SUPPLEMENTARY MATERIALS: LARGE TIME STEP HLL AND HLLC SCHEMES*

MARIN PREBEG†, TORE FLÅTTEN‡, AND BERNHARD MÜLLER†

S1. Error estimates and convergence rates. Here we include error estimates and convergence rates for the Large Time Step (LTS) extensions of the HLL (Harten-Lax-van Leer) and the HLLC (Harten-Lax-van Leer Contact) schemes for the Sod shock tube problem (5.4) in subsection 5.1 and the Woodward-Colella blast-wave problem (5.5) in subsection 5.2.

The acronym *+WL* stands for *wave limiter* [S1], indicating that the scheme is second order accurate for smooth solutions. In the present paper we used the superbee wave limiter [S1]. Further, C denotes the Courant number and n denotes the number of grid cells.

S1.1. Error estimates and convergence rates for Sod shock tube (5.4).

S1.1.1. HLL and LTS-HLL scheme.

Table S1: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-2}$) and convergence rates $\mathcal{L}$ of **density**

	HLL		HLL+WL		LTS-HLL		LTS-HLL		LTS-HLL	
$C=$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	2.886	–	1.553	–	3.781	–	5.836	–	9.802	–
200	1.916	0.591	0.998	0.638	2.399	0.656	3.415	0.773	5.801	0.757
400	1.202	0.672	0.609	0.713	1.429	0.747	2.054	0.734	3.360	0.788
800	0.753	0.675	0.381	0.677	0.873	0.711	1.220	0.750	2.005	0.745
1600	0.484	0.638	0.259	0.557	0.561	0.638	0.763	0.678	1.203	0.737
3200	0.307	0.655	0.172	0.587	0.363	0.627	0.483	0.659	0.743	0.695

Table S2: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-2}$) and convergence rates $\mathcal{L}$ of **velocity**

	HLL		HLL+WL		LTS-HLL		LTS-HLL		LTS-HLL	
$C=$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	4.004	–	1.619	–	4.675	–	10.68	–	25.69	–
200	2.302	0.798	0.870	0.896	2.397	0.963	3.823	1.483	10.27	1.322
400	1.274	0.854	0.412	1.078	1.148	1.062	1.906	1.004	3.865	1.410
800	0.713	0.838	0.195	1.080	0.601	0.933	0.938	1.025	1.871	1.046
1600	0.388	0.879	0.094	1.054	0.308	0.965	0.483	0.955	0.926	1.015
3200	0.222	0.804	0.048	0.965	0.164	0.904	0.250	0.953	0.486	0.930

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†Department of Energy and Process Engineering, Norwegian University of Science and Technology, Kolbjørn Hejes vei 2, NO-7491 Trondheim, Norway (marin.prebeg@ntnu.no, bernhard.muller@ntnu.no)

‡SINTEF Materials and Chemistry, P. O. Box 4760 Sluppen, NO-7465 Trondheim, Norway (toref@math.uio.no).

Table S3: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-2}$) and convergence rates $\mathcal{L}$ of **pressure**

	HLL		HLL+WL		LTS-HLL		LTS-HLL		LTS-HLL	
$C =$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	2.346	–	0.870	–	2.802	–	5.418	–	12.34	–
200	1.407	0.738	0.444	0.972	1.425	0.975	2.295	1.234	5.208	1.247
400	0.819	0.780	0.214	1.052	0.710	1.005	1.169	0.973	2.258	1.204
800	0.474	0.788	0.103	1.059	0.364	0.964	0.583	1.004	1.135	0.992
1600	0.272	0.802	0.052	0.973	0.190	0.941	0.299	0.961	0.566	1.003
3200	0.155	0.814	0.027	0.970	0.101	0.911	0.153	0.965	0.289	0.968

Table S4: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-1}$) and convergence rates $\mathcal{L}$ of **internal energy**

	HLL		HLL+WL		LTS-HLL		LTS-HLL		LTS-HLL	
$C =$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	1.002	–	0.675	–	1.424	–	2.137	–	3.669	–
200	0.719	0.479	0.507	0.413	0.979	0.540	1.301	0.716	2.198	0.739
400	0.470	0.612	0.333	0.604	0.644	0.605	0.840	0.630	1.279	0.781
800	0.310	0.602	0.223	0.578	0.431	0.577	0.562	0.581	0.824	0.635
1600	0.210	0.562	0.156	0.519	0.296	0.541	0.382	0.554	0.555	0.570
3200	0.141	0.569	0.107	0.535	0.204	0.541	0.261	0.552	0.376	0.561

S1.1.2. HLLC and LTS-HLLC scheme.

