CONVERGENCE OF THE GODUNOV SCHEME FOR A SCALAR CONSERVATION LAW WITH TIME AND SPACE FLUX DISCONTINUITIES

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ABSTRACT. We consider the Godunov scheme as applied to a scalar conservation law whose flux has discontinuities in both space and time. The time and space dependence of the flux occurs through a positive multiplicative coefficient. That coefficient has a spatial discontinuity along a fixed interface at $x = 0$. Time discontinuities occur in the coefficient independently on either side of the interface. This setup applies to the LWR traffic model in the case where different time-varying speed limits are imposed on different segments of a road. We prove that approximate solutions produced by the Godunov scheme converge to the unique entropy solution, as defined in G.M. COCLITE AND N.H. RISEBRO, *Conservation Laws with time dependent discontinuous coefficients*, SIAM J. Math. Anal., 36 (2005), pp. 1293–1309. Convergence of the Godunov scheme in the presence of spatial flux discontinuities alone is a well established fact. The novel aspect of this paper is convergence in the presence of additional temporal flux discontinuities.

1. INTRODUCTION

We consider the Godunov scheme as applied to the Cauchy problem for a scalar conservation law whose flux function depends on a coefficient $k(x,t)$ that is discontinuous in both space and time:

$$\begin{cases} u_t + (k(x,t)f(u))_x = 0 & \text{for } (x,t) \in \Pi_T := \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Regarding the data of the problem we make the following assumptions:

(**A**.1) The function $u \mapsto f(u)$ is Lipschitz continuous, satisfies

$$f(0) = f(u_{\max}) = 0, \tag{1.2}$$

and is unimodal, meaning that there is a number $u^* \in (0, u_{\max})$ such that $f(u^*) > 0$, f is strictly increasing on $(0, u^*)$, strictly decreasing on $(u^*, u_{\max})$.

(**A**.2) The coefficient $k \in L^\infty(\Pi_T)$ has the form

$$k(x,t) = \begin{cases} k_1(t), & x < 0, \\ k_2(t), & x > 0. \end{cases}$$

For k_1 and k_2, each $k_i \in \mathrm{BV}(\mathbb{R}^+)$, and there is a number $k_{\min} > 0$ such that $k_i(t) \geq k_{\min}$ for all $t \in [0,T]$.

(**A**.3) The initial data satisfies

$$u_0(x) \in [0, u_{\max}] \text{ for all } x \in \mathbb{R}, \quad u_0 \in L^1(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R}). \tag{1.3}$$

An example where our setup applies is the Lighthill-Witham-Richards (LWR) model of traffic flow on a unidirectional road, where u models vehicle density, and $kv(u)$ models the vehicle velocity [17]. Here $u \mapsto v(u)$ is strictly decreasing. The simplest example is $v(u) = u_{\max} - u$, in which case $kf(u) = ku(u_{\max} - u)$. The parameter k controls the maximum vehicle velocity. The case of spatially varying, discontinuous $k = k(x)$ has been studied extensively. It models an abrupt

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change in road conditions, for example speed limit. Our setup $k = k(x,t)$ allows for additional temporal changes, which occur in problems where time varying speed limit controls are applied [18, 29, 30, 31].

Another application where the flux has both spatial and temporal discontinuities is the clarifier-thickener model, with time varying controls [8, 15]. For this application the time dependence is not the simple multiplicative one that we are assuming, and so our analysis does not apply.

For the case of smooth or constant k, the existence and uniqueness theory is classical, culminating in 1970 with the work of Kružkov. For constant k, the theory of monotone difference schemes such as the Godunov scheme has also been essentially complete for a long time. Starting in the 1990's, the case where $k = k(x)$ is discontinuous but with no time dependence has attracted much interest. A partial list of the many papers on this subject can be found in the bibliographies of [4, 5, 12, 16]. In particular, the Godunov scheme described in Section 2 is known to converge when the flux has only spatial discontinuities [1, 5, 11]. The main result of this paper is a proof of convergence of the Godunov scheme when $k = k(x,t)$ also has temporal discontinuities. We know of no previous convergence result for the Godunov scheme in this case.

Coclite and Risebro [13] proved well-posedness for a problem similar to (1.1). They used a front tracking algorithm to construct approximate solutions, which converge to the unique entropy solution as the refinement parameter approaches zero. In [13], the flux has the form $f(u,x,t) = f(u, a(x), g(t))$, where $f(u,a,g)$ is smooth and $a(x) \in \mathrm{BV}(\mathbb{R})$, $g(t) \in \mathrm{BV}(\mathbb{R}^+)$. Also the mapping $u \mapsto f(u,a,g)$ has the unimodal shape that we are assuming. Compared to that setup, our problem is somewhat different since we are assuming the multiplicative form $k(x,t)f(u)$, and we are assuming only one spatial discontinuity. It will become clear that our results can be extended to any finite number of spatial discontinuities, but our analysis depends in a critical way on the multiplicative structure $kf(u)$.

In [25] the compensated compactness method was used to prove that the Lax-Friedrichs scheme converges (along a subsequence) to a weak solution of a problem like (1.1), under more general assumptions about the shape of the flux and its time and space dependence.

The Godunov scheme is only first order accurate, but it is widely used for approximating solutions of the LWR model of traffic flow [7, 17, 18, 21, 27]. In the transportation science literature it is often referred to as the Cell Transmission Model (CTM) [14, 29, 31]. It is easily implemented, it has an intuitive interpretation in terms of upstream/downstream demand/supply of capacity, and the numerical flux comes directly from the solution of the relevant Riemann problem. As a result of the supply/demand interpretation, and its derivation from the Riemann problem, it is readily adapted to deal with junctions. Our problem (1.1) has the simplest type of junction; one incoming and one outgoing road. But the Godunov scheme can also be extended to junctions where there are multiple incoming and outgoing roads.

