(\mathbb{R}^d)$ for inequality (4.3) to hold true. Recall that $\frac{d}{dt}\Gamma(t) = -M$ for all $0 \leq t \leq \tau$ and observe that

$$\partial_t \varphi(t,x) = -J_\delta^+(\tau - t)H_\delta(\Gamma(t), |x|)\phi(x) - M\psi_\delta(t)J_\delta(\Gamma(t) - |x|)\phi(x),$$

$$\nabla \varphi(t,x) = -\psi_\delta(t)J_\delta(\Gamma(t) - |x|)\frac{x}{|x|}\phi(x) + \psi_\delta(t)H_\delta(\Gamma(t), |x|)\nabla \phi(x).$$

Hence,

$$\int_{\mathbb{R}} S(v^0(x))H_\delta(R,|x|)\phi(x)\,dx \geq \iint_{\Pi_T} S(v(t,x))J_\delta^+(\tau-t)H_\delta(\Gamma(t),|x|)\phi(x)\,dxdt$$

$$+ \underbrace{\iint_{\Pi_T} \left(Q(v(t,x))\cdot\frac{x}{|x|} + MS(v(t,x)) \right)\psi_\delta(t)J_\delta(\Gamma(t)-|x|)\phi(x)\,dxdt}_{\mathscr{T}^1}$$

$$- \underbrace{\iint_{\Pi_T} Q(v(t,x))\psi_\delta(t)H_\delta(\Gamma(t),|x|)\cdot\nabla\phi(x)\,dxdt}_{\mathscr{T}^2}.$$

$$(4.4)$$

Suppose $S'(0) = S(0) = 0$. Then

$$|Q(v)| = \left| \int_0^v S'(z)f'(z)\,dz \right| \leq \|f\|_{\mathrm{Lip}}\,S(v).$$

It follows as $M \geq \|f\|_{\mathrm{Lip}}$ that $\mathscr{T}^1 \geq 0$. Due to the assumption on ϕ,

$$\left| \mathscr{T}^2 \right| \leq \|f\|_{\mathrm{Lip}}\,C_\phi \iint_{\Pi_T} S(v)\psi_\delta(t)H_\delta(R,|x|)\phi(x)\,dxdt.$$

Sending $\delta \downarrow 0$, inequality (4.4) then takes the form

$$X(\tau) \leq X(0) + \|f\|_{\mathrm{Lip}}\,C_\phi \int_0^\tau X(r)\,dr,$$

where

$$X(t) = \int_{B(0,\Gamma(t))} S(v(t,x))\phi(x)\,dx.$$

Next, apply Grönwall's inequality. The estimate (4.2) follows upon letting $S \to |\cdot|^p$ and applying the dominated convergence theorem.

2. Stochastic step. We want to prove the following: Fix $2 \leq p < \infty$. Suppose $w(s) \in L^p(\Omega,\mathscr{F}_s,P;L^p_{\mathrm{loc}}(\mathbb{R}))$ and take $w(t) = \mathcal{S}_{\mathrm{SDE}}(t,s)w(s)$ for $s \leq t$. For any $R > 0$ there exist constants C_3 and C_2 depending only on p and σ such that

$$E\left[\int_B |w(t,x)|^p\,\phi(x)\,dx \right] \leq e^{C_3(t-s)}\Big(E\left[\int_B |w(s,x)|^p\,\phi(x)\,dx \right]$$

$$+ C_2(t-s)\int_B \phi(x)\,dx \Big). \quad (4.5)$$

If $\sigma(x,0) = 0$, then $C_2 = 0$.

By Ito's lemma,

$$dS(w) = \frac{1}{2}S''(w)\sigma(x,w)^2\,dt + S'(w)\sigma(x,w)\,dB,$$

for any $S \in C^2$. Without loss of generality, we can assume $p = 2,4,6,\ldots$. Taking $S(u) = |u|^p$, multiplying by ϕ, and integrating over $B = B(0,R)$, we arrive at

$$E\left[\int_B |w(t,x)|^p\,\phi(x)\,dx \right] - E\left[\int_B |w(s,x)|^p\,\phi(x)\,dx \right]$$

$$\leq \frac{p(p-1)}{2}\int_s^t E\left[\int_B w(r,x)^{p-2}\sigma(x,w(r,x))^2\phi(x)\,dx \right]\,dr.$$

Recall that $\sigma(x,w) \leq |\sigma(x,0)| + \|\sigma\|_{\mathrm{Lip}}\,|w|$. Hence, according to assumption $(\mathcal{A}_\sigma)$,

$$\mathscr{T}^3 := \frac{p(p-1)}{2}E\left[\int_B w(r,x)^{p-2}\sigma(x,w(r,x))^2\phi(x)\,dx \right]$$

$$\leq p(p-1)\Big(\|\sigma(\cdot,0)\|_\infty^2 E\left[\int_B |w(r,x)|^{p-2}\,\phi(x)\,dx\right]$$
$$+\|\sigma\|_{\mathrm{Lip}}^2 E\left[\int_B |w(r,x)|^p\,\phi(x)\,dx\right]\Big).$$

Applying Hölder's inequlity with $\theta=\frac{p}{p-2}$ and $\theta'=\frac{p}{2}$,

$$\int_B \underbrace{(|w(r,x)|^p\,\phi(x))^{\frac{1}{\theta}}\,\phi(x)^{\frac{1}{\theta'}}}_{|w(r,x)|^{p-2}\phi(x)}\,dx \leq \underbrace{\left(\int_B |w(r,x)|^p\,\phi(x)\,dx\right)^{\frac{1}{\theta}}}_{A}\underbrace{\left(\int_B \phi(x)\,dx\right)^{\frac{1}{\theta'}}}_{B}$$

Due to Young's inequality $AB\leq\frac{1}{\theta}A^\theta+\frac{1}{\theta'}B^{\theta'}$. It follows that

$$\int_B |w(r,x)|^{p-2}\,\phi(x)\,dx \leq \frac{p-2}{p}\int_B |w(r,x)|^p\,\phi(x)\,dx + \frac{2}{p}\int_B \phi(x)\,dx.$$

Consequently,

$$\mathscr{T}^3 \leq \underbrace{(p-1)\left((p-2)\|\sigma(\cdot,0)\|_\infty + p\,\|\sigma\|_{\mathrm{Lip}}^2\right)}_{C_3}E\left[\int_B |w(r,x)|^p\,\phi(x)\,dx\right]$$
$$+\underbrace{2(p-1)\|\sigma(\cdot,0)\|_\infty^2}_{C_2}\int_B \phi(x)\,dx.$$

It follows that

$$E\left[\int_B |w(t,x)|^p\,\phi(x)\,dx\right] \leq E\left[\int_B |w(s,x)|^p\,\phi(x)\,dx\right]$$
$$+C_3\int_s^t E\left[\int_B |w(r,x)|^p\,\phi(x)\,dx\right]\,dr + C_2\left(\int_B \phi(x)\,dx\right)(t-s).$$

This inequality is of the general form

$$X(t) \leq X(s) + \int_s^t K(r)X(r)\,dr + \int_s^t H(r)\,dr. \tag{4.6}$$

Appealing to Grönwall's inequality,

$$X(t) \leq \exp\left[\int_s^t K(r)\,dr\right]X(s) + \int_s^t \exp\left[\int_r^t K(u)\,du\right]H(r)\,dr. \tag{4.7}$$

Identifying $K = C_3$ and $H = C_2\,\|\phi\|_{L^1(B)}$, it follows that

$$E\left[\int_B |w(t,x)|^p\,\phi(x)\,dx\right] \leq e^{C_3(t-s)}E\left[\int_B |w(s,x)|^p\,\phi(x)\,dx\right]$$
$$+C_2\,\|\phi\|_{L^1(B)}\int_s^t e^{C_3(t-r)}\,dr.$$

Next, observe that $e^{C_3(t-r)} \leq e^{C_3(t-s)}$ for all $s \leq r \leq t$, and so (4.5) follows.

3. *Inductive step.* Let P_n be the statement that (4.1) is true, and note that P_0 is trivially true. We must show that P_n implies P_{n+1}. By (3.3), $u^{n+1} = \mathcal{S}_{\mathrm{SDE}}(t_{n+1}, t_n)\mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$. Recall that $v_{\Delta t}((t_{n+1})-) = \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$. By (4.2),

$$E\left[\int_{B(0,\Gamma(t_{n+1}))} |v_{\Delta t}((t_{n+1})-,x)|^p\,\phi(x)\,dx\right]$$
$$\leq e^{\|f\|_{\mathrm{Lip}}C_\phi\Delta t}E\left[\int_{B(0,\Gamma(t_n))} |u^n(x)|^p\,\phi(x)\,dx\right].$$

Since $u^{n+1} = \mathcal{S}_{\mathrm{SDE}}(t_{n+1}, t_n)v_{\Delta t}((t_{n+1})-)$ it follows from (4.5) that

$$
\begin{aligned}
E\left[\int_{B(0,\Gamma(t_{n+1}))} |u^{n+1}(x)|^p \,\phi(x)\,dx\right] &\leq e^{C_3 \Delta t} \\
\times \Bigg(E\left[\int_{B(0,\Gamma(t_{n+1}))} |v_{\Delta t}((t_{n+1})-, x)|^p\, \phi(x)\,dx\right] &+ C_2 \int_{B(0,\Gamma(t_{n+1}))} \phi(x)\,dx\Delta t\Bigg).
\end{aligned}
$$

Combining the two previous estimates,

$$
\begin{aligned}
E\left[\int_{B(0,\Gamma(t_{n+1}))} |u^{n+1}(x)|^p\,\phi(x)\,dx\right] & \\
\leq e^{C_3\Delta t}\Bigg(e^{\|f\|_{\mathrm{Lip}} C_\phi \Delta t} E\left[\int_{B(0,\Gamma(t_n))} |u^n(x)|^p\,\phi(x)\,dx\right] & \\
+ C_2\Delta t \int_{B(0,\Gamma(t_{n+1}))} \phi(x)\,dx\Bigg) & \\
\leq e^{C_1\Delta t}\Bigg(E\left[\int_{B(0,\Gamma(t_n))} |u^n(x)|^p\,\phi(x)\,dx\right] & \\
+ C_2\Delta t \int_{B(0,R)} \phi(x)\,dx\Bigg), \qquad C_1 = \|f\|_{\mathrm{Lip}} C_\phi + C_3. &
\end{aligned}
$$

Inserting the induction hypothesis brings to an end the proof of (4.1). $\square$

Corollary 4.3. *Let $u_{\Delta t}$ and $v_{\Delta t}$ be defined by (3.4) and (3.5), respectively, and suppose u^0 belongs to $L^q(\Omega, \mathscr{F}_0, P; L^q(\mathbb{R}^d, \phi))$, $2 \leq q < \infty$, $\phi \in \mathfrak{N}$. Then, for each $1 \leq p \leq q$, there exists a finite constant C independent of Δt (but dependent on $T, p, \phi, f, \sigma, u^0$) such that*

$$
\max\left\{ E\left[\|u_{\Delta t}(t)\|_{p,\phi}^p\right], E\left[\|v_{\Delta t}(t)\|_{p,\phi}^p\right]\right\} \leq C, \qquad t \in [0, T].
$$

Proof. It suffices to prove the result for $p = q$. To this end, suppose $1 \leq p < q$ and $w \in L^q(\mathbb{R}^d, \phi)$. Let $r = q/p$, $r' = q/(q-p)$, so that $\frac{1}{r} + \frac{1}{r'} = 1$. Take $f = |u|^p \phi^{1/r}$, $g = \phi^{1/r'}$ and apply Hölder's inequality. The result is

$$
\int_{\mathbb{R}^d} |w(x)|^p\,\phi(x)\,dx \leq \left(\int_{\mathbb{R}^d} |w(x)|^q\,\phi(x)\,dx\right)^{p/q} \left(\int_{\mathbb{R}^d} \phi(x)\,dx\right)^{1-p/q}. \tag{4.8}
$$

Consider the case $p = q$. By Proposition 4.1, there exists a constant $C > 0$ depending only on $q, f, \sigma, u^0, T, \phi$ such that

$$
E\left[\|u^n\|_{q,\phi}^q\right] \leq C, \qquad 0 \leq n \leq N.
$$

Let $t \in [t_n, t_{n+1})$. By (4.2),

$$
E\Big[\|\underbrace{\mathcal{S}_{\mathrm{CL}}(t - t_n)u^n}_{v_{\Delta t}(t)}\|_{q,\phi}^q\Big] \leq e^{\|f\|_{\mathrm{Lip}} C_\phi \Delta t} E\left[\|u^n\|_{q,\phi}^q\right].
$$

This finishes the proof for $v_{\Delta t}$. For $u_{\Delta t}$ the result follows by (4.5). $\square$

The next result should be compared to [19, Proposition 5.2] and [6, § 6]. It can be turned into a fractional BV_x estimate (L^1 space translation estimate) along the lines of [6], but we will not need this fact here.

Proposition 4.4 (fractional BV_x estimates). *Suppose $(\mathcal{A}_f)$, $(\mathcal{A}_{f,1})$, $(\mathcal{A}_\sigma)$, and $(\mathcal{A}_{\sigma,1})$ are satisfied. Let $\phi \in \mathfrak{N}$. Suppose $u^0 \in L^2(\Omega, \mathscr{F}_0, P; L^2(\mathbb{R}^d, \phi))$. Let $u_{\Delta t}$ and $v_{\Delta t}$ be defined by (3.4) and (3.5), respectively. Then there exists a constant C_T, independent of Δt, such that*

$$E\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_{\Delta t}(t, x+z) - u_{\Delta t}(t, x-z)| \, J_r(z)\phi(x) \, dx\, dz \right]$$

$$\leq e^{C_\phi \|f\|_{\mathrm{Lip}} t} E\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| u^0(x+z) - u^0(x-z) \right| J_r(z)\phi(x)\, dx\, dz \right] + C_T r^{\kappa_\sigma},$$

for any $t \in (0, T)$. Here $\kappa_\sigma \in (0, 1/2]$ is defined in $(\mathcal{A}_{\sigma,1})$. If $\sigma(x, u) = \sigma(u)$, then we may take $C_T = 0$. The same result holds with $u_{\Delta t}$ replaced by $v_{\Delta t}$.

Remark 4.5. In the deterministic case or whenever $\sigma = \sigma(u)$ is independent of the spatial location x, we recover the usual BV bound. To this end, note that $C_T = 0$, apply the weight $\phi_\rho(x) = e^{-\rho\sqrt{1+|x|^2}}$ ($\rho > 0$), and then send $\rho \downarrow 0$.

Before we proceed to the proof, we fix some notation and make a few observations. Let us define C^2-approximations $\{S_\delta\}_{\delta > 0}$ of the absolute value function by asking that

$$S_\delta'(\sigma) = 2\int_0^\sigma J_\delta(z)\, dz, \qquad S_\delta(0) = 0. \tag{4.9}$$

Then

$$|r| - \delta \leq S_\delta(r) \leq |r|, \qquad |S_\delta''(r)| \leq \frac{2}{\delta}\|J\|_\infty \, \mathbf{1}_{|r| < \delta}. \tag{4.10}$$

Given S_δ, we define Q_δ by

$$Q_\delta(u, v) = \int_v^u S_\delta'(\xi - v)f'(\xi)\, d\xi, \qquad u, v \in \mathbb{R}. \tag{4.11}$$

This function satisfies

$$|\partial_u (Q_\delta(u, v) - Q_\delta(v, u))| \leq \|f''\|_{L^\infty} \delta \tag{4.12}$$

and

$$|Q_\delta(u, v)| \leq \|f\|_{\mathrm{Lip}} S_\delta(u - v). \tag{4.13}$$

Let us state two convenient identities. First, for $h = h(\cdot, \cdot) \in L^1_{\mathrm{loc}}$,

$$\frac{1}{2^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} h(x, y)\phi\left(\frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) dx\, dy$$

$$= \iint_{\mathbb{R}^d \times \mathbb{R}^d} h(\tilde{x} + z, \tilde{x} - z)\phi(\tilde{x})J_r(z)\, d\tilde{x}\, dz. \tag{4.14}$$

This follows by a change of variables: $(\tilde{x}, z) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$, $dy = 2^d dz$. Next,

$$\frac{1}{2^d} \int_{\mathbb{R}^d} \phi\left(\frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) dy = (\phi \star J_r)(x). \tag{4.15}$$

Proof of Proposition 4.4. Given $u = u(t) = u(t, x; \omega)$, we introduce the quantity

$$\mathcal{D}_r^u(t) := E\left[\frac{1}{2^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u(t, x) - u(t, y)| \, J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\, dx\, dy \right].$$

Actually, at first we are not going to work with this quantity but rather

$$\mathcal{D}_{r,\delta}^u(t) := E\left[\frac{1}{2^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(u(t, x) - u(t, y))J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\, dx\, dy \right],$$

where the regularized entropy S_δ is defined in (4.9). In view of (4.10) and (4.15),

$$\left| \mathcal{D}_r^u(t) - \mathcal{D}_{r,\delta}^u(t) \right| \leq \|\phi\|_{L^1(\mathbb{R}^d)} \delta, \qquad t > 0. \tag{4.16}$$

1. Deterministic step. Let $v(t,x)$ be the unique entropy solution of (3.1). We want to prove the following claim: There exists a constant C_1 depending only on J and C_ϕ such that for all $0 < r \leq 1$,

$$\mathcal{D}_{r,\delta}^v(t) \leq e^{C_\phi \|f\|_{\mathrm{Lip}} t} \left(\mathcal{D}_{r,\delta}^v(0) + C_1 \|f''\|_\infty E\left[\|v_0\|_{1,\phi} \right] t \left(\frac{\delta}{r} \right) \right). \tag{4.17}$$

Let Q_δ be defined in (4.11). Using the entropy inequalities and Kružkov's method of doubling the variables, it follows in a standard way that for $t > 0$

$$\frac{1}{2^d} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(v(t,x) - v(t,y)) J_r(\tfrac{x-y}{2}) \phi(\tfrac{x+y}{2}) \, dx \, dy$$

$$- \frac{1}{2^d} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(v_0(x) - v_0(y)) J_r(\tfrac{x-y}{2}) \phi(\tfrac{x+y}{2}) \, dx dy$$

$$\leq \frac{1}{2^d} \int_0^t \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} Q_\delta(v(s,x), v(s,y)) \cdot \nabla \phi(\tfrac{x+y}{2}) J_r(\tfrac{x-y}{2}) \, dx dy \, ds$$

$$+ \frac{1}{2^d} \int_0^t \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \left(Q_\delta(v(s,y), v(s,x)) - Q_\delta(v(s,x), v(s,y)) \right)$$

$$\cdot \nabla_y \left(\phi(\tfrac{x+y}{2}) J_r(\tfrac{x-y}{2}) \right) dx dy \, ds$$

$$=: \mathscr{T}_{\mathrm{CL}}^1 + \mathscr{T}_{\mathrm{CL}}^2.$$

By (4.13),

$$\left| \mathscr{T}_{\mathrm{CL}}^1 \right| \leq C_\phi \|f\|_{\mathrm{Lip}} \frac{1}{2^d} \int_0^t \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} S_\delta(v(s,x) - v(s,y)) J_r(\tfrac{x-y}{2}) \phi(\tfrac{x+y}{2}) \, dx dy \, ds.$$

