The Cauchy problem for the Aw-Rascle-Zhang traffic model with locally constrained flow

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Abstract

We study the Cauchy problem for the Aw-Rascle-Zhang model for traffic flow with a flux constraint at $x = 0$. More precisely we consider the Riemann solver, conserving the number of cars at $x = 0$ but not the generalized momentum, introduced in [9] for the problem with flux constrained. For such a Riemann solver, we prove existence of a solution for the Cauchy problem. The proof is based on the wave-front tracking method. For the other Riemann solver in [9], existence of solution to the Cauchy problem was proved in [1].

Key Words: Aw-Rascle-Zhang model, traffic models, unilateral constraint, Cauchy problem.

AMS Subject Classifications: 90B20, 35L65.

1 Introduction

The paper studies the Aw-Rascle-Zhang vehicular traffic model [2, 19]

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0, \\ \partial_t y + \partial_x (yv) = 0, \end{cases} \tag{1.1}$$

with the following constraint on the first component of the flux at $x = 0$:

$$\rho(t, 0)v(t, 0) \leq q, \tag{1.2}$$

where $q > 0$ is a given constant. Here ρ, v and y denote respectively the density, the average speed and a generalized momentum of cars in a road. The generalized momentum y is related to the density ρ and the speed v

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through the relation $y = \rho\,(v + p(\rho))$, where $p \in C^2([0,+\infty[;[0,+\infty[)$ is a *pressure* function satisfying

$$\begin{cases} p(0) = 0, \\ p'(\rho) > 0 \text{ for every } \rho > 0, \\ p''(\rho) \geq 0 \text{ for every } \rho > 0. \end{cases} \tag{1.3}$$

The Aw-Rascle-Zhang system (1.1) is a second-order fluido dynamic model for describing car traffic in a road. Fluido dynamic models treat traffic from a macroscopic point of view: just the evolution of macroscopic variables, such as density and average velocity of cars, is considered. The prototype of such models is the Lighthill-Whitham-Richards one [14, 16], which is based on the conservation of the number of cars and consists of a single partial differential equation in conservation form. From 1975 several second order models were considered, see for example [2] [15] [18] [19], while a third order model was presented in [11]. Various extensions can be found in [3, 5, 7, 10].

System (1.1) can also be written in the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0. \end{cases} \tag{1.4}$$

The first equation in (1.4) states the conservation of the number of vehicles, moving with flow rate ρv. The second equation is derived from the former one and from the evolution equation of the quantity $w = v + p(\rho)$ (often referred to as "Lagrangian marker"), which moves with velocity v:

$$\partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0.$$

The system in conservative form (1.4) belongs to the Temple class [17], i.e. systems for which shock and rarefaction curves in the unknowns' space coincide. In particular, for such systems the interaction of two waves of the same family can only give rise to a wave of the same family.

Problem (1.1), (1.2) models the presence of a constraint on traffic flow at the point $x = 0$, such as a toll gate, a traffic light, a construction site, etc. All these situations limit the flow at a specific location along the road. Conservation laws with unilateral constraints as (1.2) have been first introduced in [6], where the scalar Lighthill-Whitham [14] and Richards [16] traffic model is coupled with a (possibly time-dependent) constraint on the flow, as in (1.2). As regards the Aw-Rascle-Zhang model, problem (1.1)-(1.2) was first considered in [9].

The aim of the present paper is to study the Cauchy problem for (1.1), (1.2). We remark that in [9] two different solutions with flux constraints have been introduced: one which conserves both the number of cars and the generalized momentum, and one which conserves only the number of cars. The existence of a solution to the Cauchy problem using the Riemann solver which conserves both conserved quantities has been proved by Andreianov, Donadello, and Rosini in [1]. Here we prove the existence of a solution for the other Riemann solver. The proof is based on the wave-front tracking method; see for example [4, 8, 13]. This method consists in approximating the solution by a sequence of piecewise constant functions, in tracking the waves, and in monitoring the interactions between waves. As usual, the approach relies on three estimates: the number of waves, the number of wave interactions and the total variation of the solution. By Helly Theorem, the previous estimates permit to extract a converging subsequence. The limit function is indeed a solution to the Cauchy problem.

The paper is organized as follows. In Section 2 we introduce the basic quantities for the Aw-Rascle-Zhang model. Moreover we recall the definition of Riemann solver with flux constraints, introduced in [9], the shape of invariant domains for such Riemann solver, and, finally, the definition of solution to the Cauchy problem with a flux constraint at $x = 0$. Section 3 contains the proof of the existence of a solution to the Cauchy problem. More precisely, in Subsection 3.1, we introduce the definition of a wave-front tracking approximate solution and several functionals dealing with the total variation. Subsections 3.2 and 3.3 contain respectively the interaction estimates, and the proof of existence of an approximate wave-front tracking solution. Finally Subsection 3.4 concludes the proof of the existence of a solution to the Cauchy problem.

2 Basic definitions

In this section we briefly recall the basic definitions and the construction of a Riemann solver introduced in [9].

The Cauchy problem for the Aw-Rascle-Zhang model (1.1) with flux constraint (1.2) consists in the following system

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, \\ \rho(t, 0)v(t, 0) \leq q \\ (\rho, v)(0, x) = (\rho_o, v_o)(x) \end{cases} \tag{2.1}$$

where $q > 0$, and $(\rho_o, v_o) \in BV\left(\mathbb{R}; (\mathbb{R}^+)^2\right)$. It is convenient to denote by $f(\rho, v)$ the flux for system (1.4), and with $f_1(\rho, v)$, $f_2(\rho, v)$ its components, i.e.

$$f(\rho, v) = \begin{pmatrix} f_1(\rho, v) \\ f_2(\rho, v) \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho v(v + p(\rho)) \end{pmatrix}. \tag{2.2}$$

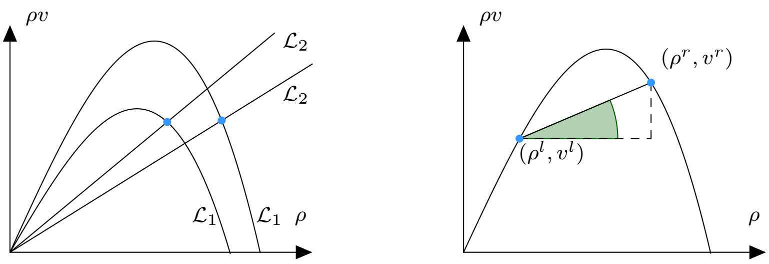

Figure 1: Left, the representation of the Lax curves of the first and second family in the $(\rho, \rho v)$ plane, passing through two points. Right, the geometric interpretation of the Rankine-Hugoniot speed of a shock wave of the first family. The speed corresponds to the tangent of the angle.

We recall here the relevant quantities concerning the system (2.1):

$$
\begin{array}{ll}
\lambda_1 = v - \rho p'(\rho) & \lambda_2 = v \\
r_1 = \begin{pmatrix} -1 \\ p'(\rho) \end{pmatrix} & r_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\nabla \lambda_1 \cdot r_1 = 2p'(\rho) + \rho p''(\rho) > 0 & \nabla \lambda_2 \cdot r_2 = 0 \\
L_1(\rho; \rho_0, v_0) = v_0 + p(\rho_0) - p(\rho) & L_2(\rho; \rho_0, v_0) = v_0 \\
z = v & w = v + p(\rho)
\end{array}
$$

where λ_1 and λ_2 are the eigenvalues of the Jacobian matrix Df, r_1 and r_2 the corresponding right eigenvectors, L_1 and L_2 the first and the second Lax curve, z and w the 1- and 2-Riemann invariant respectively.

Note that the system is strictly hyperbolic away from $\rho = 0$ (i.e. $\lambda_1 < \lambda_2$). Moreover, since $\nabla \lambda_1 \cdot r_1 > 0$, the first characteristic speed is genuinely nonlinear, with characteristic speed λ_1 that can change sign, and, since $\nabla \lambda_2 \cdot r_2 = 0$, the second characteristic field is linearly degenerate with strictly positive characteristic speed λ_2.