When proving compactness of approximate solutions u^Δ for $k =$ constant, a key ingredient is a spatial variation bound. Such a bound is not available when k is discontinuous. One well-known way around this difficulty, and the one we will employ, is the singular mapping technique [1, 9, 13, 19, 26, 32, 33]. The so-called singular mapping $\Psi(u,k)$ is

$$\Psi(u,k) = \frac{k}{f(u^*)}\operatorname{sign}(u-u^*)\left(f(u^*) - f(u)\right) = \frac{k}{f(u^*)}\int_{u^*}^{u} |f'(w)|\, dw.$$

For each fixed k, the mapping $u \mapsto \Psi(u,k)$ is strictly increasing and Lipschitz continuous, but $\Psi_u(u^*,k) = 0$, whence the term "singular" . The singular mapping technique consists of proving a spatial variation bound for $\Psi(u^\Delta, k^\Delta)$. That bound, along with some more standard ones, yields compactness for $\Psi(u^\Delta, k^\Delta)$. After that, compactness of the conserved quantity u^Δ follows with the observation that $u \mapsto \Psi(u,k)$ has a continuous inverse. Other methods that have been used to deal with a spatially discontinuous flux are compensated compactness [22, 25] and the $\mathrm{BV}_{\mathrm{loc}}$ method [4, 10, 11, 12].

As applied to difference schemes, the singular mapping method (and also the $\mathrm{BV}_{\mathrm{loc}}$ method) requires a discrete time continuity estimate, or equivalently a bound on the spatial variation of the numerical flux. When $k = k(x)$, and the scheme is monotone, such an estimate is straightforward, but not when $k = k(x,t)$. In fact, assuming the presence of a spatial discontinuity in k, the analytical difficulty is already present if there is a single time discontinuity in k, or even if k varies smoothly in time. The novel aspect of our analysis is Lemma 3.2, which provides a spatial variation bound on the numerical flux, making it possible to carry out the rest of the analysis using known results, with small modifications where necessary.

Even when $k(x,t) =$ constant and the initial data is smooth, solutions of (1.1) develop discontinuities, and so a weak definition of solution is required. The presence of discontinuities causes another difficulty, namely lack of uniqueness, and so an additional condition, a so-called entropy condition, is required in order to single out the unique physically relevant solution. We will use the following notion of solution, which is essentially the definition of entropy solution proposed in [13].

Definition 1.1 (Entropy solution). *A measurable function* $u : \Pi_T \to [0, u_{\max}]$ *is an* entropy solution *of the initial value problem* (1.1) *if it satisfies the following conditions:*

(**D**.1) $u \in L^1(\Pi_T)$*, and the map* $(0,T) \ni t \mapsto u(\cdot,t) \in L^1(\mathbb{R})$ *is Lipschitz continuous.*

(**D**.2) *For any test function* $\phi \in \mathcal{D}(\Pi_T)$*,*

$$\iint_{\Pi_T} \Big(u\phi_t + k(x,t) f(u) \phi_x \Big)\, dx\, dt = 0.$$

(**D**.3) *For any test function* $0 \le \phi \in \mathcal{D}(\Pi_T)$*, and any* $c \in [0, u_{\max}]$*,*

$$\iint_{\Pi_T} \Big(|u-c|\phi_t + F(u,x,t,c)\phi_x \Big)\, dx\, dt + \int_0^T |k_2(t) f(c) - k_1(t) f(c)|\, \phi(0,t)\, dt \ge 0,$$

where $F(u,x,t,c) = \operatorname{sign}(u-c)\,(k(x,t)f(u) - k(x,t)f(c))$*.*

(**D**.4) *The initial condition is satisfied in the* L^1 *sense:*

$$u(\cdot,t) \to u_0 \text{ in } L^1(\mathbb{R}) \text{ as } t \downarrow 0.$$

(**D**.5) $\Psi(u(\cdot,t), k(\cdot,t)) \in BV(\mathbb{R})$ *for a.e.* $t \in (0,T)$*.*

Remark 1.1. Condition (**D**.2) is actually implied by condition (**D**.3). Taking $c = 0$ and $c = u_{\max}$ in (**D**.3), one gets (**D**.2) for $\phi \ge 0$, and hence for all ϕ. We include condition (**D**.2) in the definition because it is the weak formulation of the conservation law. By itself, (**D**.2) is not sufficient to guarantee uniqueness. Condition (**D**.3) is the so-called entropy condition, which ensures uniqueness. It generalizes the classical Kružkov entropy condition to accommodate the spatial flux discontinuity.

Remark 1.2. Since $u \mapsto \Psi(u,k)$ has a continuous inverse, condition (**D**.5) guarantees the existence of the limits

$$\lim_{x \to 0^{\pm}} u(x,t) \text{ for a.e. } t \in (0,T).$$

Remark 1.3. Even when $k = k(x)$, with a single spatial discontinuity and no time-dependence, (1.1) admits infinitely many L^1-contractive semigroups of solutions [2], one for each so-called connection (A,B) . The definition above singles out the entropy solution that is understood to be the correct one for traffic modeling. It also corresponds to the optimal entropy solution [2], the entropy solution of [24], and the vanishing viscosity solution [6]. Moreover, it satisfies the Γ-condition [16] and the minimal jump condition [19]. All of these different notions of entropy solution agree for this particular problem. In more complicated situations, some of them may yield different solutions [5].

Remark 1.4. When $k = k(x)$, an alternative to (**D**.3) is the so-called adapted entropy formulation [5, 6, 12], which simplifies certain aspects of the subsequent analysis. For the present setup where $k = k(x,t)$, the adapted entropies would become time dependent. This seems feasible, but for our present purposes the most direct approach is to use condition (**D**.3).

Although our problem is slightly different from that of [13], the uniqueness portion of their well-posedness theorem applies directly to our setup.

Theorem 1.1 (Uniqueness of entropy solutions [13])**.** *If u and v are two entropy solutions having initial data u_0 and v_0 satisfying* (1.3)*, then*

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u_0-v_0\|_{L^1(\mathbb{R})}.$$

The remainder of this paper consists of Section 2, where the details of the Godunov scheme are laid out, and Section 3, which contains our convergence theorem and its proof.

2. The Godunov scheme

For a fixed spatial mesh size Δx, let $x_j = (j-1/2)\Delta x$. Let $x_{j+\frac{1}{2}} = x_j + \Delta x/2$. Define the grid cells $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. Note that with this setup the junction $x=0$ is located at $x_{\frac{1}{2}}$.

Similarly, for a fixed temporal mesh size Δt, define $N = N(\Delta t) \in \mathbb{Z}^+$ such that $T \in [N\Delta t, (N+1)\Delta t)$. The positive time axis is discretized via $t^n = n\Delta t$ for $0 \le n \le N$, resulting in the time strips $I^n = [t^n, t^{n+1})$.