Consider $\mathscr{T}_{\mathrm{CL}}^2$. Thanks to (4.12),

$$|Q_\delta(v,u) - Q_\delta(u,v)| = \left| \int_v^u \partial_\xi \left(Q_\delta(\xi, v) - Q_\delta(v, \xi) \right) d\xi \right| \leq \|f''\|_\infty |u - v| \delta,$$

so that

$$\left| \mathscr{T}_{\mathrm{CL}}^2 \right| \leq \frac{\|f''\|_\infty}{2} \delta \frac{1}{2^d} \int_0^t \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} |v(s,x) - v(s,y)| \left| \nabla J_r(\tfrac{x-y}{2}) \right| \phi(\tfrac{x+y}{2}) \, dx dy \, ds$$

$$+ \frac{\|f''\|_\infty}{2} \delta \frac{1}{2^d} \int_0^t \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} |v(s,x) - v(s,y)| J_r(\tfrac{x-y}{2}) \left| \nabla \phi(\tfrac{x+y}{2}) \right| dx dy \, ds$$

$$=: \mathscr{T}_{\mathrm{CL}}^{2,1} + \mathscr{T}_{\mathrm{CL}}^{2,2}.$$

Consider $\mathscr{T}_{\mathrm{CL}}^{2,1}$. Setting $\varphi_r(z) = \|\nabla J\|_1^{-1} \frac{1}{r^d} \left| \nabla J(\tfrac{z}{r}) \right|$, we write

$$\left| \nabla J_r \left(\frac{x-y}{2} \right) \right| = \|\nabla J\|_1 \frac{1}{r} \varphi_r \left(\frac{x-y}{2} \right).$$

By the triangle inequality and (4.15),

$$\frac{1}{2^d} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} |v(s,x) - v(s,y)| \left| \nabla J_r(\tfrac{x-y}{2}) \right| \phi(\tfrac{x+y}{2}) \, dx dy$$

$$\leq \|\nabla J\|_1 \frac{2}{r} \int_{\mathbb{R}^d} |v(s,x)| \, (\phi \star \varphi_r)(x) \, dx = \|\nabla J\|_1 \frac{2}{r} \|v(s)\|_{1,\phi\star\varphi_r}.$$

Considering $\mathscr{T}_{\mathrm{CL}}^{2,2}$, with $\phi \in \mathfrak{N}$,

$$\frac{1}{2^d} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} |v(s,x) - v(s,y)| \, J_r(\tfrac{x-y}{2}) \, |\nabla\phi(\tfrac{x+y}{2})| \, dxdy$$

$$\leq 2C_\phi \int_{\mathbb{R}^d} |v(s,x)| \, (\phi \star J_r)(x) \, dx = 2C_\phi \|v(s)\|_{1,\phi\star J_r}.$$

By Lemma 7.3,

$$\max\left\{ \|v(s)\|_{1,\phi\star\varphi_r}, \|v(s)\|_{1,\phi\star J_r} \right\} \leq \|v(s)\|_{1,\phi} \, (1 + w_{1,\phi}(r)),$$

where $w_{1,\phi}$ is defined in Lemma 7.2. Hence,

$$\left| \mathscr{T}_{\mathrm{CL}}^2 \right| \leq \|f''\|_\infty \, (1 + w_{1,\phi}(r)) \left(\int_0^t \|v(s)\|_{1,\phi} \, ds \right) \left(\|\nabla J\|_1 \frac{1}{r} + C_\phi \right) \delta.$$

In view of (4.2), $\|v(s)\|_{1,\phi} \leq e^{\|f\|_{\mathrm{Lip}} C_\phi s} \|v_0\|_{1,\phi}$. Summarizing,

$$\frac{1}{2^d} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta(v(t,x) - v(t,y)) J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2}) \, dx \, dy$$

$$- \frac{1}{2^d} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta(v_0(x) - v_0(y)) J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2}) \, dxdy$$

$$\leq \underbrace{C_\phi \|f\|_{\mathrm{Lip}}}_{K} \int_0^t \frac{1}{2^d} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta(v(s,x) - v(s,y)) J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2}) \, dx \, dy \, ds \tag{4.18}$$

$$+ \int_0^t \underbrace{C_1 \|f''\|_\infty \|v_0\|_{1,\phi} \, e^{\|f\|_{\mathrm{Lip}} C_\phi s} \left(\frac{\delta}{r}\right)}_{H(s)} \, ds,$$

where $C_1 = (1 + w_{1,\phi}(1))(\|\nabla J\|_1 + C_\phi)$. This inequality is of the form (4.6). By Grönwall's inequality (4.7),

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta(v(t,x) - v(t,y)) J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2}) \, dxdy$$

$$\leq e^{C_\phi \|f\|_{\mathrm{Lip}} t} \Bigg(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta(v_0(x) - v_0(y)) J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2}) \, dxdy$$

$$+ C_1 \|f''\|_\infty \|v_0\|_{1,\phi} \, t\left(\frac{\delta}{r}\right) \Bigg).$$

This proves the claim (4.17)

2. Stochastic step. Let $w(t) = \mathcal{S}_{\mathrm{SDE}}(t,s)w(s)$. We will now derive an estimate for w similar to (4.18): There exist constants C_1 and C_2, depending only on J, σ, ϕ, such that

$$\mathcal{D}_{r,\delta}^w(t) \leq \mathcal{D}_{r,\delta}^w(s) + C_1 \frac{r^{2\kappa+1}}{\delta} \int_s^t E\left[\|1 + |w(\tau)|\|_{2,\phi}^2\right] \, d\tau + C_2(t-s)\delta, \tag{4.19}$$

for all $0 \leq r \leq 1$. If $M_\sigma = 0$, then $C_1 = 0$.

Since $w(t,x) - w(t,y)$ solves

$$d(w(t,x) - w(t,y)) = \big(\sigma(x, w(t,x)) - \sigma(y, w(t,y))\big) \, dB(t),$$

applying Ito's formula to $S_\delta(w(t,x) - w(t,y))$ yields

$$dS_\delta(w(t,x) - w(t,y))$$
$$= \frac{1}{2}S_\delta''(w(t,x) - w(t,y))\big(\sigma(x,w(t,x)) - \sigma(y,w(t,y))\big)^2 dt,$$
$$+ S_\delta'(w(t,x) - w(t,y))\big(\sigma(x,w(t,x)) - \sigma(y,w(t,y))\big) dB(t).$$

Integrating against the test function $\frac{1}{2^d}J_r(\frac{x-y}{2})\phi(\frac{x+y}{2})$, we arrive at

$$\frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta(w(t,x) - w(t,y))J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\,dxdy$$

$$-\frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta(w(s,x) - w(s,y))J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})dxdy$$

$$= \int_s^t \frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{2}S_\delta''(w(\tau,x) - w(\tau,y))$$

$$\times \big(\sigma(x,w(\tau,x)) - \sigma(y,w(\tau,y))\big)^2 J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\,dxdy\,d\tau$$

$$+ \int_s^t \frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} S_\delta'(w(\tau,x) - w(\tau,y))\big(\sigma(x,w(\tau,x)) - \sigma(y,w(\tau,y))\big)\,dxdy\,dB(\tau)$$

$$= \mathscr{T}_{\text{SDE}}^1 + \mathscr{T}_{\text{SDE}}^2,$$

where the $\mathscr{T}_{\text{SDE}}^2$-term has zero expectation. Note that

$$(\sigma(x,u) - \sigma(y,v))^2 \le 2\,(\sigma(x,u) - \sigma(x,v))^2 + 2\,(\sigma(x,v) - \sigma(y,v))^2,$$

for any $u,v \in \mathbb{R}$. We estimate the $\mathscr{T}_{\text{SDE}}^1$-term as follows:

$$E\Big[\big|\mathscr{T}_{\text{SDE}}^1\big|\Big] \le 2E\Bigg[\int_s^t \frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} J_\delta(w(\tau,x) - w(\tau,y))\big(\sigma(x,w(\tau,x)) - \sigma(y,w(\tau,x))\big)^2$$

$$\times J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\,dxdy\,d\tau\Bigg]$$

$$+ 2E\Bigg[\int_s^t \frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} J_\delta(w(\tau,x) - w(\tau,y))\big(\sigma(y,w(\tau,x)) - \sigma(y,w(\tau,y))\big)^2$$

$$\times J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\,dxdy\,d\tau\Bigg] =: S_1 + S_2.$$

Regarding S_1, recall that $|J_\delta| \le \|J\|_\infty /\delta$. By $(\mathcal{A}_{\sigma,1})$,

$$|S_1| \le \|J\|_\infty \frac{2}{\delta}E\Bigg[\int_s^t \frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} |\sigma(x,w(\tau,x)) - \sigma(y,w(\tau,x))|^2$$

$$\times J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\,dxdy\,d\tau\Bigg]$$

$$\le \|J\|_\infty M_\sigma^2 \frac{2}{\delta}E\Bigg[\int_s^t \frac{1}{2^d}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^{2\kappa_\sigma+1}\,(1 + |w(\tau,x)|)^2$$

$$\times J_r(\tfrac{x-y}{2})\phi(\tfrac{x+y}{2})\,dxdy\,d\tau\Bigg]$$

$$\leq 2 \|J\|_\infty M_\sigma^2 \frac{(2r)^{2\kappa_\sigma+1}}{\delta} \int_s^t E\left[\||1+|w(\tau)|\|_{2,\phi\star J_r}^2\right] d\tau.$$

By Lemma 7.3,

$$\||1+|w(\tau)|\|_{2,\phi\star J_r}^2 \leq \||1+|w(\tau)|\|_{2,\phi}^2 (1+w_{1,\phi}(r)),$$

where $w_{1,\phi}$ is defined in Lemma 7.2. It follows that

$$|S_1| \leq \underbrace{2^{2(\kappa_\sigma+1)} \|J\|_\infty M_\sigma^2(1+w_{1,\phi}(1))}_{C_1} \int_s^t E\left[\||1+|w(\tau)|\|_{2,\phi}^2\right] d\tau \frac{r^{2\kappa_\sigma+1}}{\delta},$$

for all $0 < r \leq 1$. Consider S_2. Due to assumption $(\mathcal{A}_\sigma)$,

$$J_\delta(w(\tau,x) - w(\tau,y)) \left(\sigma(y,w(\tau,x)) - \sigma(y,w(\tau,y))\right)^2 \leq \|\sigma\|_{\mathrm{Lip}}^2 \|J\|_\infty \delta.$$

Hence,

$$|S_2| \leq \underbrace{2 \|\sigma\|_{\mathrm{Lip}}^2 \|J\|_\infty \|\phi\|_{L^1(\mathbb{R}^d)}}_{C_2}(t-s)\delta.$$

This proves (4.19)

3. Inductive step. Let P_n be the following claim: There exist constants C_1, C_2, C_3 depending only on J, ϕ, σ such that for all $0 < r \leq 1$,

$$\mathcal{D}_{r,\delta}^{u^n} \leq e^{C_\phi\|f\|_{\mathrm{Lip}}t_n}\left(\mathcal{D}_{r,\delta}^{u^0} + C_3 \|f''\|_\infty \left(\Delta t \sum_{k=0}^{n-1} E\left[\|u^k\|_{1,\phi}\right]\right)\frac{\delta}{r}\right.$$

$$\left. + C_1 \frac{r^{2\kappa_\sigma+1}}{\delta} \int_0^{t_n} E\left[\||1+|u_{\Delta t}(t)|\|_{2,\phi}^2\right] dt + C_2 t_n\delta\right). \quad (4.20)$$

If $M_\sigma = 0$, then $C_1 = 0$. Note that P_0 is trivially true. Assuming that P_n is true, we want to verify P_{n+1}. Recall that $u^{n+1} = \mathcal{S}_{\mathrm{SDE}}(t_{n+1}, t_n)\mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$. Let $w^n = \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$ and note that $\mathcal{S}_{\mathrm{SDE}}(t, t_n)w^n = u_{\Delta t}(t)$ for $t_n \leq t < t_{n+1}$. As $1 \leq e^{C_\phi\|f\|_{\mathrm{Lip}}\Delta t}$, (4.19) yields

$$\mathcal{D}_{r,\delta}^{u^{n+1}} \leq \mathcal{D}_{r,\delta}^{w^n} + e^{C_\phi\|f\|_{\mathrm{Lip}}\Delta t}\left(C_1 \frac{r^{2\kappa_\sigma+1}}{\delta} \int_{t_n}^{t_{n+1}} E\left[\||1+|u_{\Delta t}(t)|\|_{2,\phi}^2\right] dt + C_2\Delta t\delta\right).$$

By (4.17),

$$\mathcal{D}_{r,\delta}^{w^n} \leq e^{C_\phi\|f\|_{\mathrm{Lip}}\Delta t}\left(\mathcal{D}_{r,\delta}^{u^n} + C_3 \|f''\|_\infty E\left[\|u^n\|_{1,\phi}\right]\Delta t\left(\frac{\delta}{r}\right)\right).$$

Hence,

$$\mathcal{D}_{r,\delta}^{u^{n+1}} \leq e^{C_\phi\|f\|_{\mathrm{Lip}}\Delta t}\left(\mathcal{D}_{r,\delta}^{u^n} + C_3 \|f''\|_\infty E\left[\|u^n\|_{1,\phi}\right]\Delta t\left(\frac{\delta}{r}\right)\right.$$

$$\left. + C_1 \frac{r^{2\kappa_\sigma+1}}{\delta} \int_{t_n}^{t_{n+1}} E\left[\||1+|u_{\Delta t}(t)|\|_{2,\phi}^2\right] dt + C_2\Delta t\delta\right),$$

and inserting the hypothesis P_n yields P_{n+1}.

4. Concluding the proof. Consider (4.20). By Corollary 4.3, there exists a constant C, independent of Δt, such that

$$\mathcal{D}_{r,\delta}^{u^n} \leq e^{C_\phi\|f\|_{\mathrm{Lip}}t_n}\left(\mathcal{D}_{r,\delta}^{u^0} + C t_n \left(\frac{\delta}{r} + \delta + \frac{r^{2\kappa_\sigma+1}}{\delta}\right)\right).$$

Due to (4.16), this translates into

$$\mathcal{D}_r^{u^n} \leq e^{C_\phi\|f\|_{\mathrm{Lip}}t_n}\left(\mathcal{D}_r^{u^0} + C t_n \left(\frac{\delta}{r} + \delta + \frac{r^{2\kappa_\sigma+1}}{\delta}\right) + 2 \|\phi\|_{L^1(\mathbb{R}^d)} \delta\right), \qquad 0 \leq n \leq N.$$

We can argue via (4.19) to obtain

$$\mathcal{D}_r^{u_{\Delta t}}(t) \leq e^{C_\phi \|f\|_{\mathrm{Lip}} t} \left(\mathcal{D}_r^{u^0} + Ct \left(\frac{\delta}{r} + \delta + \frac{r^{2\kappa_\sigma + 1}}{\delta} \right) + 2 \|\phi\|_{L^1(\mathbb{R}^d)} \delta \right), \qquad t \in [0, T].$$

Note that the same holds true if we replace $u_{\Delta t}$ by $v_{\Delta t}$, thanks to (4.17). Viewing $r > 0$ as fixed, we can choose $\delta = r^{\kappa_\sigma + 1}$ to arrive at the bound

$$\mathcal{D}_r^{u_{\Delta t}}(t) \leq e^{C_\phi \|f\|_{\mathrm{Lip}} t} \mathcal{D}_r^{u^0} + C_T r^{\kappa_\sigma}.$$

The result follows by (4.14). In the case that $M_\sigma = 0$, we have

$$\mathcal{D}_r^{u_{\Delta t}}(t) \leq e^{C_\phi \|f\|_{\mathrm{Lip}} t} \mathcal{D}_r^{u^0} + C_T \left(\frac{\delta}{r} + \delta \right), \qquad t \in (0, T),$$

and we may send $\delta \downarrow 0$ independently of r. $\qquad\qquad\square$

In Proposition 4.4 the spatial regularity of $u_{\Delta t}, v_{\Delta t}$ is characterized in terms of averaged L^1 space translates. In the BV context, this is equivalently characterized by integration against the divergence of a smooth bounded function. Restricting to one dimension ($d = 1$) and $u \in C^1(\mathbb{R})$, we have

$$\sup_{h > 0} \left\{ \frac{1}{h} \int_{\mathbb{R}} |u(x + h) - u(x)| \, dx \right\}$$
$$= \int_{\mathbb{R}} |u'(x)| \, dx = \sup \left\{ \int_{\mathbb{R}} u(x)\beta'(x) \, dx \ : \ \beta \in C_c^\infty(\mathbb{R}), \|\beta\|_\infty \leq 1 \right\}.$$

Fix $\kappa \in (0, 1]$. The left-hand side has a natural generalization to the fractional BV setting by considering $u \in L^1(\mathbb{R})$ satisfying

$$\sup_{h > 0} \left\{ \frac{1}{h^\kappa} \int_{\mathbb{R}} |u(x + h) - u(x)| \, dx \right\} < \infty. \tag{4.21}$$

A possible generalization of the right-hand side reads

$$\sup \left\{ \delta^{1-\kappa} \int_{\mathbb{R}} u(x)(J_\delta \star \beta)'(x) \, dx \ : \ \delta > 0, \|\beta\|_\infty \leq 1 \right\} < \infty, \tag{4.22}$$

where $\{J_\delta\}_{\delta > 0}$ is a suitable family of symmetric mollifiers. Loosely speaking, the next lemma shows that (4.22) may be bounded in terms of (4.21). The lemma plays a key role in obtaining the optimal L^1 time continuity estimates in Proposition 4.8.