Table S5: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-2}$) and convergence rates $\mathcal{L}$ of **density**

	HLLC		HLLC+WL		LTS-HLLC		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	2.610	–	0.753	–	2.456	–	3.762	–	8.243	–
200	1.749	0.577	0.392	0.941	1.399	0.812	1.981	0.925	3.865	1.093
400	1.104	0.663	0.182	1.109	0.761	0.879	1.027	0.947	1.943	0.992
800	0.689	0.680	0.087	1.068	0.434	0.810	0.536	0.938	0.977	0.992
1600	0.443	0.638	0.049	0.805	0.266	0.704	0.295	0.861	0.517	0.917
3200	0.280	0.663	0.023	1.080	0.159	0.744	0.162	0.868	0.275	0.911

Table S6: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-2}$) and convergence rates $\mathcal{L}$ of **velocity**

	HLLC		HLLC+WL		LTS-HLLC		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	3.724	–	1.447	–	4.303	–	9.866	–	24.77	–
200	2.188	0.767	0.668	1.115	2.028	1.085	3.475	1.505	10.29	1.268
400	1.250	0.808	0.280	1.255	0.958	1.082	1.533	1.181	3.706	1.473
800	0.686	0.865	0.147	0.926	0.530	0.854	0.769	0.994	1.560	1.248
1600	0.380	0.851	0.073	1.004	0.284	0.902	0.386	0.957	0.816	0.935
3200	0.216	0.818	0.040	0.859	0.151	0.906	0.209	0.883	0.409	0.997

Table S7: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-2}$) and convergence rates $\mathcal{L}$ of **pressure**

	HLCL		HLLC+WL		LTS-HLLC		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	2.210	–	0.694	–	2.324	–	4.632	–	10.26	–
200	1.343	0.719	0.330	1.072	1.150	1.014	1.949	1.249	4.754	1.110
400	0.795	0.755	0.148	1.156	0.576	0.999	0.921	1.081	1.996	1.274
800	0.460	0.791	0.075	0.986	0.312	0.882	0.462	0.997	0.906	1.139
1600	0.266	0.789	0.039	0.926	0.173	0.853	0.236	0.965	0.462	0.971
3200	0.151	0.813	0.020	1.000	0.093	0.885	0.124	0.934	0.234	0.981

Table S8: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-1}$) and convergence rates $\mathcal{L}$ of **internal energy**

	HLLC		HLLC+WL		LTS-HLLC		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5		10	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	0.851	–	0.223	–	0.677	–	1.116	–	2.402	–
200	0.626	0.444	0.138	0.694	0.444	0.610	0.534	1.062	1.219	0.979
400	0.413	0.598	0.064	1.109	0.251	0.821	0.284	0.910	0.504	1.274
800	0.270	0.614	0.030	1.063	0.155	0.693	0.158	0.849	0.276	0.868
1600	0.183	0.557	0.017	0.865	0.100	0.629	0.093	0.760	0.160	0.782
3200	0.123	0.577	0.008	1.127	0.065	0.634	0.054	0.774	0.088	0.862

S1.2. Error estimates and convergence rates for Woodward-Colella blast-wave problem (5.5).

S1.2.1. HLL and LTS-HLL scheme.

Table S9: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-1}$) and convergence rates $\mathcal{L}$ of **density**

	HLL		HLL+WL		LTS-HLL		LTS-HLL	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	3.711	–	3.032	–	4.266	–	4.713	–
200	3.267	0.184	2.236	0.439	3.555	0.263	4.085	0.206
400	2.715	0.267	1.582	0.499	2.836	0.326	3.329	0.295
800	2.152	0.335	1.038	0.608	2.148	0.400	2.555	0.382
1600	1.629	0.402	0.691	0.588	1.580	0.443	1.888	0.436
3200	1.172	0.475	0.450	0.617	1.126	0.487	1.356	0.478

Table S10: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-1}$) and convergence rates $\mathcal{L}$ of **velocity**

	HLL		HLL+WL		LTS-HLL		LTS-HLL	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	6.846	–	5.386	–	7.066	–	7.404	–
200	6.052	0.177	3.859	0.481	6.122	0.207	6.578	0.170
400	4.526	0.419	2.487	0.634	5.120	0.258	6.399	0.397
800	2.984	0.601	1.378	0.852	3.691	0.472	4.988	0.359
1600	1.873	0.672	0.746	0.885	2.376	0.636	3.306	0.593
3200	1.179	0.667	0.4150	0.845	1.439	0.723	2.008	0.719