For grid points away from the junction, the numerical flux is $k\bar{f}(v,u)$, where $\bar{f}(v,u)$ is the classical Godunov flux:

$$\bar{f}(v,u) = \left\{ \begin{array}{ll} \min\limits_{[u,v]} f(w), & u \le v \\ \max\limits_{[v,u]} f(w), & u \ge v \end{array} \right\} = \min\left\{ f\left(\max(v,u^*)\right), f\left(\min(u,u^*)\right)\right\}. \tag{2.1}$$

The first formula above is the standard one [28], while the second one results from our assumptions about $f(u)$. The Godunov flux $\bar{f}$ is Lipschitz continuous with respect to both variables, and consistent with f, meaning that $\bar{f}(u,u) = f(u)$. Also $\bar{f}$ is monotone, i.e., nonincreasing with respect to the first variable, nondecreasing with respect to the second variable.

The Godunov flux at the spatial interface is [1, 3, 11, 14, 21, 27, 34]:

$$g(v,u,k_2,k_1) = \min\left\{ k_2 f\left(\max(v,u^*)\right), k_1 f\left(\min(u,u^*)\right)\right\}. \tag{2.2}$$

The function g is Lipschitz continuous in all variables, and like $\bar{f}$ it is monotone, i.e., nonincreasing with respect to v and nondecreasing with respect to u.

The data $u_0(x)$ and $k_i(t)$ are discretized via

$$U_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)\,dx, \quad k_i^n = \frac{1}{\Delta t}\int_{I^n} k_i(t)\,dt.$$

We use Δ_+ and Δ_- to denote the difference operators in the x direction, e.g.,

$$\Delta_+ Z_j = Z_{j+1} - Z_j, \quad \Delta_- Z_j = Z_j - Z_{j-1}.$$

Let $\lambda = \Delta t/\Delta x$. The time marching equation is

$$\begin{cases} U_j^{n+1} = U_j^n - \lambda\Delta_-\left(k_1^n \bar{f}(U_{j+1}^n, U_j^n)\right), & j<0, \\ U_0^{n+1} = U_0^n - \lambda\left(g(U_1^n, U_0^n, k_2^n, k_1^n) - k_1^n \bar{f}(U_0^n, U_{-1}^n)\right), & j=0, \\ U_1^{n+1} = U_1^n - \lambda\left(k_2^n \bar{f}(U_2^n, U_1^n) - g(U_1^n, U_0^n, k_2^n, k_1^n)\right), & j=1, \\ U_j^{n+1} = U_j^n - \lambda\Delta_-\left(k_2^n \bar{f}(U_{j+1}^n, U_j^n)\right), & j>1. \end{cases} \tag{2.3}$$

With the notation

$$h_{j+\frac{1}{2}}^m(v,u) = \begin{cases} k_1^m \bar{f}(v,u), & j<0, \\ g(v,u,k_2^m,k_1^m), & j=0, \\ k_2^m \bar{f}(v,u), & j>0, \end{cases} \tag{2.4}$$

one can write the scheme more compactly as

$$U_j^{n+1} = U_j^n - \lambda\Delta_- h_{j+\frac{1}{2}}^n(U_{j+1}^n, U_j^n). \tag{2.5}$$

From (2.1) and (2.2), it is readily verified that

$$\left|\frac{\partial}{\partial u} h_{j+\frac{1}{2}}^n(v,u)\right|, \left|\frac{\partial}{\partial v} h_{j+\frac{1}{2}}^n(v,u)\right| \le L_u, \text{ for } u,v \in [0,u_{\max}], \tag{2.6}$$

where

$$L_u = \|k\|_\infty \cdot \max\{|f'(u)| : u \in [0,u_{\max}]\}.$$

Let $\Delta = (\Delta x, \Delta t)$. For our convergence analysis we will assume that $\Delta \to 0$ with λ fixed, and satisfying the following CFL condition:

$$\lambda L_u \le 1/2. \tag{2.7}$$

Let $\chi_j(x)$ ($\chi^n(t)$) denote the indicator function for the interval I_j (I^n). The finite difference solution $\{U_j^n\}$ is extended to all of Π_T by defining

$$u^\Delta(x,t) = \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \chi_j(x)\chi^n(t)U_j^n, \qquad (x,t) \in \Pi_T.$$

Similarly, with $\chi_-(x)$ ($\chi_+(x)$) denoting the indicator function of $(-\infty,0)$ ($[0,\infty)$), we define

$$k^\Delta(x,t) = \sum_{n=0}^{N} \chi^n(t)\left(\chi_-(x)k_1^n + \chi_+(x)k_2^n\right), \qquad (x,t) \in \Pi_T.$$

3. CONVERGENCE

This section contains our convergence result, stated in Theorem 3.1. With the exception of Lemma 3.2, our analysis requires only straightforward modifications of known results concerning the case $k = k(x)$. Lemma 3.2 deals with the only significant new difficulty, the time dependence of $k(x,t)$, and we give a detailed proof of that lemma. In order to keep the presentation mostly self-contained, we also include all of the other relevant lemmas.

Theorem 3.1. *Let $\Delta = (\Delta x, \Delta t)$ denote a sequence of grid refinements approaching zero with $\lambda = \Delta t/\Delta x$ fixed. The approximations $u^\Delta(x,t)$ produced by the Godunov scheme of Section 2 converge to $u(x,t)$ boundedly a.e., and in $L^1(\Pi_T)$, where u is the unique entropy solution to* (1.1) *in the sense of Definition 1.1.*

A finite difference scheme such as the scheme (2.3) is monotone [20, 28] if

$$U_j^n \le W_j^n \quad \forall j \in \mathbb{Z} \quad \Longrightarrow \quad U_j^{n+1} \le W_j^{n+1} \quad \forall j \in \mathbb{Z}.$$

Using the monotonicity of the numerical flux function, the CFL condition, and the assumption (1.2), a standard calculation [11, 12] gives the following lemma.