Lemma 4.6. *Let $\rho \in C_c^\infty((0, 1))$ satisfy $\int_0^1 \rho(r) \, dr = 1$ and $\rho \geq 0$. For $x \in \mathbb{R}^d$ define*

$$U(x) = \frac{1}{\alpha(d) M_d} \left(1 - \int_0^{|x|} \rho(r) \, dr \right), \qquad V(x) = \frac{1}{d\alpha(d) M_{d-1}} \rho(|x|),$$

where $M_n = \int_0^\infty r^n \rho(r) \, dr$, $n \geq 0$ and $\alpha(d)$ denotes the volume of the unit ball in $\mathbb{R}^d$. Then U, V are symmetric mollifiers on $\mathbb{R}^d$ with support in $B(0, 1)$. For $\phi \in \mathfrak{N}$, $u \in L^1(\mathbb{R}^d, \phi)$, and $\delta > 0$, define

$$\mathcal{V}_\delta(u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u(x + z) - u(x - z)| \, V_\delta(z) \phi(x) \, dz dx,$$

where $V_\delta(z) = \delta^{-d} V(\delta^{-1} z)$. Similarly, for $\beta \in L^\infty(\mathbb{R}^d)$ let

$$\mathcal{U}_\delta^i(u, \beta) = \int_{\mathbb{R}^d} u(x) \partial_{x_i} (U_\delta \star \beta)(x) \phi(x) \, dx, \qquad 1 \leq i \leq d,$$

where $U_\delta(z) = \delta^{-d} U(\delta^{-1} z)$. Then

$$\left| \mathcal{U}_\delta^i(u, \beta) \right| \leq \frac{d M_{d-1}}{2 M_d} \left(\frac{1}{\delta} \mathcal{V}_\delta(u) + 2 \|u\|_{1,\phi} \frac{w_{1,\phi}(\delta)}{\delta} \right) \|\beta\|_{L^\infty},$$

for each $1 \leq i \leq d$, where $w_{1,\phi}$ is defined in Lemma 7.2.

Remark 4.7. We note that Lemma 4.6 covers the BV case. If there is a constant $C \geq 0$ such that $\mathcal{V}_\delta(u) \leq C\delta$ (the BV case), then

$$\int_{\mathbb{R}^d} u(\nabla \cdot \beta)\phi \, dx = \lim_{\delta \downarrow 0} \sum_{i=1}^d \mathcal{U}_\delta^i(u, \beta^i) \leq \frac{d^2 M_{d-1}}{2M_d} \left(C + 2C_\phi \, \|u\|_{1,\phi} \right),$$

for any $\beta = (\beta^1, \ldots, \beta^d) \in C_c^1(\mathbb{R}^d; \mathbb{R}^d)$ satisfying $\|\beta\|_\infty \leq 1$. It follows that

$$\int_{\mathbb{R}^d} |\nabla u| \, \phi \, dx \leq \sup_{|\beta| \leq 1} \int_{\mathbb{R}^d} (\nabla u \cdot \beta)\phi \, dx$$

$$= \sup_{|\beta| \leq 1} \int_{\mathbb{R}^d} u(\nabla \cdot \beta)\phi + u(\beta \cdot \nabla \phi) \, dx$$

$$\leq \frac{d^2 M_{d-1}}{2M_d} \left(C + 2C_\phi \, \|u\|_{1,\phi} \right) + C_\phi \, \|u\|_{1,\phi},$$

and so $|\nabla u|$ is a finite measure with respect to $\phi \, dx$.

Proof. Let us first show that U is a symmetric mollifier. It is clearly symmetric, furthermore it is smooth since $\{0\} \notin \mathrm{cl}(\mathrm{supp}(\rho))$. Change to polar coordinates and integrate by parts to obtain

$$\int_{\mathbb{R}^d} \left(1 - \int_0^{|x|} \rho(\sigma)d\sigma \right) dx = \alpha(d) \int_0^\infty dr^{d-1} \left(1 - \int_0^r \rho(\sigma)d\sigma \right) dr$$

$$= \alpha(d) \int_0^\infty r^d \rho(r) \, dr = \alpha(d) M_d.$$

Similarly for V,

$$\int_{\mathbb{R}^d} \rho(|x|) \, dx = d\alpha(d) \int_0^\infty r^{d-1} \rho(r) dr = d\alpha(d) M_{d-1}.$$

Note that

$$\mathcal{U}_\delta^i(u, \beta) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} u(x) \partial_{x_i} U_\delta(x - y)\beta(y)\phi(x) \, dydx.$$

Next, we differentiate to obtain

$$\partial_{x_i} U_\delta(x) = -\frac{1}{\alpha(d)M_d} \frac{1}{\delta^d} \rho \left(\frac{|x|}{\delta} \right) \frac{1}{\delta} \mathrm{sign}\,(x_i) = -\frac{dM_{d-1}}{M_d} V_\delta(x) \frac{1}{\delta} \mathrm{sign}\,(x_i).$$

Hence

$$\mathcal{U}_\delta^i(u, \beta) = -\frac{dM_{d-1}}{M_d} \frac{1}{\delta} \iint_{\mathbb{R}^d \times \mathbb{R}^d} u(x) V_\delta(x - y) \mathrm{sign}\,(x_i - y_i) \, \beta(y) \, dydx.$$

This integral may be reformulated according to

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} u(x) V_\delta(x - y) \mathrm{sign}\,(x_i - y_i) \, \beta(y)\phi(x) \, dydx$$

$$= \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} u(x) V_\delta(x - y) \mathrm{sign}\,(x_i - y_i) \, \beta(y)\phi(x) \, dydx$$

$$- \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} u(x) V_\delta(y - x) \mathrm{sign}\,(y_i - x_i) \, \beta(y)\phi(x) \, dydx$$

$$= \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y - z)\phi(y - z) - u(y + z)\phi(y + z)) V_\delta(z) \mathrm{sign}\,(z_i) \, \beta(y) \, dzdy,$$

where we made the substitution $x = y - z$ and $x = y + z$ respectively. Since

$$u(y - z)\phi(y - z) - u(y + z)\phi(y + z) = (u(y - z) - u(y + z))\phi(y)$$

$$+ u(y - z)(\phi(y - z) - \phi(y)) - u(y + z)(\phi(y + z) - \phi(y)),$$

it follows that

$$\mathcal{U}_\delta^i(u, \beta) = \frac{dM_{d-1}}{2M_d} \frac{1}{\delta} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y + z) - u(y - z))V_\delta(z)\mathrm{sign}\,(z_i)\,\beta(y)\phi(y)\,dzdy$$

$$+ \frac{dM_{d-1}}{2M_d} \frac{1}{\delta} \iint_{\mathbb{R}^d \times \mathbb{R}^d} u(y + z)(\phi(y + z) - \phi(y))V_\delta(z)\mathrm{sign}\,(z_i)\,\beta(y)\,dzdy$$

$$+ \frac{dM_{d-1}}{2M_d} \frac{1}{\delta} \iint_{\mathbb{R}^d \times \mathbb{R}^d} u(y - z)(\phi(y) - \phi(y - z))V_\delta(z)\mathrm{sign}\,(z_i)\,\beta(y)\,dzdy$$

$$=: \mathscr{Z}_\delta^1 + \mathscr{Z}_\delta^2 + \mathscr{Z}_\delta^3.$$

Clearly

$$\left|\mathscr{Z}_\delta^1\right| \leq \frac{dM_{d-1}}{2M_d} \frac{1}{\delta} \underbrace{\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u(y + z) - u(y - z)|\,V_\delta(z)\phi(y)\,dzdy}_{\mathcal{V}_\delta(u)}\,\|\beta\|_{L^\infty}\,.$$

Consider $\mathscr{Z}_\delta^2$, the term $\mathscr{Z}_\delta^3$ is treated similarly. By Lemma 7.2

$$|\phi(y + z) - \phi(y)| \leq w_{1,\phi}(|z|)\phi(y + z).$$

Hence, by Young's inequality for convolutions,

$$\left|\mathscr{Z}_\delta^2\right| \leq \frac{dM_{d-1}}{2M_d} \frac{w_{1,\phi}(\delta)}{\delta} \int_{\mathbb{R}^d} (|u\phi| \star V_\delta)(y)\,dy\,\|\beta\|_{L^\infty} \leq \frac{dM_{d-1}}{2M_d} \frac{w_{1,\phi}(\delta)}{\delta}\,\|u\|_{1,\phi}\,.$$

This concludes the proof of the lemma.   □

Next, we consider the time continuity of the splitting approximations. Recall that the interpolants $u_{\Delta t}, v_{\Delta t}$ are discontinuous at $t_n = n\Delta t$. Hence, the result must somehow quantify the size of the jumps as $\Delta t \downarrow 0$. The idea of the proof is to "transfer à la Kružkov" spatial regularity to temporal continuity [22, 23]. Given a bounded variation bound, or some spatial L^1 modulus of continuity, this approach has been applied to miscellaneous splitting methods for deterministic problems, cf. [18] (and references therein). At variance with [18], we quantify spatial regularity differently, namely in terms of averaged (weighted) L^1 translates. Combined with Lemma 4.6, we deduce L^1 time continuity estimates that recover the optimal estimates in the BV_x case ($\kappa = 1$).

Proposition 4.8 (L^1 *time continuity*). *Assume that* $(\mathcal{A}_f)$, $(\mathcal{A}_{f,1})$, $(\mathcal{A}_\sigma)$, *and* $(\mathcal{A}_{\sigma,1})$ *hold. Fix* $\phi \in \mathfrak{N}$, *and let* $u^0 \in L^2(\Omega, \mathscr{F}_0, P; L^2(\mathbb{R}^d, \phi))$ *satisfy*

$$E\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left|u^0(x + z) - u^0(x - z))\right| J_r(z)\phi(x)\,dx\,dz \right] = \mathcal{O}(r^{\kappa_0}), \qquad (4.23)$$

for any symmetric mollifier J *and some* $0 < \kappa_0 \leq 1$. *Set*

$$\kappa := \begin{cases} \min\{\kappa_0, \kappa_\sigma\} & \text{if } \sigma = \sigma(x, u), \\ \kappa_0 & \text{if } \sigma = \sigma(u). \end{cases}$$

Let $u_{\Delta t}$ *and* $v_{\Delta t}$ *be defined in* (3.4) *and* (3.5), *respectively. Then:*

(i) *Suppose* $0 < \tau_1 < \tau_2 \leq T$ *satisfy* $\tau_1 \in (t_k, t_{k+1}]$ *and* $\tau_2 \in (t_l, t_{l+1}]$. *Then there exists a finite constant* $C_{T,\phi}$, *independent of* Δt, *such that*

$$E\left[\int_{\mathbb{R}^d} |u_{\Delta t}(\tau_2, x) - u_{\Delta t}(\tau_1, x)|\,\phi(x)\,dx \right] \leq C_{T,\phi}\left(|(l - k)\Delta t|^\kappa + \sqrt{\tau_2 - \tau_1} \right).$$

(ii) *Suppose $0 \leq \tau_1 \leq \tau_2 < T$ satisfy $\tau_1 \in [t_k, t_{k+1})$ and $\tau_2 \in [t_l, t_{l+1})$. Then there exists a finite constant $C_{T,\phi}$, independent of Δt, such that*

$$E\left[\int_{\mathbb{R}^d} |v_{\Delta t}(\tau_2, x) - v_{\Delta t}(\tau_1, x)| \, \phi(x) \, dx\right] \leq C_{T,\phi}\left(\sqrt{(l-k)\Delta t} + |\tau_2 - \tau_1|^\kappa\right).$$

Proof of Proposition 4.8. We shall first quantify weak continuity in the mean of $t \mapsto u_{\Delta t}(t)$, $t \mapsto v_{\Delta t}(t)$, and then turn this into fractional L^1 time continuity in the mean. The reason for first exhibiting a weak estimate is that the splitting steps do not produce functions that are Lipschitz continuous in time, thereby preventing a direct "inductive argument", see [22].

1. *Weak estimate.* Let $t_n = n\Delta t$. Suppose $0 < \tau_1 \leq \tau_2 \leq T$ satisfies $\tau_1 \in (t_k, t_{k+1}]$ and $\tau_2 \in (t_l, t_{l+1}]$. Suppose β belongs to $L^\infty(\Omega \times \mathbb{R}^d, \mathscr{F} \otimes \mathscr{B}(\mathbb{R}^d), dP \otimes dx)$ and let $\beta_\delta = \beta \star U_\delta$, where U_δ is defined in Lemma 4.6. We claim that there is a constant $C > 0$, independent of Δt, such that

$$E\left[\int_{\mathbb{R}^d} \left(u_{\Delta t}(\tau_2, x) - u_{\Delta t}(\tau_1, x)\right)(\beta_\delta \phi)(x) \, dx\right] \tag{4.24}$$
$$\leq C\left(\delta^{\kappa-1}(l-k)\Delta t + \sqrt{\tau_2 - \tau_1}\right)\|\beta\|_{L^\infty}.$$

Consider the case $l \geq k + 1$. We continue as follows:

$$\mathscr{T} = E\left[\int_{\mathbb{R}^d} \left(u_{\Delta t}(\tau_2, x) - u_{\Delta t}(\tau_1, x)\right)(\beta_\delta \phi)(x) \, dx\right]$$
$$= E\left[\int_{\mathbb{R}^d} \left(u_{\Delta t}(\tau_2, x) - u_{\Delta t}((t_l)+, x)\right)(\beta_\delta \phi)(x) \, dx\right]$$
$$+ E\left[\int_{\mathbb{R}^d} \left(u_{\Delta t}(t_{k+1}, x) - u_{\Delta t}(\tau_1, x)\right)(\beta_\delta \phi)(x) \, dx\right]$$
$$+ E\left[\int_{\mathbb{R}^d} \left(u_{\Delta t}((t_l)+, x) - u_{\Delta t}(t_{k+1}, x)\right)(\beta_\delta \phi)(x) \, dx\right]$$
$$=: \mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3.$$

Recall that $u_{\Delta t}((t_n)+) = v_{\Delta t}((t_{n+1})-) = \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$. Regarding the last term,

$$u_{\Delta t}((t_l)+, x) - u_{\Delta t}(t_{k+1}, x) = v_{\Delta t}((t_{l+1})-, x) - v_{\Delta t}(t_l, x)$$
$$+ \sum_{n=k+1}^{l-1} u_{\Delta t}(t_{n+1}, x) - u_{\Delta t}(t_n, x),$$

where the sum is empty for the case $l = k + 1$. Furthermore, we note that

$$u_{\Delta t}(t_{n+1}, x) - u_{\Delta t}(t_n, x) = (u_{\Delta t}(t_{n+1}, x) - u_{\Delta t}((t_n)+, x))$$
$$+ (v_{\Delta t}((t_{n+1})-, x) - v_{\Delta t}(t_n, x)).$$

This yields

$$\mathscr{T}_3 = E\left[\sum_{n=k+1}^{l-1} \int_{\mathbb{R}^d} \left(u_{\Delta t}(t_{n+1}, x) - u_{\Delta t}((t_n)+, x)\right)(\beta_\delta \phi)(x) \, dx\right]$$
$$+ E\left[\sum_{n=k+1}^{l} \int_{\mathbb{R}^d} \left(v_{\Delta t}((t_{n+1})-, x) - v_{\Delta t}(t_n, x)\right)(\beta_\delta \phi)(x) \, dx\right]$$
$$= E\left[\int_{\mathbb{R}^d} \left(\int_{t_{k+1}}^{t_l} \sigma(x, u_{\Delta t}(t, x)) \, dB(t)\right)(\beta_\delta \phi)(x) \, dx\right]$$

$$+ E\left[\sum_{n=k+1}^{l}\int_{\mathbb{R}^d}\Big(\mathcal{S}_{\mathrm{CL}}(\Delta t)u^n(x) - u^n(x)\Big)(\beta_\delta\phi)(x)\,dx\right].$$

It follows that $\mathscr{T} = \mathscr{T}^{\mathrm{CL}} + \mathscr{T}^{\mathrm{SDE}}$, where

$$\mathscr{T}^{\mathrm{CL}} := E\left[\sum_{n=k+1}^{l}\int_{\mathbb{R}^d}\Big(\mathcal{S}_{\mathrm{CL}}(\Delta t)u^n(x) - u^n(x)\Big)(\beta_\delta\phi)(x)\,dx\right],$$

$$\mathscr{T}^{\mathrm{SDE}} := E\left[\iint_{\mathbb{R}^d}\int_{\tau_1}^{\tau_2}\sigma(x, u_{\Delta t}(t,x))\,dB(t)(\beta_\delta\phi)(x)\,dx\right].$$

Note that this holds true for $k = l$ as $\mathscr{T}^{\mathrm{CL}} = 0$ in this case. As $v_{\Delta t}(t,x)$ is a weak solution of the conservation law (3.1) on $[t_n, t_{n+1})$

$$\left|\int_{\mathbb{R}^d}\big(v_{\Delta t}((t_{n+1})-, x) - v_{\Delta t}(t_n, x)\big)(\beta_\delta\phi)(x)\,dx\right|$$

$$\leq \left|\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d} f(v_{\Delta t}(r,x))\cdot\nabla(\beta_\delta\phi)(x)\,dxdr\right|$$

$$\leq \left|\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d} f(v_{\Delta t}(r,x))\cdot\nabla\beta_\delta(x)\,\phi(x)\,dxdr\right|$$

$$+ \left|\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^d} f(v_{\Delta t}(r,x))\cdot(\beta_\delta(x)\nabla\phi(x))\,dxdr\right|$$

$$=: \mathscr{Z}_\delta^1 + \mathscr{Z}_\delta^2.$$

By Proposition 4.4, there exists a constant $C > 0$ such that

$$E\left[\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v_{\Delta t}(r, x+z) - v_{\Delta t}(r, x-z)|\,V_\delta(z)\phi(x)\,dzdx\right] \leq C\delta^\kappa.$$

Consequently, taking expectations in Lemma 4.6 yields

$$E\left[\mathscr{Z}_\delta^1\right] \leq \frac{d^2 M_{d-1}}{2M_d}\|f\|_{\mathrm{Lip}}\left(\Delta t C\delta^{\kappa-1}\right.$$

$$\left. + 2E\left[\int_{t_n}^{t_{n+1}}\|v_{\Delta t}(r)\|_{1,\phi}\,dr\right]\delta^{-1}w_{1,\phi}(2\delta)\right)\|\beta\|_{L^\infty}.$$

As $\phi \in \mathfrak{N}$,

$$\mathscr{Z}_\delta^2 \leq \|f\|_{\mathrm{Lip}}\,C_\phi E\left[\int_{t_{k+1}}^{t_{l+1}}\|v_{\Delta t}(t)\|_{1,\phi}\,dt\right]\|\beta\|_{L^\infty}.$$

Summarizing, there exists a constant C such that

$$\left|\mathscr{T}^{\mathrm{CL}}\right| \leq C\delta^{\kappa-1}(l-k)\Delta t\,\|\beta\|_{L^\infty},$$

for all $0 < \delta \leq 1$.