In the $(\rho, \rho v)$ plane, the Lax curves of the first and the second family are

$$
\mathcal{L}_1(\rho; \rho_0, v_0) = (v_0 + p(\rho_0) - p(\rho))\, \rho, \qquad \mathcal{L}_2(\rho; \rho_0, v_0) = v_0 \rho; \tag{2.3}
$$

see Figure 1, left. By hypothesis (1.3) on the pressure p, the function $\mathcal{L}_1$ is concave. Note moreover that the Rankine-Hugoniot speed of a shock wave of the first family is given by the slope of the segment in the $(\rho, \rho v)$ plane, connecting the left and right states; see Figure 1, right.

In the following, by $\mathcal{RS}$ we denote the classical Riemann solver for the Aw-Rascle-Zhang model, i.e. the Riemann solver without the constraint (1.2); see for example [2, 19]. Moreover by $\mathcal{RS}^q$ we denote the Riemann solver, introduced in [9, Section 2.2].

2.1 The constrained Riemann solver $\mathcal{RS}^q$

Here we recall the definition of $\mathcal{RS}^q$ and its corresponding invariant domain.

For $(\rho^l, v^l) \in (\mathbb{R}^+)^2$, $(\rho^r, v^r) \in (\mathbb{R}^+)^2$, and $q > 0$, let us consider the set

$$\begin{aligned} I_q &= \left\{ \rho \in [0, +\infty[\; : \rho L_1(\rho; \rho^l, v^l) = q \right\} \\ &= \left\{ \rho \in [0, +\infty[\; : \rho(v^l + p(\rho^l) - p(\rho)) = q \right\}, \end{aligned} \tag{2.4}$$

which contains the densities of all the points (ρ, v) belonging to the Lax curve of the first family passing through (ρ^l, v^l) and such that $f_1(\rho, v) = q$. If $I_q \neq \emptyset$, then we denote by $\hat{\rho}$, $\hat{v}$, respectively

$$\hat{\rho} = \max I_q, \quad \hat{v} = \frac{q}{\hat{\rho}}. \tag{2.5}$$

Moreover, define $\check{\rho}$ and $\check{v}$ by

$$\check{\rho} L_2(\check{\rho}; \rho^r, v^r) = q, \quad \check{v} = \frac{q}{\check{\rho}}; \tag{2.6}$$

i.e. $(\check{\rho}, \check{v})$ belongs to the Lax curve of the second family passing through (ρ^r, v^r) and satisfies $f_1(\check{\rho}, \check{v}) = q$. In particular, note that $\check{v} = v^r$ and $\check{\rho} = q/v^r$. Clearly, $\hat{\rho}$ and $\hat{v}$ depend on q, on ρ^l, and on v^l; $\check{\rho}$ and $\check{v}$ depend on q, on ρ^r, and on v^r.

The Riemann solver $\mathcal{RS}^q$ is defined as follows.

1. If $f_1(\mathcal{RS}\left((\rho^l, v^l), (\rho^r, v^r)\right)(0)) \leq q$, then we put

$$\mathcal{RS}^q \left((\rho^l, v^l), (\rho^r, v^r) \right)(x) = \mathcal{RS} \left((\rho^l, v^l), (\rho^r, v^r) \right)(x) \tag{2.7}$$

for every $x \in \mathbb{R}$.

2. If $f_1(\mathcal{RS}\left((\rho^l, v^l), (\rho^r, v^r)\right)(0)) > q$, then

$$\mathcal{RS}^q \left((\rho^l, v^l), (\rho^r, v^r) \right)(x) = \begin{cases} \mathcal{RS}\left((\rho^l, v^l), (\hat{\rho}, \hat{v})\right)(x), & \text{if } x < 0, \\ \mathcal{RS}\left((\check{\rho}, \check{v}), (\rho^r, v^r)\right)(x), & \text{if } x > 0. \end{cases} \tag{2.8}$$

2.2 Invariant domains

Fix v_1, v_2, w_1 and w_2 in $\mathbb{R}$ such that $0 < v_1 < v_2$, $0 < w_1 < w_2$ and $v_2 < w_2$. The set

$$\mathcal{D}_{v_1, v_2, w_1, w_2} = \{(\rho, v) \in \mathbb{R}^+ \times \mathbb{R}^+ : w_1 \leq v + p(\rho) \leq w_2, \; v_1 \leq v \leq v_2\}$$

is an invariant domain of the classical Riemann solver for the Aw-Rascle-Zhang model; see [12].

Before considering invariant domains for $\mathcal{RS}^q$, we introduce, for $q > 0$, the following function:

$$\begin{array}{rcl} h_q : (0, +\infty) & \longrightarrow & \mathbb{R} \\ v & \longmapsto & v + p\left(\dfrac{q}{v}\right), \end{array} \tag{2.9}$$

which gives the value of the Riemann invariant w of the point $(\tilde{\rho}, v)$ such that $\tilde{\rho}\, v = q$. The shape of invariant domains for the Riemann solver $\mathcal{RS}^q$ is given by the next proposition. We complete the proof given in [9].

Proposition 2.1 *Assume (1.3). Fix v_1, v_2, w_1 and w_2 in $\mathbb{R}$ such that $0 < v_1 < v_2$, $0 < w_1 < w_2$ and $v_2 < w_2$ and $q > 0$.*

(i) *If $h_q(v) \geq w_2$ for every $v \in [v_1, v_2]$, then the domain $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for the Riemann solver $\mathcal{RS}^q$.*

(ii) *Assume that there exists $\bar{v} \in [v_1, v_2]$ for which $h_q(\bar{v}) < w_2$. The set $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for the Riemann solver $\mathcal{RS}^q$ if and only if*

$$h_q(v_1) \geq w_2, \quad h_q(v_2) \leq w_2 \quad \text{and} \quad h_q(v) \geq w_1 \quad \forall v \in [v_1, v_2]. \tag{2.10}$$

Proof. The proof of (i) is contained in [9, Proposition 3.1]. Thus we consider only the case (ii). The proof is divided into two parts.

Part 1. Assume that $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for $\mathcal{RS}^q$. Hence $h_q(v_1) \geq w_2$ and $h_q(v) \geq w_1$ for every $v \in [v_1, v_2]$, by Lemmas 3.2 and 3.3 of [9]. Suppose, by contradiction, that $h_q(v_2) > w_2$. Let (ρ^l, v^l) and (ρ^r, v^r) be the points of $\mathcal{D}_{v_1,v_2,w_1,w_2}$ defined respectively by

$$\begin{cases} v + p(\rho) = w_2, \\ v = v_1, \end{cases} \qquad \text{and} \qquad \begin{cases} v + p(\rho) = w_2, \\ v = v_2. \end{cases}$$

The classical solution connects (ρ^l, v^l) to (ρ^r, v^r) with a rarefaction wave, because $v^l < v^r$. Let $(\check{\rho}_1, \check{v}_1)$ be the point defined by

$$\begin{aligned} \check{\rho}_1 &= \min I_q = \min \left\{ \rho \in [0, +\infty[\, : \rho(v^l + p(\rho^l) - p(\rho)) = q \right\} \\ \check{v}_1 &= \frac{q}{\check{\rho}_1}. \end{aligned}$$

We have $\rho^l > \hat{\rho}$ and $\rho^r < \check{\rho}_1$; see Figure 2. Since the function $\rho \mapsto \mathcal{L}_1(\rho; \rho^l, v^l)$ is strictly concave, then the classical solution in $x = 0$ does not satisfy the flux constraint. Therefore the right trace of $\mathcal{RS}^q((\rho^l, v^l), (\rho^r, v^r))$ at $x = 0$ is given by $(\check{\rho}, \check{v})$. Since $\check{v} = v^r = v_2$, we deduce that

$$h_q(\check{v}) = h_q(v_2) > w_2.$$

Therefore $(\check{\rho}, \check{v})$ does not belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$, which is a contradiction; see Figure 2.