Lemma 3.1. *The Godunov scheme of Section 2 is monotone. The computed solution satisfies*

$$U_j^n \in [0, u_{\max}], \quad j \in \mathbb{Z}, \quad n = 0, \dots, N.$$

Lemma 3.2. *The spatial variation of the numerical flux is bounded, i.e.,*

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ h_{j+\frac12}^n(U_{j+1}^n, U_j^n)\right| \le C_1, \quad n = 0, \dots, N,$$

where the constant C_1 is independent of the mesh size Δ.

Proof. We start by applying $h_{j+\frac12}^{n+1}(\cdot,\cdot)$ to (2.5):

$$\begin{aligned} &h_{j+\frac12}^{n+1}(U_{j+1}^{n+1}, U_j^{n+1}) \\ &= h_{j+\frac12}^{n+1}(U_{j+1}^n, U_j^n) + \int_0^1 \frac{d}{d\theta} h_{j+\frac12}^{n+1}(U_{j+1}^n + \theta(U_{j+1}^{n+1} - U_{j+1}^n), U_j^n + \theta(U_j^{n+1} - U_j^n))\, d\theta \\ &= h_{j+\frac12}^{n+1}(U_{j+1}^n, U_j^n) + \alpha_{j+\frac12}^n \Delta_+ h_{j+\frac12}^n(U_{j+1}^n, U_j^n) - \beta_{j-\frac12}^n \Delta_+ h_{j-\frac12}^n(U_j^n, U_{j-1}^n). \end{aligned} \tag{3.1}$$

where

$$\alpha_{j+\frac12}^n = -\lambda \int_0^1 \frac{\partial}{\partial v} h_{j+\frac12}^{n+1}(U_{j+1}^n + \theta(U_{j+1}^{n+1} - U_{j+1}^n), U_j^n + \theta(U_j^{n+1} - U_j^n))\, d\theta,$$

$$\beta_{j-\frac12}^n = \lambda \int_0^1 \frac{\partial}{\partial u} h_{j+\frac12}^{n+1}(U_{j+1}^n + \theta(U_{j+1}^{n+1} - U_{j+1}^n), U_j^n + \theta(U_j^{n+1} - U_j^n))\, d\theta.$$

Due to the monotonicity of the numerical flux, the CFL condition (2.7), and the bounds (2.6), we have

$$0 \le \alpha_{j+\frac12}^n \le 1/2, \quad 0 \le \beta_{j-\frac12}^n \le 1/2. \tag{3.2}$$

Next, we write (3.1) in the form:

$$h^{n+1}_{j+\frac12}(U^{n+1}_{j+1},U^{n+1}_j) = \underbrace{h^n_{j+\frac12}(U^n_{j+1},U^n_j) + \alpha^n_{j+\frac12}\Delta_+ h^n_{j+\frac12}(U^n_{j+1},U^n_j) - \beta^n_{j-\frac12}\Delta_+ h^n_{j-\frac12}(U^n_j,U^n_{j-1})}_{\mathcal{H}_{j+\frac12}}$$
$$+ \left(h^{n+1}_{j+\frac12}(U^n_{j+1},U^n_j) - h^n_{j+\frac12}(U^n_{j+1},U^n_j)\right).$$

We now apply Δ_+, take absolute values, use the triangle inequality, and then sum over j, yielding

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left|\Delta_+ h^{n+1}_{j+\frac12}(U^{n+1}_{j+1},U^{n+1}_j)\right| &\le \underbrace{\sum_{j\in\mathbb{Z}} |\Delta_+\mathcal{H}_{j+\frac12}|}_{S_1} \\ &\quad + \underbrace{\sum_{j\in\mathbb{Z}} \left|\Delta_+\left(h^{n+1}_{j+\frac12}(U^n_{j+1},U^n_j) - h^n_{j+\frac12}(U^n_{j+1},U^n_j)\right)\right|}_{S_2}.\end{aligned} \tag{3.3}$$

The formula for $\mathcal{H}_{j+\frac12}$ is written in incremental form, and the inequalities (3.2) make it possible to invoke Theorem 16.3 of [28], which yields

$$S_1 \le \sum_{j\in\mathbb{Z}} \left|\Delta_+ h^n_{j+\frac12}(U^n_{j+1},U^n_j)\right|. \tag{3.4}$$

We turn our attention to S_2. First consider contributions that do not involve the interface at $j=0$. For $j\neq 0$, it follows from (2.4) that

$$\begin{aligned}h^{n+1}_{j+\frac12}(U^n_{j+1},U^n_j) - h^n_{j+\frac12}(U^n_{j+1},U^n_j) &= \left(k^{n+1}_i - k^n_i\right)\bar f(U^n_{j+1},U^n_j) \\ &= \frac{1}{k^n_i}\left(k^{n+1}_i - k^n_i\right) h^n_{j+\frac12}(U^n_{j+1},U^n_j).\end{aligned} \tag{3.5}$$

Here $i=1$ for $j<0$, and $i=2$ for $j>0$. So for $j\neq -1,0$,

$$\begin{aligned}&\left|\Delta_+\left(h^{n+1}_{j+\frac12}(U^n_{j+1},U^n_j) - h^n_{j+\frac12}(U^n_{j+1},U^n_j)\right)\right| \\ &\qquad = \frac{1}{k^n_i}\left|k^{n+1}_i - k^n_i\right|\left|\Delta_+ h^n_{j+\frac12}(U^n_{j+1},U^n_j)\right| \\ &\qquad \le \frac{1}{k_{\min}}\left(\left|k^{n+1}_1 - k^n_1\right| + \left|k^{n+1}_2 - k^n_2\right|\right)\left|\Delta_+ h^n_{j+\frac12}(U^n_{j+1},U^n_j)\right|.\end{aligned} \tag{3.6}$$

We now consider the contributions to S_2 that involve the interface. Take the case where $j=0$. Referring to (2.4), we get

$$h^{n+1}_{\frac12}(U^n_1,U^n_0) - h^n_{\frac12}(U^n_1,U^n_0) = g(U^n_1,U^n_0,k^{n+1}_2,k^{n+1}_1) - g(U^n_1,U^n_0,k^n_2,k^n_1),$$

and since g is Lipschitz continuous in k_1 and k_2, we have (with L_k denoting a Lipschitz constant)

$$\left|h^{n+1}_{\frac12}(U^n_1,U^n_0) - h^n_{\frac12}(U^n_1,U^n_0)\right| \le L_k\cdot\left(\left|k^{n+1}_1 - k^n_1\right| + \left|k^{n+1}_2 - k^n_2\right|\right). \tag{3.7}$$