By (4.8), Jensen's inequality, and the Itô isometry,

$$\left|\mathscr{T}^{\mathrm{SDE}}\right| \leq \|\beta\|_{L^\infty}\int_{\mathbb{R}^d} E\left[\left|\int_{\tau_1}^{\tau_2}\sigma(x, u_{\Delta t}(t,x))\,dB(t)\right|\right]\phi(x)\,dx$$

$$\leq \|\beta\|_{L^\infty}\|\phi\|_{L^1(\mathbb{R}^d)}^{1/2}\left(\int_{\mathbb{R}^d} E\left[\left|\int_{\tau_1}^{\tau_2}\sigma(x, u_{\Delta t}(t,x))\,dB(t)\right|\right]^2\phi(x)\,dx\right)^{1/2}$$

$$\leq \|\beta\|_{L^\infty}\|\phi\|_{L^1(\mathbb{R}^d)}^{1/2}\left(\int_{\mathbb{R}^d} E\left[\int_{\tau_1}^{\tau_2}\sigma^2(x, u_{\Delta t}(t,x))\,dt\right]\phi(x)\,dx\right)^{1/2}$$

$$= \|\beta\|_{L^\infty} \|\phi\|^{1/2}_{L^1(\mathbb{R}^d)} \left(\int_{\tau_1}^{\tau_2} E\left[\|\sigma(\cdot, u_{\Delta t}(t, \cdot))\|^2_{2,\phi} \right] dt \right)^{1/2}$$

$$\leq C \|\beta\|_{L^\infty} \|\phi\|^{1/2}_{L^1(\mathbb{R}^d)} \sqrt{\tau_2 - \tau_1},$$

since, in view of $(\mathcal{A}_\sigma)$ and Corollary 4.3, $E\left[\|\sigma(\cdot, u_{\Delta t}(t, \cdot))\|^2_{2,\phi} \right]^{1/2} \leq C$ for some constant C independent of $t \in [0, T]$. Summarizing, the above estimates imply the existence of a constant C, independent of $\Delta t, \delta$ and β, such that

$$|\mathscr{T}| \leq C \left(\delta^{\kappa-1}(k-l)\Delta t + \sqrt{\tau_2 - \tau_1} \right) \|\beta\|_{L^\infty},$$

which yields (4.24).

Let us consider $v_{\Delta t}$. Suppose $0 \leq \tau_1 \leq \tau_2 < T$, with $\tau_1 \in [t_k, t_{k+1})$, $\tau_2 \in [t_l, t_{l+1})$. We claim there is a constant $C > 0$, independent of $\Delta t, \delta$ and β, such that

$$E\left[\int_{\mathbb{R}^d} \left(v_{\Delta t}(\tau_2, x) - v_{\Delta t}(\tau_1, x) \right) (\beta_\delta \phi)(x)\, dx \right]$$
$$\leq C \left(\delta^{\kappa-1} |\tau_2 - \tau_1| + \sqrt{(l-k)\Delta t} \right) \|\beta\|_{L^\infty}. \tag{4.25}$$

To prove this claim, note that

$$v_{\Delta t}(\tau_2, x) - v_{\Delta t}(\tau_1, x) = v_{\Delta t}(\tau_2, x) - v_{\Delta t}(t_l, x)$$
$$+ \sum_{n=k+1}^{l} v_{\Delta t}(t_n, x) - v_{\Delta t}((t_n)-, x)$$
$$+ \sum_{n=k+1}^{l-1} v_{\Delta t}((t_{n+1})-, x) - v_{\Delta t}(t_n, x)$$
$$+ v_{\Delta t}((t_{k+1})-, x) - v_{\Delta t}(\tau_1, x),$$

and so

$$E\left[\int_{\mathbb{R}^d} \left(v_{\Delta t}(\tau_2, x) - v_{\Delta t}(\tau_1, x) \right) (\beta_\delta \phi)(x)\, dx \right] = \mathscr{T}^{\mathrm{CL}} + \mathscr{T}^{\mathrm{SDE}},$$

where

$$\mathscr{T}^{\mathrm{CL}} := E\left[\int_{\mathbb{R}^d} \left(\mathcal{S}_{\mathrm{CL}}(\tau_2 - t_l) u^l(x) - u^l(x) \right) (\beta_\delta \phi)(x)\, dx \right]$$
$$+ \sum_{n=k+1}^{l-1} E\left[\int_{\mathbb{R}^d} \left(\mathcal{S}_{\mathrm{CL}}(\Delta t) u^n(x) - u^n(x) \right) (\beta_\delta \phi)(x)\, dx \right]$$
$$+ E\left[\int_{\mathbb{R}^d} \left(\mathcal{S}_{\mathrm{CL}}(\Delta t) u^k - \mathcal{S}_{\mathrm{CL}}(\tau_1 - t_k) u^k \right) (\beta_\delta \phi)(x)\, dx \right],$$

$$\mathscr{T}^{\mathrm{SDE}} := \sum_{n=k+1}^{l} E\left[\int_{\mathbb{R}^d} \left(\int_{t_{n-1}}^{t_n} \sigma(x, u_{\Delta t}(t, x))\, dB(t) \right) (\beta_\delta \phi)(x)\, dx \right]$$
$$= E\left[\int_{\mathbb{R}^d} \left(\int_{t_k}^{t_l} \sigma(x, u_{\Delta t}(t, x))\, dB(t) \right) (\beta_\delta \phi)(x)\, dx \right].$$

Combining the above estimates yields (4.25).

2. *Strong estimate.* Let $d(x) = u_{\Delta t}(\tau_2) - u_{\Delta t}(\tau_1)$, $\beta(x) = \operatorname{sign}(d(x))$. By the triangle inequality,

$$E\left[\int_{\mathbb{R}^d} |u_{\Delta t}(\tau_2, x) - u_{\Delta t}(\tau_1, x)|\, \phi(x)\, dx \right]$$
$$\leq \left| E\left[\int_{\mathbb{R}^d} \beta_\delta(x) d(x) \phi(x)\, dx \right] \right| + E\left[\int_{\mathbb{R}^d} ||d(x)| - \beta_\delta(x) d(x)|\, \phi(x)\, dx \right]$$

$$=: \mathscr{T}_1 + \mathscr{T}_2.$$

By (4.24),

$$\mathscr{T}_1 = \mathcal{O}\left(\delta^{\kappa-1}(l-k)\Delta t + \sqrt{\tau_2 - \tau_1}\right).$$

Consider $\mathscr{T}_2$. Following, e.g. [23, Lemma 1],

$$\left||d(x)| - \beta_\delta(x)d(x)\right| \le \int_{\mathbb{R}^d} \left||d(x)| - d(x)\mathrm{sign}\,(d(y))\right| V_\delta(x-y)\,dy$$

$$\le 2\int_{\mathbb{R}^d} |d(x) - d(y)|\,V_\delta(x-y)\,dy.$$

Upon adding and subtracting identical terms and changing variables $2\tilde{x} = x + y$, $2z = x - y$, it follows (after relabeling $\tilde{x}$ by x)

$$\int_{\mathbb{R}^d} \left||d(x)| - \beta_\delta(x)d(x)\right|\phi(x)\,dx \le 2\iint_{\mathbb{R}^d \times \mathbb{R}^d} |d(x+z) - d(x-z)|$$

$$\times V_{\delta/2}(z)\,|\phi(x+z) - \phi(x)|\;dzdx$$

$$+ 2\iint_{\mathbb{R}^d \times \mathbb{R}^d} |d(x+z) - d(x-z)|\,V_{\delta/2}(z)\phi(x)\,dzdx$$

$$=: \mathscr{T}_2^1 + \mathscr{T}_2^2.$$

Consider $\mathscr{T}_2^1$. By Lemma 7.2,

$$|\phi(x+z) - \phi(x)| \le w_{1,\phi}(|z|)\phi(x).$$

Hence, by the symmetry of V and the triangle inequality,

$$\left|\mathscr{T}_2^1\right| \le 4\iint_{\mathbb{R}^d \times \mathbb{R}^d} |d(x-z)|\,V_{\delta/2}(z)w_{1,\phi}(|z|)\phi(x)\,dzdx$$

$$\le 4w_{1,\phi}(\delta/2)\iint_{\mathbb{R}^d \times \mathbb{R}^d} |d(y)|\,V_{\delta/2}(x-y)\phi(x)\,dydx$$

$$\le 2w_{1,\phi}(\delta)\,\|u_{\Delta t}(\tau_2) - u_{\Delta t}(\tau_1)\|_{1,\phi\star V_{\delta/2}}.$$

By Lemma 7.3 and Corollary 4.3, $E\left[\left|\mathscr{T}_2^1\right|\right] = \mathcal{O}(\delta)$. By Proposition 4.4, it follows in view of assumption (4.23) that $E\left[\left|\mathscr{T}_2^2\right|\right] = \mathcal{O}(\delta^\kappa)$. Consequently,

$$\mathscr{T}_1 + \mathscr{T}_2 = \mathcal{O}\left(\delta^{\kappa-1}(l-k)\Delta t + \sqrt{\tau_2 - \tau_1} + \delta^\kappa\right).$$

Choosing $\delta = ((l-k)\Delta t)$ concludes the proof of (i). The result (ii) follows analogously due to (4.25). $\qquad\square$

5. CONVERGENCE

Equipped with Δt-uniform a priori estimates, we are now prepared to study the limiting behavior of $u_{\Delta t}, v_{\Delta t}$ as $\Delta t \downarrow 0$. As discussed in the introduction, we will apply the framework of Young measures. We refer to the appendix (Section 7.5) for some background material on Young measures and weak compactness.

We start by establishing an approximate entropy inequality for the operator splitting solutions.

Lemma 5.1. *Suppose $u^0 \in L^2(\Omega, \mathscr{F}_0, P; L^2(\mathbb{R}^d, \phi))$, $\phi \in \mathfrak{N}$. Let $u_{\Delta t}$ and $v_{\Delta t}$ be defined by (3.4) and (3.5), respectively. For any $(S, Q) \in \mathscr{E}$, any $V \in \mathcal{S}$, and any*

non-negative $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$,

$$
\begin{aligned}
0 \leq\ & E\left[\int_{\mathbb{R}^d} S(u^0(x) - V)\varphi(0, x)\,dx\right] \\
& + E\left[\iint_{\Pi_T} S(u_{\Delta t}(t, x) - V)\partial_t \varphi(t, x) + Q(v_{\Delta t}(t, x), V) \cdot \nabla \varphi(t, x)\,dxdt\right] \\
& - E\left[\iint_{\Pi_T} S''(u_{\Delta t}(t, x) - V)D_t V\sigma(x, u_{\Delta t}(t, x))\varphi(t, x)\,dxdt\right] \\
& + E\left[\frac{1}{2}\iint_{\Pi_T} S''(u_{\Delta t}(t, x) - V)\sigma^2(x, u_{\Delta t}(t, x))\varphi(t, x)\,dxdt\right] \\
& + \sum_{n=0}^{N-1} E\left[\int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} (S(v_{\Delta t}(t, x) - V) - S(v_{\Delta t}((t_{n+1})-, x) - V)\,\partial_t \varphi(t, x)\,dxdt\right].
\end{aligned}
$$

$$(5.1)$$

Proof. Let us for the moment assume that $u^0 \in L^p(\Omega, \mathscr{F}_0, P; L^p(\mathbb{R}^d, \phi))$ for all $2 \leq p < \infty$. By definition, $v_{\Delta t}$ satisfies

$$
\begin{aligned}
\int_{\mathbb{R}^d} S(u^n(x) - V)\varphi(t_n, x)\,dx &- \int_{\mathbb{R}^d} S(v_{\Delta t}((t_{n+1})-, x) - V)\varphi(t_{n+1}, x)\,dx \\
&+ \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} S(v_{\Delta t}(t, x) - V)\partial_t \varphi(t, x)\,dx\,dt \\
&+ \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} Q(v_{\Delta t}(t, x), V) \cdot \nabla \varphi(t, x)\,dx\,dt \geq 0.
\end{aligned}
$$

For fixed $x \in \mathbb{R}^d$, apply Theorem 7.1 with $F(\zeta, \lambda, t) = S(\zeta - \lambda)\varphi(t, x)$ and

$$
\underbrace{u_{\Delta t}(t, x)}_{X(t)} = \underbrace{u_{\Delta t}((t_n)+, x)}_{X_0} + \int_{t_n}^{t} \underbrace{\sigma(x, u_{\Delta t}(s, x))}_{u(s)}\,dB(s).
$$

This yields, after integrating in space,

$$
\begin{aligned}
\int_{\mathbb{R}^d} S(u^{n+1}(x) - V)\varphi(t_{n+1}, x)\,dx &= \int_{\mathbb{R}^d} S(u_{\Delta t}((t_n)+) - V)\varphi(t_n, x)\,dx \\
&+ \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S(u_{\Delta t}(t, x) - V)\partial_t \varphi(t, x)\,dt\,dx \\
&+ \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S'(u_{\Delta t}(t, x) - V)\sigma(x, u_{\Delta t}(t, x))\varphi(t, x)\,dB(t)\,dx \\
&- \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S''(u_{\Delta t}(t, x) - V)D_t V\sigma(x, u_{\Delta t}(t, x))\varphi(t, x)\,dt\,dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S''(u_{\Delta t}(t, x) - V)\sigma^2(x, u_{\Delta t}(t, x))\varphi(t, x)\,dt\,dx,
\end{aligned}
$$

where the stochastic integral is a Skorohod integral. Note that

$$
\begin{aligned}
\int_{\mathbb{R}^d} S(u_{\Delta t}((t_n)+, x) - V)\varphi(t_n, x)\,dx &- \int_{\mathbb{R}^d} S(v_{\Delta t}((t_{n+1})-, x) - V)\varphi(t_{n+1}, x)\,dx \\
&= -\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S(v_{\Delta t}((t_{n+1})-, x) - V)\partial_t \varphi(t, x)\,dtdx.
\end{aligned}
$$

Adding the two equations and taking expectations we attain

$$E\left[\int_{\mathbb{R}^d} S(u^n(x) - V)\varphi(t_n, x)\, dx\right] - E\left[\int_{\mathbb{R}^d} S(u^{n+1}(x) - V)\varphi(t_{n+1}, x)\, dx\right]$$

$$+ E\left[\int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} \left(S(v_{\Delta t}(t, x) - V) - S(v_{\Delta t}((t_{n+1})-, x) - V)\right) \partial_t \varphi(t, x)\, dx\, dt\right]$$

$$+ E\left[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S(u_{\Delta t}(t, x) - V)\partial_t \varphi(t, x)\, dt\, dx\right]$$

$$+ E\left[\int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} Q(v_{\Delta t}(t, x), V) \cdot \nabla \varphi(t, x)\, dx\, dt\right]$$

$$- E\left[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S''(u_{\Delta t}(t, x) - V)D_t V \sigma(x, u_{\Delta t}(t, x))\varphi(t, x)\, dt\, dx\right]$$

$$+ E\left[\frac{1}{2} \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} S''(u_{\Delta t}(t, x) - V)\sigma^2(x, u_{\Delta t}(t, x))\varphi(t, x)\, dt\, dx\right] \geq 0,$$

where we applied the fact that the Skorohod integral has zero expectation. Next we sum over $n = 0, 1, \ldots, N - 1$. This yields (5.1). The result follows for general $u^0 \in L^2(\Omega, \mathscr{F}_0, P; L^2(\mathbb{R}^d, \phi))$ by approximation. $\square$

Theorem 5.1. *Suppose $(\mathcal{A}_f)$, $(\mathcal{A}_{f,1})$, $(\mathcal{A}_\sigma)$, and $(\mathcal{A}_{\sigma,1})$ hold. Let $\phi \in \mathfrak{N}$ and $2 \leq p < \infty$. Suppose $u^0 \in L^p(\Omega, \mathscr{F}_0, P; L^p(\mathbb{R}^d, \phi))$ satisfies (4.23). Let $u_{\Delta t}$ and $v_{\Delta t}$ be defined by (3.4) and (3.5), respectively. Then there exists a subsequence $\{\Delta t_j\}$ and a predictable $u \in L^p([0, T] \times \Omega; L^p(\mathbb{R}^d \times [0, 1], \phi))$ such that both $u_{\Delta t_j} \to u$ and $v_{\Delta t_j} \to u$ in the following sense: For any Carathéodory function $\Psi : \mathbb{R} \times \Pi_T \times \Omega \to \mathbb{R}$ such that $\Psi(u_{\Delta t_j}, \cdot) \rightharpoonup \overline{\Psi}$ (respectively $\Psi(v_{\Delta t_j}, \cdot) \rightharpoonup \overline{\Psi}$) in $L^1(\Pi_T \times \Omega, \phi\, dx \otimes dt \otimes dP)$,*

$$\overline{\Psi}(t, x, \omega) = \int_0^1 \Psi(u(t, x, \alpha, \omega), t, x, \omega)\, d\alpha. \tag{5.2}$$

The process $\tilde{u} = \int_0^1 u\, d\alpha$ is an entropy solution in the sense of Definition 2.1 with initial condition u^0.

Proof. 1. Existence of limits. Let us investigate the limit behavior of $u_{\Delta t}$, noting that the same considerations apply to $v_{\Delta t}$. We argue as in [19, Theorem 4.1, Step 1] (see also [3, § A.3.3]). We apply Theorem 7.2 to $\{u_{\Delta t}\}$ on the measure space

$$(X, \mathscr{A}, \mu) = (\Omega \times \Pi_T, \mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d), dP \otimes dt \otimes \phi\, dx).$$

By Corollary 4.3,

$$\sup_{\Delta t > 0} \left\{ E\left[\iint_{\Pi_T} |u_{\Delta t}|^2 \phi(x)\, dx dt\right] \right\} < \infty.$$

Hence there exists a Young measure $\nu = \nu_{t,x,\omega}$ such that for any Carathéodory function Ψ satisfying $\Psi(u_{\Delta t_j}, \cdot) \rightharpoonup \overline{\Psi}$ in $L^1(\Pi_T \times \Omega, \phi\, dx \otimes dt \otimes dP)$, it follows that

$$\overline{\Psi}(t, x, \omega) = \int_{\mathbb{R}} \Psi(\xi, t, x, \omega)\, d\nu_{t,x,\omega}(\xi).$$

Define [11, 30]

$$u(t, x, \alpha, \omega) := \inf \{\xi \in \mathbb{R} : \nu_{t,x,\omega}((-\infty, \xi]) > \alpha\}.$$

The representation (5.2) follows from the fact that $\mathcal{L} \circ u^{-1}(t, x, \cdot, \omega) = \nu_{t,x,\omega}$, where $\mathcal{L}$ denotes the Lebesgue measure on $[0, 1]$. For predictability and the fact that $u \in L^p([0, T] \times \Omega; L^p(\mathbb{R}^d \times [0, 1], \phi))$, see [19, Theorem 4.1] and also [3, § A.3.3], [30, § 3].