Part 2. Assume that conditions in (2.10) hold. We show that $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is invariant for $\mathcal{RS}^q$. Let (ρ^l, v^l) and (ρ^r, v^r) be two arbitrary points in $\mathcal{D}_{v_1,v_2,w_1,w_2}$. It is sufficient to prove that the solution connecting (ρ^l, v^l) to (ρ^r, v^r) is contained in $\mathcal{D}_{v_1,v_2,w_1,w_2}$. Let $(\rho^m, v^m) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ be the intermediate state produced by the classical Riemann solver $\mathcal{RS}$.

If the Riemann solver $\mathcal{RS}^q$ produces the classical solution (i.e. a wave connecting (ρ^l, v^l) to (ρ^m, v^m) and/or a wave connecting (ρ^m, v^m) to (ρ^r, v^r)), then the solution is clearly contained in $\mathcal{D}_{v_1,v_2,w_1,w_2}$. Suppose therefore that $\mathcal{RS}^q$ does not produce the classical solution. Denote by $(\hat{\rho}, \hat{v})$ and by $(\check{\rho}, \check{v})$ the states defined in (2.5) and in (2.6). Proposition 3.3 in [9] implies that $(\hat{\rho}, \hat{v})$ belongs to $\mathcal{D}_{v_1,v_2,w_1,w_2}$. So, it remains to prove that $(\check{\rho}, \check{v}) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$.

Suppose, by contradiction, that $(\check{\rho}, \check{v}) \notin \mathcal{D}_{v_1,v_2,w_1,w_2}$. Since $\check{v} = v^r$ and $h_q(v) \geq w_1$ for every $v \in [v_1, v_2]$, we deduce that $h_q(\check{v}) > w_2$. Moreover, since $h_q(\check{v}) > w_2$, then every point (ρ^*, v^*) belonging to $\mathcal{D}_{v_1,v_2,w_1,w_2}$ and to the Lax curve of the second family through (ρ^r, v^r) satisfies $\rho^* v^* < q$. In particular $\rho^m v^m < q$. The following cases happen.

1. $v^l > v^r$. Since the classical Riemann problem connecting (ρ^l, v^l) to (ρ^m, v^m) is solved by a shock wave of the first family and since the solution produced is the non classical one, then we deduce that $\rho^l v^l > q$ and the shock wave has positive speed. Since $\rho^l < \rho^m$, the Rankine-Hugoniot condition implies that

$$\rho^l v^l < \rho^m v^m < q$$

which is a contradiction.
2. $v^l = v^r$. In this situation, $\left(\rho^l, v^l\right) = (\rho^m, v^m)$ and so the Riemann solver $\mathcal{RS}^q$ produces the classical solution. This is a contradiction.
3. $v^l < v^r$. The classical Riemann problem connecting (ρ^l, v^l) to (ρ^m, v^m) is solved by a rarefaction wave of the first family, and all the states (ρ^*, v^*) of this rarefaction wave satisfy $v^* < v^m = v^r$. By [9, Lemma 3.1], we deduce that all the states (ρ^*, v^*) satisfy $h_q(v^*) > w_2$ and hence $\rho^* v^* < q$. Therefore the Riemann solver $\mathcal{RS}^q$ produces the classical solution. This is a contradiction.

Thus $(\check{\rho}, \check{v}) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$.

The proof is so completed. $\square$

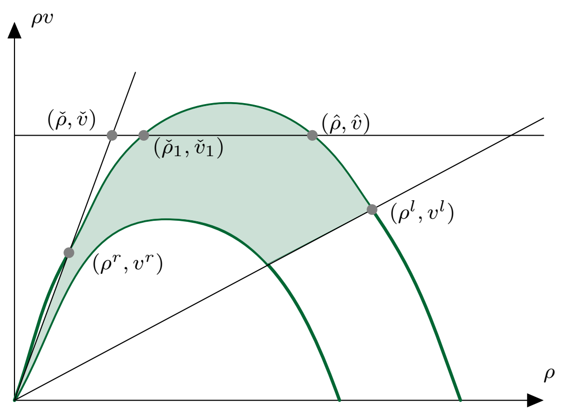

Figure 2: The situation described in the proof of Proposition 2.1: if $h_q(v_2) > w_2$ the point $(\check\rho, \check v)$ does not belong to $\mathcal{D}_{v_1,v_2,w_1,w_2}$ (the shaded area).

2.3 Definition of solution to (2.1)

Here we give the definition of solution to the constrained Cauchy problem (2.1).

Definition 2.1 *A couple $(\rho, v) \in C^0\left([0,+\infty); BV\left(\mathbb{R}; (\mathbb{R}^+)^2\right)\right)$ provides a solution to the Cauchy problem (2.1) if*

1. *(ρ, v) is a weak entropy solution to (1.4) in $(0,+\infty) \times (-\infty, 0)$ and in $(0,+\infty) \times (0,+\infty)$;*
2. *$(\rho, v)(0, x) = (\rho_o, v_o)(x)$ for a.e. $x \in \mathbb{R}$;*
3. *$\mathcal{RS}^q\left((\rho, v)(t, 0-), (\rho, v)(t, 0+)\right)(0-) = (\rho, v)(t, 0-)$ for a.e. $t \in [0, T]$;*
4. *$\mathcal{RS}^q\left((\rho, v)(t, 0-), (\rho, v)(t, 0+)\right)(0+) = (\rho, v)(t, 0+)$ for a.e. $t \in [0, T]$.*

3 The Cauchy problem

In this section we prove that, under suitable assumptions, the Cauchy problem (2.1) admits a solution, in the sense of Definition 2.1. Fix $q > 0$, $0 < v_1 < v_2$, $0 < w_1 < w_2$ and $v_2 < w_2$ such that $\mathcal{D}_{v_1,v_2,w_1,w_2}$ is an invariant domain for the Riemann solver $\mathcal{RS}^q$ such that

$$\lambda_1(\rho, v) < 0 \qquad \forall\,(\rho, v) \in \mathcal{D}_{v_1,v_2,w_1,w_2}. \tag{3.1}$$

We have the following result.

Theorem 3.1 *Assume that $(\rho_o, v_o) \in BV\left(\mathbb{R}; \mathcal{D}_{v_1,v_2,w_1,w_2}\right)$. Then there exists*

$$(\rho, v) \in C^0\left([0,+\infty); BV\left(\mathbb{R}; \mathcal{D}_{v_1,v_2,w_1,w_2}\right)\right)$$

a solution to the Cauchy problem (2.1) in the sense of Definition 2.1.

The proof is contained in the next subsections. For a later use, we define the densities

$$\rho_{min} = \min\left\{\rho > 0 : \ (\rho, v) \in \mathcal{D}_{v_1,v_2,w_1,w_2} \text{ for some } v \in [v_1, v_2]\right\} \tag{3.2}$$
$$\rho_{max} = \max\left\{\rho > 0 : \ (\rho, v) \in \mathcal{D}_{v_1,v_2,w_1,w_2} \text{ for some } v \in [v_1, v_2]\right\}. \tag{3.3}$$

Clearly ρ_{min} and ρ_{max} exist and $0 < \rho_{min} < \rho_{max}$.