Note that $0\le \bar f(v,u)\le f(u^*)$ for $u,v\in[0,u_{\max}]$, and let $D = L_k + f(u^*)$. Then (3.7) yields

$$\begin{aligned}&\left|\Delta_+\left(h^{n+1}_{\frac12}(U^n_1,U^n_0) - h^n_{\frac12}(U^n_1,U^n_0)\right)\right| \\ &\qquad \le \left|h^{n+1}_{\frac12}(U^n_1,U^n_0) - h^n_{\frac12}(U^n_1,U^n_0)\right| + \left|h^{n+1}_{\frac32}(U^n_2,U^n_1) - h^n_{\frac32}(U^n_2,U^n_1)\right| \\ &\qquad \le L_k\cdot\left(\left|k^{n+1}_1 - k^n_1\right| + \left|k^{n+1}_2 - k^n_2\right|\right) + \left|k^{n+1}_2 - k^n_2\right|\bar f(U^n_2,U^n_1) \\ &\qquad \le D\cdot\left(\left|k^{n+1}_1 - k^n_1\right| + \left|k^{n+1}_2 - k^n_2\right|\right).\end{aligned} \tag{3.8}$$

Similarly, when $j=-1$, we find that

$$\begin{aligned}&\left|\Delta_+\left(h^{n+1}_{-\frac12}(U^n_0,U^n_{-1}) - h^n_{-\frac12}(U^n_0,U^n_{-1})\right)\right| \\ &\qquad \le L_k\cdot\left(\left|k^{n+1}_1 - k^n_1\right| + \left|k^{n+1}_2 - k^n_2\right|\right) + \left|k^{n+1}_1 - k^n_1\right|\bar f(U^n_0,U^n_{-1}) \\ &\qquad \le D\cdot\left(\left|k^{n+1}_1 - k^n_1\right| + \left|k^{n+1}_2 - k^n_2\right|\right).\end{aligned} \tag{3.9}$$

Combining (3.6), (3.8), (3.9), we get

$$\begin{aligned} S_2 \leq \frac{1}{k_{\min}} \left(\left|k_1^{n+1} - k_1^n\right| + \left|k_2^{n+1} - k_2^n\right| \right) \sum_{j\in\mathbb{Z}} \left|\Delta_+ h^n_{j+\frac{1}{2}}(U^n_{j+1}, U^n_j)\right| \\ + 2D \cdot \left(\left|k_1^{n+1} - k_1^n\right| + \left|k_2^{n+1} - k_2^n\right| \right). \end{aligned} \tag{3.10}$$

For the remainder of the proof, we will use the abbreviation $h^n_{j+\frac{1}{2}} = h^n_{j+\frac{1}{2}}(U^n_{j+1}, U^n_j)$. Referring back to (3.3), and using (3.4), (3.10), we have

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ h^{n+1}_{j+\frac{1}{2}}\right| \leq (1 + p^n) \sum_{j\in\mathbb{Z}} \left|\Delta_+ h^n_{j+\frac{1}{2}}\right| + r^n, \tag{3.11}$$

where

$$p^n = \frac{1}{k_{\min}} \left(\left|k_1^{n+1} - k_1^n\right| + \left|k_2^{n+1} - k_2^n\right| \right), \quad r^n = 2D \cdot \left(\left|k_1^{n+1} - k_1^n\right| + \left|k_2^{n+1} - k_2^n\right| \right). \tag{3.12}$$

Our goal now is to show that (3.11) and (3.12) imply the discrete Gronwall inequality (3.16) below. Due to the total variation bounds on k_1 and k_2, there are constants G_1 and G_2, independent of the mesh size Δ, such that

$$\sum_{n=0}^{N-1} p^n \leq G_1, \quad \sum_{n=0}^{N-1} r^n \leq G_2.$$

Starting from (3.11), a straightforward induction proof results in the following inequality, with the agreement that $\prod_{m=n}^{n-1} (1 + p^m) = 1$:

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ h^n_{j+\frac{1}{2}}\right| \leq \prod_{m=0}^{n-1} (1 + p^m) \cdot \sum_{j\in\mathbb{Z}} \left|\Delta_+ h^0_{j+\frac{1}{2}}\right| + \sum_{\nu=0}^{n-1} r^\nu \cdot \prod_{m=\nu+1}^{n-1} (1 + p^m). \tag{3.13}$$

Using the arithmetic/geometric mean inequality, we get

$$\prod_{m=0}^{n-1} (1 + p^m) \leq \prod_{m=0}^{N-1} (1 + p^m) \leq \left(1 + \frac{1}{N} \sum_{m=0}^{N-1} p^m \right)^N \leq \left(1 + \frac{G_1}{N} \right)^N \leq e^{G_1 T}. \tag{3.14}$$

Similarly,

$$\sum_{\nu=0}^{n-1} r^\nu \cdot \prod_{m=\nu+1}^{n-1} (1 + p^m) \leq \sum_{\nu=0}^{N-1} r^\nu \cdot \prod_{m=0}^{N-1} (1 + p^m) \leq G_2 e^{G_1 T}. \tag{3.15}$$

Combining (3.13), (3.14), (3.15), we have

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ h^n_{j+\frac{1}{2}}\right| \leq e^{G_1 T} \sum_{j\in\mathbb{Z}} \left|\Delta_+ h^0_{j+\frac{1}{2}}\right| + G_2 e^{G_1 T}. \tag{3.16}$$

Finally, due to the assumption that $u_0 \in \mathrm{BV}(\mathbb{R})$, and the Lipschitz continuity of the numerical flux, we have

$$\sum_{j\in\mathbb{Z}} \left|\Delta_+ h^0_{j+\frac{1}{2}}\right| \leq B,$$

where B is independent of the mesh size. Inserting B into (3.16), the proof is complete. □

Remark 3.1. Our assumption that k appears as a multiplicative coefficient (and also that k is bounded away from zero) was used in (3.5).

