The Riemann Problem at a Junction for a Phase Transition Traffic Model

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Abstract

We extend the Phase Transition model for traffic proposed in [7], by Colombo, Marcellini, and Rascle to the network case. More precisely, we consider the Riemann problem for such a system at a general junction with n incoming and m outgoing roads. We propose a Riemann solver at the junction which conserves both the number of cars and the maximal speed of each vehicle, which is a key feature of the Phase Transition model. For special junctions, we prove that the Riemann solver is well defined.

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1 Introduction

The paper deals with Riemann problems at junctions for a macroscopic phase transition traffic model. More precisely, we consider the 2-Phase Traffic Model, proposed by Colombo, Marcellini and Rascle in [7], given by the system in conservation form

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} \quad \text{with } v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}, \tag{1.1}$$

where ρ denotes the car traffic density, η is a generalized momentum, $v \in [0, V_{max}]$ is the speed of cars, and ψ is a decreasing function. This model has been derived as an extension of the famous Lighthill-Whitham-Richards

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(LWR) model (see [15, 17]), by assuming that different typologies of drivers have different maximal speed w, where $\eta = \rho w$. A key feature of this model is that there are two different traffic regimes: the free one and the congested one. Consequently, the fundamental diagram is composed by the *Free* phase F and the *Congested* phase C. In the free phase the model is the classical LWR one, while in the congested phase it consists on a system of two differential equations.

The phase transitions traffic models belong to the class of macroscopic second order models, started by the Aw-Rascle-Zhang (ARZ) model, see [1] and [18]. The first phase transition model for traffic has been introduced by Colombo in 2002, see [4, 5]. For other phase transition models, see [2, 11, 14, 16] and the references therein.

More recently, a growing attention was devoted to the extension of these models to road networks, see [3, 6, 8, 10, 13]. A complex network consists in a finite set of arcs and nodes connected by vertices or junctions. In this paper we deal with a network composed by a single junction; due to the finite speed of waves, this simple case is indeed general, see [10, Theorem 4.3.9].

In the present paper, we consider a Riemann problem at a junction and we propose a Riemann solver, which conserves both the number of cars and the maximal speed w of each driver, a key feature of (1.1). This is in the same spirit as the Riemann solver, proposed by Herty and Rascle in [12] for the ARZ model, even if the procedure in [12] can not be directly applied in our case. We prove that the Riemann solver is well defined in the cases of $1 \times m$ and 2×1 junctions (i.e. with one incoming and m outgoing roads or with two incoming and one outgoing roads). The case of the 2×1 junction presents some technical problems. These are due to the fact that the conservation of the maximal speed w produces some nonlinear constraints in the set of admissible fluxes.

The paper is organized as follow. In the next section we describe the 2-Phases Traffic Model introduced in [7]. In Section 3 we propose the junction conditions and we describe in details the admissible states at the junction for a solution to the Riemann problem. In Section 4, we treat the case of a junction $1 \times m$. More precisely we introduce a Riemann solver and we prove that it is well defined. In Subsection 4.1, we point out with an explicit example that the procedure introduced by Herty and Rascle in [12] can not be directly used in our case. Finally, in Section 5 we deal with the 2×1 junction.

2 The Phase Transition Model

We recall at first the Phase Transition model, introduced in [7] as an extension of the LWR model, since it allows different speeds for different typology of drivers. The LWR model is given by the following scalar conservation law

$$\partial_t \rho + \partial_x (\rho V) = 0\,, \tag{2.1}$$

where ρ is the traffic density and $V = V(t, x, \rho)$ is the speed. Assume now that $V = w\,\psi(\rho)$, where $\psi = \psi(\rho)$ is a $\mathbf{C^2}$ function and $w = w(t, x)$ is the maximal speed of a driver, located at position x at time t. Introducing a uniform bound $V_{\max} > 0$ on the speed of vehicles, we obtain the model

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v\,\partial_x w = 0 \end{cases} \qquad \text{with} \qquad v = \min\left\{V_{\max},\, w\,\psi(\rho)\right\}. \tag{2.2}$$

With the change of variables $\eta = \rho w$, the former system can be written in conservation form (1.1), where the conserved quantities are ρ and η.

As in [7], we introduce the following assumptions.

(H-1) $R, \check{w}, \hat{w}, V_{\max}$ are positive constants, with $\check{w} < \hat{w}$.

(H-2) $\psi \in \mathbf{C^2}\left([0, R]; [0, 1]\right)$ is such that

$$\begin{aligned} &\psi(0) = 1, && \psi(R) = 0, \\ &\psi'(\rho) \le 0, && \frac{d^2}{d\rho^2}\left(\rho\,\psi(\rho)\right) \le 0 \quad \text{for all } \rho \in [0, R]\,. \end{aligned}$$

(H-3) $\check{w} > V_{\max}$.

Here, R is the maximal possible density, while $\check{w}$, respectively, $\hat{w}$, is the minimum, respectively, maximum, of the maximal speeds of each vehicle.

The two phases, free and congested, are described by the sets

$$F = \left\{(\rho, w) \in [0, R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = V_{\max}\right\}, \tag{2.3}$$

$$C = \left\{(\rho, w) \in [0, R] \times [\check{w}, \hat{w}] \colon v(\rho, \rho w) = w\,\psi(\rho)\right\}. \tag{2.4}$$

see Figure 1. Both F and C are closed sets and $F \cap C \neq \emptyset$. Note also that F is one-dimensional in the $(\rho, \rho v)$ plane, while it is two-dimensional in the (ρ, η) coordinates. Figure 1, left, also contains the curves $\eta = \check{w}\rho$, $\eta = \hat{w}\rho$, and the curve $\eta = \frac{V_{\max}}{\psi(\rho)}\rho$ that separates the two phases. Note that, in the free phase F, the system (1.1) reduces to

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, V_{\max}) = 0 \\ \partial_t \eta + \partial_x (\eta V_{\max}) = 0\,, \end{cases} \tag{2.5}$$

while, in the congested phase C, it is given by

$$\begin{cases} \partial_t \rho + \partial_x \left(\eta\,\psi(\rho)\right) = 0 \\ \partial_t \eta + \partial_x \left(\frac{\eta^2}{\rho}\,\psi(\rho)\right) = 0\,. \end{cases} \tag{2.6}$$

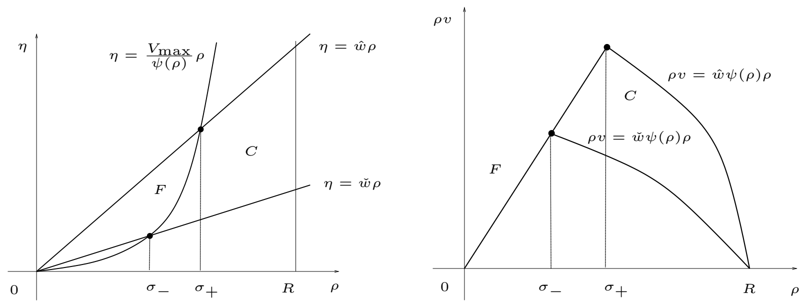

Figure 1: The free phase F and the congested phase C resulting from (1.1) in the coordinates, from left to right, (ρ, η) and $(\rho, \rho v)$. In the (ρ, η) plane, the curves $\eta = \check{w}\rho$, $\eta = \hat{w}\rho$ and the curve $\eta = \frac{V_{\max}}{\psi(\rho)}\rho$ that divides the two phases are represented. The densities σ_- and σ_+ are given by the interesections between the previous curves. Similarly in the $(\rho, \rho v)$ plane, the curves $\rho v = \check{w}\psi(\rho)\rho$, $\rho v = \hat{w}\psi(\rho)\rho$ and the densities σ_- and σ_+ are represented.

By **(H-1)**, **(H-2)**, and **(H-3)**, system (2.6) is strictly hyperbolic in C, see [7], and

$$\begin{aligned}
\lambda_1(\rho,\eta) &= \eta\,\psi'(\rho) + v(\rho,\eta), & \lambda_2(\rho,\eta) &= v(\rho,\eta),\\
r_1(\rho,\eta) &= \begin{bmatrix} -\rho \\ -\eta \end{bmatrix}, & r_2(\rho,\eta) &= \begin{bmatrix} 1 \\ \eta\left(\frac{1}{\rho} - \frac{\psi'(\rho)}{\psi(\rho)}\right) \end{bmatrix},\\
\nabla\lambda_1 \cdot r_1 &= -\frac{d^2}{d\rho^2}\left[\rho\,\psi(\rho)\right], & \nabla\lambda_2 \cdot r_2 &= 0,\\
\mathcal{L}_1(\rho;\rho_o,\eta_o) &= \eta_o\frac{\rho}{\rho_o}, & \mathcal{L}_2(\rho;\rho_o,\eta_o) &= \frac{\rho\,v(\rho_o,\eta_o)}{\psi(\rho)},\ \rho_o < R,
\end{aligned}$$

where λ_i and r_i are respectively the eigenvalues and right eigenvectors of the Jacobian matrix of the flux, and $\mathcal{L}_i$ are the Lax curves. When $\rho_o = R$, the 2-Lax curve through (ρ_o, η_o) is given by the segment $\rho = R$, $\eta \in [R\check{w}, R\hat{w}]$.

Introduce also the following technical assumption:

(H-4) the waves of the first family in C have negative speed.