2. Independence of interpolation. Denote by v the limit of $\{v_{\Delta t}\}$, see Step 1. We want to show that $v = u$. By [31, Lemma 6.3], this holds true if

$$\mathscr{T}(\Delta t) := E\left[\iint_{\Pi_T} |u_{\Delta t}(t,x) - v_{\Delta t}(t,x)| \, \phi(x) \, dt dx\right] \to 0 \text{ as } \Delta t \downarrow 0. \qquad (5.3)$$

To see this, observe that

$$\mathscr{T} \le E\left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} |u_{\Delta t}(t,x) - u_{\Delta t}((t_n)+, x)| \, \phi(x) \, dt dx\right]$$

$$+ E\left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} |v_{\Delta t}((t_{n+1})-, x) - v_{\Delta t}(t,x)| \, \phi(x) \, dt dx\right]$$

$$=: \mathscr{T}_1 + \mathscr{T}_2.$$

By Proposition 4.8 (i),

$$\mathscr{T}_1 \le C_{T,\phi} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sqrt{t - t_n} \, dt = \frac{2}{3} C_{T,\phi} T \sqrt{\Delta t}.$$

By Proposition 4.8 (ii),

$$\mathscr{T}_2 \le C_{T,\phi} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (t_{n+1} - t)^\kappa \, dt \le C_{T,\phi} T \Delta t^\kappa,$$

where κ is defined in Proposition 4.8. This proves (5.3).

3. Entropy inequality. We need to prove that u is a Young measure-valued entropy solution in the sense of [19, Definition 2.2]. The result then follows from [19, Theorem 5.1]. Let S, V, φ be as in Lemma 5.1 and define

$$\mathscr{T}_{\Delta t} := \sum_{n=0}^{N-1} E\left[\int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} \big(S(v_{\Delta t}(t,x) - V) - S(v_{\Delta t}((t_{n+1})-, x) - V)\big) \partial_t \varphi \, dx dt\right].$$

We want to show that $\mathscr{T}_{\Delta t} \to 0$ as $\Delta t \downarrow 0$. Recall the definition of the weighted L^∞-norm (2.1). By Proposition 4.8,

$$|\mathscr{T}_{\Delta t}| \le \|S\|_{\mathrm{Lip}} \sup_{t \in [0,T]} \left\{\|\partial_t \varphi\|_{\infty, \phi^{-1}}\right\}$$

$$\times \sum_{n=0}^{N-1} E\left[\int_{t_n}^{t_{n+1}} \int_{\mathbb{R}^d} |v_{\Delta t}(t,x) - v_{\Delta t}((t_{n+1})-, x)| \, \phi(x) \, dx dt\right]$$

$$\le \|S\|_{\mathrm{Lip}} \sup_{t \in [0,T]} \left\{\|\partial_t \varphi\|_{\infty, \phi^{-1}}\right\} C_{T,\phi} T \Delta t^\kappa,$$

as in the proof of Step 2. Concerning the remaining terms in Lemma 5.1, the limit $\Delta t \downarrow 0$ is treated exactly as in [19, Proof of Theorem 4.1, Step 2]. It follows that u is a Young measure-valued entropy solution. $\qquad \square$

6. Error estimate

We now restrict our attention to the case

$$\sigma(x, u) = \sigma(u), \qquad \sigma \in L^\infty. \qquad (\mathcal{A}_{\sigma,2})$$

As mentioned in the introduction, for homogeneous noise functions $\sigma = \sigma(u)$, whenever $E\left[\|\nabla u_0\|_{1,\phi}\right] < \infty$, the entropy solution u to (1.1) satisfies a spatial BV estimate of the form

$$E\left[\int_{\mathbb{R}^d} |\nabla u(t,x)| \, \phi(x) \, dx\right] \le C, \qquad (0 \le t \le T), \qquad (6.1)$$

for some finite constant C (depending on u_0, f, ϕ, σ, T). Here $\nabla u(t, \cdot)$ is a (locally finite) measure and $\phi \in \mathfrak{N}$. This can be seen as a consequence of the fractional space translation estimate (2.3) and Remark 4.7. A direct verification of (6.1) can also be found in [6, Theorem 2.1] (when $\phi \equiv 1$). The same estimate is available for the operator splitting solution, cf. Proposition 4.4.

For the error estimate, we consider yet another time interpolation $\eta_{\Delta t}$ of the operator splitting $\{u^n\}_{n=0}^N$. Inspired by [24], let

$$\eta_{\Delta t}(t) := \underbrace{(\mathcal{S}_{\text{SDE}}(t, t_n) - \mathcal{I})\mathcal{S}_{\text{CL}}(\Delta t)u^n}_{u_{\Delta t}(t) - \mathcal{S}_{\text{CL}}(\Delta t)u^n} + \underbrace{\mathcal{S}_{\text{CL}}(t - t_n)u^n}_{v_{\Delta t}(t)}, \qquad t \in [t_n, t_{n+1}]. \quad (6.2)$$

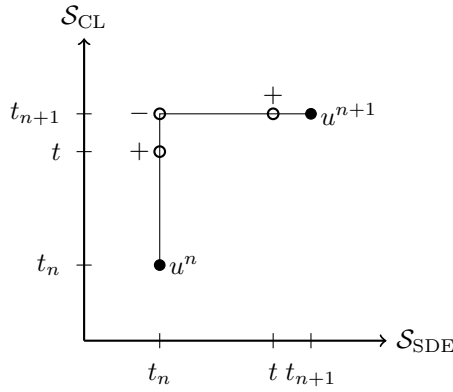A graphical representation of the interpolation $\eta_{\Delta t}$ is given in Figure 2.

FIGURE 2. A graphical representation of $\eta_{\Delta t}$. The value of $\eta_{\Delta t}(t)$ corresponds to summing (with signs) the values taken at the unfilled dots.

Theorem 6.1. *Fix $\phi \in \mathfrak{N}$. Suppose $(\mathcal{A}_f)$, $(\mathcal{A}_{f,1})$, $(\mathcal{A}_\sigma)$, and $(\mathcal{A}_{\sigma,2})$ are satisfied. Suppose also that $u^0, u_0 \in L^2(\Omega, \mathscr{F}_0, P; L^2(\mathbb{R}^d, \phi))$ satisfy (4.23) with $\kappa_0 = 1$. Let u be the entropy solution of (1.1)–(1.2) according to Definition 2.1 with initial condition u_0, and let $\eta_{\Delta t}$ be defined by (6.2). Then there exists a constant C, independent of Δt but dependent on $\sigma, f, T, \phi, u_0, u^0$, such that*

$$E\left[\|u(t) - \eta_{\Delta t}(t)\|_{1,\phi}\right] \le e^{C_\phi \|f\|_{\text{Lip}} t} \left(E\left[\|u_0 - u^0\|_{1,\phi}\right] + C\Delta t^{\frac{1}{3}}\right), \quad t \in [0, T].$$

The proof is split into several parts, the results of which are gathered towards the end of the section. To help motivate the upcoming technical arguments, let us outline a "high-level" overview of the main idea, assuming that all relevant functions are smooth in x and the spatial dimension is $d = 1$.

The function $\eta_{\Delta t}$ defined in (6.2) ought to satisfy an "approximate" entropy inequality. Formally, we have

$$d\eta_{\Delta t} + \partial_x f(v_{\Delta t}) \, dt = \sigma(u_{\Delta t}) \, dB, \qquad (6.3)$$

indicating that the error terms can be expressed as perturbations of the coefficients f, σ. Let u be a smooth (in x) solution of (1.1). By (6.3),

$$d(\eta_{\Delta t} - u) = -\partial_x(f(v_{\Delta t}) - f(u)) \, dt + (\sigma(u_{\Delta t}) - \sigma(u)) \, dB,$$

and thus the Itô formula gives

$$dS(\eta_{\Delta t} - u) = -S'(\eta_{\Delta t} - u)\partial_x(f(v_{\Delta t}) - f(u)) \, dt + S'(\eta_{\Delta t} - u)(\sigma(u_{\Delta t}) - \sigma(u)) \, dB$$
$$+ \frac{1}{2}S''(\eta_{\Delta t} - u)(\sigma(u_{\Delta t}) - \sigma(u))^2 \, dt,$$

for any $S \in C^2(\mathbb{R})$. Upon adding and subtracting identical terms and taking expectations, we arrive at

$$
\begin{aligned}
E\left[dS(\eta_{\Delta t} - u)\right] = & - E\left[S'(\eta_{\Delta t} - u)\partial_x(f(\eta_{\Delta t}) - f(u))\,dt\right] \\
& + \frac{1}{2}E\left[S''(\eta_{\Delta t} - u)(\sigma(\eta_{\Delta t}) - \sigma(u))^2\,dt\right] \\
& + E\left[S'(\eta_{\Delta t} - u)\partial_x(f(\eta_{\Delta t}) - f(v_{\Delta t}))\,dt\right] \\
& + E\left[S''(\eta_{\Delta t} - u)\left(\int_{\eta_{\Delta t}}^{u_{\Delta t}}(\sigma(z) - \sigma(u))\sigma'(z)\,dz\right)dt\right].
\end{aligned}
$$

The first two terms vanish as $S \to |\cdot|$. Note that these terms also appear in the uniqueness argument, when two exact solutions are compared. Accordingly, they should not be thought of as error terms originating from the splitting procedure. The last two terms, however, are genuine error terms associated with the operator splitting and the interpolation $\eta_{\Delta t}$. All of the above terms may be recognized in the forthcoming Lemma 6.2. The above simplified representation provides intuition on how to estimate these error terms. This is in particular the case concerning the third term on the right-hand side. To this end, note that

$$
\eta_{\Delta t} - v_{\Delta t} = u_{\Delta t} - \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n = \int_{t_n}^t \sigma(u_{\Delta t}(s))\,dB(s),
$$

for $t_n \le t < t_{n+1}$. Consequently,

$$
\begin{aligned}
\partial_x(f(\eta_{\Delta t}) - f(v_{\Delta t})) = & \left(f'(\eta_{\Delta t}) - f'(v_{\Delta t})\right)\partial_x v_{\Delta t} \\
& + f'(\eta_{\Delta t})\int_{t_n}^t \partial_x \sigma(u_{\Delta t}(s))\,dB(s).
\end{aligned}
\tag{6.4}
$$

Furthermore,

$$
\begin{aligned}
E\left[|\left(f'(\eta_{\Delta t}) - f'(v_{\Delta t})\right)\partial_x v_{\Delta t}|\right] & \\
& \le \|f'\|_{\mathrm{Lip}} E\left[E\left[\left|\int_{t_n}^t \sigma(u_{\Delta t}(s))\,dB(s)\right|\,\Big|\,\mathscr{F}_{t_n}\right]|\partial_x v_{\Delta t}|\right],
\end{aligned}
$$

which provides a way to estimate the term since $v_{\Delta t}(t) \in BV$ and $\sigma \in L^\infty$.

Due to the lack of regularity we will work with an approximation of $\eta_{\Delta t}$. Given $\{w^n = w^n(x)\}_{n=0}^{N-1}$, we set

$$
\psi(t) := (\mathcal{S}_{\mathrm{SDE}}(t, t_n) - \mathcal{I})w^n, \qquad t \in [t_n, t_{n+1}),
\tag{6.5}
$$

and $\tilde{\eta} := \psi + v_{\Delta t}$. Note that $\eta_{\Delta t} = \psi + v_{\Delta t}$ whenever $w^n = \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$ for $n = 0, \ldots, N-1$. However, due to the lack of differentiability of $\mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$, we will work with a sequence $\{w_k^n\}_{k \ge 1}$ of smooth functions satisfying $w_k^n \to \mathcal{S}_{\mathrm{CL}}(\Delta t)u^n$ in $L^1(\Omega; L^1(\mathbb{R}^d, \phi))$ as $k \to \infty$. To simplify notation we suppress the dependence on k and write $w^n = w_k^n$.

Proposition 6.1. *Suppose $(\mathcal{A}_f)$, $(\mathcal{A}_\sigma)$, and $(\mathcal{A}_{\sigma,2})$ are satisfied. Let $\tilde{\eta} = \psi + v_{\Delta t}$, where ψ and $v_{\Delta t}$ are defined in (6.5) and (3.5), respectively. Then, for all nonnegative $\phi \in C_c^\infty([t_n, t_{n+1}] \times \mathbb{R}^d)$, any $V \in \mathbb{D}^{1,2}$, and all entropy/entropy-flux pairs $(S, Q) \in \mathscr{E}$,*

$$
\begin{aligned}
& E\left[\int_{\mathbb{R}^d} S(\tilde{\eta}(t_n, x) - V)\phi(t_n, x)\,dx\right] \\
& \quad - E\left[\int_{\mathbb{R}^d} S(\tilde{\eta}((t_{n+1})-, x) - V)\phi(t_{n+1}, x)\,dx\right] \\
& \quad + E\left[\iint_{\Pi_n} S(\tilde{\eta} - V)\partial_t \phi + Q(\tilde{\eta}, V) \cdot \nabla\phi\,dxdt\right]
\end{aligned}
$$

$$+ E\left[\iint_{\Pi_n} \int_V^{\tilde{\eta}} S'(z - V)\left(f'(z - \psi) - f'(z)\right) dz \cdot \nabla\phi\, dxdt\right]$$

$$+ E\left[\iint_{\Pi_n} \int_V^{\tilde{\eta}} S''(z - V)f'(z - \psi)\, dz \cdot \nabla\psi\, \phi\, dxdt\right]$$

$$- E\left[\iint_{\Pi_n} S''(\tilde{\eta} - V)D_t V \sigma(\psi + w^n)\phi\, dxdt\right]$$

$$+ \frac{1}{2} E\left[\iint_{\Pi_n} S''(\tilde{\eta} - V)\sigma^2(\psi + w^n)\phi\, dxdt\right] \geq 0,$$

where $\Pi_n = [t_n, t_{n+1}] \times \mathbb{R}$.

The proof of Proposition 6.1 is deferred to Section 7.1. To ensure that the relevant quantities are Malliavin differentiable, we replace the entropy solution u by the viscous approximation u^ε, which solves

$$du^\varepsilon + \nabla \cdot f(u^\varepsilon)dt = \sigma(x, u^\varepsilon)dB(t) + \varepsilon\Delta u^\varepsilon dt, \qquad u^\varepsilon(0) = u_0,$$

and then send $\varepsilon \downarrow 0$ at a later stage. Let us recall that $\{D_r u^\varepsilon(t)\}_{t>r}$ is a predictable weak solution to the linear problem

$$dw + \nabla \cdot (f'(u^\varepsilon)w)dt = \sigma'(x, u^\varepsilon)wdB(t) + \varepsilon\Delta w dt, \qquad w(r) = \sigma(u^\varepsilon(r)),$$

for almost all $r \in [0, T]$, cf. [19, § 3]. Furthermore,

$$\operatorname*{ess\,sup}_{r \in [0,T]} \left\{ \sup_{t \in [0,T]} E\left[\|D_r u^\varepsilon(t)\|_{2,\phi}^2\right] \right\} < \infty.$$

As a consequence of [19, Theorem 5.1] and [31, Proposition 6.12], we have $u^\varepsilon \to u$ in $L^1([0,T] \times \Omega; L^1(\mathbb{R}^d, \phi))$ as $\varepsilon \downarrow 0$. In fact, under the assumptions of Theorem 6.1, $u^\varepsilon \to u$ with rate $1/2$ [6, Theorem 5.2].

We may now proceed with the doubling-of-the-variables argument.

Lemma 6.2. *Fix $\phi \in \mathfrak{N}$. Let $u^\varepsilon = u^\varepsilon(s, y)$ be the viscous approximation of (1.1). Take $w(t, x) = w^n(x)$ for $t \in [t_n, t_{n+1})$, and let $\psi = \psi(t, x)$, $v_{\Delta t} = v_{\Delta t}(t, x)$, and $\tilde{\eta} = \tilde{\eta}(t, x)$ be defined in Proposition 6.1. Let $t_0 \in [0, T)$, and pick $\gamma, r_0, r > 0$ such that $t_0 \leq T - 2(\gamma + r_0)$. Define*

$$\xi_\gamma(t) = 1 - \int_0^t J_\gamma^+(s - t_0)\, ds.$$

Furthermore, let

$$\varphi(t, x, s, y) = \frac{1}{2^d}\phi\left(\frac{x + y}{2}\right) J_r\left(\frac{x - y}{2}\right) J_{r_0}^+(s - t)\xi_\gamma(t),$$

and S_δ be defined in (4.9). Then

$$L - R + F + \mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3 + \mathscr{T}_4 + \mathscr{T}_5 + \mathscr{T}_6 \geq 0, \tag{6.6}$$

where

$$L = E\left[\iint_{\Pi_T} \int_{\mathbb{R}^d} S_\delta(\tilde{\eta}(0, x) - u^\varepsilon(s, y))\varphi(0, x, s, y)\, dxdsdy\right],$$

$$R = -E\left[\iiiint_{\Pi_T^2} S_\delta(\tilde{\eta} - u^\varepsilon)(\partial_t + \partial_s)\varphi\, dX\right],$$

$$F = E\left[\iiiint_{\Pi_T^2} Q(u^\varepsilon, \tilde{\eta}) \cdot \nabla_y \varphi + Q(\tilde{\eta}, u^\varepsilon) \cdot \nabla_x \varphi\, dX\right],$$

$$\mathscr{T}_1 = \frac{1}{2} E\left[\iiiint_{\Pi_T^2} S_\delta''(u^\varepsilon - \tilde{\eta})(\sigma(u^\varepsilon) - \sigma(\tilde{\eta}))^2 \varphi \, dX \right],$$

$$\mathscr{T}_2 = E\left[\iiiint_{\Pi_T^2} S_\delta''(u^\varepsilon - \tilde{\eta})(\sigma(u^\varepsilon) - D_t u^\varepsilon)\sigma(\psi + w)\varphi \, dX \right],$$

$$\mathscr{T}_3 = E\left[\iiiint_{\Pi_T^2} S_\delta''(u^\varepsilon - \tilde{\eta}) \left(\int_{\tilde{\eta}}^{\psi+w} (\sigma(z) - \sigma(u^\varepsilon))\sigma'(z) \, dz \right) \varphi \, dX \right],$$

$$\mathscr{T}_4 = E\left[\iiiint_{\Pi_T^2} \int_{u^\varepsilon}^{\tilde{\eta}} S_\delta'(z - u^\varepsilon)\left(f'(z - \psi) - f'(z)\right) dz \cdot \nabla_x \varphi \, dX \right]$$
$$+ E\left[\iiiint_{\Pi_T^2} \int_{u^\varepsilon}^{\tilde{\eta}} S_\delta''(z - u^\varepsilon)f'(z - \psi) \, dz \cdot \nabla_x \psi \varphi \, dX \right],$$

$$\mathscr{T}_5 = \varepsilon E\left[\iiiint_{\Pi_T^2} S_\delta(u^\varepsilon - \tilde{\eta})\Delta_y \varphi \, dX \right],$$

$$\mathscr{T}_6 = \sum_{n=0}^{N-1} E\left[\iint_{\Pi_T} \int_{\mathbb{R}^d} \Big(S_\delta(\tilde{\eta}((t_{n+1}), x) - u^\varepsilon(s, y)) \right.$$
$$\left. - S_\delta(\tilde{\eta}((t_{n+1})-, x) - u^\varepsilon(s, y)) \Big) \varphi(t_{n+1}, x, s, y) \, dxdsdy \right],$$

where $dX = dxdtdsdy$.