3.1 Wave-front tracking

Definition 3.1 *Given $\varepsilon > 0$, we say that the map $\bar{u}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{v}_\varepsilon)$ is an ε-approximate wave-front tracking solution to (2.1) if the following conditions hold.*

1. *$\bar{u}_\varepsilon \in C((0, +\infty); L^1(\mathbb{R}; \mathcal{D}_{v_1,v_2,w_1,w_2}))$.*
2. *$(\bar{\rho}_\varepsilon, \bar{v}_\varepsilon)$ is piecewise constant, with discontinuities occurring along finitely many straight lines in $(0, +\infty) \times \mathbb{R}$. Moreover the jumps can be at $x = 0$, or of the first family, or of the second family. They are indexed, respectively, by $\mathcal{J}(t) = \mathcal{J}_0(t) \cup \mathcal{J}_1(t) \cup \mathcal{J}_2(t)$.*
3. *It holds that*
$$\begin{cases} \|(\bar{\rho}_\varepsilon(0, \cdot), \bar{v}_\varepsilon(0, \cdot)) - (\rho_o(\cdot), v_o(\cdot))\|_{L^1(\mathbb{R})} < \varepsilon \\ TV\left(\bar{\rho}_\varepsilon(0, \cdot), \bar{v}_\varepsilon(0, \cdot)\right) \leq TV\left(\rho_o(\cdot), v_o(\cdot)\right). \end{cases}$$
4. *For a.e. $t > 0$,*
$$\mathcal{RS}^q\left(\bar{u}_\varepsilon\left(t, 0-\right), \bar{u}_\varepsilon\left(t, 0+\right)\right)(0-) = \bar{u}_\varepsilon\left(t, 0-\right).$$
5. *For a.e. $t > 0$,*
$$\mathcal{RS}^q\left(\bar{u}_\varepsilon\left(t, 0-\right), \bar{u}_\varepsilon\left(t, 0+\right)\right)(0+) = \bar{u}_\varepsilon\left(t, 0+\right).$$

We construct a sequence of wave-front tracking approximate solutions in the following way. First consider a sequence $(\rho_{o,\nu}, v_{o,\nu})$, of piecewise constant functions with a finite number of discontinuities such that

1. $(\rho_{o,\nu}, v_{o,\nu}) : \mathbb{R} \to \mathcal{D}_{v_1,v_2,w_1,w_2}$;
2. the following limit holds
$$\lim_{\nu \to +\infty} (\rho_{o,\nu}, v_{o,\nu}) = (\rho_o, v_o) \qquad \text{in } L^1(\mathbb{R}; \mathcal{D}_{v_1,v_2,w_1,w_2});$$
3. the following inequality holds
$$\mathrm{TV}(\rho_{o,\nu}, v_{o,\nu}) \leq \mathrm{TV}(\rho_o, v_o).$$

For every $\nu \in \mathbb{N} \setminus \{0\}$, we apply the following procedure. At time $t = 0$, we solve all the Riemann problems for $x \in \mathbb{R}$ with $x \neq 0$, by using the classical Riemann solver, while at $x = 0$ we solve the corresponding Riemann problem by using the Riemann solver $\mathcal{RS}^q$. We approximate every rarefaction wave of the first family with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ traveling with the Rankine-Hugoniot speed. Here we mean that a rarefaction shock connects two states whose 2-Riemann invariant w differs at most by $\frac{1}{\nu}$. At every interaction between two waves, we solve the corresponding Riemann problem. Finally, when a wave interacts with the interface $x = 0$, we solve the corresponding Riemann problem by using the Riemann solver $\mathcal{RS}^q$.

Remark 1 *As usual, by slightly modifying the speed of waves, we may assume that, at every positive time t, at most one of the following possibilities happens:*

1. *two waves interact together at a point $x \in \mathbb{R} \setminus \{0\}$;*
2. *a wave interacts with the interface $x = 0$.*

Remark 2 *For interactions at a point $x \in \mathbb{R} \setminus \{0\}$, we split rarefaction waves into rarefaction fans just at time $t = 0$. At the interface $x = 0$, instead, we allow the formation of rarefaction fans only when the interacting wave is of the second family.*

Given an ε-approximate wave-front tracking solution $\bar{u}_\varepsilon = (\bar{\rho}_\varepsilon, \bar{v}_\varepsilon)$ define, for a.e. $t > 0$, the following functionals

$$W_0(t) = \left|\bar{w}_\varepsilon(t, 0^+) - \bar{w}_\varepsilon(t, 0^-)\right| + \left|\bar{v}_\varepsilon(t, 0^+) - \bar{v}_\varepsilon(t, 0^-)\right|, \tag{3.4}$$

$$W_1(t) = \sum_{x \in \mathcal{J}_1(t)} \left|\bar{v}_\varepsilon(t, x^+) - \bar{v}_\varepsilon(t, x^-)\right|, \tag{3.5}$$

$$W_2(t) = \sum_{x \in \mathcal{J}_2(t)} \left|\bar{w}_\varepsilon(t, x^+) - \bar{w}_\varepsilon(t, x^-)\right|, \tag{3.6}$$

$$W(t) = W_0(t) + W_1(t) + W_2(t), \tag{3.7}$$

$$TV_\rho(t) = \left|\bar{\rho}_\varepsilon(t, 0^+) - \bar{\rho}_\varepsilon(t, 0^-)\right| + \sum_{x \in \mathcal{J}_1(t) \cup \mathcal{J}_2(t)} \left|\bar{\rho}_\varepsilon(t, x^+) - \bar{\rho}_\varepsilon(t, x^-)\right| \tag{3.8}$$

where the function w stands for the 2-Riemann invariant, $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$ contains the point of discontinuity for $\bar{u}_\varepsilon$ respectively for the waves of the first and second family. Note that the previous functionals may vary only at times $\tilde{t}$ when two waves interact or a wave reaches $x = 0$. Moreover we introduce the functional

$$\mathcal{N}(t) = \#\left(\mathcal{J}_0(t) \cup \mathcal{J}_1(t) \cup \mathcal{J}_2(t)\right), \tag{3.9}$$

where # denotes the cardinality of a set, while

$$\mathcal{J}_0(t) = \begin{cases} 0, & \text{if } \bar{u}_\varepsilon(t, 0^-) = \bar{u}_\varepsilon(t, 0^+) \\ 1, & \text{if } \bar{u}_\varepsilon(t, 0^-) \neq \bar{u}_\varepsilon(t, 0^+). \end{cases}$$

3.2 Interaction estimates

In this subsection we collect various results concerning the interactions between waves of a ε-approximate wave-front tracking solution. Define the constant

$$K_\varepsilon = \max\left\{ \left\lfloor \frac{w_2 - w_1}{\varepsilon} \right\rfloor + 1, 2 \right\}, \tag{3.10}$$

where $\lfloor\cdot\rfloor$ denote the integer value, and w_1 and w_2 define the invariant domain $\mathcal{D}_{v_1,v_2,w_1,w_2}$. Note that K_ε provides an upper bound for the number of rarefaction shocks in a rarefaction fan. First consider the case of the interaction between waves of the first family.

Proposition 3.1 *Assume that a wave of the first family joining (ρ^l, v^l) to (ρ^m, v^m) interacts with a wave of the first family connecting (ρ^m, v^m) to (ρ^r, v^r) at time $\tilde{t}$ and at position $\tilde{x} \neq 0$. Then, at time $\tilde{t}$, a single shock wave of the first family is generated and*

$$\Delta W_0(\tilde{t}) = 0 \qquad \Delta W_1(\tilde{t}) \leq 0 \qquad \Delta W_2(\tilde{t}) = 0 \qquad \Delta\mathcal{N}(\tilde{t}) = -1.$$

Therefore $\Delta W(\tilde{t}) \leq 0$.

Proof. Since $\tilde{x} \neq 0$, we have $\Delta W_0(\tilde{t}) = 0$.

The fact that the interacting waves are of the first family implies that

$$v^l + p(\rho^l) = v^m + p(\rho^m) = v^r + p(\rho^r)$$

and so the Riemann problem with initial data (ρ^l, v^l) and (ρ^r, v^r) is solved by a wave of the first family; hence $\Delta\mathcal{N}(\tilde{t}) = -1$ and $\Delta W_2(\tilde{t}) = 0$. Moreover

$$\Delta W_1(\tilde{t}) = |v^l - v^r| - |v^l - v^m| - |v^m - v^r| \leq 0$$

by the triangular inequality.

Finally we prove that the wave, generated at time $\tilde{t}$, is a shock wave. Assume, by contradiction, that the wave connecting (ρ^l, v^l) to (ρ^r, v^r) is a rarefaction wave, so that $\rho^l > \rho^r$. We have the following three possibilities.

1. $\rho^m < \rho^r$. In this case the velocity of the wave, connecting $\left(\rho^l, v^l\right)$ with (ρ^m, v^m) is strictly less than the velocity of the wave, connecting (ρ^m, v^m) with (ρ^r, v^r); see Figure 3, left. Therefore the two waves can not interact together.