Remark 2.1 *It is possible to choose the parameters such that* **(H-4)** *is satisfied. Indeed* $\lambda_1 = \eta\psi' + \eta\frac{\psi}{\rho} < 0$ *in* C *if and only if* $\rho\psi'(\rho) + \psi(\rho) < 0$ *for every* $(\rho,\eta) \in C$*. The assumption* $\frac{d^2}{d\rho^2}\left(\rho\,\psi(\rho)\right)$ *implies that the function* $\rho \mapsto \rho\psi'(\rho) + \psi(\rho)$ *is decreasing, so that* $\rho\psi'(\rho) + \psi(\rho) < 0$ *holds if and only if* $\bar{\rho}^*\psi'(\bar{\rho}^*) + \psi(\bar{\rho}^*) < 0$ *where* $\bar{\rho}^*$ *solves the following system*

$$\begin{cases} \eta = \check{w}\rho \\ \eta = \frac{\rho V_{max}}{\psi(\rho)}. \end{cases}$$

In particular, if $\psi(\rho) = 1 - \rho$*, then* $\lambda_1 < 0$ *in* C *if and only if* $\check{w} > 2V_{max}$*.*

For simplicity, we use the following notation.

- *Linear wave*: a wave connecting two states in the free phase.
- *Phase transition wave*: a wave connecting a left state $(\rho_l, \eta_l) \in F$ with a right state $(\rho_r, \eta_r) \in C$ satisfying $\frac{\eta_l}{\rho_l} = \frac{\eta_r}{\rho_r}$.
- *First family wave*: a wave connecting a left state $(\rho_l, \eta_l) \in C$ with a right state $(\rho_r, \eta_r) \in C$ such that $\frac{\eta_l}{\rho_l} = \frac{\eta_r}{\rho_r}$.
- *Second family wave*: a wave connecting a left state $(\rho_l, \eta_l) \in C$ with a right state $(\rho_r, \eta_r) \in C$ such that $v(\rho_l, \eta_l) = v(\rho_r, \eta_r)$.

3 The Riemann Problem at a Generic Node

Consider a node J with n incoming arcs $I_1, ..., I_n$ and m outgoing arcs $I_{n+1}, ..., I_{n+m}$, where each incoming arc is modeled by $I_i =]-\infty, 0]$ and each outgoing arc by $I_j = [0, +\infty[$. On each arc we consider the phase transition model in (1.1).

A Riemann problem at J is the following Cauchy problem

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x \big(\rho\, v(\rho, \eta)\big) = 0 \\ \partial_t \eta + \partial_x \big(\eta\, v(\rho, \eta)\big) = 0 \end{cases} & (\rho, \eta) \in I_i \\ \begin{cases} \partial_t \rho + \partial_x \big(\rho\, v(\rho, \eta)\big) = 0 \\ \partial_t \eta + \partial_x \big(\eta\, v(\rho, \eta)\big) = 0 \end{cases} & (\rho, \eta) \in I_j \\ (\rho_i, \eta_i)(0, x) = (\bar{\rho}_i, \bar{\eta}_i) \\ (\rho_j, \eta_j)(0, x) = (\bar{\rho}_j, \bar{\eta}_j)\,, \end{cases} \tag{3.1}$$

where $(\bar{\rho}_i, \bar{\eta}_i) \in F \cup C$ are the initial data in each incoming arc I_i, $i = 1, \ldots, n$, and $(\bar{\rho}_j, \bar{\eta}_j) \in F \cup C$ are the initial data in each outgoing arc I_j, $j = n+1, \ldots, n+m$. Next, we analyze all the possible traces, and the corresponding flows, at $x = 0$ for self-similar solutions, separately in the incoming arcs and in the outgoing arcs.

Incoming Arc. We define $\mathcal{T}_{inc}(\bar{\rho}, \bar{\eta})$ as the set of all the possible traces at $x = 0$ of a solution in the incoming arc when the initial condition is $(\bar{\rho}, \bar{\eta})$. More precisely, the set $\mathcal{T}_{inc}(\bar{\rho}, \bar{\eta})$ is composed by all the points $(\rho^*, \eta^*) \in F \cup C$ such that the classical Riemann problem

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x \big(\rho\, v(\rho, \eta)\big) = 0 \\ \partial_t \eta + \partial_x \big(\eta\, v(\rho, \eta)\big) = 0 \end{cases} & t > 0,\ x \in \mathbb{R} \\ (\rho, \eta)(0, x) = (\bar{\rho}, \bar{\eta}) & x < 0 \\ (\rho, \eta)(0, x) = (\rho^*, \eta^*) & x > 0 \end{cases} \tag{3.2}$$

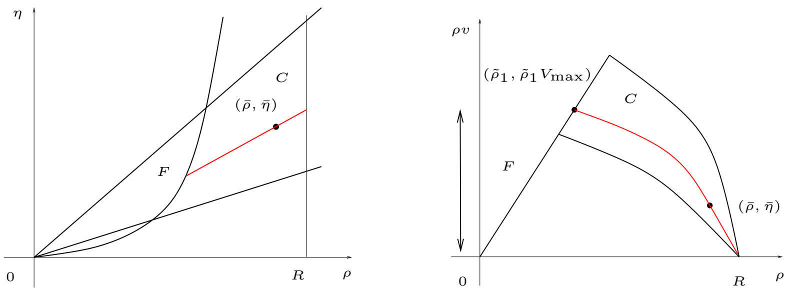

Figure 2: The case $(\bar\rho, \bar\eta) \in C$. The set $\mathcal{T}_{inc}(\bar\rho, \bar\eta)$ it is represented in red in the coordinates, from left to right, (ρ, η) and $(\rho, \rho v)$. The set $\mathcal{T}^f_{inc}(\bar\rho, \bar\eta)$ is represented on the ρv axis in the $(\rho, \rho v)$ plane.

is solved with waves with negative speed, i.e., by **(H-4)** with waves of the first family or with phase transition waves with negative speed. Moreover we define the corresponding set of flows

$$\mathcal{T}^f_{inc}(\bar\rho, \bar\eta) = \left\{\rho v(\rho, \eta) : (\rho, \eta) \in \mathcal{T}_i(\bar\rho, \bar\eta)\right\} .$$

The following result holds.

Proposition 3.1 *Assume* **(H-1)**, **(H-2)**, **(H-3)**, *and* **(H-4)**. *Fix* $(\bar\rho, \bar\eta) \in F \cup C$. *All the points* $(\rho^*, \eta^*) \in \mathcal{T}_{inc}(\bar\rho, \bar\eta)$ *have maximal speed* w^* *equal to* $\bar w$. *The following cases hold.*

1. *Case* $(\bar\rho, \bar\eta) \in C$. *The set* $\mathcal{T}_{inc}(\bar\rho, \bar\eta)$ *consists of all the points in the congested phase* C *belonging to the Lax curve of the first family passing through* $(\bar\rho, \bar\eta)$. *Moreover* $\mathcal{T}^f_{inc}(\bar\rho, \bar\eta) = [0, \tilde\rho_1 V_{\max}]$, *see Figure 2, where* $\tilde\rho_1 \in [0, R]$ *is uniquely defined by* $\bar w = \frac{V_{max}}{\psi(\tilde\rho_1)}$.

2. *Case* $(\bar\rho, \bar\eta) \in F$. *There exists a unique point* $(\hat\rho_1, \hat\eta_1) \in C$ *such that the set* $\mathcal{T}_{inc}(\bar\rho, \bar\eta)$ *consists of the point* $(\bar\rho, \bar\eta)$ *itself and of all the points in the congested phase* C *belonging to the Lax curve of the first family passing through* $(\hat\rho_1, \hat\eta_1)$, *with density strictly bigger than* $\hat\rho_1$. *Moreover* $\mathcal{T}^f_{inc}(\bar\rho, \bar\eta) = [0, \bar\rho V_{\max}]$, *see Figure 3.*

Proof. The waves with negative speed could be wave of the first family (see assumption **(H-4)**) and phase-transition waves. Thus, since $\frac{\bar\eta}{\bar\rho} = \bar w$, we deduce that $w^* = \bar w$.

Case 1. Since $(\bar\rho, \bar\eta) \in C$, phase transitions waves do not appear. Therefore the set $\mathcal{T}_{inc}(\bar\rho, \bar\eta)$ consists of all the points in the congested phase C of the Lax curve of the first family passing through $(\bar\rho, \bar\eta)$, that is

$$\mathcal{T}_{inc}(\bar\rho, \bar\eta) = \left\{ \left(\rho, \bar\eta \frac{\rho}{\bar\rho}\right) : \rho \in [0, R] \text{ and } \left(\rho, \bar\eta \frac{\rho}{\bar\rho}\right) \in C \right\};$$

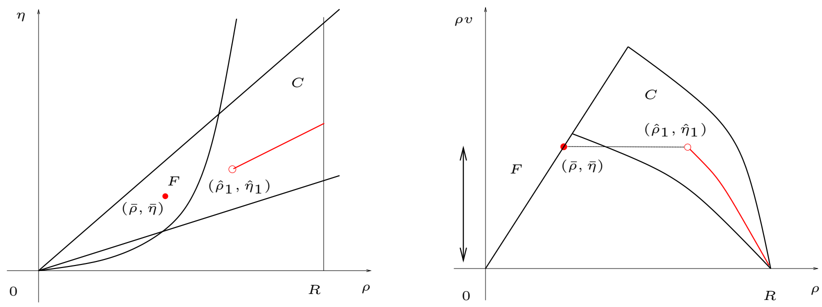

Figure 3: The case $(\bar\rho, \bar\eta) \in F$. The set $\mathcal{T}_{inc}(\bar\rho, \bar\eta)$ it is represented in red in the coordinates, from left to right, (ρ, η) and $(\rho, \rho v)$. The set $\mathcal{T}^{f}_{inc}(\bar\rho, \bar\eta)$ is represented on the ρv axis in the $(\rho, \rho v)$ plane.

see Figure 2, left. Next, in the $(\rho, \rho v)$ plane, the Lax curve passing through $(\bar\rho, \bar\eta)$ is the graph of the function $\rho \mapsto \frac{\bar\eta}{\bar\rho}\rho\psi(\rho)$. By imposing $\rho V_{\max} = \frac{\bar\eta}{\bar\rho}\rho\psi(\rho)$, we obtain the point of maximum flow $(\tilde\rho_1, \tilde\rho_1 V_{\max})$, where $\tilde\rho_1 = \psi^{-1}\left(V_{\max}\frac{\bar\rho}{\bar\eta}\right)$, see Figure 2, right. Thus, $\mathcal{T}^{f}_{inc}(\bar\rho, \bar\eta) = [0, \tilde\rho_1 V_{\max}]$.