Proof. Let us first assume $\phi \in C_c^\infty(\mathbb{R}^d)$, as the result for $\phi \in \mathfrak{N}$ then follows from an approximation argument. After a standard application of Itô's formula to $u^\varepsilon(s, y) \mapsto S_\delta(u^\varepsilon(s, y) - \tilde{\eta}(t, x))\varphi(s)$ for $s \geq t$, we arrive at

$$E\left[\iiiint_{\Pi_T^2} S_\delta(u^\varepsilon - \tilde{\eta})\partial_s \varphi + Q(u^\varepsilon, \tilde{\eta}) \cdot \nabla_y \varphi \, dX \right]$$
$$+ \frac{1}{2} E\left[\iiiint_{\Pi_T^2} S_\delta''(u^\varepsilon - \tilde{\eta})\sigma^2(u^\varepsilon)\varphi \, dX \right] + \varepsilon E\left[\iiiint_{\Pi_T^2} S_\delta(u^\varepsilon - \tilde{\eta})\Delta_y \varphi \, dX \right] \geq 0,$$

cf. [19, Lemma 5.3]. Take $V = u^\varepsilon(s, y)$ in Proposition 6.1, integrate in $(s, y) \in \Pi_T$, and sum over $n = 0, \ldots, N - 1$. The outcome is

$$E\left[\iint_{\Pi_T} \int_{\mathbb{R}^d} S_\delta(\tilde{\eta}(0, x) - u^\varepsilon(s, y))\varphi(0, x, s, y) \, dxdsdy \right]$$
$$+ E\left[\iiiint_{\Pi_T^2} S_\delta(\tilde{\eta} - u^\varepsilon)\partial_t \varphi + Q(\tilde{\eta}, u^\varepsilon) \cdot \nabla_x \varphi \, dX \right]$$
$$+ E\left[\iiiint_{\Pi_T^2} \int_{u^\varepsilon}^{\tilde{\eta}} S_\delta'(z - u^\varepsilon)\left(f'(z - \psi) - f'(z)\right) dz \cdot \nabla_x \varphi \, dX \right]$$
$$+ E\left[\iiiint_{\Pi_T^2} \int_{u^\varepsilon}^{\tilde{\eta}} S_\delta''(z - u^\varepsilon)f'(z - \psi) \, dz \cdot \nabla_x \psi \varphi \, dX \right]$$
$$- E\left[\iiiint_{\Pi_T^2} S_\delta''(\tilde{\eta} - u^\varepsilon)D_t u^\varepsilon \sigma(\psi + w)\varphi \, dX \right]$$
$$+ \frac{1}{2} E\left[\iiiint_{\Pi_T^2} S_\delta''(\tilde{\eta} - u^\varepsilon)\sigma^2(\psi + w)\varphi \, dX \right]$$

$$+ \sum_{n=0}^{N-1} E\left[\iint_{\Pi_T} \int_{\mathbb{R}^d} \Big(S_\delta(\tilde{\eta}((t_{n+1}), x) - u^\varepsilon(s, y))\right.$$
$$\left. - S_\delta(\tilde{\eta}((t_{n+1})-, x) - u^\varepsilon(s, y))\Big)\varphi(t_{n+1}, x, s, y)\, dxdsdy\right] \geq 0.$$

The lemma follows upon adding the two previous inequalities, noting that

$$\frac{1}{2}\sigma^2(u^\varepsilon) - D_t u^\varepsilon \sigma(\psi + w) + \frac{1}{2}\sigma^2(\psi + w)$$
$$= \frac{1}{2}(\sigma(\psi + w) - \sigma(u^\varepsilon))^2 + (\sigma(u^\varepsilon) - D_t u^\varepsilon)\sigma(\psi + w)$$
$$= \frac{1}{2}(\sigma(\tilde{\eta}) - \sigma(u^\varepsilon))^2 + \int_{\tilde{\eta}}^{\psi + w} (\sigma(z) - \sigma(u^\varepsilon))\sigma'(z)\, dz$$
$$+ (\sigma(u^\varepsilon) - D_t u^\varepsilon)\sigma(\psi + w).$$
$$\square$$

In the following we estimate the terms appearing in Lemma 6.2. The underlying assumptions are the ones made in Theorem 6.1. We let C denote a generic constant, meaning that it is independent of the "small" parameters $\Delta t, r, r_0, \gamma, \varepsilon, \delta$. Furthermore, given a term $\mathscr{T}$, we write $\mathscr{T} = \mathcal{O}(g(\Delta t, \ldots, \delta))$ whenever $|\mathscr{T}| \leq Cg(\Delta t, \ldots, \delta)$ for some nonnegative function g.

Estimate 6.1. *Let L be defined in Lemma 6.2. Then*

$$\limsup_{r_0 \downarrow 0} L \leq E\left[\left\|u_0 - u^0\right\|_{1,\phi}\right] + \mathcal{O}\left(\delta + r\right).$$

Proof. By (4.10),

$$\left|S_\delta(\tilde{\eta}(0, x) - u^\varepsilon(s, y)) - |\tilde{\eta}(0, x) - u^\varepsilon(s, y)|\right| \leq \delta.$$

By the reverse triangle inequality

$$\left|\,|\tilde{\eta}(0, x) - u^\varepsilon(s, y)| - |\tilde{\eta}(0, x) - u_0(y)|\,\right| \leq |u^\varepsilon(s, y) - u_0(y)|,$$
$$\left|\,|\tilde{\eta}(0, x) - u_0(y)| - |\tilde{\eta}(0, x) - u_0(x)|\,\right| \leq |u_0(y) - u_0(x)|.$$

Hence, after adding and subtracting identical terms, noting that $\tilde{\eta}(0) = u^0$, it follows by the triangle inequality that

$$\left|S_\delta(\tilde{\eta}(0, x) - u^\varepsilon(s, y)) - |u^0(x) - u_0(x)|\,\right|$$
$$\leq \delta + |u^\varepsilon(s, y) - u_0(y)| + |u_0(y) - u_0(x)|.$$

By (4.15),

$$\left|L - E\left[\left\|u^0 - u_0\right\|_{1,\phi \star J_r}\right]\right|$$
$$\leq \delta \left\|\phi\right\|_{L^1(\mathbb{R}^d)} + \underbrace{\int_0^T E\left[\left\|u^\varepsilon(s) - u_0\right\|_{1,\phi \star J_r}\right] J_{r_0}^+(s)\, ds}_{\mathscr{Z}_1}$$
$$+ \underbrace{E\left[\frac{1}{2^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_0(y) - u_0(x)| \phi\left(\frac{x + y}{2}\right) J_r\left(\frac{x - y}{2}\right) dxdy\right]}_{\mathscr{Z}_2}.$$

Thanks to [19, Lemma 2.3], $\mathscr{Z}_1 \to 0$ as $r_0 \to 0$. Regarding $\mathscr{Z}_2$ we apply (4.14). As u_0 satisfies (4.23) with $\kappa_0 = 1$,

$$\mathscr{Z}_2 = E\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_0(x + z) - u_0(x - z)|\, \phi(x) J_r(z)\, dxdz\right] = \mathcal{O}(r).$$

Finally, we apply Lemma 7.3 to conclude that

$$\left| E\left[\left\| u^0 - u_0 \right\|_{1,\phi \star J_r} - \left\| u^0 - u_0 \right\|_{1,\phi} \right] \right| = \mathcal{O}(r).$$

$\square$

Estimate 6.2. *Let R be defined in Lemma 6.2. Then*

$$\liminf_{\varepsilon,r_0 \downarrow 0} R \geq E\left[\int_0^T \|\tilde\eta(t) - u(t)\|_{1,\phi} \, J_\gamma^+(t - t_0) \, dt \right] + \mathcal{O}\left(\delta + r \right).$$

Proof. It is easy to check that

$$R = E\left[\iiiint_{\Pi_T^2} S_\delta(\tilde\eta(t,x) - u^\varepsilon(s,y)) \frac{1}{2^d} \phi\left(\frac{x+y}{2} \right) \right.$$
$$\left. \times J_r\left(\frac{x-y}{2} \right) J_{r_0}^+(s-t) J_\gamma^+(t-t_0) \, dX \right].$$

Moreover, adding and subtracting identical terms, we obtain

$$|S_\delta(\tilde\eta(t,x) - u^\varepsilon(s,y)) - |\tilde\eta(t,x) - u^\varepsilon(t,x)|| \leq$$
$$\delta + |u^\varepsilon(s,y) - u^\varepsilon(t,y)| + |u^\varepsilon(t,y) - u^\varepsilon(t,x)|,$$

and so

$$\left| R - E\left[\int_0^T \|\tilde\eta(t) - u^\varepsilon(t)\|_{1,\phi \star J_r} \, J_\gamma^+(t-t_0) \, dt \right] \right|$$

$$\leq \delta \left\| \phi \right\|_{L^1(\mathbb{R}^d)} + \underbrace{E\left[\iint_{[0,T]^2} \|u^\varepsilon(s) - u^\varepsilon(t)\|_{\phi \star J_r} \, J_{r_0}^+(s-t) J_\gamma^+(t-t_0) \, dsdt \right]}_{\mathscr{Z}_1}$$

$$+ \underbrace{E\left[\iint_{\Pi_T} \int_{\mathbb{R}^d} |u^\varepsilon(t,y) - u^\varepsilon(t,x)| \frac{1}{2^d} \phi\left(\frac{x+y}{2} \right) J_r\left(\frac{x-y}{2} \right) J_\gamma^+(t-t_0) \, dxdydt \right]}_{\mathscr{Z}_2}.$$

Owing to Lemma 7.4, $\lim_{r_0 \downarrow 0} \mathscr{Z}_1 = 0$. Next, we utilize the strong convergence $u^\varepsilon \to u$ in $L^1([0,T] \times \Omega; L^1(\mathbb{R}^d, \phi))$ and (4.14) to conclude that

$$\lim_{\varepsilon,r_0 \downarrow 0} \mathscr{Z}_2 = \int_0^T E\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} |u(t,x+z) - u(t,x-z)| \, \phi(x) J_r(z) \, dxdz \right] J_\gamma^+(t-t_0) dt.$$

It follows from [19, Proposition 5.2] and the assumption (4.23) with $\kappa_0 = 1$ that $|\lim_{\varepsilon,r_0 \downarrow 0} \mathscr{Z}_2| = \mathcal{O}(r)$. The claim is now a consequence of Lemma 7.3. $\square$

Estimate 6.3. *Let F be defined in Lemma 6.2. Then*

$$\limsup_{\varepsilon,r_0 \downarrow 0} F \leq C_\phi \|f\|_{\mathrm{Lip}} E\left[\int_0^T \|u(t) - \tilde\eta(t)\|_{1,\phi} \, \xi_\gamma(t) \, dt \right] + \mathcal{O}\left(\delta \left(1 + \frac{1}{r} \right) + r \right).$$

Proof. Observe that

$$F = F_1 + F_2 + F_3, \tag{6.7}$$

where

$$F_1 := E\left[\iiiint_{\Pi_T^2} S_\delta'(u^\varepsilon - \tilde\eta)(f(u^\varepsilon) - f(\tilde\eta))(\nabla_x + \nabla_y)\varphi \, dX \right],$$

$$F_2 := -E\left[\iiiint_{\Pi_T^2} \int_{u^\varepsilon}^{\tilde\eta} S_\delta''(z - u^\varepsilon)(f(z) - f(u^\varepsilon)) \, dz \cdot \nabla_x \varphi \, dX \right],$$

$$F_3 := -E\left[\iiiint_{\Pi_T^2}\int_{\tilde\eta}^{u^\varepsilon}S_\delta''(z-\tilde\eta)(f(z)-f(\tilde\eta))\,dz\cdot\nabla_y\varphi\,dX\right].$$

The decomposition (6.7) follows from the identities

$$Q_\delta(u^\varepsilon,\tilde\eta) = S_\delta'(u^\varepsilon-\tilde\eta)(f(u^\varepsilon)-f(\tilde\eta))-\int_{\tilde\eta}^{u^\varepsilon}S_\delta''(z-\tilde\eta)(f(z)-f(\tilde\eta))\,dz,$$

$$Q_\delta(\tilde\eta,u^\varepsilon) = S_\delta'(\tilde\eta-u^\varepsilon)(f(\tilde\eta)-f(u^\varepsilon))-\int_{u^\varepsilon}^{\tilde\eta}S_\delta''(z-u^\varepsilon)(f(z)-f(u^\varepsilon))\,dz,$$

derived using integration by parts.

Next, we claim that

$$|F_2|+|F_3| = \mathcal{O}\left(\delta\left(1+\frac{1}{r}\right)\right). \tag{6.8}$$

We consider F_2; the F_3 term is estimated likewise. Note that

$$\left|\int_{u^\varepsilon}^{\tilde\eta}S_\delta''(z-u^\varepsilon)(f(z)-f(u^\varepsilon))\,dz\right| \le \|f\|_{\mathrm{Lip}}\,\delta.$$

Hence,

$$|F_2| \le \|f\|_{\mathrm{Lip}}\,\delta E\left[\iiiint_{\Pi_T^2}|\nabla_x\varphi|\,dX\right].$$

By a straightforward computation,

$$\iiiint_{\Pi_T^2}|\nabla_x\varphi|\,dX \le \frac{1}{2}T\left(C_\phi+\|\nabla J\|_{L^1(\mathbb{R}^d)}\frac{1}{r}\right)\|\phi\|_{L^1(\mathbb{R}^d)}.$$

This proves (6.8).

Next, we claim that

$$\limsup_{\varepsilon,r_0\downarrow 0}F_1 \le C_\phi\,\|f\|_{\mathrm{Lip}}\,E\left[\int_0^T\|u(t)-\tilde\eta(t)\|_{1,\phi*J_r}\,\xi_\gamma(t)\,dt\right]+\mathcal{O}\left(\delta+r\right). \tag{6.9}$$

Set

$$\mathcal{F}_\delta(b,a) = S_\delta'(b-a)(f(b)-f(a)).$$

Then

$$|\mathcal{F}_\delta(b,a)-\mathcal{F}_\delta(c,a)| = \left|\int_c^b\partial_z\big(S_\delta'(z-a)(f(z)-f(a))\big)\,dz\right|$$
$$\le 2\,\|f\|_{\mathrm{Lip}}\,\delta+\|f\|_{\mathrm{Lip}}\,|b-c|\,;$$

whence

$$|\mathcal{F}_\delta(u^\varepsilon(s,y),\tilde\eta(t,x))-\mathcal{F}_\delta(u^\varepsilon(t,x),\tilde\eta(t,x))|$$
$$\le \|f\|_{\mathrm{Lip}}\left(2\delta+|u^\varepsilon(s,y)-u^\varepsilon(t,y)|+|u^\varepsilon(t,y)-u^\varepsilon(t,x)|\right),$$

and so

$$\left|F_1-E\left[\iint_{\Pi_T}\mathcal{F}_\delta(u^\varepsilon(t,x),\tilde\eta(t,x))\cdot(\nabla\phi*J_r)(x)\xi_\gamma(t)\,dxdt\right]\right|$$

$$\le C_\phi\,\|f\|_{\mathrm{Lip}}\,E\left[\iint_{[0,T]^2}\|u^\varepsilon(s)-u^\varepsilon(t)\|_{1,\phi\star J_r}\,J_{r_0}^+(s-t)\xi_\gamma(t)\,dsdt\right]$$

$$+C_\phi\,\|f\|_{\mathrm{Lip}}\,E\left[\int_0^T\iint_{\mathbb{R}^d\times\mathbb{R}^d}|u^\varepsilon(t,x+z)-u^\varepsilon(t,x-z)|\,J_r(z)\xi_\gamma(t)\phi(x)\,dxdzdt\right]$$

$$+2\delta\,\|f\|_{\mathrm{Lip}}\,T\,\|\nabla\phi\|_{L^1(\mathbb{R}^d)}\,,$$

where we have made a change of variables as in Estimate 6.2. Following the same reasoning as in that estimate we arrive at

$$\limsup_{\varepsilon, r_0 \downarrow 0} F_1 \le E\left[\iint_{\Pi_T} \mathcal{F}_\delta(u(t,x), \tilde{\eta}(t,x)) \cdot (\nabla \phi * J_r)(x) \xi_\gamma(t) \, dx dt \right] + \mathcal{O}\left(\delta + r \right).$$

Inequality (6.9) follows from $\mathcal{F}_\delta(a,b) \le \|f\|_{\mathrm{Lip}} |a - b|$ and $|\nabla \phi| \le C_\phi \phi$. Combining the above estimates for F_1, F_2, F_3 concludes the proof of the claim. $\qquad \square$

Estimate 6.4. *Let $\mathscr{T}_1$ be defined in Lemma 6.2. Then*

$$|\mathscr{T}_1| \le C\delta.$$

Proof. Since $S_\delta'' = 2J_\delta$,

$$S_\delta''(u^\varepsilon - \tilde{\eta})(\sigma(u^\varepsilon) - \sigma(\tilde{\eta}))^2 \le 2 \|\sigma\|_{\mathrm{Lip}}^2 J_\delta(u^\varepsilon - \tilde{\eta}) |u^\varepsilon - \tilde{\eta}|^2 \le 2 \|\sigma\|_{\mathrm{Lip}}^2 \|J\|_\infty \delta.$$

Due to (4.15) and Young's inequality for convolutions,

$$\iiiint_{\Pi_T^2} \varphi \, dX = \left(\int_0^T \int_0^T J_{r_0}^+(s-t)\xi_\gamma(t) \, ds dt \right) \left(\int_{\mathbb{R}^d} \phi \star J_r(x) \, dx \right) \le T \|\phi\|_{L^1(\mathbb{R}^d)}.$$

The result follows. $\qquad \square$

Estimate 6.5. *Let $\mathscr{T}_2$ be defined in Lemma 6.2. Then*

$$\lim_{r_0 \downarrow 0} \mathscr{T}_2 = 0.$$

Proof. This follows exactly as in [19, Limit 5]. However, the assumption $\sigma \in L^\infty$ simplifies the analysis and allows for $\phi \in \mathfrak{N}$ instead of $C_c^\infty(\mathbb{R}^d)$. $\qquad \square$

Estimate 6.6. *Let $\mathscr{T}_3$ be defined in Lemma 6.2. Then*

$$|\mathscr{T}_3| \le C\frac{1}{\delta} E\left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|w^n - v_{\Delta t}(t)\|_{\phi \star J_r} \, dt \right].$$