Table S11: 1-norm error estimates $\mathcal{E}$ ($\times 10$) and convergence rates $\mathcal{L}$ of **pressure**

	HLL		HLL+WL		LTS-HLL		LTS-HLL	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	1.679	–	0.886	–	1.737	–	2.080	–
200	1.252	0.423	0.490	0.853	1.142	0.605	1.429	0.541
400	0.852	0.555	0.316	0.632	0.784	0.544	0.986	0.535
800	0.556	0.615	0.182	0.800	0.511	0.615	0.648	0.605
1600	0.374	0.570	0.103	0.818	0.345	0.566	0.434	0.579
3200	0.255	0.553	0.054	0.926	0.236	0.571	0.295	0.558

Table S12: 1-norm error estimates $\mathcal{E}$ ($\times 10^2$) and convergence rates $\mathcal{L}$ of **internal energy**

	HLL		HLL+WL		LTS-HLL		LTS-HLL	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	2.168	–	1.790	–	3.024	–	3.570	–
200	1.659	0.386	1.261	0.505	2.252	0.425	2.732	0.386
400	1.217	0.447	0.870	0.535	1.668	0.433	2.039	0.422
800	0.870	0.483	0.602	0.531	1.187	0.491	1.485	0.457
1600	0.615	0.501	0.418	0.528	0.825	0.524	1.042	0.511
3200	0.432	0.508	0.291	0.521	0.572	0.528	0.722	0.528

19 **S1.2.2. HLLC and LTS-HLLC scheme.**

Table S13: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-1}$) and convergence rates $\mathcal{L}$ of **density**

	HLLC		HLLC+WL		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	3.603	–	2.160	–	2.658	–	2.334	–
200	3.207	0.168	1.373	0.654	2.253	0.239	1.953	0.257
400	2.649	0.275	0.684	1.004	1.795	0.327	1.490	0.390
800	2.068	0.357	0.350	0.968	1.358	0.403	1.082	0.462
1600	1.541	0.425	0.194	0.854	1.005	0.435	0.796	0.423
3200	1.095	0.492	0.093	1.064	0.713	0.494	0.561	0.504

Table S14: 1-norm error estimates $\mathcal{E}$ ($\times 10^{-1}$) and convergence rates $\mathcal{L}$ of **velocity**

	HLLC		HLLC+WL		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	5.640	–	2.956	–	6.369	–	7.113	–
200	4.944	0.190	1.430	1.047	4.065	0.648	4.108	0.792
400	3.896	0.344	0.767	0.899	2.601	0.644	2.524	0.703
800	2.726	0.515	0.368	1.060	1.647	0.659	1.578	0.677
1600	1.764	0.628	0.207	0.830	1.072	0.619	0.979	0.689
3200	1.109	0.669	0.112	1.885	0.711	0.593	0.622	0.653

Table S15: 1-norm error estimates $\mathcal{E}$ ($\times 10$) and convergence rates $\mathcal{L}$ of **pressure**

	HLLC		HLLC+WL		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	1.559	–	0.597	–	1.322	–	1.534	–
200	1.175	0.408	0.299	0.997	0.807	0.712	0.745	1.042
400	0.809	0.539	0.155	0.947	0.525	0.621	0.450	0.727
800	0.542	0.577	0.074	1.060	0.363	0.531	0.297	0.601
1600	0.366	0.565	0.037	0.987	0.256	0.506	0.200	0.572
3200	0.248	0.565	0.017	1.100	0.181	0.501	0.137	0.542

Table S16: 1-norm error estimates $\mathcal{E}$ ($\times 10^2$) and convergence rates $\mathcal{L}$ of **internal energy**

	HLLC		HLLC+WL		LTS-HLLC		LTS-HLLC	
$C =$	1		1		3		5	
n	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$	$\mathcal{E}_n$	$\mathcal{L}_n$
100	1.781	–	0.625	–	0.751	–	0.704	–
200	1.368	0.381	0.324	0.950	0.556	0.432	0.550	0.353
400	1.016	0.428	0.158	1.032	0.400	0.476	0.403	0.447
800	0.737	0.464	0.074	1.031	0.288	0.474	0.290	0.477
1600	0.525	0.489	0.037	1.077	0.206	0.480	0.206	0.493
3200	0.371	0.501	0.016	1.148	0.147	0.492	0.146	0.498

REFERENCES

[S1] R. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2002, doi:10.1017/CBO9780511791253.