Case 2. Since $(\bar\rho, \bar\eta) \in F$, one can use only phase transition waves with negative speed. By the Rankine-Hugoniot condition, a phase transition wave connecting $(\rho_l, \eta_l) \in F$ and $(\rho_r, \eta_r) \in C$ has strictly negative speed if and only if $\rho_l V_{max} > \eta_r\psi(\rho_r)$ and has zero speed if and only if $\rho_l V_{max} = \eta_r\psi(\rho_r)$. Define $(\hat\rho_1, \hat\eta_1) \in C$ by the unique solution to

$$\begin{cases} \hat\eta_1 = \frac{V_{\max}\bar\rho}{\psi(\hat\rho_1)} \\ \frac{\hat\eta_1}{\hat\rho_1} = \frac{\bar\eta}{\bar\rho}\,. \end{cases} \tag{3.3}$$

In particular the first equation, by the Rankine-Hugoniot conditions, means that the wave between $(\bar\rho, \bar\eta)$ and $(\hat\rho_1, \hat\eta_1)$ has zero speed. The set $\mathcal{T}_{inc}(\bar\rho, \bar\eta)$ consists of $(\bar\rho, \bar\eta)$ and of all the points in the congested phase C of the Lax curve of the first family passing through $(\bar\rho, \bar\eta)$, with $\rho > \hat\rho_1$; that is

$$\mathcal{T}_{inc}(\bar\rho, \bar\eta) = \left\{\left(\rho, \bar\eta\frac{\rho}{\bar\rho}\right) : \rho \in [\hat\rho_1, R] \text{ and } \left(\rho, \bar\eta\frac{\rho}{\bar\rho}\right) \in C\right\} \cup \left\{(\bar\rho, \bar\eta)\right\};$$

see Figure 3, left. Clearly, the set of flows in the $(\rho, \rho v)$ plane is $\mathcal{T}^{f}_{inc}(\bar\rho, \bar\eta) = [0, \bar\rho V_{\max}]$, see Figure 3, right. $\square$

Outgoing Arc. We define $\mathcal{T}_{out}(w, \bar\rho, \bar\eta)$ as the set of all the possible traces at $x = 0$ of a solution, having w as maximal speed, in the outgoing arc when the initial condition is $(\bar\rho, \bar\eta)$. More precisely, the set $\mathcal{T}_{out}(w, \bar\rho, \bar\eta)$ is composed by all the points $(\rho^*, \eta^*) \in F \cup C$ such that $\eta^* = w\rho^*$ and the

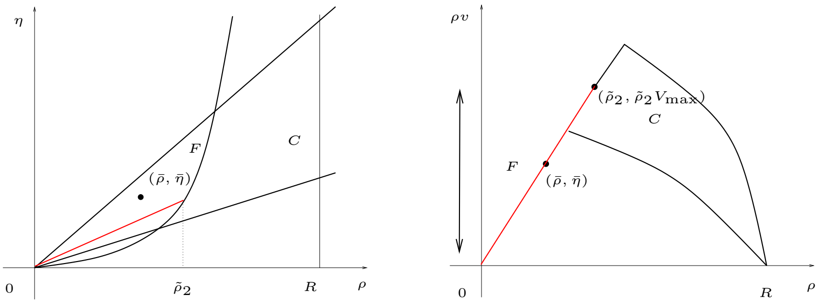

Figure 4: The case $(\bar{\rho}, \bar{\eta}) \in F$. The set $\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta})$ it is represented in red in the coordinates, from left to right, (ρ, η) and $(\rho, \rho v)$. The set $\mathcal{T}^f_{out}(w, \bar{\rho}, \bar{\eta})$ is represented on the ρv axis in the $(\rho, \rho v)$ plane.

classical Riemann problem

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, \eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho, \eta)\right) = 0 \end{cases} & t > 0,\ x \in \mathbb{R} \\ (\rho, \eta)(0, x) = (\rho^*, \eta^*) & x < 0 \\ (\rho, \eta)(0, x) = (\bar{\rho}, \bar{\eta}) & x > 0 \end{cases} \tag{3.4}$$

is solved with waves with positive speed, i.e., with waves of the second family, with phase transition waves with positive speed or with linear waves connecting two states in F. Moreover we define the corresponding set of flows

$$\mathcal{T}^f_{out}(w, \bar{\rho}, \bar{\eta}) = \left\{ \rho v(\rho, \eta) : (\rho, \eta) \in \mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta}) \right\} .$$

The following result holds.

Proposition 3.2 *Assume* **(H-1)**, **(H-2)**, **(H-3)**, *and* **(H-4)**. *Fix* $(\bar{\rho}, \bar{\eta}) \in F \cup C$ *and the maximal speed* $w \in [\check{w}, \hat{w}]$. *The following cases hold.*

1. *Case* $(\bar{\rho}, \bar{\eta}) \in F$. *The set* $\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta})$ *consists of all the points* (ρ^*, η^*) *of the free phase* F *such that* $\eta^*/\rho^* = w$. *Moreover* $\mathcal{T}^f_{out}(w, \bar{\rho}, \bar{\eta}) = [0, \tilde{\rho}_2 V_{\max}]$ *for a suitable* $\tilde{\rho}_2 \in [\sigma^-, \sigma^+]$, *see Figure 4.*

2. *Case* $(\bar{\rho}, \bar{\eta}) \in C$. *There exists a unique point* $(\hat{\rho}_2, \hat{\eta}_2) \in F$ *such that the set* $\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta})$ *consists of all the points* (ρ^*, η^*) *of the free phase* F *such that* $\eta^*/\rho^* = w$, *with* $\rho < \hat{\rho}_2$, *and of the point* (ρ^+, η^+) *of the congested phase* C, *where* $(\rho^+, \eta^+) \in C$ *is uniquely defined by* $v(\bar{\rho}, \bar{\eta}) = v(\rho^+, \eta^+)$ *and* $\eta^+ = w\rho^+$. *Moreover* $\mathcal{T}^f_{out}(w, \bar{\rho}, \bar{\eta}) = \left[0, \rho^+ v(\rho^+, \eta^+)\right]$; *see Figure 5.*

Proof. Case 1. Since $(\bar{\rho}, \bar{\eta}) \in F$, phase transitions waves do not appear and we use only linear waves. Once fixed the maximal speed w, since $w = \eta^*/\rho^*$,

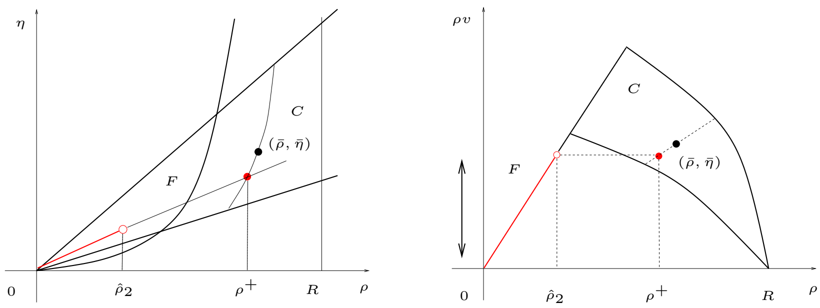

Figure 5: The case $(\bar{\rho}, \bar{\eta}) \in C$. The set $\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta})$ it is represented in red in the coordinates, from left to right, (ρ, η) and $(\rho, \rho v)$. The set $\mathcal{T}^f_{out}(w, \bar{\rho}, \bar{\eta})$ is represented on the ρv axis in the $(\rho, \rho v)$ plane.

we have

$$\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta}) = \left\{ \left(\rho^*, \eta^*\right) \in F : \frac{\eta^*}{\rho^*} = w \right\};$$

see Figure 4, left.

Next, by imposing $\rho w = \frac{V_{\max}}{\psi(\rho)}\rho$ we obtain the density $\tilde{\rho}_2 = \psi^{-1}\left(\frac{V_{\max}}{w}\right)$ in the (ρ, η) plane. Thus, in the $(\rho, \rho v)$ plane, $\mathcal{T}^f_{out}(w, \bar{\rho}, \bar{\eta}) = [0, \tilde{\rho}_2 V_{\max}]$, see Figure 4, right.

Case 2. Since waves of the second family have positive speed, then all the points in C of the Lax curve of the second family through $(\bar{\rho}, \bar{\eta})$ should belong to $\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta})$. Since we fixed w, we consider only the point (ρ^+, η^+), which is the point of intersection between $w = \eta^*/\rho^*$ and the Lax curve of the second family through $(\bar{\rho}, \bar{\eta})$; that is $\rho^+ = \psi^{-1}\left(\frac{v(\bar{\rho},\bar{\eta})}{w}\right)$.

Moreover $\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta})$ contains also points in F which belong to the curve $w = \eta^*/\rho^*$ and which can be connected by a phase transition wave with positive speed to the point (ρ^+, η^+). By the Rankine-Hugoniot condition, a phase transition wave connecting $(\rho_l, \eta_l) \in F$ and $(\rho_r, \eta_r) \in C$ has strictly positive speed if and only if $\rho_l V_{max} < \eta_r \psi(\rho_r)$ and has zero speed if and only if $\rho_l V_{max} = \eta_r \psi(\rho_r)$. In particular, define $(\hat{\rho}_2, \hat{\eta}_2) \in F$ by the unique solution to

$$\begin{cases} \hat{\eta}_2 = \frac{V_{\max}\rho^+}{\psi(\hat{\rho}_2)} \\ \frac{\hat{\eta}_2}{\hat{\rho}_2} = \frac{\eta^+}{\rho^+}\,. \end{cases} \tag{3.5}$$

The first equation above, by the Rankine-Hugoniot conditions, means that the wave between (ρ^+, η^+) and $(\hat{\rho}_2, \hat{\eta}_2)$ has zero speed.