2. $\rho^r < \rho^m < \rho^l$. In this case both the interacting waves are rarefaction waves. This is not possible, since rarefaction waves can not interact together.

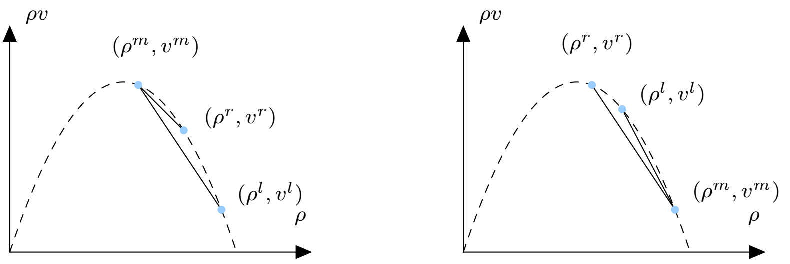

Figure 3: The cases 1 and 3 in the proof of Proposition 3.1. The slopes of the segments are related to the velocity of the waves.

3. $\rho^m > \rho^l$. In this case the velocity of the wave, connecting (ρ^l, v^l) with (ρ^m, v^m), is strictly less than the velocity of the wave, connecting (ρ^m, v^m) with (ρ^r, v^r); see Figure 3, right. Thus the two waves can not interact together.

The proof is so concluded. □

Proposition 3.2 *Assume that a wave of the second family joining (ρ^l, v^l) to (ρ^m, v^m) interacts with a wave of the first family connecting (ρ^m, v^m) to (ρ^r, v^r) at time $\tilde{t}$ and at position $\tilde{x} \neq 0$. Then, at time $\tilde{t}$, a wave of the first family and a wave of the second family are generated. Moreover*

$$\Delta W_0(\tilde{t}) = 0 \qquad \Delta W_1(\tilde{t}) = 0 \qquad \Delta W_2(\tilde{t}) = 0 \qquad \Delta \mathcal{N}(\tilde{t}) = 0.$$

Therefore $\Delta W(\tilde{t}) = 0$.

Proof. Since $\tilde{x} \neq 0$, we have $\Delta W_0(\tilde{t}) = 0$.

In this case, at time $\tilde{t}$, two waves are produced. More precisely, a wave of the first family connecting (ρ^l, v^l) to (ρ^i, v^i), followed by a wave of the second family connecting (ρ^i, v^i) to (ρ^r, v^r), where the previous states satisfy

$$v^l = v^m, \quad v^i = v^r, \quad v^l + p(\rho^l) = v^i + p(\rho^i), \quad v^m + p(\rho^m) = v^r + p(\rho^r).$$

Thus we deduce that $\Delta \mathcal{N}(\tilde{t}) = 0$ and

$$\begin{aligned} \Delta W_1(\tilde{t}) &= \left| v^l - v^i \right| - \left| v^r - v^m \right| = 0 \\ \Delta W_2(\tilde{t}) &= \left| p(\rho^r) - p(\rho^i) \right| - \left| p(\rho^l) - p(\rho^m) \right| = 0, \end{aligned}$$

concluding the proof. □

We pass now to consider the case when the interaction happens at $\tilde{x} = 0$. First we need the following technical lemma.

Lemma 3.1 *There exist constants* $0 < L_1 < L_2$ *such that*

$$L_1 \left|v^m - v^r\right| \le \left|w^l - w^m\right| \le L_2 \left|v^m - v^r\right| \tag{3.11}$$

for every $\left(\rho^l, v^l\right) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$, $\left(\rho^m, v^m\right) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$, $\left(\rho^r, v^r\right) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ *satisfying* $v^l = v^m$, $w^m = w^r$, *and* $\rho^l v^l = \rho^r v^r < \rho^m v^m$.

Proof. Fix $\left(\rho^l, v^l\right) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$, $\left(\rho^m, v^m\right) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$, $\left(\rho^r, v^r\right) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ satisfying $v^l = v^m$, $w^m = w^r$, and $\rho^l v^l = \rho^r v^r < \rho^m v^m$. By assumption (1.3), there exist $0 < K_1 < K_2$ such that

$$K_1 \left|\rho_1 - \rho_2\right| \le \left|p\left(\rho_1\right) - p\left(\rho_2\right)\right| \le K_2 \left|\rho_1 - \rho_2\right| \tag{3.12}$$

for every $\rho_1, \rho_2 \in [\rho_{min}, \rho_{max}]$, where ρ_{min} and ρ_{max} are defined in (3.2) and in (3.3) respectively. Note that $\rho^m v^m - \rho^l v^l = v^l\left(\rho^m - \rho^l\right)$ and so

$$v_1\left(\rho^m - \rho^l\right) \le \rho^m v^m - \rho^l v^l \le v_2\left(\rho^m - \rho^l\right). \tag{3.13}$$

Moreover

$$\begin{aligned}\rho^m v^m - \rho^l v^l &= \rho^m v^m - \rho^r v^r = \mathcal{L}_1\left(\rho^m; \rho^r, v^r\right) - \mathcal{L}_1\left(\rho^r; \rho^r, v^r\right)\\ &= \frac{\partial}{\partial \rho}\mathcal{L}_1\left(\rho; \rho^r, v^r\right)_{|\rho=\rho_1}\left(\rho^m - \rho^r\right)\end{aligned}$$

for some $\rho_1 \in (\rho^m, \rho^r)$. By (3.1), we deduce that there exist $K_3 < K_4 < 0$, which depend only on v_1, v_2, w_1, and w_2, such that

$$K_3\left(\rho^r - \rho^m\right) \le \rho^m v^m - \rho^l v^l \le K_4\left(\rho^r - \rho^m\right). \tag{3.14}$$

By (3.12), (3.13), and (3.14), and since $w^m = w^r$, we have that

$$\begin{aligned}\left|w^l - w^m\right| &= w^m - w^l = v^m - v^l + p(\rho^m) - p(\rho^l) = p(\rho^m) - p(\rho^l)\\ &\le K_2\left|\rho^m - \rho^l\right| \le \frac{K_2}{v_1}\left(\rho^m v^m - \rho^l v^l\right)\\ &\le \frac{K_2 K_4}{v_1}\left(\rho^r - \rho^m\right) \le \frac{K_2 K_4}{K_1 v_1}\left(p(\rho^r) - p(\rho^m)\right)\\ &= \frac{K_2 K_4}{K_1 v_1}\left(v^m - v^r\right)\end{aligned}$$

proving the second inequality in (3.11).

By $w^m = w^r$, by (3.12), (3.13), and (3.14), we deduce that

$$\begin{aligned}\left|v^r - v^m\right| &= v^m - v^r = p(\rho^r) - p(\rho^m) \le K_2\left(\rho^r - \rho^m\right) \le \frac{K_2}{K_3}\left(\rho^m v^m - \rho^l v^l\right)\\ &\le \frac{K_2 v_2}{K_3}\left(\rho^m - \rho^l\right) \le \frac{K_2 v_2}{K_1 K_3}\left(p(\rho^m) - p(\rho^l)\right) = \frac{K_2 v_2}{K_1 K_3}\left(w^m - w^l\right)\end{aligned}$$

proving the first inequality in (3.11). This concludes the proof. □

Proposition 3.3 *Assume that a wave of the second family, connecting* (ρ^l, v^l) *with* (ρ^k, v^k)*, interacts at time* $\tilde{t}$ *with* $\tilde{x} = 0$*.*