Lemma 3.3. *There exists a constant C_2, independent of Δ, such that*

$$\Delta x \sum_{j\in\mathbb{Z}} \left|U_j^{n+1} - U_j^n\right| \leq C_2 \Delta t, \quad n = 0, \ldots, N, \tag{3.17}$$

and the approximate solutions are bounded in L^1:

$$\Delta x \sum_{j\in\mathbb{Z}} \left|U_j^n\right| \leq \Delta x \sum_{j\in\mathbb{Z}} \left|U_j^0\right| + C_2 T, \quad n = 0, \ldots, N. \tag{3.18}$$

Proof. From the marching formula (2.5) we get

$$\sum_{j\in\mathbb{Z}} \left|U_j^{n+1} - U_j^n\right| = \lambda \sum_{j\in\mathbb{Z}} \left|\Delta_- h^n_{j+\frac{1}{2}}(U^n_{j+1}, U^n_j)\right|.$$

The bound (3.17) follows by invoking Lemma 3.2 and multiplying by Δx. An application of the reverse triangle inequality to (3.17) then yields

$$\Delta x \sum_{j\in\mathbb{Z}} \left|U_j^n\right| \le \Delta x \sum_{j\in\mathbb{Z}} \left|U_j^{n-1}\right| + C_2 \Delta t,$$

and (3.18) follows by induction. □

The next two lemmas provide the bounds required for compactness of $\Psi(u^\Delta, k^\Delta)$. Lemma 3.5 is basically Theorem 3.4 of [33]. We provide a proof for the convenience of the reader.

Lemma 3.4. *Let $z^\Delta(x,t) = \Psi(u^\Delta(x,t), k^\Delta(x,t))$. We have the bounds*

$$\left|z^\Delta(x,t)\right| \le \|k\|_\infty, \quad (x,t)\in\Pi_T, \tag{3.19}$$

$$\int_{\mathbb{R}} \left|z^\Delta(x,t^n) - z^\Delta(x,t^m)\right| dx \le C_3 \left|n-m\right| \Delta t, \quad 0\le n,m\le N, \tag{3.20}$$

where C_3 is a Δ-independent constant.

Proof. The bound (3.19) results from Lemma 3.1 and the observation that $\Psi(u,k)\in[-k,k]$ when $u\in[0,u_{\max}]$. The estimate (3.20) results from Lemma 3.3, along with the fact that $\Psi(u,k)$ is Lipschitz continuous. □

Lemma 3.5. *Let $z^n = z^\Delta(\cdot,t^n)$. There is a constant C_4, independent of the mesh refinement Δ, such that*

$$TV(z^n) \le C_4, \quad n = 0,\dots,N.$$

Proof. Define

$$\psi(u) := \operatorname{sign}(u-u^*)\left(f(u^*) - f(u)\right) = f(u^*)\Psi(u,k)/k.$$

Note that

$$\mathrm{TV}(z^n) = \sum_{j\in\mathbb{Z}} \left|\Delta_+ z_j^n\right|, \quad \text{where } z_j^n = \begin{cases} k_1^n \psi(U_j^n)/f(u^*), & j\le 0,\\ k_2^n \psi(U_j^n)/f(u^*), & j\ge 1.\end{cases}$$

Let $a^+ = \max(a,0)$, $a^- = \min(a,0)$, and $\operatorname{sign}^\pm(a) = (\operatorname{sign}(a))^\pm$. Lemma 3.3 of [33] provides the following inequality:

$$\begin{aligned}\left(\psi(U_j^n) - \psi(U_{j+1}^n)\right)^+ \le -&\operatorname{sign}^-(f'(U_j^n))\left|\bar f(U^n_{j+1},U^n_j) - \bar f(U^n_j, U^n_{j-1})\right| \\ &+ \operatorname{sign}^+(f'(U^n_{j+1}))\left|\bar f(U^n_{j+2},U^n_{j+1}) - \bar f(U^n_{j+1}, U^n_j)\right|.\end{aligned}$$

For $j\le -2$ ($j\ge 2$) we multiply by k_1^n (k_2^n), yielding

$$f(u^*)\left(z_j^n - z_{j+1}^n\right)^+ \le \underbrace{-\operatorname{sign}^-(f'(U_j^n))\left|h^n_{j+\frac{1}{2}} - h^n_{j-\frac{1}{2}}\right| + \operatorname{sign}^+(f'(U^n_{j+1}))\left|h^n_{j+\frac{3}{2}} - h^n_{j+\frac{1}{2}}\right|}_{=:Q^n_{j+\frac{1}{2}}\ge 0}, \quad |j|\ge 2.$$

Here we are using the abbreviation $h^n_{j+\frac{1}{2}} = h^n_{j+\frac{1}{2}}(U^n_{j+1}, U^n_j)$. Summing over $j\in\mathbb{Z}$, we get

$$\begin{aligned} f(u^*)\sum_{j\in\mathbb{Z}}\left(z_j^n - z_{j+1}^n\right)^+ &= f(u^*)\sum_{|j|\ge 2}\left(z_j^n - z_{j+1}^n\right)^+ + f(u^*)\sum_{|j|<2}\left(z_j^n - z_{j+1}^n\right)^+ \\ &\le \sum_{|j|\ge 2} Q^n_{j+\frac{1}{2}} + E \le \sum_{j\in\mathbb{Z}} Q^n_{j+\frac{1}{2}} + E \\ &= \sum_{j\in\mathbb{Z}}\left|\Delta_+ h^n_{j+\frac{1}{2}}\right| + E.\end{aligned} \tag{3.21}$$

Here E is a Δ-independent constant, whose existence follows from the fact that $\left|z_j^n\right| \le \|k\|_\infty$ (Lemma 3.4). Recalling Lemma 3.2, we have in (3.21) a Δ-independent bound on the negative variation of $\{z^n\}$. Since $\{z^n\}$ is bounded, this implies that the total variation is also bounded, and the proof is complete. □

Remark 3.2. In the proof above we referred to Lemma 3.3 of [33]. That paper assumes that f is strictly concave, but the cited lemma only requires the assumptions about f stated in Section 1.