Therefore

$$\mathcal{T}_{out}(w, \bar{\rho}, \bar{\eta}) = \left\{ \left(\rho^*, \eta^*\right) \in F : \frac{\eta^*}{\rho^*} = w \ , \rho < \hat{\rho}_2 \right\} \cup \left\{ \left(\rho^+, \eta^+\right) \right\},$$

see Figure 5, left.

Finally, we obtain that the maximum flow is attained at the point $\left(\rho^+, \rho^+ v\left(\rho^+, \eta^+\right)\right)$, thus $\mathcal{T}^f_{out}\left(w, \bar\rho, \bar\eta\right) = \left[0, \rho^+ v\left(\rho^+, \eta^+\right)\right]$, see Figure 5, right. $\square$

Admissible Solutions at J. Define $\Gamma_i = \max \mathcal{T}^f_{inc}\left(\bar\rho_i, \bar\eta_i\right)$, for $i = 1, \dots, n$ in the incoming arcs and, for every $w \in [\check w, \hat w]$, $\Gamma^w_j = \max \mathcal{T}^f_{out}\left(w, \bar\rho_j, \bar\eta_j\right)$ for $j = n+1, \dots, n+m$ in the outgoing arcs. Fix a matrix $A \in \mathcal{A}$, where

$$\mathcal{A} := \left\{ A = \left\{\alpha_{i,j}\right\}_{i=1,\dots,n,\ j=n+1,\dots,n+m} : 0 < \alpha_{i,j} < 1\ \forall i, j,\ \sum_{j=n+1}^{n+m} \alpha_{i,j} = 1\ \forall i \right\},$$

where $\left\{\alpha_{i,j}\right\}_{i=1,\dots,n,\ j=n+1,\dots,n+m}$ indicates the percentage of traffic that passes from I_i to I_j.

Consider the set

$$\Omega = \left\{ (\gamma_1, \dots, \gamma_n) \in \prod_{i=1}^{n} [0, \Gamma_i] : A\left(\gamma_1, \dots, \gamma_n\right)^T \in \prod_{j=n+1}^{n+m} [0, \Gamma^{w_j}_j] \right\}, \tag{3.6}$$

where the maximal speeds w_j are defined by

$$\begin{aligned} w_{n+1} &= \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+1}\gamma_i} \left[\alpha_{1,n+1}\gamma_1 w_1 + \ldots + \alpha_{n,n+1}\gamma_n w_n\right], \\ &\vdots \\ w_{n+m} &= \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+m}\gamma_i} \left[\alpha_{1,n+m}\gamma_1 w_1 + \ldots + \alpha_{n,n+m}\gamma_n w_n\right]. \end{aligned} \tag{3.7}$$

Note that every point in the set Ω is a tuple of admissible fluxes at the junction.

We define the concept of Riemann solver at a generic node.

Definition 3.3 *A Riemann solver at the node is a function*

$$\begin{aligned} \mathcal{RS}_J : \textstyle\prod_{i=1}^{n+m} (F \cup C) &\longrightarrow \textstyle\prod_{i=1}^{n+m} (F \cup C) \\ \left((\rho_1, \eta_1), \cdots, (\rho_{n+m}, \eta_{n+m})\right) &\longmapsto \left((\rho^*_1, \eta^*_1), \cdots, (\rho^*_{n+m}, \eta^*_{n+m})\right) \end{aligned}$$

satisfying the following properties.

1. *The consistency condition*

$$\mathcal{RS}_J\left((\rho^*_1, \eta^*_1), \cdots, (\rho^*_{n+m}, \eta^*_{n+m})\right) = \left((\rho^*_1, \eta^*_1), \cdots, (\rho^*_{n+m}, \eta^*_{n+m})\right)$$

holds.

2. *For every $i \in \{1, \ldots, n\}$, the classical Riemann problem*

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, \eta)\right) = 0, \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho, \eta)\right) = 0, \end{cases} & t > 0, x \in \mathbb{R} \\ (\rho, \eta)(0, x) = (\rho_i, \eta_i), & x < 0 \\ (\rho, \eta)(0, x) = (\rho_i^*, \eta_i^*), & x > 0 \end{cases}$$

is solved with waves with negative speed.

3. *For every $i \in \{n+1, \ldots, n+m\}$, the classical Riemann problem*

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, \eta)\right) = 0, \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho, \eta)\right) = 0, \end{cases} & t > 0, x \in \mathbb{R} \\ (\rho, \eta)(0, x) = (\rho_i^*, \eta_i^*), & x < 0 \\ (\rho, \eta)(0, x) = (\rho_i, \eta_i), & x > 0 \end{cases}$$

is solved with waves with positive speed.

4. *The constraint*

$$A \left(\gamma_1^*, \ldots \gamma_n^*\right)^T = \left(\gamma_{n+1}^*, \ldots \gamma_{n+m}^*\right)^T$$

holds, where $\gamma_i^* = \rho_i^* v\left(\rho_i^*, \eta_i^*\right)$ for every $i \in \{1, \ldots, n+m\}$.

5. *The mass conservation*

$$\sum_{i=1}^{n} \rho_i^* v\left(\rho_i^*, \eta_i^*\right) = \sum_{i=n+1}^{n+m} \rho_i^* v\left(\rho_i^*, \eta_i^*\right)$$

holds.

6. *The conservation of the maximal speed holds, i.e.:*

$$\begin{aligned} w_{n+1}^* &= \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+1} \gamma_i^*} \left[\alpha_{1,n+1} \gamma_1^* w_1^* + \ldots + \alpha_{n,n+1} \gamma_n^* w_n^*\right], \\ &\vdots \\ w_{n+m}^* &= \frac{1}{\sum_{i=1}^{n} \alpha_{i,n+m} \gamma_i^*} \left[\alpha_{1,n+m} \gamma_1^* w_1^* + \ldots + \alpha_{n,n+m} \gamma_n^* w_n^*\right], \end{aligned}$$

where $w_i^* = \frac{\eta_i^*}{\rho_i^*}$ and $\gamma_i^* = \rho_i^* v\left(\rho_i^*, \eta_i^*\right)$ for every $i \in \{1, \ldots, n+m\}$.

4 The Riemann Problem for the $1 \times m$ Junction

Here we consider a junction with $n = 1$ incoming arc and m outgoing arcs ($m \geq 2$) and the corresponding Riemann problem (3.1). Fix a matrix $A \in \mathcal{A}$, which assumes the form

$$A = \begin{pmatrix} \alpha_{1,2} & \cdots & \alpha_{1,m+1} \end{pmatrix}^T$$

whose coefficients are positive and satisfy

$$\alpha_{1,2} + \cdots + \alpha_{1,m+1} = 1.$$

We construct a particular Riemann solver $\mathcal{RS}_J$ with the following procedure.

1. Define the maximal speed $\bar{w}_1 = \frac{\bar{\eta}_1}{\bar{\rho}_1}$ in the incoming road.
2. Define $\Gamma_1 = \max \mathcal{T}_{inc}^f(\bar{\rho}_1, \bar{\eta}_1)$, according to Proposition 3.1.
3. Define $\Gamma_j^{\bar{w}_1} = \max \mathcal{T}_{out}^f(\bar{w}_1, \bar{\rho}_j, \bar{\eta}_j)$, for every $j = 1, \ldots, 1+m$, according to Proposition 3.2.
4. Consider the set in (3.6), which, in this situation, becomes

$$\begin{aligned} \Omega &= \left\{ \gamma_1 \in [0, \Gamma_1] : A\gamma_1 \in \prod_{j=2}^{1+m} [0, \Gamma_j^{\bar{w}_1}] \right\} \\ &= \left\{ \gamma_1 \in [0, \Gamma_1] : \alpha_{1,2}\gamma_1 \leq \Gamma_2^{\bar{w}_1}, \cdots, \alpha_{1,1+m}\gamma_1 \leq \Gamma_{1+m}^{\bar{w}_1} \right\}. \end{aligned} \tag{4.1}$$

   Note that Ω is a closed, non empty real interval.
5. Define $\gamma_1^* = \max \Omega$.
6. Define $\left(\gamma_2^*, \ldots, \gamma_{1+m}^*\right)^T = A\gamma_1^* = \left(\alpha_{1,2}\gamma_1^* \cdots \alpha_{1,1+m}\gamma_1^*\right)^T$
7. Define $\left(\rho_1^*, \eta_1^*\right) \in \mathcal{T}_{inc}(\bar{\rho}_1, \bar{\eta}_1)$ in such a way $\rho_1^* v\left(\rho_1^*, \eta_1^*\right) = \gamma_1^*$.
8. Define $\left(\rho_j^*, \eta_j^*\right) \in \mathcal{T}_{out}(\bar{w}_1, \bar{\rho}_j, \bar{\eta}_j)$ in such a way $\rho_j^* v\left(\rho_j^*, \eta_j^*\right) = \gamma_j^*$, for every $j = 2, \ldots, 1+m$.