Proof. Now, as $\tilde{\eta} = \psi + v_{\Delta t}$,

$$\left| \int_{\tilde{\eta}}^{\psi + w} (\sigma(z) - \sigma(u^\varepsilon))\sigma'(z) \, dz \right| \le 2 \|\sigma\|_\infty \|\sigma\|_{\mathrm{Lip}} |w - v_{\Delta t}|.$$

Keep in mind that $w(t) = w^n$ for $t \in [t_n, t_{n+1})$. The estimate then follows from (4.10) and (4.15). $\qquad \square$

Estimate 6.7. *Let $\mathscr{T}_4$ be defined in Lemma 6.2. Then*

$$|\mathscr{T}_4| \le C\sqrt{\Delta t} \left(1 + E\left[\int_0^T \|\nabla w(t)\|_{1, \phi \star J_r} \right] \right).$$

Proof. The estimate is established under the assumption that $v_{\Delta t}$ is smooth in x. The general result follows by an approximation argument. Integrating by parts and using the chain rule,

$$\mathscr{T}_4 = E\left[\iiiint_{\Pi_T^2} \int_{u^\varepsilon}^{\tilde{\eta}} S_\delta'(z - u^\varepsilon) \left(f'(z - \psi) - f'(z) \right) dz \cdot \nabla_x \varphi \, dX \right]$$

$$= -E\left[\iiiint_{\Pi_T^2} S_\delta'(\tilde{\eta} - u^\varepsilon) \left(f'(v_{\Delta t}) - f'(\tilde{\eta}) \right) \cdot \nabla_x \tilde{\eta} \, \varphi \, dX \right]$$

$$+ E\left[\iiiint_{\Pi_T^2} \int_{u^\varepsilon}^{\tilde{\eta}} S_\delta'(z - u^\varepsilon) f''(z - \psi) \, dz \cdot \nabla_x \psi \, \varphi \, dX \right].$$

Next, we observe that

$$\int_{u^\varepsilon}^{\tilde\eta} S'_\delta(z - u^\varepsilon) f''(z - \psi)\, dz = -\int_{u^\varepsilon}^{\tilde\eta} S''_\delta(z - u^\varepsilon) f'(z - \psi)\, dz + S'_\delta(\tilde\eta - u^\varepsilon) f'(v_{\Delta t}).$$

Therefore,

$$\mathscr{T}_4 = \underbrace{E\left[\iiiint_{\Pi_T^2} S'_\delta(\tilde\eta - u^\varepsilon) f'(\tilde\eta) \cdot \nabla_x \psi\, \varphi\, dX\right]}_{\mathscr{Z}_1}$$

$$+ \underbrace{E\left[\iiiint_{\Pi_T^2} S'_\delta(\tilde\eta - u^\varepsilon)\left(f'(\tilde\eta) - f'(v_{\Delta t})\right) \cdot \nabla_x v_{\Delta t}\, \varphi\, dX\right]}_{\mathscr{Z}_2},$$

cf. (6.4). Consider $\mathscr{Z}_2$. Since $v_{\Delta t}(t)$ is $\mathscr{F}_{t_n}$-measurable for all $t \in [t_n, t_{n+1})$,

$$|\mathscr{Z}_2| \le E\left[\iiiint_{\Pi_T^2} |f'(\tilde\eta) - f'(v_{\Delta t})|\, |\nabla_x v_{\Delta t}|\, \varphi\, dX\right]$$

$$\le \|f'\|_{\mathrm{Lip}} \sum_{n=0}^{N-1} \iiiint_{\Pi_T \times \Pi_n} E\left[E\left[|\psi|\,\Big|\,\mathscr{F}_{t_n}\right] |\nabla_x v_{\Delta t}|\right] \varphi\, dX.$$

By definition,

$$\psi(t, x) = \int_{t_n}^t \sigma(\psi(r, x) + w^n(x))\, dB(r), \qquad t_n \le t < t_{n+1}. \tag{6.10}$$

Set

$$\tilde\psi(t, \lambda) = \int_{t_n}^t \sigma(\tilde\psi(r, \lambda) + \lambda)\, dB(r),$$

so that $\psi(t, x) = \tilde\psi(t, w^n(x))$. Consequently,

$$E\left[|\psi(t, x)|\,\Big|\,\mathscr{F}_{t_n}\right](\omega) = E\left[|\tilde\psi(t, \lambda)|\right]_{\lambda = w^n(x; \omega)}.$$

By the Burkholder-Davies-Gundy inequality, there is a constant $c_1 > 0$ such that

$$E\left[|\tilde\psi(t, \lambda)|\right] \le c_1 E\left[\left(\int_{t_n}^t \sigma^2(\tilde\psi(r, \lambda) + \lambda)\, dr\right)^{1/2}\right] \le c_1 \|\sigma\|_\infty \sqrt{t - t_n},$$

independent of $\lambda \in \mathbb{R}$. It follows from Proposition 4.4 that

$$|\mathscr{Z}_2| \le c_1 \|\sigma\|_\infty \|f'\|_{\mathrm{Lip}} \sqrt{\Delta t}\, E\left[\int_0^T \|\nabla_x v_{\Delta t}(t)\|_{1, \phi \star J_r}\, dt\right] \le C\sqrt{\Delta t}.$$

Consider $\mathscr{Z}_1$. In view of (4.15),

$$|\mathscr{Z}_1| \le \|f\|_{\mathrm{Lip}} E\left[\iiiint_{\Pi_T^2} |\nabla_x \psi|\, \varphi\, dX\right] \le \|f\|_{\mathrm{Lip}} E\left[\iint_{\Pi_T} |\nabla_x \psi|\, (\phi \star J_r)\, dx dt\right].$$

Differentiating (6.10) yields, for $t_n \le t < t_{n+1}$,

$$\nabla_x \psi(t, x) = \int_{t_n}^t \sigma'(\psi(r, x) + w^n(x))(\nabla_x \psi(r, x) + \nabla_x w^n(x))\, dB(r).$$

By Lemma 6.3 below there is a constant $C > 0$, depending only on σ, such that

$$E\left[|\nabla_x \psi(t, x)|\right] \le C\sqrt{t - t_n}\, E\left[|\nabla w^n(x)|\right], \qquad t_n \le t < t_{n+1}.$$

We conclude that

$$|\mathscr{Z}_1| \leq C \left(E \left[\int_0^T \|\nabla w(t)\|_{1,\phi * J_r} \right] dt \right) \sqrt{\Delta t}.$$

$\square$

Lemma 6.3. *Suppose* $h : [t_n, t_{n+1}] \times \Omega \to \mathbb{R}^d$ *is predictable and*

$$P\Big[\int_{t_n}^t |h(s)|^2 \, ds < \infty \Big] = 1.$$

Suppose $X(t_n) \in L^p(\Omega, \mathscr{F}_{t_n}, P; \mathbb{R}^d)$, $1 \leq p < \infty$, *and let* $X : [t_n, t_{n+1}] \times \Omega \to \mathbb{R}^d$ *satisfy*

$$X(t) = X(t_n) + \int_{t_n}^t h(s) \, dB(s), \qquad t \in [t_n, t_{n+1}].$$

Suppose there exist a constant K *and* $Y \in L^p(\Omega, \mathscr{F}_{t_n}, P)$ *such that*

$$|h(t;\omega)| \leq Y(\omega) + K|X(t)|, \qquad t \in [t_n, t_{n+1}]. \tag{6.11}$$

Then, for all $t \in [t_n, t_{n+1}]$ *and* $\beta > p(c_p^{1/p}K)^2/2$,

$$\sup_{t_n \leq s \leq t} E\left[|X(s)|^p\right]^{1/p} \leq C(\beta) e^{\beta(t-t_n)} \left(E\left[|X(t_n)|^p\right]^{1/p} + c_p^{1/p}\sqrt{t-t_n} E\left[|Y|^p\right]^{1/p} \right),$$

where $C(\beta) = \left(1 - c_p^{1/p} K \sqrt{p/2\beta} \right)^{-1}$ *and* c_p *is the constant from the Burkholder-Davies-Gundy inequality.*

Proof. Set

$$\|X\|_{\beta,p,\tau} := \left(\sup_{t_n \leq t \leq \tau} e^{-\beta(t-t_n)} E\left[|X(t)|^p\right] \right)^{1/p}.$$

The triangle inequality yields

$$E\left[|X(t)|^p\right]^{1/p} \leq E\left[\left| \int_{t_n}^t h(s) \, dB(s) \right|^p \right]^{1/p} + E\left[|X(t_n)|^p\right]^{1/p}.$$

By the Burkholder-Davies-Gundy inequality,

$$E\left[\left| \int_{t_n}^t h(s) \, dB(s) \right|^p \right]^{1/p} \leq c_p^{1/p} E\left[\left(\int_{t_n}^t h^2(s) \, ds \right)^{p/2} \right]^{1/p}.$$

Due to (6.11) and the triangle inequality on $L^p(\Omega; L^2([t_n, t]))$,

$$E\left[\left(\int_{t_n}^t |h(s)|^2 \, ds \right)^{p/2} \right]^{1/p} \leq \sqrt{t - t_n} E\left[|Y|^p\right]^{1/p} + K E\left[\left(\int_{t_n}^t |X(s)|^2 \, ds \right)^{p/2} \right]^{1/p}.$$

By Minkowski's integral inequality,

$$E\left[\left(\int_{t_n}^t |X(s)|^2 \, ds \right)^{p/2} \right]^{2/p} \leq \int_{t_n}^t E\left[|X(s)|^p\right]^{2/p} \, ds.$$

Furthermore,

$$\int_{t_n}^t E\left[|X(s)|^p\right]^{2/p} \, ds = e^{2\beta(t-t_n)/p} \int_{t_n}^t \left(e^{-\beta(t-s)} e^{-\beta(s-t_n)} E\left[|X(s)|^p\right] \right)^{2/p} \, ds$$

$$\leq e^{2\beta(t-t_n)/p} \|X\|_{\beta,p,t}^2 \int_{t_n}^t e^{-2\beta(t-s)/p} \, ds$$

$$= \frac{p}{2\beta} \left(e^{2\beta(t-t_n)/p} - 1 \right) \|X\|_{\beta,p,t}^2.$$

Summarizing, we arrive at

$$E\left[|X(t)|^p\right]^{1/p} \le E\left[|X(t_n)|^p\right]^{1/p} + c_p^{1/p}\sqrt{t-t_n}E\left[|Y|^p\right]^{1/p}$$
$$+ c_p^{1/p}K\sqrt{\frac{p}{2\beta}}\left(e^{2\beta(t-t_n)/p}-1\right)^{1/2}\|X\|_{\beta,p,t}.$$

Multiplying by $e^{-\beta(t-t_n)/p}$ and taking the supremum over $t_n \le t \le \tau$, we obtain

$$\|X\|_{\beta,p,\tau} \le E\left[|X(t_n)|^p\right]^{1/p} + c_p^{1/p}\sqrt{\tau-t_n}E\left[|Y|^p\right]^{1/p} + c_p^{1/p}K\sqrt{\frac{p}{2\beta}}\|X\|_{\beta,p,\tau}.$$

Choosing β sufficiently large, i.e. $c_p^{1/p}K\sqrt{p/2\beta} < 1$, we secure the bound

$$\|X\|_{\beta,p,\tau} \le \frac{1}{1-c_p^{1/p}K\sqrt{p/2\beta}}\left(c_p^{1/p}\sqrt{\tau-t_n}E\left[|Y|^p\right]^{1/p} + E\left[|X(t_n)|^p\right]^{1/p}\right).$$

The result follows upon multiplication by $e^{\beta(\tau-t_n)/p}$, since

$$e^{\beta(\tau-t_n)/p}\|X\|_{\beta,p,\tau} = \left(\sup_{t_n\le t\le\tau}e^{\beta(\tau-t)}E\left[|X(t)|^p\right]\right)^{1/p} \ge \sup_{t_n\le t\le\tau}E\left[|X(t)|^p\right]^{1/p}.$$

$\square$

Estimate 6.8. *Let $\mathscr{T}_5$ be defined in Lemma 6.2. Then*

$$\mathscr{T}_5 = \mathcal{O}(\varepsilon).$$

Proof. This follows as in [19, Limit 6]. $\square$

Estimate 6.9. *Let $\mathscr{T}_6$ be defined in Lemma 6.2. Then*

$$|\mathscr{T}_6| \le 2\sum_{n=0}^{N-1}E\left[\|\mathcal{S}_{CL}(\Delta t)u^n - w^n\|_{1,\phi\star J_r}\right].$$

Proof. First, we note that $|S_\delta(b) - S_\delta(a)| \le |b-a|$. This and (4.15) yields

$$|\mathscr{T}_6| \le \sum_{n=0}^{N-1}E\left[\|\tilde{\eta}(t_{n+1}) - \tilde{\eta}((t_{n+1})-)\|_{1,\phi\star J_r}\right].$$

Since

$$\tilde{\eta}(t_{n+1}) - \tilde{\eta}((t_{n+1})-) = \mathcal{S}_{\text{SDE}}(t_{n+1},t_n)(\mathcal{S}_{\text{CL}}(\Delta t)u^n - w^n) + \mathcal{S}_{\text{CL}}(\Delta t)u^n - w^n,$$

the result follows from (3.2). $\square$

Proof of Theorem 6.1. Consider Lemma 6.2, and take the upper limits in (6.6) as $r_0 \downarrow 0, \varepsilon \downarrow 0$, and $\gamma \downarrow 0$ (in that order). Next we recall that $w^n = w_k^n$. Letting $k \to \infty$, $w_k^n \to \mathcal{S}_{\text{CL}}(\Delta t)u^n$ in $L^1(\Omega, L^1(\mathbb{R}^d, \phi))$. Due to the L^1-Lipschitz continuity of $\mathcal{S}_{\text{CL}}$ (cf. Proposition 4.8) and the uniform BV-bound on the splitting approximation, it follows from Estimates 6.1–6.9 that

$$E\left[\|u_0 - u^0\|_{1,\phi}\right] + C_\phi\|f\|_{\text{Lip}}\int_0^{t_0}E\left[\|\eta_{\Delta t}(t) - u(t)\|_{1,\phi}\right]dt$$
$$+ \mathcal{O}\left(\delta + r + \sqrt{\Delta t} + \frac{\delta}{r} + \frac{\Delta t}{\delta}\right) \ge E\left[\|\eta_{\Delta t}(t_0) - u(t_0)\|_{1,\phi}\right].$$

Finally, we apply Grönwall's inequality, and then choose $\delta = \Delta t^{2/3}$ and $r = \Delta t^{1/3}$.

$\square$

7. Appendix

7.1. Proof of Proposition 6.1.

The proof of Proposition 6.1 is based on the following result.

Lemma 7.1. *Suppose $u, w \in L^2(\Omega, P, \mathscr{F}_{t_n}; L^2(\mathbb{R}^d))$ and w is smooth. Set*

$$\psi(t) = (\mathcal{S}_{SDE}(t, t_n) - \mathcal{I})w, \quad v(t) = \mathcal{S}_{CL}(t - t_n)u, \qquad t \in [t_n, t_{n+1}].$$

Then for all $(S, Q) \in \mathscr{E}$, all nonnegative $\varphi \in C_c^\infty(\Pi_n^2)$, and all $V \in \mathcal{S}$,

$$R - L + \mathscr{T}_1 + \mathscr{T}_2 - \mathscr{T}_3 + \mathscr{T}_4 \geq 0,$$

where

$$L = E\left[\iint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t_{n+1}, x) + \psi(s, y) - V)\varphi(t_{n+1}, x, s, y)\, dxdyds\right]$$
$$+ E\left[\iint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t, x) + \psi(t_{n+1}, y) - V)\varphi(t, x, t_{n+1}, y)\, dydxdt\right],$$

$$R = E\left[\iint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t_n, x) + \psi(s, y) - V)\varphi(t_n, x, s, y)\, dxdyds\right]$$
$$+ E\left[\iint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t, x) + \psi(t_n, y) - V)\varphi(t, x, t_n, y)\, dydxdt\right],$$

$$\mathscr{T}_1 = E\left[\iiiint_{\Pi_n^2} S(v(t, x) + \psi(s, y) - V)(\partial_t + \partial_s)\varphi\, dX\right],$$

$$\mathscr{T}_2 = E\left[\iiiint_{\Pi_n^2} Q(v(t, x), V - \psi(s, y)) \cdot \nabla_x \varphi\, dX\right],$$

$$\mathscr{T}_3 = E\left[\iiiint_{\Pi_n^2} S''(v(t, x) + \psi(s, y) - V) D_s V \sigma(\psi(s, y) + w(y))\varphi\, dX\right],$$

$$\mathscr{T}_4 = \frac{1}{2}E\left[\iiiint_{\Pi_n^2} S''(v(t, x) + \psi(s, y) - V)\sigma^2(\psi(s, y) + w(y))\varphi\, dX\right],$$

and $\Pi_n = [t_n, t_{n+1}] \times \mathbb{R}^d$.