For each $c \in [0, u_{\max}]$, define the numerical entropy flux:

$$H^n_{j+\frac{1}{2}}(v,u) = h^n_{j+\frac{1}{2}}(v \vee c, u \vee c) - h^n_{j+\frac{1}{2}}(v \wedge c, u \wedge c),$$

where $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$. Applied to (2.4), this definition yields

$$H^n_{j+\frac{1}{2}}(v,u) = \begin{cases} k_1^n \bar{f}(v \vee c, U_j^n \vee c) - k_1^n \bar{f}(v \wedge c, u \wedge c), & j < 0, \\ g(U_1^n \vee c, U_0^n \vee c, k_2^n, k_1^n) - g(U_1^n \wedge c, U_0^n \wedge c, k_2^n, k_1^n), & j = 0, \\ k_2^n \bar{f}(v \vee c, u \vee c) - k_2^n \bar{f}(v \wedge c, u \wedge c), & j > 0. \end{cases}$$

Lemma 3.6. *For any $c \in [0, u_{\max}]$, the approximate solutions satisfy the following discrete entropy inequality:*

$$\left|U_j^{n+1} - c\right| \le \left|U_j^n - c\right| - \lambda \Delta_- H^n_{j+\frac{1}{2}}\left(U^n_{j+1}, U^n_j\right) + \lambda R^n_j, \quad n = 0, \ldots, N, \tag{3.22}$$

where

$$R_j^n = \begin{cases} 0, & j \ne 0, 1, \\ |h^n_{\frac{1}{2}}(c,c) - k_1^n f(c)|, & j = 0, \\ |k_2^n f(c) - h^n_{\frac{1}{2}}(c,c)|, & j = 1, \end{cases} \tag{3.23}$$

and for R_0^n, R_1^n we have

$$R_0^n + R_1^n = |k_2^n f(c) - k_1^n f(c)|. \tag{3.24}$$

Proof. Formulas (3.22) and (3.23) result by repeating Lemma 5.1 of [23] or Lemma 4.1 of [24], with small modifications where necessary. We omit the details.

For the proof of (3.24), we use (2.4) to write

$$R_0^n + R_1^n = |g(c,c,k_2^n,k_1^n) - k_1^n f(c)| + |k_2^n f(c) - g(c,c,k_2^n,k_1^n)|, \tag{3.25}$$

where

$$g(c,c,k_2^n,k_1^n) = \min\left\{k_2^n f\left(\max(c,u^*)\right), k_1^n f\left(\min(c,u^*)\right)\right\}. \tag{3.26}$$

Take the case where $c \le u^*$. The proof when $c \ge u^*$ is similar, and we omit it. With $c \le u^*$, (3.26) becomes

$$g(c,c,k_2^n,k_1^n) = \min\left\{k_2^n f(u^*), k_1^n f(c)\right\}.$$

If $g(c,c,k_2^n,k_1^n) = k_1^n f(c)$, then substituting this into (3.25) we get (3.24). The remaining possibility is that $g(c,c,k_2^n,k_1^n) = k_2^n f(u^*)$. In this case we have

$$k_2^n f(c) \le k_2^n f(u^*) \le k_1^n f(c).$$

Substituting $k_2^n f(u^*)$ into (3.25), and then using the inequalities above, we get (3.24) again. □

At this point we have assembled all of the ingredients required for Theorem 3.1, whose proof we sketch below.

Proof. With the bounds stated in Lemmas 3.4 and 3.5, a standard compactness argument gives a subsequence z^{Δ_i} such that $z^{\Delta_i} \to z$ in $L^1(\Pi_T)$ and boundedly a.e. We define

$$u(x,t) := \Psi^{-1}(z(x,t), k(x,t)) = \psi^{-1}(f(u^*) z(x,t)/k(x,t)).$$

Then,

$$\iint_{\Pi_T} \left|u^{\Delta_i} - u\right| \, dx \, dt = \iint_{\Pi_T} \left|\psi^{-1}(f(u^*) z^{\Delta_i}/k^{\Delta_i}) - \psi^{-1}(f(u^*) z/k)\right| \, dx \, dt.$$

The integrand on the right side is bounded, and since ψ^{-1} is continuous an application of Lebesgue's dominated convergence theorem gives $u^{\Delta_i} \to u$ in $L^1(\Pi_T)$. By extracting a further subsequence if necessary, we also have $u^{\Delta_i} \to u$ a.e. in Π_T.

It is straightforward to conclude that the limit u satisfies (**D**.1), (**D**.4) and (**D**.5) of Definition 1.1. (See eg., the proof Theorem 3.1 of [9].) Starting from the discrete entropy inequality of Lemma 3.6, and proceeding as in the proof of Lemma 4.2 of [24], we find that the entropy condition (**D**.3) of Definition 1.1 also holds for the limit function u. As mentioned in Section 1, condition (**D**.2) is implied by condition (**D**.3).

Thus the limit solution u is an entropy solution. Uniqueness of the entropy solution, guaranteed by Theorem 1.1, implies that the entire computed sequence u^{Δ}, not just a subsequence, converges to u. □

Remark 3.3. We have used the singular mapping technique to prove compactness, but one could also use a local spatial variation bound on $u^{\Delta}(x,t)$, i.e., the $\mathrm{BV}_{\mathrm{loc}}$ method mentioned in Section 1. The following is basically Lemma 4.3 of [10] (see also Lemma 5.3 of [12]), which provides such a bound. We omit the proof, but mention that it relies on Lemma 3.3 and the fact that away from the interface at $x=0$, the scheme can be written in incremental form:

$$U_j^{n+1} = U_j^n + C_{j+\frac{1}{2}}^n \Delta_+ U_j^n - D_{j-\frac{1}{2}}^n \Delta_- U_j^n,$$

where

$$C_{j+\frac{1}{2}}^n, D_{j+\frac{1}{2}}^n \in [0,1], \quad C_{j+\frac{1}{2}}^n + D_{j+\frac{1}{2}}^n \le 1.$$

Proposition 3.1. *Let $V_a^b(q)$ denote the total variation of the function $x \mapsto q(x)$ over the interval $[a,b]$. For each $a>0$, there is a constant $C_5(a)$ such that*

$$V_{-\infty}^{-a}(u^{\Delta}(\cdot,t)) \le C_5(a), \quad V_a^{\infty}(u^{\Delta}(\cdot,t)) \le C_5(a),$$

where $C_5(a)$ is independent of Δ.

References

[1] Adimurthi, J. Jaffré and G.D. Veerappa Gowda, *Godunov-type methods for conservation laws with a flux function discontinuous in space*, SIAM J. Numer. Anal., 42 (2004), pp. 179–208.