Remark 4.1 *Note that the choice of $\left(\rho_1^*, \eta_1^*\right)$ is unique. In fact, once selected a unique point $\gamma_1^* \in \mathcal{T}_{inc}^f(\bar{\rho}_1, \bar{\eta}_1)$, there exists a unique $\left(\rho_1^*, \eta_1^*\right) \in \mathcal{T}_{inc}(\bar{\rho}_1, \bar{\eta}_1)$ with that given flow $\rho_1^* v(\rho_1^*, \eta_1^*) = \gamma_1^*$, as we can see in Figure 2 and Figure 3. Analogously the choice of $\left(\rho_j^*, \eta_j^*\right)$, for every $j = 2, \ldots, 1+m$, is unique, see Figure 4 and Figure 5.*

Remark 4.2 *With this setting, the maximal speed $\bar{w}_1$ of the incoming arc is conserved through the junction and we have*

$$\bar{w}_1 = w_2^* = \ldots = w_{1+m}^*.$$

Now we can state and prove the following result.

Theorem 4.3 *Under assumptions* **(H-1)**, **(H-2)**, **(H-3)**, **(H-4)**, *the Riemann solver $\mathcal{RS}_J$ constructed in this section satisfies all the conditions of Definition 3.3 and produces a solution to the Riemann problem (3.1).*

Proof. We only have to verify the consistency condition for $\mathcal{RS}_J$, the other conditions being obvious by construction. To this aim, we fix $(\bar\rho_i, \bar\eta_i) \in F \cup C$ for every $i \in \{1, \ldots, 1+m\}$ and define

$$\left((\rho_1^*, \eta_1^*), \cdots, (\rho_{1+m}^*, \eta_{1+m}^*)\right) = \mathcal{RS}_J\left((\bar\rho_1, \bar\eta_1), \cdots, (\bar\rho_{1+m}, \bar\eta_{1+m})\right).$$

We need to prove that

$$\mathcal{RS}_J\left((\rho_1^*, \eta_1^*), \cdots, (\rho_{1+m}^*, \eta_{1+m}^*)\right) = \left((\rho_1^*, \eta_1^*), \cdots, (\rho_{1+m}^*, \eta_{1+m}^*)\right).$$

By points 2 and 3 of the construction of $\mathcal{RS}_J$, $\Gamma_1 = \max \mathcal{T}_{inc}^f(\bar\rho_1, \bar\eta_1)$, and $\Gamma_j^{\bar w_1} = \max \mathcal{T}_{out}^f\left(\bar w_1, \bar\rho_j, \bar\eta_j\right)$. Hence by Proposition 3.1 and Proposition 3.2,

$$\Gamma_1 = \begin{cases} \tilde\rho_1 V_{\max} & \text{if} \quad (\bar\rho_1, \bar\eta_1) \in C \\ \bar\rho_1 V_{\max} & \text{if} \quad (\bar\rho_1, \bar\eta_1) \in F \end{cases}$$

and

$$\Gamma_j^{\bar w_1} = \begin{cases} \rho^+ v(\bar\rho_j, \bar\eta_j) & \text{if} \quad (\bar\rho_j, \bar\eta_j) \in C \\ \tilde\rho_2 V_{\max} & \text{if} \quad (\bar\rho_j, \bar\eta_j) \in F. \end{cases}$$

where $\tilde\rho_1$, ρ^+, $\tilde\rho_2$ are defined as in propositions 3.1 and 3.2. In a similar way, we define $\Gamma_1^* = \max \mathcal{T}_{inc}^f(\rho_1^*, \eta_1^*)$, and, for every $j \in \{2, \ldots, 1+m\}$, $\Gamma_j^{*,\bar w_1} = \max \mathcal{T}_{out}^f\left(\bar w_1, \rho_j^*, \eta_j^*\right)$. Moreover the sets Ω and Ω^* are defined in (4.1) respectively for the states $(\bar\rho_i, \bar\eta_i)$ and for $\left(\rho_i^*, \eta_i^+\right)$.

For simplicity, we consider the following two cases.

1. $\sup \Omega = \Gamma_1$. If $(\bar\rho_1, \bar\eta_1)$ is in the free phase F, then also (ρ_1^*, η_1^*) is in the free phase F. Thus $\Gamma_1 = \bar\rho_1 V_{\max} = \rho_1^* V_{\max} = \Gamma_1^*$, by Proposition 3.1.

   If $(\bar\rho_1, \bar\eta_1)$ is in the congested phase C, then (ρ_1^*, η_1^*) is in the intersection between the free phase F and the congested phase C and $\Gamma_1 = \Gamma_1^*$, by Proposition 3.1.

   For the outgoing arcs ($j = 2, ..., 1+m$), if $(\bar\rho_j, \bar\eta_j)$ is in the free phase F, then also (ρ_j^*, η_j^*) is in the free phase F. Thus $\Gamma_j^{\bar w_1} = \tilde\rho_2 V_{\max} = \Gamma_j^{*,\bar w_1}$, by Proposition 3.2.

   If $(\bar\rho_j, \bar\eta_j)$ is in the congested phase C, then if also (ρ_j^*, η_j^*) is in the congested phase C, then $\Gamma_j^{\bar w_1} = \rho^+ v(\bar\rho_j, \bar\eta_j) = \Gamma_j^{*,\bar w_1}$, by Proposition 3.2.

Otherwise, if (ρ_j^*, η_j^*) is in the free phase F, then $\Gamma_j^{\bar{w}_1} = \tilde{\rho}_2 V_{\max} < \Gamma_j^{*,\bar{w}_1}$, by Proposition 3.2.

In every case $\Gamma_1 = \Gamma_1^*$ and $\Gamma_j^{*,\bar{w}_1} \geq \Gamma_j^{\bar{w}_1}$, for $j = 2, ..., 1+m$; thus $\Omega = \Omega^*$.

2. $\sup \Omega = \frac{\Gamma_2^{\bar{w}_1}}{\alpha_{1,2}}$. If $(\bar{\rho}_2, \bar{\eta}_2)$ is in the free phase F, then also (ρ_2^*, η_2^*) is in the free phase F. Thus $\Gamma_2^{\bar{w}_1} = \tilde{\rho}_2 V_{\max} = \rho_2^* V_{\max} = \Gamma_2^{*,\bar{w}_1}$, by Proposition 3.2.

   If $(\bar{\rho}_2, \bar{\eta}_2)$ is in the congested phase C, then also (ρ_2^*, η_2^*) is in the congested phase C. Thus $\Gamma_2^{\bar{w}_1} = \rho^+ v(\rho^+, \eta^+) = \rho_2^* v(\rho^+, \eta^+) = \Gamma_2^{*,\bar{w}_1}$, by Proposition 3.2.

   The case of $\Gamma_j^{\bar{w}_1}$ and $\Gamma_j^{*,\bar{w}_1}$, for $j = 3, ..., 1+m$, can be treated, as in the previous case $\sup \Omega = \Gamma_1$, and we have $\Gamma_j^{*,\bar{w}_1} \geq \Gamma_j^{\bar{w}_1}$, for $j = 3, ..., 1+m$.

   Finally, for the incoming arc, if $(\bar{\rho}_1, \bar{\eta}_1)$ is in the free phase F, then if also (ρ_1^*, η_1^*) is in the free phase F, then $\Gamma_1 = \bar{\rho}_1 V_{\max} = \Gamma_1^*$, by Proposition 3.1. Otherwise, if (ρ_1^*, η_1^*) is in the congested phase C, then $\Gamma_1 = \bar{\rho}_1 V_{\max} < \Gamma_1^*$, by Proposition 3.1.

   If $(\bar{\rho}_1, \bar{\eta}_1)$ is in the congested phase C then (ρ_1^*, η_1^*) is in the congested phase C, and so $\Gamma_1 = \tilde{\rho}_1 V_{\max} = \Gamma_1^*$, by Proposition 3.1.

   In all cases we have that $\Gamma_2^{\bar{w}_1} = \Gamma_2^{*,\bar{w}_1}$, $\Gamma_1^* \geq \Gamma_1$ and $\Gamma_j^{*,\bar{w}_1} \geq \Gamma_j^{\bar{w}_1}$, for $j = 3, ..., 1+m$, thus $\Omega = \Omega^*$.

The other cases, that is $\sup \Omega = \frac{\Gamma_j}{\alpha_{1,j}}$, for $j = 3, ..., 1+m$, can be treated as in the previous case 2. □

4.1 A not-working approach

In this subsection, we outline the fact that it is fundamental to impose the constraint $w_j = \bar{w}_1$ ($j \in \{2, \ldots, 1+m\}$) before calculating the admissible fluxes in the outgoing roads. Indeed in the point 3. of the construction of the Riemann solver, the number $\Gamma_j^{\bar{w}_1}$ depends explicitly on that constraint. The approach, similar to that of Garavello and Piccoli [9] or Herty and Rascle [12] in the case of the Aw-Rascle-Zhang traffic model (see [1, 18]), consisting of first calculating all the possible admissible fluxes at the junction and then imposing the constraint on the maximum speed is not working for the phase transition model, considered in this paper.