Then $v^k = v^l$ *and* $w^l \neq w^k$*. Moreover the following statements hold.*

1. *If* $w^l < w^k$*, then*

$$\Delta\mathcal{N}(\tilde{t}) \leq K_\varepsilon - 1, \qquad \Delta W(\tilde{t}) \leq 0,$$

where K_ε *is defined in (3.10).*

2. *If* $w^l > w^k$*, then*

$$\Delta\mathcal{N}(\tilde{t}) \leq 0, \qquad \Delta W(\tilde{t}) \leq \frac{2}{L_1}\left(w^l - w^k\right),$$

where L_1 *is the constant defined in Lemma 3.1.*

Proof. The interacting wave is of the second family, then $v^l = v^k$ and $w^l \neq w^k$. Denote with (ρ^r, v^r) the state at $x = 0+$ before the interaction. Assume first $w^l < w^k$, so that $\rho^l v^l < \rho^k v^k$. If $\left(\rho^k, v^k\right) = (\rho^r, v^r)$, then after the interaction time $\tilde{t}$, only the wave of the second family, connecting $\left(\rho^l, v^l\right)$ to $\left(\rho^k, v^k\right)$, emerges from $\tilde{x} = 0$. Hence

$$\Delta\mathcal{N}(\tilde{t}) = 0, \qquad \Delta W(\tilde{t}) = 0.$$

Assume therefore that $\left(\rho^k, v^k\right) \neq (\rho^r, v^r)$. In this case, at $\tilde{x} = 0$, the solution to the Riemann problem is given by the case (2.8), and so we necessarily have

$$\rho^k v^k = \rho^r v^r = q, \qquad \rho^k > \rho^r, \qquad v^k < v^r.$$

At time $\tilde{t}$ we have to consider $\mathcal{RS}^q\left(\left(\rho^l, v^l\right), (\rho^r, v^r)\right)$. We have the following possibilities.

1. $w^l < w^r$. Define the state (ρ^m, v^m) satisfying $w^l = w^m$ and $v^m = v^r$. The solution to the Riemann problem consists of a (fan) wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^m, v^m), and by a wave of the second family connecting (ρ^m, v^m) to (ρ^r, v^r); see Figure 4, left. Hence $\Delta\mathcal{N}(\tilde{t}) \leq K_\varepsilon - 1$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= -\left|w^k - w^r\right| - \left|v^k - v^r\right| = w^r - w^k - v^r + v^k \\
\Delta W_1(\tilde{t}) &= \left|v^l - v^m\right| = v^m - v^l = v^r - v^k \\
\Delta W_2(\tilde{t}) &= |w^m - w^r| - \left|w^l - w^k\right| = w^r - w^k \\
\Delta W(\tilde{t}) &= 2\left(w^r - w^k\right) < 0.
\end{aligned}$$

2. $w^l = w^r$. In this case $\rho^l v^l < \rho^r v^r = q$ and so the solution to the Riemann problem consists of a (fan) wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^r, v^r). Hence $\Delta\mathcal{N}(\tilde{t}) \le K_\varepsilon - 2$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= -\left|w^k - w^r\right| - \left|v^k - v^r\right| = w^r - w^k + v^k - v^r \\
\Delta W_1(\tilde{t}) &= \left|v^l - v^r\right| = v^r - v^l \\
\Delta W_2(\tilde{t}) &= -\left|w^l - w^k\right| = w^l - w^k \\
\Delta W(\tilde{t}) &= 2\left(w^l - w^k\right) < 0.
\end{aligned}$$

3. $w^l > w^r$. Define the states (ρ^m, v^m) and (ρ^{m_1}, v^{m_1}) satisfying $w^l = w^m = w^{m_1}$, $v^m = v^r > v^{m_1}$, and $\rho^{m_1} v^{m_1} = q$. In this case $\rho^m v^m > q$ and so the solution to the Riemann problem consists of a (fan) wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^{m_1}, v^{m_1}), and by a stationary wave connecting (ρ^{m_1}, v^{m_1}) to (ρ^r, v^r); see Figure 4, right. Hence $\Delta\mathcal{N}(\tilde{t}) \le K_\varepsilon - 1$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= |w^{m_1} - w^r| + |v^{m_1} - v^r| - \left|w^k - w^r\right| - \left|v^k - v^r\right| \\
&= w^l - w^k - v^{m_1} + v^k \\
\Delta W_1(\tilde{t}) &= \left|v^l - v^{m_1}\right| = v^{m_1} - v^l \\
\Delta W_2(\tilde{t}) &= -\left|w^l - w^k\right| = w^l - w^k \\
\Delta W(\tilde{t}) &= 2\left(w^l - w^k\right) < 0.
\end{aligned}$$

Assume now $w^l > w^k$, so that $\rho^l v^l > \rho^k v^k$. If $\left(\rho^k, v^k\right) \neq (\rho^r, v^r)$, then at $\tilde{x} = 0$, the solution to the Riemann problem is given by the case (2.8), and so we necessarily have $\rho^k v^k = \rho^r v^r = q$, $\rho^k > \rho^r$, and $v^k < v^r$. Define the state (ρ^m, v^m) satisfying $w^l = w^m$, $v^m < v^l$, and $\rho^m v^m = q$. In this case the solution to the Riemann problem consists of a shock wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^m, v^m), and by a stationary wave connecting (ρ^m, v^m) to (ρ^r, v^r); see Figure 5, right. Hence $\Delta\mathcal{N}(\tilde{t}) = 0$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= |w^m - w^r| + |v^m - v^r| - \left|w^k - w^r\right| - \left|v^k - v^r\right| \\
&= w^l - w^k - v^m + v^k \\
\Delta W_1(\tilde{t}) &= \left|v^l - v^m\right| = v^l - v^m \\
\Delta W_2(\tilde{t}) &= -\left|w^l - w^k\right| = w^k - w^l \\
\Delta W(\tilde{t}) &= 2\left(v^l - v^m\right) \le \frac{2}{L_1}\left(w^l - w^k\right),
\end{aligned}$$

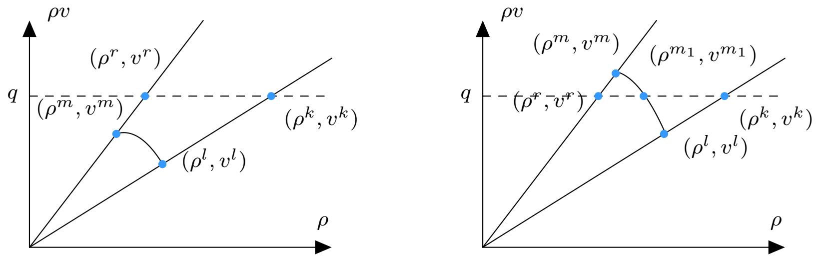

Figure 4: The interaction, described in Proposition 3.3, of a wave of the second family with $x = 0$ in the case $w^l < w^k$. At left the case $w^l < w^r$, at right the case $w^l > w^r$.

where $L_1 > 0$ is defined in (3.11).

Consider now the case $\left(\rho^k, v^k\right) = (\rho^r, v^r)$. We have the following possibilities.

1. $\rho^l v^l \le q$. In this case, the solution at $\tilde{x} = 0$ and at time $\tilde{t}$ consists on a single wave of the second family connecting $\left(\rho^l, v^l\right)$ with (ρ^r, v^r). Hence
$$\Delta\mathcal{N}(\tilde{t}) = \Delta W_0(\tilde{t}) = \Delta W_1(\tilde{t}) = \Delta W_2(\tilde{t}) = \Delta W(\tilde{t}) = 0.$$

2. $\rho^l v^l > q$. Define the states (ρ^m, v^m) and (ρ^s, v^s) satisfying $w^l = w^m$, $v^m < v^l = v^s$, and $\rho^m v^m = \rho^s v^s = q$. The solution to the Riemann problem consists of a shock wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^m, v^m), by a stationary wave, connecting (ρ^m, v^m) to (ρ^s, v^s), and by a wave of the second family connecting (ρ^s, v^s) to (ρ^r, v^r); see Figure 5, left. Hence $\Delta\mathcal{N}(\tilde{t}) = 2$ and
$$\begin{aligned}
\Delta W_0(\tilde{t}) &= |w^m - w^s| + |v^m - v^s| = w^l - w^s - v^m + v^s \\
\Delta W_1(\tilde{t}) &= \left|v^l - v^m\right| = v^l - v^m \\
\Delta W_2(\tilde{t}) &= |w^s - w^r| - \left|w^l - w^r\right| = w^s - w^l \\
\Delta W(\tilde{t}) &= 2\left(v^l - v^m\right) \le \frac{2}{L_1}\left(w^l - w^r\right),
\end{aligned}$$
where $L_1 > 0$ is defined in (3.11).