[2] Adimurthi, S. Mishra and G.D. Veerappa Gowda, *Optimal entropy solutions for conservation laws with discontinuous flux functions*, J. Hyperbolic Differ. Equ., 2 (2005), pp. 783–837.

[3] B. Andreianov and C. Cances, *The Godunov scheme for scalar conservation laws with discontinuous bell-shaped flux functions*, Appl. Math. Lett., 25 (2012), pp. 1844–1848.

[4] B. Andreianov and C. Cances, *On interface transmission conditions for conservation laws with discontinuous flux or general shape*, J. Hyper. Differential Equations, 12 (2015), pp. 343–384.

[5] B. Andreianov, K.H. Karlsen and N.H. Risebro, *A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux*, Arch. Ration. Mech. Anal., 201 (2011), pp. 27–86.

[6] B. Andreianov, K.H. Karlsen and N.H. Risebro, *On vanishing viscosity approximation of conservation laws with discontinuous flux*, Netw. Heterog. Media, 5(3) (2010), pp. 617–633.

[7] G. Bretti, R. Natalini and B. Piccoli, *Numerical approximations of a traffic flow model on networks*, Netw Heterog Media 1(1) (2006), pp. 57– 84.

[8] R. Bürger, K.H. Karlsen, C. Klingenberg and N.H. Risebro, *A front tracking approach to a model of continuous sedimentation in ideal clarifier-thickener units*, Nonlin. Anal. Real World Appl. 4 (2003), pp. 457–481.

[9] R. Bürger, K.H. Karlsen, N.H. Risebro and J.D. Towers, *Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units*, Numer. Math., 97 (2004), pp. 25–65.

[10] R. Bürger, A. García, K.H. Karlsen and J.D. Towers, *A family of schemes for kinematic flows with discontinuous flux*, J. Engrg. Math., 60 (2008), pp. 387–425.

[11] R. Bürger, A. García, K.H. Karlsen and J.D. Towers, *Difference schemes, entropy solutions, and speedup impulse for an inhomogeneous kinematic traffic flow model*, Netw. Heterog. Media, 3 (2008), pp. 1–41.

[12] R. Bürger, K.H. Karlsen and J.D. Towers, *An Engquist-Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections*, SIAM J. Numer. Anal. 47(3) (2009), pp. 1684–1712.

[13] G.M. Coclite and N.H. Risebro, *Conservation Laws with time dependent discontinuous coefficients*, SIAM J. Math. Anal., 36 (2005), pp. 1293–1309.

[14] C.F. Daganzo, *The cell transmission model, part II: network traffic*, Transp. Res. B, 29 (1995), pp. 79–93.

[15] S. Diehl, *A regulator for continuous sedimentation in ideal clarifier-thickener units*, J. Eng. Math., 60 (2008), pp. 265–291.

[16] S. Diehl, *A uniqueness condition for non-linear convection-diffusion equations with discontinuous coefficients*, J. Hyperbolic Differ. Equ. 6(1) (2009), pp. 127–159.

[17] M Garavello and B. Piccoli, *Traffic flow on networks*, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006.

[18] P.G. Goatin, S. Göttlich and O. Kolb, *Speed limit and ramp meter control for traffic flow networks.* Eng. Optim., published online: 08 Nov 2015.

[19] T. Gimse and N.H. Risebro, *Solution of the Cauchy problem for a conservation law with a discontinuous flux function*, SIAM J. Math. Anal., 23 (1992), pp. 635–648.
[20] H. Holden and N.H. Risebro *Front Tracking for Hyperbolic Conservation Laws*, Springer, New York, NY, 2002.
[21] W. Jin and H. Zhang, *The inhomogeneous kinematic wave traffic flow model as a resonant nonlinear system*, Transp. Sci., 37 (2003), pp. 294–311.
[22] K.H. Karlsen, N.H. Risebro and J.D. Towers, *On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient*, Electron. J. Differential Equations, 2002 (2002), pp. 1–23.
[23] K.H. Karlsen, N.H. Risebro and J.D. Towers, *Upwind difference approximations for degenerate parabolic convection-diffusion equations with a discontinuous coefficient*, IMA J. Numer. Anal., 22 (2002), pp. 623–644.
[24] K.H. Karlsen, N.H. Risebro and J.D. Towers, L^1 *stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vid. Selsk. (2003), 49 pp.
[25] K.H. Karlsen and J.D. Towers, *Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontinuous space-time dependent flux*, Chin. Ann. Math., 25B (2004), pp. 287–318.
[26] C. Klingenberg and N.H. Risebro, *Convex conservation laws with discontinuous coefficients, existence, uniqueness and asymptotic behavior*, Comm. Partial Differential Equations, 20 (1995), pp. 1959–1990.
[27] J.P. Lebacque, *The Godunov scheme and what it means for first order traffic flow models*, in Transportation and Traffic Theory, Proceedings of the 13th ISTTT, J.B. Lesort (ed.), Pergamon Press, Oxford, UK, 1996, pp. 647–677.
[28] R.J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhauser Verlag, Basel, 1990.
[29] H. Liu, L. Zhang, D. Sun and D. Wang, *Optimize the Settings of Variable Speed Limit System to Improve the Performance of Freeway Traffic*. Intelligent Transportation Systems, IEEE Transactions on.,16(6) (2015), pp. 3249–57.
[30] M.L.D. Monache, B. Piccoli and F. Rossi, *Traffic regulation via controlled speed limit*, arXiv preprint arXiv:1603.04785, 2016 - arxiv.org
[31] A. Muralidharan and R. Horowitz, *Optimal control of freeway networks based on the link node cell transmission model*, in Proc. Amer. Control Conf., Jun. 2012, pp. 5769–5774.
[32] N. Seguin and J. Vovelle, *Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients*, Math. Models Methods Appl. Sci., 13 (2003), pp. 221–257.
[33] J.D. Towers, *Convergence of a difference scheme for conservation laws with a discontinuous flux*, SIAM J. Numer. Anal., 38 (2000), pp. 681–698.
[34] M. Zhang, C.W. Shu, G.C.K. Wong and S.C. Wong *A weighted essentially non-oscillatory numerical scheme for a multi-class Lighthill-Whitham-Richards traffic flow model.*, J. Comput. Phys., 191 (2003), pp. 639–659.