We propose the following example. Choose the constants $R = 1$, $V_{max} = 1$, $\check{w} = 2$, $\hat{w} = 3$, and the function $\psi(\rho) = 1 - \rho$. In this way the hypothesis **(H-1)**, **(H-2)**, **(H-3)**, and **(H-4)** are all satisfied. Moreover, consider a junction J with one incoming I_1 and two outgoing arcs I_2, I_3, and fix the

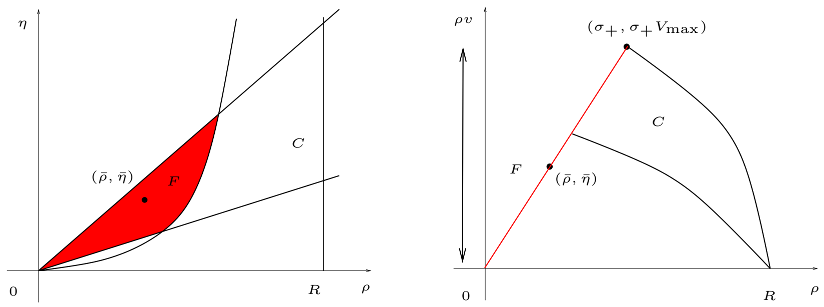

Figure 6: The case $(\bar{\rho}, \bar{\eta}) \in F$ in an outgoing road for the approach in Subsection 4.1. The set of all the possible traces it is represented in red in the coordinates, from left to right, (ρ, η) and $(\rho, \rho v)$. The corresponding set of flows is represented on the ρv axis in the $(\rho, \rho v)$ plane.

distribution matrix $A = (3/10, 7/10)^T$. Consider the Riemann problem at J with initial data

$$
\begin{aligned}
(\bar{\rho}_1, \bar{\eta}_1) &= \left(\frac{745}{1000}, \frac{18625}{10000}\right) \\
(\bar{\rho}_2, \bar{\eta}_2) &= \left(\frac{255}{1000}, \frac{51}{100}\right) \\
(\bar{\rho}_3, \bar{\eta}_3) &= \left(\frac{745}{1000}, \frac{149}{100}\right) .
\end{aligned}
$$

We can easily check that $(\bar{\rho}_1, \bar{\eta}_1) \in C$, $(\bar{\rho}_2, \bar{\eta}_2) \in F$, and $(\bar{\rho}_3, \bar{\eta}_3) \in C$.

For the incoming arc I_1, we find that the maximum flow that can pass through the junction is equal to $3/5$ according to the case 1. of Proposition 3.1, that is $\Gamma_1 = \max \mathcal{T}_{inc}^{f} (\bar{\rho}_1, \bar{\eta}_1) = 3/5$, see Figure 2.

For the case of the outgoing arcs I_2 and I_3, without imposing a constraint on w, the set of all possible fluxes at J is different from those of Proposition 3.2. More precisely, if $(\bar{\rho}, \bar{\eta})$ denotes the initial datum in an outgoing arc, then the following cases hold.

1. Case $(\bar{\rho}, \bar{\eta}) \in F$. The set of all the possible traces consists of all the points of the free phase F. Moreover the corresponding set of flows is $[0, \sigma_+ V_{\max}]$, see Figure 6.

2. Case $(\bar{\rho}, \bar{\eta}) \in C$. There exists a unique curve $\gamma(\rho)$, with support in F, such that the set of all the possible traces consists of all the points $\{(\rho, \eta) \in F : \eta > \gamma(\rho)\}$ in the free phase F and of all the points in the congested phase C belonging to the Lax curve of the second family passing through $(\bar{\rho}, \bar{\eta})$. Moreover the corresponding set of flows is $\left[0, \tilde{\rho}_2 v\left(\bar{\rho}, \bar{\eta}\right)\right]$; see Figure 7.

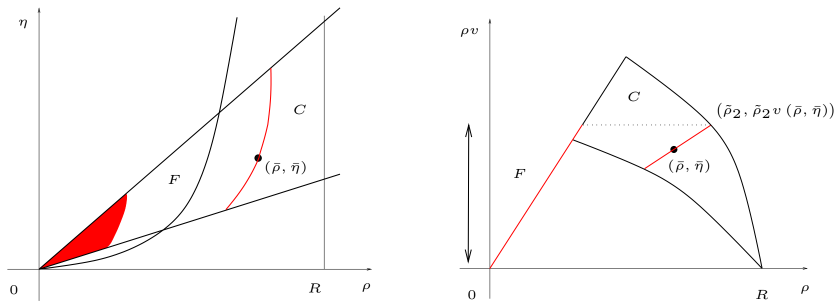

Figure 7: The case $(\bar\rho,\bar\eta) \in C$ in an outgoing road for the approach in Subsection 4.1. The set of all the possible traces it is represented in red in the coordinates, from left to right, (ρ,η) and $(\rho,\rho v)$. The corresponding set of flows is represented on the ρv axis in the $(\rho,\rho v)$ plane.

Thus, following these cases, we find that the maximum flows that can enter in I_2, I_3 are equal respectively to $2/3$ and to $4233/10000$. Therefore the set Ω should be equal to $\left[0, 3/5\right]$ and, consequently, the fluxes of the solution are

$$\gamma_1^* = 3/5, \qquad \gamma_2^* = 3/5 \times 3/10 = 9/50, \qquad \gamma_3^* = 3/5 \times 7/10 = 21/50.$$

Imposing now the constraints $w_2 = w_3 = \bar w_1$ we obtain the solution

$$(\rho_1^*,\eta_1^*) = \left(\frac{3}{5},\frac{3}{2}\right), \quad (\rho_2^*,\eta_2^*) = \left(\frac{9}{50},\frac{9}{20}\right), \quad (\rho_3^*,\eta_3^*) = \left(\frac{21}{50},\frac{21}{20}\right),$$

where $(\rho_1^*,\eta_1^*) \in C$, $(\rho_2^*,\eta_2^*) \in F$, and $(\rho_3^*,\eta_3^*) \in F$.

In the case of the outgoing arc I_3, for connecting the left state $(\rho_3^*,\eta_3^*) \in F$ with the right state $(\bar\rho_3,\bar\eta_3) \in C$, we need a phase transition wave joining (ρ_3^*,η_3^*) with (ρ^m,η^m), and a wave of the second family joining (ρ^m,η^m) with $(\bar\rho_3,\bar\eta_3)$. With simple computations, we find that $(\rho^m,\eta^m) = \left(\frac{199}{250},\frac{199}{100}\right) \in C$. By the Rankine-Hugoniot condition, we deduce that the phase transition wave, connecting (ρ_3,η_3) to (ρ^m,η^m), has strictly negative speed equal to $-351/9400$; this can not happen in an outgoing arc.

5 The Riemann Problem for the 2×1 Junction

Here we consider a junction J with $n = 2$ incoming arcs and $m = 1$ outgoing arc and the corresponding Riemann problem (3.1). Fix $P = (p_1,p_2) \in \mathbb{R}^2$, with $p_1, p_2 > 0$.

We construct a Riemann solver $\mathcal{RS}_J$ with the following procedure.

1. Define the maximal speeds $\bar w_1 = \frac{\bar\eta_1}{\bar\rho_1}$, $\bar w_2 = \frac{\bar\eta_2}{\bar\rho_2}$.

2. Define $\Gamma_i = \max \mathcal{T}^f_{inc}(\bar{\rho}_i, \bar{\eta}_i)$, for every $i = 1, 2$, according to Proposition 3.1.

3. For every maximal speed w, define $\Gamma^w_3 = \max \mathcal{T}^f_{out}(w, \bar{\rho}_3, \bar{\eta}_3)$, according to Proposition 3.2.

4. Consider the set in (3.6). In this situation, given $\bar{w}_1$, $\bar{w}_2$, it becomes

$$\Omega = \left\{ (\gamma_1, \gamma_2) \in \prod_{i=1}^{2} [0, \Gamma_i] : \begin{array}{l} \gamma_1 + \gamma_2 \in [0, \Gamma^{w_3}_3] \\ w_3 = \frac{\gamma_1}{\gamma_1+\gamma_2}\bar{w}_1 + \frac{\gamma_2}{\gamma_1+\gamma_2}\bar{w}_2 \end{array} \right\}. \tag{5.1}$$

   This is a subset of $\mathbb{R}^2$, convex and not empty. See Lemma 5.3 for the proof.

5. Define $(\gamma^*_1, \gamma^*_2) \in \Omega$ in such a way $\Pi_\Omega(P) = (\gamma^*_1, \gamma^*_2)$, where Π_Ω is the orthogonal projection on the convex set Ω.

6. Define $\gamma^*_3 = \gamma^*_1 + \gamma^*_2$.

7. Define $\left(\rho^*_i, \eta^*_i\right) \in \mathcal{T}_{inc}(\bar{\rho}_i, \bar{\eta}_i)$ such that $\rho^*_i v\left(\rho^*_i, \eta^*_i\right) = \gamma^*_i$, for $i = 1, 2$.

8. Define $\left(\rho^*_3, \eta^*_3\right) \in \mathcal{T}_{out}(w_3, \bar{\rho}_3, \bar{\eta}_3)$ in such a way $\rho^*_3 v\left(\rho^*_3, \eta^*_3\right) = \gamma^*_3$.

Remark 5.1 *The function $\Pi_\Omega : \mathbb{R}^2 \to \Omega$ is unique since Ω is a closed convex and not empty set, see Lemma 5.3.*

Remark 5.2 *Note that the choice of $\left(\rho^*_i, \eta^*_i\right)$, for every $i = 1, 2$, is unique. In fact, once selected a unique point $\gamma^*_i \in \mathcal{T}^f_{inc}(\bar{\rho}_i, \bar{\eta}_i)$, there exists a unique $\left(\rho^*_i, \eta^*_i\right) \in \mathcal{T}_{inc}(\bar{\rho}_i, \bar{\eta}_i)$ with that given flow $\rho^*_i v(\rho^*_i, \eta^*_i) = \gamma^*_i$, for every $i = 1, 2$, as we can see in Figure 2 and Figure 3. Analogously the choice of $\left(\rho^*_3, \eta^*_3\right)$ is unique, see Figure 4 and Figure 5.*

Lemma 5.3 *The set Ω in (5.1) is convex and not empty.*

Proof. Clearly $\Omega \neq \emptyset$, since $(0, 0) \in \Omega$.