Proof of Lemma 7.1. The entropy inequality reads

$$\int_{\mathbb{R}^d} S(v(t_n, x) - c)\varphi(t_n, x, s, y) - S(v(t_{n+1}, x) - c)\varphi(t_{n+1}, x, s, y)\, dx$$
$$+ \iint_{\Pi_n} S(v - c)\partial_t \varphi + Q(v, c) \cdot \nabla_x \varphi\, dtdx \geq 0, \quad (7.1)$$

for all $c \in \mathbb{R}$ and all $s, y \in \Pi_n$. Specify $c = V - \psi(s, y)$ in (7.1), integrate in (s, y), and take expectations, to obtain

$$E\left[\iint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t_n, x) + \psi(s, y) - V)\varphi(t_n, x, s, y)\, dxdsdy\right]$$
$$- E\left[\iint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t_{n+1}, x) + \psi(s, y) - V)\varphi(t_{n+1}, x, s, y)\, dxdsdy\right] \quad (7.2)$$
$$+ E\left[\iiiint_{\Pi_n^2} S(v + \psi - V)\partial_t \varphi + Q(v, V - \psi) \cdot \nabla_x \varphi\, dX\right] \geq 0.$$

Note that $v(t)$ is $\mathscr{F}_{t_n}$-adapted for all $t \in [t_n, t_{n+1}]$. To reveal the equation satisfied by ψ, let $\zeta(t) = \mathcal{S}_{\mathrm{SDE}}(t, t_n)w$. By definition,

$$\zeta(t, x) = w(x) + \int_{t_n}^{t} \sigma(\zeta(r, x)) \, dB(r).$$

Since $\psi(t) = \zeta(t) - w$,

$$\psi(t, x) = \int_{t_n}^{t} \sigma(\psi(r, x) + w(x)) \, dB(r), \qquad t \in [t_n, t_{n+1}]. \tag{7.3}$$

Fix $t, x \in \Pi_n, y \in \mathbb{R}^d$ and set

$$X(s) := v(t, x) + \psi(s, y), \quad F(X(s), V, s) := S(X(s) - V)\varphi(t, x, s, y), \qquad s \in [t_n, t_{n+1}].$$

By (7.3),

$$X(s) = v(t, x) + \int_{t_n}^{s} \sigma(\psi(r, y) + w(y)) \, dB(r).$$

By Theorem 7.1,

$$S(X(t_{n+1}) - V)\varphi(t, x, t_{n+1}, y) = S(X(t_n) - V)\varphi(t, x, t_n, y)$$

$$+ \int_{t_n}^{t_{n+1}} S(X(s) - V)\partial_s\varphi \, ds$$

$$+ \int_{t_n}^{t_{n+1}} S'(X(s) - V)\sigma(\psi(s) + w)\varphi \, dB(s)$$

$$- \int_{t_n}^{t_{n+1}} S''(X(s) - V)D_s V\sigma(\psi(s) + w)\varphi \, ds$$

$$+ \frac{1}{2} \int_{t_n}^{t_{n+1}} S''(X(s) - V)\sigma^2(\psi(s) + w)\varphi \, ds,$$

where the stochastic integral is interpreted as a Skorohod integral. Upon integrating in t, x, y and taking expectations,

$$E\left[\iiint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t, x) + \psi(t_n, y) - V)\varphi(t, x, t_n, y) \, dy \, dt \, dx \right]$$

$$- E\left[\iiint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t, x) + \psi(t_{n+1}, y) - V)\varphi(t, x, t_{n+1}, y) \, dy \, dt \, dx \right]$$

$$+ E\left[\iiiint_{\Pi_n^2} S(v(t, x) + \psi(s, y) - V)\partial_s\varphi \, dX \right] \tag{7.4}$$

$$+ \frac{1}{2} E\left[\iiiint_{\Pi_n^2} S''(v(t, x) + \psi(s, y) - V)(\sigma(\psi(s, y) + w(y)))^2\varphi \, dX \right]$$

$$- E\left[\iiiint_{\Pi_n^2} S''(v(t, x) + \psi(s, y) - V)D_s V\sigma(\psi(s, y) + w(y))\varphi \, dX \right] = 0.$$

Adding (7.2) and (7.4) concludes the proof. $\qquad\square$

Proof of Proposition 6.1. We use

$$\varphi(t, x, s, y) = \frac{1}{2^d}\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) J_{r_0}(t-s) \tag{7.5}$$

in Lemma 7.1 and then send r_0, r to zero (in that order). The sought result for $V \in \mathcal{S}$ is a consequence of Limits 1–5 below. The extension to $V \in \mathbb{D}^{1,2}$ follows by an approximation argument as in [19, Lemma 2.2]. $\qquad\square$

Limit 1. *Let L, R be defined in Lemma 7.1 and φ in (7.5). Then*

$$\lim_{r,r_0 \downarrow 0} L(r, r_0) = E\left[\int_{\mathbb{R}^d} S(v(t_{n+1}, x) + \psi(t_{n+1}, x))\phi(t_{n+1}, x)\, dx\right],$$

$$\lim_{r,r_0 \downarrow 0} R(r, r_0) = E\left[\int_{\mathbb{R}^d} S(v(t_n, x) + \psi(t_n, x) - V)\phi(t_n, x)\, dx\right].$$

Proof. Let us only consider the term

$$E\left[\iint_{\Pi_n} \int_{\mathbb{R}^d} S(v(t_{n+1}, x) + \psi(s, y) - V)\varphi(t_{n+1}, x, s, y)\, dxdyds\right] =: \mathscr{Z}.$$

The remaining terms can be treated in the same way. As a consequence of the dominated convergence theorem and Lemma 7.4,

$$\lim_{r_0 \downarrow 0} \mathscr{Z} = \frac{1}{2} E\left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S(v(t_{n+1}, x) + \psi(t_{n+1}, y) - V)\right.$$

$$\left. \times \frac{1}{2^d}\phi\left(t_{n+1}, \frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) dxdy\right].$$

Moreover,

$$\lim_{r,r_0 \downarrow 0} \mathscr{Z} = \frac{1}{2} E\left[\int_{\mathbb{R}^d} S(v(t_{n+1}, x) + \psi(t_{n+1}, x) - V)\phi(t_{n+1}, x)\right].$$

$\square$

Limit 2. *Let $\mathscr{T}_1$ be defined in Lemma 7.1 and φ in (7.5). Then*

$$\lim_{r,r_0 \downarrow 0} \mathscr{T}_1 = E\left[\iint_{\Pi_n} S(u(t, x) - V)\partial_t \phi(t, x)\, dxdt\right].$$

Proof. Observe that

$$(\partial_t + \partial_s)\varphi(t, x, s, y) = \frac{1}{2^d}\partial_1\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) J_{r_0}(t-s).$$

The result follows by the dominated convergence theorem and Lemma 7.4, consult the proof of Limit 1. $\square$

Limit 3. *Let $\mathscr{T}_2$ be defined in Lemma 7.1 and φ in (7.5). Then*

$$\lim_{r,r_0 \downarrow 0} \mathscr{T}_2 = E\left[\iint_{\Pi_n} Q(v + \psi, V) \cdot \nabla \phi\, dxdt\right]$$

$$+ E\left[\iint_{\Pi_n} \left(\int_V^{v+\psi} S'(z - V)\left(f'(z - \psi) - f'(z)\right) dz\right) \cdot \nabla \phi\, dxdt\right]$$

$$+ E\left[\iint_{\Pi_n} \left(\int_V^{v+\psi} S''(z - V)f'(z - \psi)\, dz\right) \cdot \nabla \psi\, \phi\, dxdt\right].$$

Proof. First observe that

$$(\nabla_x + \nabla_y)\varphi(t, x, s, y) = \frac{1}{2^d}\partial_2\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) J_{r_0}(t-s).$$

Integration by parts results in

$$\mathscr{T}_2 = E\left[\iiiint_{\Pi_n^2} Q(v(t, x), V - \psi(s, y))\right.$$

$$\left. \cdot \frac{1}{2^d}\partial_2\phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) J_r\left(\frac{x-y}{2}\right) J_{r_0}(t-s)\, dxdtdyds\right]$$

$$+ E \left[\iiiint_{\Pi_n^2} \nabla_y \cdot Q(v(t,x), V - \psi(s,y))\varphi(t,x,s,y)\,dxdtdyds \right]$$

$$=: \mathscr{T}_2^1 + \mathscr{T}_2^2.$$

It is straightforward to show that

$$\lim_{r,r_0 \downarrow 0} \mathscr{T}_2^1 = E \left[\iint_{\Pi_n} Q(v(t,x), V - \psi(t,x)) \cdot \nabla\phi(t,x)\,dxdt \right].$$

Finally, we apply the identity

$$Q(v, V - \psi) = Q(v + \psi, V) + \int_V^{v+\psi} S'(z - V)\left(f'(z - \psi) - f'(z)\right)\,dz.$$

Consider $\mathscr{T}_2^2$. By the chain rule,

$$\mathscr{T}_2^2 = -E \left[\iiiint_{\Pi_n^2} \partial_2 Q(v(t,x), V - \psi(s,y)) \cdot \nabla_y \psi(s,y)\,\varphi(t,x,s,y)\,dxdtdyds \right].$$

Sending r_0, r to zero, we arrive at

$$\lim_{r,r_0 \downarrow 0} \mathscr{T}_2^2 = -E \left[\iint_{\Pi_n} \partial_2 Q(v(t,x), V - \psi(t,x)) \cdot \nabla_x \psi(t,x)\,\phi(t,x)\,dxdt \right].$$

Finally, note that

$$\partial_2 Q(v, V - \psi) = - \int_{V-\psi}^v S''(z - V + \psi)f'(z)\,dz$$

$$= - \int_V^{v+\psi} S''(z - V)f'(z - \psi)\,dz.$$

This concludes the proof. □

Limit 4. *Let $\mathscr{T}_3$ be defined in Lemma 7.1 and φ in (7.5). Then*

$$\lim_{r,r_0 \downarrow 0} \mathscr{T}_3 = E \left[\iint_{\Pi_n} S''(v(t,x) + \psi(t,x) - V)D_t V\sigma(\psi(t,x) + w(x))\phi(t,x)\,dxdt \right].$$

Proof. The proof is a straightforward application of the dominated convergence theorem and Lemma 7.4. □

Limit 5. *Let $\mathscr{T}_4$ be defined in Lemma 7.1 and φ by (7.5). Then*

$$\lim_{r,r_0 \downarrow 0} \mathscr{T}_4 = \frac{1}{2} E \left[\iint_{\Pi_n} S''(v(t,x) + \psi(t,x) - V)\sigma^2(\psi(t,x) + w(x))\phi(t,x)\,dxdt \right].$$

Proof. This term may be treated similarly as $\mathscr{T}_3$. □

7.2. **Weighted L^p spaces.** In the next two lemmas we collect a few elementary properties of (weight) functions in $\mathfrak{N}$. For proofs, see [19].

Lemma 7.2. *Suppose $\phi \in \mathfrak{N}$ and $0 < p < \infty$. Then, for $x, z \in \mathbb{R}^d$,*

$$\left| \phi^{1/p}(x + z) - \phi^{1/p}(x) \right| \leq w_{p,\phi}(|z|)\phi^{1/p}(x),$$

where

$$w_{p,\phi}(r) = \frac{C_\phi}{p} r \left(1 + \frac{C_\phi}{p} r e^{C_\phi r/p} \right),$$

which is defined for all $r \geq 0$. As a consequence it follows that if $\phi(x_0) = 0$ for some $x_0 \in \mathbb{R}^d$, then $\phi \equiv 0$ (and by definition $\phi \notin \mathfrak{N}$).

Lemma 7.3. *Fix $\phi \in \mathfrak{N}$, and let $w_{p,\phi}$ be defined in Lemma 7.2. Let J be a mollifier as defined in Section 2 and take $\phi_\delta = \phi \star J_\delta$ for $\delta > 0$. Then*

(i) $\phi_\delta \in \mathfrak{N}$ *with* $C_{\phi_\delta} = C_\phi$.
(ii) *For any* $u \in L^p(\mathbb{R}^d, \phi)$,

$$\left| \|u\|_{p,\phi}^p - \|u\|_{p,\phi_\delta}^p \right| \leq w_{1,\phi}(\delta) \min \left\{ \|u\|_{p,\phi}^p, \|u\|_{p,\phi_\delta}^p \right\}.$$

(iii)

$$|\Delta \phi_\delta(x)| \leq \frac{1}{\delta} C_\phi \|\nabla J\|_{L^1(\mathbb{R}^d)} (1 + w_{1,\phi}(\delta))^2 \phi_\delta(x).$$

7.3. A "doubling of variables" tool.
The following result follows along the lines of [32, Lemma 2.7.2]. See also [19, § 6].

Lemma 7.4. *Suppose* $u, v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and* F *is Lipschitz on* $\mathbb{R}^2$. *Fix* $\psi \in C_c(\mathbb{R}^d)$ *and set*

$$\mathcal{T}_r := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(u(x), v(y)) \frac{1}{2^d} \psi \left(\frac{x+y}{2} \right) J_r \left(\frac{x-y}{2} \right) \, dy dx$$

$$- \int_{\mathbb{R}^d} F(u(x), v(x)) \psi(x) \, dx,$$

where J_r *is defined in* (2.2). *Then* $\mathcal{T}_r \to 0$ *as* $r \downarrow 0$.

Similarly, let $G : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be measurable in the first variable and Lipschitz continuous in the second variable. With* $w \in L^1([0, T])$, *set*

$$\mathcal{T}_{r_0}(s) = \int_0^T |G(s, w(t)) - G(s, w(s))| \, J_{r_0}(t - s) \, dt.$$

Then $\mathcal{T}_{r_0}(s) \to 0$ *for a.e.* s *as* $r_0 \downarrow 0$.

The above results do not rely on the the symmetry of J.

7.4. A version of Itô's formula.
Here we recall the particular anticipating Itô formula applied in the proof of Lemma 5.1 and Lemma 7.1. The proof of this follows [29, Theorem 3.2.2] closely. However, due to the particular assumptions, certain points simplifies. See [19, Theorem 6.7] for an outline of a proof.

Theorem 7.1. *Let* X *be a continuous process of the form*

$$X(t) = X_0 + \int_0^t u(s) \, dB(s) + \int_0^t v(s) \, ds,$$

where $u : [0, T] \times \Omega \to \mathbb{R}$ *and* $v : [0, T] \times \Omega \to \mathbb{R}$ *are predictable processes, satisfying*

$$E\left[\left(\int_0^T u^2(s, z) \, ds \right)^2 \right] < \infty, \qquad E\left[\int_0^T v^2(s) \, ds \right] < \infty,$$

and $X_0 \in L^2(\Omega, \mathscr{F}_0, P)$. *Let* $F : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ *be twice continuously differentiable. Suppose there exists a constant* $C > 0$ *such that for all* $(\zeta, \lambda, t) \in \mathbb{R}^2 \times [0, T]$,

$$|F(\zeta, \lambda, t)|, |\partial_3 F(\zeta, \lambda, t)| \leq C(1 + |\zeta| + |\lambda|),$$

$$|\partial_1 F(\zeta, \lambda, t)|, \left| \partial_{1,2}^2 F(\zeta, \lambda, t) \right|, \left| \partial_1^2 F(\zeta, \lambda, t) \right| \leq C.$$

Let $V \in \mathcal{S}$. *Then* $s \mapsto \partial_1 F(X(s), V, s) u(s)$ *is Skorohod integrable, and*

$$F(X(t), V, t) = F(X_0, V, 0)$$

$$+ \int_0^t \partial_3 F(X(s), V, s) \, ds$$

$$+ \int_0^t \partial_1 F(X(s), V, s) u(s, z) \, dB(s)$$

$$+ \int_0^t \partial_1 F(X(s), V, s) v(s) \, ds$$

$$+ \int_0^t \partial_{1,2}^2 F(X(s), V, s) D_s V u(s) \, ds$$

$$+ \frac{1}{2} \int_0^t \partial_1^2 F(X(s), V, s) u^2(s) \, ds, \quad dP\text{-almost surely.}$$

7.5. **Young measures.** The purpose of this section is to provide a reference for some results concerning Young measures and their use in representation formulas for weak limits. For a more general introduction, see for instance [13, 27, 34].

Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and $\mathscr{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$. In this paper, X is typically $\Pi_T \times \Omega$. A *Young measure* from X into $\mathbb{R}$ is a function $\nu : X \to \mathscr{P}(\mathbb{R})$ such that $x \mapsto \nu_x(B)$ is $\mathscr{A}$-measurable for every Borel measurable set $B \subset \mathbb{R}$. We denote by $\mathcal{Y}(X, \mathscr{A}, \mu; \mathbb{R})$, or $\mathcal{Y}(X; \mathbb{R})$ if the measure space is understood, the set of all Young measures from X into $\mathbb{R}$. The following theorem is proved in [31, Theorem 6.2] in the case that $X \subset \mathbb{R}^n$ and μ is the Lebesgue measure:

Theorem 7.2. *Fix a σ-finite measure space $(X, \mathscr{A}, \mu)$. Let $\zeta : [0, \infty) \to [0, \infty]$ be a continuous, non decreasing function satisfying $\lim_{\xi \to \infty} \zeta(\xi) = \infty$ and $\{u^n\}_{n \geq 1}$ a sequence of measurable functions such that*

$$\sup_n \int_X \zeta(|u^n|) d\mu(x) < \infty.$$

Then there exist a subsequence $\{u^{n_j}\}_{j \geq 1}$ and $\nu \in \mathcal{Y}(X, \mathscr{A}, \mu; \mathbb{R})$ such that for any Carathéodory function $\psi : \mathbb{R} \times X \to \mathbb{R}$ with $\psi(u^{n_j}(\cdot), \cdot) \rightharpoonup \overline{\psi}$ in $L^1(X)$, we have

$$\overline{\psi}(x) = \int_{\mathbb{R}} \psi(\xi, x) \, d\nu_x(\xi).$$

The proof is based on the embedding of $\mathcal{Y}(X; \mathbb{R})$ into $L_{w*}^\infty(X, \mathcal{M}(\mathbb{R}))$. Here $\mathcal{M}(\mathbb{R})$ denotes the space of Radon measures on $\mathbb{R}$. The crucial observation is that $(L^1(X, C_0(\mathbb{R})))^*$ is isometrically isomorphic to $L_{w*}^\infty(X, \mathcal{M}(\mathbb{R}))$ also in the case that $(X, \mathscr{A}, \mu)$ is an abstract σ-finite measure space. It is relatively straightforward to go through the proof and extend it to the more general case [27, Theorem 2.11]. The result then follows as an application of Alaoglu's theorem combined with the Eberlein-Šmulian theorem. Note, however, due to our use of weighted L^p spaces, it suffices with the version for finite measure spaces.

7.6. **Weak compactness in L^1.** To apply Theorem 7.2 it is necessary to know if $\{\psi(\cdot, u^n(\cdot))\}_{n \geq 1}$ has a subsequence converging weakly in $L^1(X)$. The key result is the well-known Dunford-Pettis Theorem.

Definition 7.1. Let $\mathcal{K} \subset L^1(X, \mathscr{A}, \mu)$.

(i) $\mathcal{K}$ is *uniformly integrable* if for any $\varepsilon > 0$ there exists $c_0(\varepsilon)$ such that

$$\sup_{f \in \mathcal{K}} \int_{|f| \geq c} |f| \, d\mu \leq \varepsilon \text{ whenever } c \geq c_0(\varepsilon).$$

(ii) $\mathcal{K}$ has *uniform tail* if for any $\varepsilon > 0$ there exists $E \in \mathscr{A}$ with $\mu(E) < \infty$ such that

$$\sup_{f \in \mathcal{K}} \int_{X \setminus E} |f| \, d\mu \leq \varepsilon.$$

If $\mathcal{K}$ satisfies both (i) and (ii) it is said to be *equiintegrable*.

Remark 7.5. Note that (ii) is void when μ is finite.

Theorem 7.3 (Dunford-Pettis). *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. A subset $\mathcal{K}$ of $L^1(X)$ is relatively weakly sequentially compact if and only if it is equiintegrable.*

There are a couple of well known reformulations of uniform integrability.

Lemma 7.6. *Suppose $\mathcal{K} \subset L^1(X)$ is bounded. Then $\mathcal{K}$ is uniformly integrable if and only if:*

(i) *For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$\sup_{f \in \mathcal{K}} \int_E |f| \, d\mu \le \varepsilon \quad \text{whenever } \mu(E) \le \delta(\varepsilon).$$

(ii) *There is an increasing function $\Psi : [0, \infty) \to [0, \infty)$ such that $\Psi(\zeta)/\zeta \to \infty$ as $\zeta \to \infty$ and*

$$\sup_{f \in \mathcal{K}} \int_X \Psi(|f(x)|) \, d\mu(x) < \infty.$$

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