The proof is so finished. $\square$

Proposition 3.4 *Assume that a wave of the first family, connecting (ρ^k, v^k) with (ρ^r, v^r), interacts at time $\tilde{t}$ with $\tilde{x} = 0$.*

Then $w^k = w^r$ and $v^k \neq v^r$. Moreover the following statements hold.

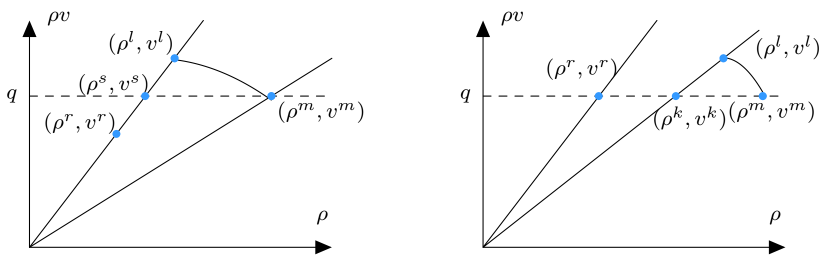

Figure 5: The interaction, described in Proposition 3.3, of a wave of the second family with $x = 0$ in the case $w^l > w^k$. At left the case $\left(\rho^k, v^k\right) = (\rho^r, v^r)$ and $\rho^l v^l > q$, at right the case $\left(\rho^k, v^k\right) \neq (\rho^r, v^r)$ and $\rho^l v^l > q$.

1. *If $v^k > v^r$, then*
$$\Delta\mathcal{N}(\tilde{t}) \leq 0, \qquad \Delta W(\tilde{t}) \leq 0.$$
2. *If $v^k < v^r$, then*
$$\Delta\mathcal{N}(\tilde{t}) \leq 2, \qquad \Delta W(\tilde{t}) \leq 2L_2\left(v^r - v^k\right),$$
where L_2 is the constant defined in Lemma 3.1.

Proof. Since the interacting wave is of the first family, then $w^k = w^r$ and $v^k \neq v^r$. Denote with $\left(\rho^l, v^l\right)$ the state at $x = 0^-$ before the interaction.

Assume first $v^k < v^r$. If $\left(\rho^l, v^l\right) \neq \left(\rho^k, v^k\right)$, then $\rho^l v^l = \rho^k v^k = q$ and $\rho^k < \rho^l$. Define the state (ρ^m, v^m) satisfying $v^m = v^r$ and $\rho^m v^m = q$. At time $\tilde{t}$, the solution to the Riemann problem is given by a stationary wave, connecting $\left(\rho^l, v^l\right)$ to (ρ^m, v^m), and by a wave of the second family, connecting (ρ^m, v^m) to (ρ^r, v^r); see Figure 6, left. Hence $\Delta\mathcal{N}(\tilde{t}) = 0$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= \left|w^l - w^m\right| + \left|v^l - v^m\right| - \left|w^k - w^l\right| - \left|v^k - v^l\right| \\
&= w^k - w^m - v^k + v^m \\
\Delta W_1(\tilde{t}) &= -\left|v^k - v^r\right| = v^k - v^r \\
\Delta W_2(\tilde{t}) &= |w^m - w^r| = w^r - w^m \\
\Delta W(\tilde{t}) &= 2\left(w^r - w^m\right) \leq 2L_2\left(v^r - v^k\right),
\end{aligned}$$

where L_2 is defined in Lemma 3.1.

If $\left(\rho^l, v^l\right) = \left(\rho^k, v^k\right)$, then we have the following possibilities.

1. $\rho^r v^r \leq q$. In this case the solution to the Riemann problem consists of the wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^r, v^r). Hence $\Delta\mathcal{N}(\tilde{t}) = 0$ and $\Delta W(\tilde{t}) = 0$.

2. $\rho^r v^r > q$. Define (ρ^m, v^m) and (ρ^s, v^s) the states satisfying $w^m = w^l$, $v^s = v^r$, $\rho^s < \rho^m$, and $\rho^s v^s = \rho^m v^m = q$. The solution to the Riemann problem consists of a wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^m, v^m), of a stationary wave, connecting (ρ^m, v^m) to (ρ^s, v^s), and of a wave of the second family, connecting (ρ^s, v^s) to (ρ^r, v^r); see Figure 6, right. Hence $\Delta\mathcal{N}(\tilde{t}) = 3 - 1 = 2$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= |w^m - w^s| + |v^m - v^s| = w^r - w^s + v^r - v^m \\
\Delta W_1(\tilde{t}) &= \left|v^l - v^m\right| - \left|v^l - v^r\right| = v^m - v^r \\
\Delta W_2(\tilde{t}) &= |w^s - w^r| = w^r - w^s \\
\Delta W(\tilde{t}) &= 2\left(w^r - w^s\right) < 2L_2\left(v^r - v^l\right),
\end{aligned}$$

where $L_2 > 0$ is defined in Lemma 3.1.

Assume now $v^k > v^r$. If $\left(\rho^l, v^l\right) = \left(\rho^k, v^k\right)$, then at time $\tilde{t}$, the solution to the Riemann problem is given by a wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^r, v^r). Hence $\Delta\mathcal{N}(\tilde{t}) = 0$ and $\Delta W(\tilde{t}) = 0$. If $\left(\rho^l, v^l\right) \neq \left(\rho^k, v^k\right)$, then we deduce that $\rho^l v^l = \rho^k v^k = q$ and $\rho^k < \rho^l$. Moreover we have the following possibilities.

1. $v^r > v^l$. Define (ρ^m, v^m) the state satisfying $v^m = v^r$ and $\rho^m v^m = q$. In this case the solution to the Riemann problem consists of a stationary wave, connecting $\left(\rho^l, v^l\right)$ to (ρ^m, v^m), and of a wave of the second family, connecting (ρ^m, v^m) to (ρ^r, v^r); see Figure 7, left. Hence $\Delta\mathcal{N}(\tilde{t}) = 0$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= \left|w^m - w^l\right| + \left|v^m - v^l\right| - \left|w^k - w^l\right| - \left|v^k - v^l\right| \\
&= w^r - w^m + v^m - v^k \\
\Delta W_1(\tilde{t}) &= -\left|v^k - v^r\right| = v^r - v^k \\
\Delta W_2(\tilde{t}) &= |w^m - w^r| = w^m - w^r \\
\Delta W(\tilde{t}) &= 2\left(v^r - v^k\right) < 0.
\end{aligned}$$

2. $v^r = v^l$. The solution to the Riemann problem consists of a wave of the second family, connecting $\left(\rho^l, v^l\right)$ to (ρ^r, v^r). Hence $\Delta\mathcal{N}(\tilde{t}) = -1$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= -\left|w^l - w^k\right| - \left|v^l - v^k\right| = w^k - w^l + v^l - v^k \\
\Delta W_1(\tilde{t}) &= -\left|v^k - v^r\right| = v^r - v^k \\
\Delta W_2(\tilde{t}) &= \left|w^l - w^r\right| = w^l - w^r \\
\Delta W(\tilde{t}) &= 2\left(v^l - v^k\right) < 0.
\end{aligned}$$

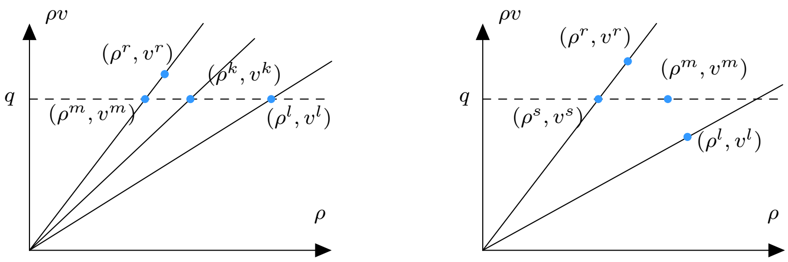

Figure 6: The interaction, described in Proposition 3.4, of a wave of the first family with $x = 0$ in the case $v^k < v^r$. At left the case $\left(\rho^l, v^l\right) \neq \left(\rho^k, v^k\right)$, at right the case $\left(\rho^l, v^l\right) = \left(\rho^k, v^k\right)$ and $\rho^r v^r > q$.