Fix now $\bar{\gamma}, \bar{\bar{\gamma}} \in \Omega$, with $\bar{\gamma} \neq \bar{\bar{\gamma}}$. We aim to prove that $\lambda\bar{\gamma} + (1-\lambda)\bar{\bar{\gamma}} \in \Omega$ for every $\lambda \in [0, 1]$. Denote $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2)$, $\bar{\bar{\gamma}} = (\bar{\bar{\gamma}}_1, \bar{\bar{\gamma}}_2)$, and assume by simplicity that

$$\bar{w}_1 \leq \bar{w}_2 \qquad \text{and} \qquad \bar{\bar{\gamma}}_1\bar{\gamma}_2 - \bar{\gamma}_1\bar{\bar{\gamma}}_2 \geq 0, \tag{5.2}$$

the other cases can be treated in a similar way. For every $\lambda \in [0, 1]$, define

$$w^\lambda_3 = \frac{\gamma^\lambda_1}{\gamma^\lambda_1 + \gamma^\lambda_2}\bar{w}_1 + \frac{\gamma^\lambda_2}{\gamma^\lambda_1 + \gamma^\lambda_2}\bar{w}_2,$$

where $\gamma^\lambda = \lambda\bar{\gamma} + (1-\lambda)\bar{\bar{\gamma}}$. Thus

$$\gamma^\lambda = \left(\gamma^\lambda_1, \gamma^\lambda_2\right) = \left(\lambda\bar{\gamma}_1 + (1-\lambda)\,\bar{\bar{\gamma}}_1, \lambda\bar{\gamma}_2 + (1-\lambda)\,\bar{\bar{\gamma}}_2\right).$$

By Proposition 3.2, we have

$$\Gamma_3^{w_3^\lambda} = \begin{cases} v\left(\bar{\rho}_3, \bar{\eta}_3\right)\left(1 - \frac{v(\bar{\rho}_3,\bar{\eta}_3)}{w_3^\lambda}\right) & \text{if } (\bar{\rho}_3, \bar{\eta}_3) \in C \\ V_{max}\left(1 - \frac{V_{max}}{w_3^\lambda}\right) & \text{if } (\bar{\rho}_3, \bar{\eta}_3) \in F. \end{cases}$$

Note that

$$\Gamma_3^{w_3^\lambda} = K\left(1 - \frac{K}{w_3^\lambda}\right)$$

for a suitable constant $K > 0$. Therefore we need to prove that

$$\gamma_1^\lambda + \gamma_2^\lambda \le K\left(1 - \frac{K}{w_3^\lambda}\right) \tag{5.3}$$

for every $\lambda \in [0,1]$. The assumptions $\bar{\gamma}, \bar{\bar{\gamma}} \in \Omega$ imply that (5.3) is satisfied for $\lambda = 0$ and $\lambda = 1$. Without loss of generalities we therefore assume that

$$\gamma_1^0 + \gamma_2^0 = K\left(1 - \frac{K}{w_3^0}\right) \qquad \text{and} \qquad \gamma_1^1 + \gamma_2^1 = K\left(1 - \frac{K}{w_3^1}\right). \tag{5.4}$$

We have that

$$\partial_\lambda w_3^\lambda = \frac{\left(\bar{\bar{\gamma}}_1\bar{\gamma}_2 - \bar{\gamma}_1\bar{\bar{\gamma}}_2\right)\left(\bar{w}_2 - \bar{w}_1\right)}{\left[\lambda\left(\bar{\gamma}_1 + \bar{\gamma}_2\right) + (1-\lambda)\left(\bar{\bar{\gamma}}_1 + \bar{\bar{\gamma}}_2\right)\right]^2}.$$

By (5.2), we deduce that $\partial_\lambda w_3^\lambda \ge 0$. In particular, if $\bar{w}_1 = \bar{w}_2$ or $\bar{\bar{\gamma}}_1\bar{\gamma}_2 - \bar{\gamma}_1\bar{\bar{\gamma}}_2 = 0$, then $\partial_\lambda w_3^\lambda = 0$ and so (5.3) holds trivially for every $\lambda \in [0,1]$. Therefore we assume

$$\bar{w}_1 < \bar{w}_2 \qquad \text{and} \qquad \bar{\bar{\gamma}}_1\bar{\gamma}_2 - \bar{\gamma}_1\bar{\bar{\gamma}}_2 > 0. \tag{5.5}$$

Define the function $g : [0,1] \to \mathbb{R}$ in the following way

$$g(\lambda) = K\left(1 - \frac{K}{w_3^\lambda}\right) - \gamma_1^\lambda - \gamma_2^\lambda.$$

By (5.4), we have that $g(0) = g(1) = 0$. We prove that g is a concave function, which permits to deduce (5.3) and, consequently, to complete the proof. We get

$$g'(\lambda) = \frac{K^2(\bar{w}_2 - \bar{w}_1)(\bar{\bar{\gamma}}_1\bar{\gamma}_2 - \bar{\gamma}_1\bar{\bar{\gamma}}_2)}{(\bar{\bar{\gamma}}_1\bar{w}_1 + \bar{\bar{\gamma}}_2\bar{w}_2 + \bar{\gamma}_2\lambda\bar{w}_2 - \bar{\bar{\gamma}}_2\lambda\bar{w}_2 + \bar{\gamma}_1\lambda\bar{w}_1 - \bar{\bar{\gamma}}_1\lambda\bar{w}_1)^2} + \bar{\bar{\gamma}}_1 + \bar{\bar{\gamma}}_2 - \bar{\gamma}_1 - \bar{\gamma}_2$$

and

$$g''(\lambda) = -\frac{2K^2(\bar{w}_2 - \bar{w}_1)(\bar{\bar{\gamma}}_1\bar{\gamma}_2 - \bar{\gamma}_1\bar{\bar{\gamma}}_2)(\bar{\gamma}_2\bar{w}_2 - \bar{\bar{\gamma}}_2\bar{w}_2 + \bar{\gamma}_1\bar{w}_1 - \bar{\bar{\gamma}}_1\bar{w}_1)}{(\bar{\bar{\gamma}}_1\bar{w}_1 + \bar{\bar{\gamma}}_2\bar{w}_2 + \bar{\gamma}_2\lambda\bar{w}_2 - \bar{\bar{\gamma}}_2\lambda\bar{w}_2 + \bar{\gamma}_1\lambda\bar{w}_1 - \bar{\bar{\gamma}}_1\lambda\bar{w}_1)^3}.$$

Note that the denominator of g'' is strictly positive. In fact, if we define

$$D(\lambda) = \bar{\bar{\gamma}}_1 \bar{w}_1 + \bar{\bar{\gamma}}_2 \bar{w}_2 + \lambda \left(\bar{\gamma}_2 \bar{w}_2 - \bar{\bar{\gamma}}_2 \bar{w}_2 + \bar{\gamma}_1 \bar{w}_1 - \bar{\bar{\gamma}}_1 \bar{w}_1 \right) ,$$

we have that D is affine with respect to λ, $D(0) = \bar{\bar{\gamma}}_1 \bar{w}_1 + \bar{\bar{\gamma}}_2 \bar{w}_2 > 0$ and $D(1) = \bar{\gamma}_1 \bar{w}_1 + \bar{\gamma}_2 \bar{w}_2 > 0$; thus $D(\lambda) > 0$ for every $\lambda \in [0, 1]$. By (5.5), $g''(\lambda) \leq 0$ for every $\lambda \in [0, 1]$ if and only if

$$\bar{\gamma}_2 \bar{w}_2 - \bar{\bar{\gamma}}_2 \bar{w}_2 + \bar{\gamma}_1 \bar{w}_1 - \bar{\bar{\gamma}}_1 \bar{w}_1 \geq 0.$$

Assume by contradiction that

$$\bar{\gamma}_2 \bar{w}_2 - \bar{\bar{\gamma}}_2 \bar{w}_2 + \bar{\gamma}_1 \bar{w}_1 - \bar{\bar{\gamma}}_1 \bar{w}_1 < 0 .$$

which is equivalent to

$$\left(\bar{\gamma}_2 - \bar{\bar{\gamma}}_2 \right) \bar{w}_2 < \left(\bar{\bar{\gamma}}_1 - \bar{\gamma}_1 \right) \bar{w}_1. \tag{5.6}$$

We claim that

$$\bar{\gamma}_1 + \bar{\gamma}_2 < \bar{\bar{\gamma}}_1 + \bar{\bar{\gamma}}_2 . \tag{5.7}$$

We have three cases.

1. $\bar{\gamma}_2 - \bar{\bar{\gamma}}_2 < 0$. We deduce that $\bar{\bar{\gamma}}_1 - \bar{\gamma}_1 > 0$. Indeed, if by contradiction $\bar{\bar{\gamma}}_1 - \bar{\gamma}_1 \leq 0$, then $\bar{\bar{\gamma}}_1 \bar{\gamma}_2 \leq \bar{\gamma}_1 \bar{\gamma}_2 < \bar{\gamma}_1 \bar{\bar{\gamma}}_2$ contradicting (5.5). Therefore (5.6) implies that

$$\left(\bar{\gamma}_2 - \bar{\bar{\gamma}}_2 \right) \bar{w}_2 < \left(\bar{\bar{\gamma}}_1 - \bar{\gamma}_1 \right) \bar{w}_1 < \left(\bar{\bar{\gamma}}_1 - \bar{\gamma}_1 \right) \bar{w}_2$$

   and so $(\bar{\gamma}_2 - \bar{\bar{\gamma}}_2) < (\bar{\bar{\gamma}}_1 - \bar{\gamma}_1)$ proving (5.7).

2. $\bar{\gamma}_2 - \bar{\bar{\gamma}}_2 = 0$. In this case (5.6) becomes

$$0 < \left(\bar{\bar{\gamma}}_1 - \bar{\gamma}_1 \right) \bar{w}_1$$

   which implies $0 < (\bar{\bar{\gamma}}_1 - \bar{\gamma}_1)$ and so (5.7).

3. $\bar{\gamma}_2 - \bar{\bar{\gamma}}_2 > 0$. In this case (5.6) implies

$$\left(\bar{\gamma}_2 - \bar{\bar{\gamma}}_2 \right) \bar{w}_1 < \left(\bar{\gamma}_2 - \bar{\bar{\gamma}}_2 \right) \bar{w}_2 < \left(\bar{\bar{\gamma}}_1 - \bar{\gamma}_1 \right) \bar{w}_1.$$

   and so $(\bar{\gamma}_2 - \bar{\bar{\gamma}}_2) < (\bar{\bar{\gamma}}_1 - \bar{\gamma}_1)$ proving (5.7).

By (5.5) and (5.7) we deduce that $g'(\lambda) > 0$ for every $\lambda \in (0, 1)$. This yields a contradiction with $g(0) = g(1) = 0$ and the proof is complete. $\square$

Now we can state the following

Theorem 5.4 *Under assumptions* **(H-1)**, **(H-2)**, **(H-3)**, **(H-4)**, *the Riemann solver $\mathcal{RS}_J$ constructed in this section satisfies all the conditions of Definition 3.3 and produces a solution to the Riemann problem (3.1).*

Proof. We only have to verify the consistency condition for $\mathcal{RS}_J$. To this aim, we fix $(\bar\rho_i, \bar\eta_i) \in F \cup C$ for every $i \in \{1, 2, 3\}$ and define

$$\left((\rho_1^*, \eta_1^*), (\rho_2^*, \eta_2^*), (\rho_3^*, \eta_3^*)\right) = \mathcal{RS}_J \left((\bar\rho_1, \bar\eta_1), (\bar\rho_2, \bar\eta_2), (\bar\rho_3, \bar\eta_3)\right) .$$

We need to prove that

$$\mathcal{RS}_J \left((\rho_1^*, \eta_1^*), (\rho_2^*, \eta_2^*), (\rho_3^*, \eta_3^*)\right) = \left((\rho_1^*, \eta_1^*), (\rho_2^*, \eta_2^*), (\rho_3^*, \eta_3^*)\right) .$$

By points 2 and 3 of the construction of $\mathcal{RS}_J$, $\Gamma_i = \max \mathcal{T}_{inc}^f(\bar\rho_i, \bar\eta_i)$, for $i = 1, 2$, and $\Gamma_3^w = \max \mathcal{T}_{out}^f(w, \bar\rho_3, \bar\eta_3)$ for every w. In a similar way, we define $\Gamma_i^* = \max \mathcal{T}_{inc}^f(\rho_i^*, \eta_i^*)$, for every $i = 1, 2$, and $\Gamma_3^{*,w} = \max \mathcal{T}_{out}^f(w, \rho_3^*, \eta_3^*)$ for every w.

We consider the following two cases. The details similar to those in proof of Theorem 5.4 are omitted.

1. $\gamma_1^* + \gamma_2^* = \Gamma_3^{w_3}$. In this case (ρ_3^*, η_3^*) is in the congested phase C and $w_3^* = \frac{\gamma_1^*}{\gamma_1^*+\gamma_2^*}\bar w_1 + \frac{\gamma_2^*}{\gamma_1^*+\gamma_2^*}\bar w_2$. We have that $\Gamma_1 \le \Gamma_1^*$, $\Gamma_2 \le \Gamma_2^*$ and $\Gamma_3^w = \Gamma_3^{*,w}$ for every maximal speed w.

   By Lemma 5.3 the curve $\gamma_1 + \gamma_2 = \Gamma_3^{\frac{\gamma_1}{\gamma_1+\gamma_2}\bar w_1 + \frac{\gamma_2}{\gamma_1+\gamma_2}\bar w_2}$ is concave in $[0, +\infty[\times [0, +\infty[$. For the properties of the projection on the convex set Ω we conclude, see Figure 8.

2. $\gamma_1^* + \gamma_2^* < \Gamma_3^{w_3}$. In this case (ρ_3^*, η_3^*) is in the free phase F and $w_3^* = \frac{\gamma_1^*}{\gamma_1^*+\gamma_2^*}\bar w_1 + \frac{\gamma_2^*}{\gamma_1^*+\gamma_2^*}\bar w_2$. We can suppose that $\gamma_1^* = \Gamma_1$ and we have that $\Gamma_1 = \Gamma_1^*$, $\Gamma_2 \le \Gamma_2^*$ and $\Gamma_3^w \le \Gamma_3^{*,w}$ for every maximal speed w.

   By Lemma 5.3 the curves $\gamma_1 + \gamma_2 = \Gamma_3^{\frac{\gamma_1}{\gamma_1+\gamma_2}\bar w_1 + \frac{\gamma_2}{\gamma_1+\gamma_2}\bar w_2}$ and $\gamma_1 + \gamma_2 = \Gamma_3^{*,\frac{\gamma_1}{\gamma_1+\gamma_2}\bar w_1 + \frac{\gamma_2}{\gamma_1+\gamma_2}\bar w_2}$ are concave in $[0, +\infty[\times [0, +\infty[$ and for the properties of the projection on the convex set Ω we conclude, see Figure 9.

The proof is so concluded. $\square$

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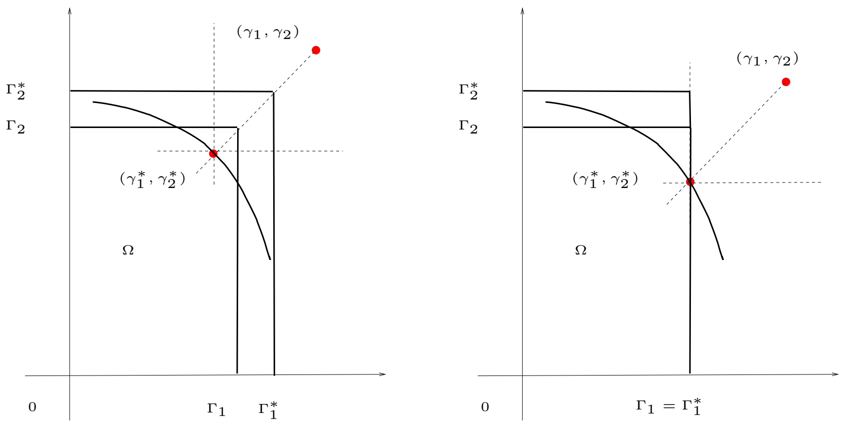

Figure 8: The case $\gamma_1^* + \gamma_2^* = \Gamma_3^{w_3}$. At left the case $\gamma_1^* < \Gamma_1$ and $\gamma_2^* < \Gamma_2$. At right the case $\gamma_1^* = \Gamma_1$.

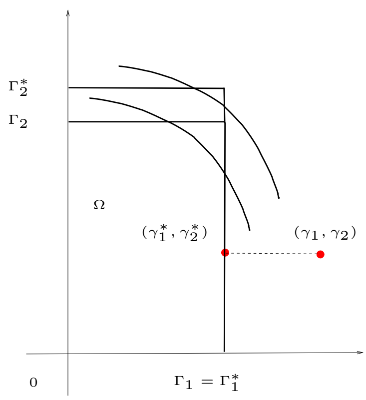

Figure 9: The case $\gamma_1^* + \gamma_2^* < \Gamma_3^{w_3}$.

References

[1] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000.

[2] S. Blandin, D. Work, P. Goatin, B. Piccoli, and A. Bayen. A general phase transition model for vehicular traffic. *SIAM J. Appl. Math.*, 71(1):107–127, 2011.

[3] G. M. Coclite, M. Garavello, and B. Piccoli. Traffic flow on a road network. *SIAM J. Math. Anal.*, 36(6):1862–1886, 2005.

[4] R. M. Colombo. Hyperbolic phase transitions in traffic flow. *SIAM J. Appl. Math.*, 63(2):708–721, 2002.

[5] R. M. Colombo. Phase transitions in hyperbolic conservation laws. In *Progress in analysis, Vol. I, II (Berlin, 2001)*, pages 1279–1287. World Sci. Publ., River Edge, NJ, 2003.

[6] R. M. Colombo, P. Goatin, and B. Piccoli. Road networks with phase transitions. *J. Hyperbolic Differ. Equ.*, 7(1):85–106, 2010.

[7] R. M. Colombo, F. Marcellini, and M. Rascle. A 2-phase traffic model based on a speed bound. *SIAM J. Appl. Math.*, 70(7):2652–2666, 2010.

[8] M. Garavello, K. Han, and B. Piccoli. *Models for Vehicular Traffic on Networks*, volume 9 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2016.

[9] M. Garavello and B. Piccoli. Traffic flow on a road network using the Aw-Rascle model. *Comm. Partial Differential Equations*, 31(1-3):243–275, 2006.

[10] M. Garavello and B. Piccoli. *Traffic flow on networks*, volume 1 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006. Conservation laws models.

[11] P. Goatin. The Aw-Rascle vehicular traffic flow model with phase transitions. *Math. Comput. Modelling*, 44(3-4):287–303, 2006.

[12] M. Herty and M. Rascle. Coupling conditions for a class of second-order models for traffic flow. *SIAM Journal on Mathematical Analysis*, 38(2):595–616, 2006.

[13] H. Holden and N. H. Risebro. A mathematical model of traffic flow on a network of unidirectional roads. *SIAM J. Math. Anal.*, 26(4):999–1017, 1995.

[14] J. P. Lebacque, X. Louis, S. Mammar, B. Schnetzlera, and H. Haj-Salem. Modélisation du trafic autoroutier au second ordre. *Comptes Rendus Mathematique*, 346(21–22):1203–1206, November 2008.

[15] M. J. Lighthill and G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.*, 229:317–345, 1955.

[16] F. Marcellini. Free-congested and micro-macro descriptions of traffic flow. *Discrete Contin. Dyn. Syst. Ser. S*, 7(3):543–556, 2014.

[17] P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956.

[18] H. Zhang. A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B: Methodological*, 36(3):275 – 290, 2002.