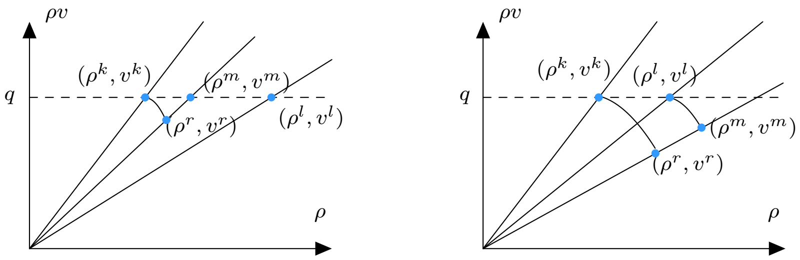

Figure 7: The interaction, described in Proposition 3.4, of a wave of the first family with $x = 0$ in the case $v^k > v^r$ and $\left(\rho^l, v^l\right) \neq \left(\rho^k, v^k\right)$. At left the case $v^r > v^l$, at right the case $v^r < v^l$.

3. $v^r < v^l$. Define (ρ^m, v^m) the state satisfying $w^m = w^l$ and $v^m = v^r$. The solution to the Riemann problem consists of a shock wave of the first family, connecting $\left(\rho^l, v^l\right)$ to (ρ^m, v^m), and of a wave of the second family, connecting (ρ^m, v^m) to (ρ^r, v^r); see Figure 7, right. Hence $\Delta\mathcal{N}(\tilde{t}) = 0$ and

$$\begin{aligned}
\Delta W_0(\tilde{t}) &= -\left|w^l - w^k\right| - \left|v^l - v^k\right| = w^k - w^l + v^l - v^k \\
\Delta W_1(\tilde{t}) &= \left|v^l - v^m\right| - \left|v^k - v^r\right| = v^l - v^k \\
\Delta W_2(\tilde{t}) &= |w^m - w^r| = w^l - w^r \\
\Delta W(\tilde{t}) &= 2\left(v^l - v^k\right) < 0.
\end{aligned}$$

The proof is so finished. □

3.3 Existence of a wave-front tracking solution

We now want to bound the number of waves and of interactions. The following proposition holds.

Proposition 3.5 *For every $\nu \in \mathbb{N} \setminus \{0\}$, the construction in Subsection 3.1 can be done for every positive time, producing a $\frac{1}{\nu}$-approximate wave-front tracking solution to (2.1).*

PROOF. For $\nu \in \mathbb{N} \setminus \{0\}$, call $u_\nu = (\rho_\nu, v_\nu)$ the function built with the procedure of Subsection 3.1. It is sufficient to prove that the number of waves and interactions, generated by the construction, is finite. As in (3.10), consider the constant $K_\nu = \lfloor \nu (w_2 - w_1) \rfloor + 1$. The functional $\mathcal{N}(t)$, which is defined in (3.9) and counts the number of discontinuities of u_ν, is locally constant in time and can vary at interaction times in the following way.

1. If at time $\tilde{t} > 0$ two waves interact at $\tilde{x} \neq 0$, then $\Delta\mathcal{N}(\tilde{t}) = 0$.
2. If at time $\tilde{t} > 0$ a wave interacts with $x = 0$ from the left, then $\Delta\mathcal{N}(\tilde{t}) \leq K_\nu - 1$; see Proposition 3.3.
3. If at time $\tilde{t} > 0$ a wave interacts with $x = 0$ from the right, then $\Delta\mathcal{N}(\tilde{t}) \leq 2$; see Proposition 3.4.

Note that the point 2. happens when a wave of the second family interact with $x = 0$; hence the number of times point 2. happens depends on the number of waves of the second family in $x < 0$. Analogous consideration holds for point 3.. Since point 1., and since the wave of the first and second family have respectively negative and positive speed, points 2. and 3. can happen at most $\mathcal{N}(0^+)$ times. This implies that

$$\mathcal{N}(t) \leq \mathcal{N}(0^+) + 2\mathcal{N}(0^+) + (K_\nu - 1)\mathcal{N}(0^+) = (2 + K_\nu)\mathcal{N}(0^+)$$

for a.e. $t > 0$.

Since a wave of the first family and a wave of the second family can interact together at most once, the previous analysis implies that also the number of interactions is finite. The proof is so concluded. $\square$

3.4 Existence of a solution

Proposition 3.6 *There exists $M > 0$ such that*

$$W(t) \leq M \qquad \textit{and} \qquad TV_\rho(t) \leq M \tag{3.15}$$

for a.e. $t > 0$.

PROOF. Clearly the functionals W_0, W_1, W_2, and W vary when two waves interact together or when a wave interacts with $x = 0$. Moreover, by Propositions 3.1 and 3.2, all the previous functionals at most decrease when two waves interact at a point $x \neq 0$. Since waves of first family have negative speed and waves of the second family have positive speed, then each wave can interact with $x = 0$ at most once. Therefore, by Propositions 3.3 and 3.4, for a.e. $t > 0$,

$$\begin{aligned} W(t) &\leq W(0^+) + \frac{2}{L_1} W(0^+) + 2L_2 W(0^+) \\ &= \left[1 + \frac{2}{L_1} + 2L_2\right] W(0^+), \end{aligned}$$

where L_1 and L_2 are the constants defined in Lemma 3.1. This implies the first inequality of (3.15).

Fix now $\left(\rho^l, v^l\right) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$, and $(\rho^r, v^r) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$. Define the point $(\rho^m, v^m) \in \mathcal{D}_{v_1,v_2,w_1,w_2}$ such that $w^l = w^m$ and $v^m = v^r$. By (1.3), there exists $K_1 > 0$ such that

$$K_1 \left|\rho_1 - \rho_2\right| \leq \left|p(\rho_1) - p(\rho_2)\right|$$

for every $\rho_1, \rho_2 \in [\rho_{min}, \rho_{max}]$; see also equation (3.12). Therefore we have

$$\begin{aligned} \left|\rho^l - \rho^m\right| &\leq \frac{1}{K_1}\left|p(\rho^l) - p(\rho^m)\right| = \frac{1}{K_1}\left|p(\rho^l) - w^l - p(\rho^m) + w^m\right| \\ &= \frac{1}{K_1}\left|v^l - v^m\right| = \frac{1}{K_1}\left|v^l - v^r\right|. \end{aligned} \tag{3.16}$$

Moreover

$$\begin{aligned} \left|\rho^m - \rho^r\right| &\leq \frac{1}{K_1}\left|p(\rho^m) - p(\rho^r)\right| = \frac{1}{K_1}\left|p(\rho^m) + v^m - p(\rho^r) - v^r\right| \\ &= \frac{1}{K_1}\left|w^m - w^r\right| = \frac{1}{K_1}\left|w^l - w^r\right|. \end{aligned} \tag{3.17}$$

Thus, by (3.16) and (3.17), we have that

$$\left|\rho^l - \rho^r\right| \leq \left|\rho^l - \rho^m\right| + \left|\rho^m - \rho^r\right| \leq \frac{1}{K_1}\left(\left|v^l - v^r\right| + \left|w^l - w^r\right|\right)$$

proving also the second inequality in (3.15). □

PROOF OF THEOREM 3.1. Fix an ε-approximate wave-front tracking solution $\bar{u}_\varepsilon$ to (2.1), in the sense of Definition 3.1. By Proposition 3.6, we deduce that there exists a constant $M > 0$, depending on the total variation of the initial datum, such that

$$W(t) \leq M \qquad \text{and} \qquad TV_\rho(t) \leq M$$

for a.e. $t > 0$. Hence, by Helly Theorem (see [4, Theorem 2.4]), there is a function $(\bar{\rho}, \bar{v})$, which is a solution to (2.1), in the sense of Definition 2.1. This permits to conclude. $\square$

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