On the Stability of a Model for the Cutting of Metal Plates by Means of Laser Beams

Francesca Marcellini

Department of Mathematics and Applications
University of Milano-Bicocca, Italy.

Abstract

In a class of systems of balance laws in several space dimensions, we prove the stability of solutions with respect to variations in the flow and in the source. This class comprises a model describing the cutting of metal plates by means of laser beam is proved to admit solutions.

Keywords: Nonlocal Balance Laws; Laser Cutting
2010 MSC: 35L65

1. Introduction

Following [3], we consider this system of n balance laws in several space dimensions:

$$\left\{\begin{array}{l} \partial_t u_i + \operatorname{div}_x \varphi_i(t,x,u_i,\vartheta * u) = \Phi_i(t,x,u_i,\vartheta * u) \\ u_i(0,x) = \bar{u}_i(x) \end{array}\right. \qquad i = 1,\ldots,n\,. \tag{1}$$

Here, $t \in [0,+\infty[$ is time, $x \in \mathbb{R}^N$ is the space coordinate and $u_1,\ldots,u_n$ are the unknowns. The function ϑ is a smooth function defined in $\mathbb{R}^N$ attaining values in $\mathbb{R}^{m\times n}$, so that

$$\vartheta \in \mathbf{C_c^2}(\mathbb{R}^N;\mathbb{R}^{m\times n}), \qquad \big(\vartheta * u(t)\big)(x) = \int_{\mathbb{R}^N} \vartheta(x-\xi)\, u(t,\xi)\,\mathrm{d}\xi\,, \qquad \big(\vartheta * u(t)\big)(x) \in \mathbb{R}^m\,.$$

Requirements on the flows φ_i, on the sources Φ_i and on the initial data $\bar{u}_i$ ensuring the well posedness of (1) are provided below.

A key property of system (1) is that the equations are coupled only through the nonlocal convolution term $\vartheta * u$. It is this feature that allows a well posedness and stability theory, although we are dealing with *systems* of balance laws in *several* space dimensions.

The driving example motivating (1) is a new model for the cutting of metal plates by means of a laser beam, presented in [3, Section 3], see also [2, 4]. However, (1) also comprises the model [7], see also [3, Section 4], devoted to the dynamics on a conveyor belt, as well as several crowd dynamics models, e.g. [1, 6, 8]. Theorem 2.3 below, applied to each of these cases, provides the stability of solutions with respect to perturbations of fluxes and sources.

2. Results

Throughout, $\operatorname{grad}_x f$ and $\operatorname{div}_x f$ denote the gradient and the divergence of f with respect to the space variable $x \in \mathbb{R}^N$. Throughout, we fix the non trivial time interval $\widehat{I} = [0,\widehat{T}]$. For any $k > 0$, we also denote $\mathcal{U}_k = [-k,\, k]$ and $\mathcal{U}_k^m = [-k,\, k]^m$.

Recall the definition of solution to (1), based on [9, Definition 1], and the well posedness result obtained in [3].

Definition 2.1 ([3, Definition 2.1]). *Let $\bar{u} \in \mathbf{L}^\infty(\mathbb{R}^N,\mathbb{R}^n)$. A map $u\colon \widehat{I} \to \mathbf{L}^\infty(\mathbb{R}^N,\mathbb{R}^n)$ is a solution on $\widehat{I}$ to (1) with initial datum $\bar{u}$ if, for $i = 1,\ldots,n$, setting for all $w \in \mathbb{R}$*

$$\widetilde{\varphi}_i(t,x,w) = \varphi_i\big(t,x,w,(\vartheta * u)(t,x)\big) \qquad \textit{and} \qquad \widetilde{\Phi}_i(t,x,w) = \Phi_i\big(t,x,w,(\vartheta * u)(t,x)\big)\,,$$

the map u_i is a Kružkov solution [9] to $\begin{cases} \partial_t u_i + \operatorname{div}_x \widetilde{\varphi}_i(t,x,u_i) = \widetilde{\Phi}_i(t,x,u_i) \\ u_i(0,x) = \bar{u}_i(x) \end{cases}$, *for* $i = 1, \ldots, n$.

Theorem 2.2. [3, Theorem 2.2] *Assume φ, Φ and ϑ satisfy the following conditions, for a given $\lambda \in (\mathbf{C^0} \cap \mathbf{L^1})(\widehat{I} \times \mathbb{R}^N \times \mathbb{R}^+; \mathbb{R}^+)$:*

($\boldsymbol{\varphi}$) *For any $U > 0$, $\varphi \in (\mathbf{C^2} \cap \mathbf{W^{2,\infty}})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{n \times N})$ and for all $t \in \widehat{I}$, $x \in \mathbb{R}^N$, $u \in \mathcal{U}_U$, $A \in \mathcal{U}_U^m$*

$$\max \left\{ \begin{array}{ll} \left\|\operatorname{grad}_x \varphi(t,x,u,A)\right\|, & \left\|\operatorname{div}_x \varphi(t,x,u,A)\right\|, \\ \left\|\operatorname{grad}_x \operatorname{div}_x \varphi(t,x,u,A)\right\|, & \left\|\operatorname{grad}_x \operatorname{grad}_A \varphi(t,x,u,A)\right\|, \\ \left\|\operatorname{grad}_A \varphi(t,x,u,A)\right\|, & \left\|\operatorname{grad}_A^2 \varphi(t,x,u,A)\right\| \end{array} \right\} \leq \lambda(t,x,U).$$

($\boldsymbol{\Phi}$) *For any $U > 0$, $\Phi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^n)$ and for all $t \in \widehat{I}$, $x \in \mathbb{R}^N$, $u \in \mathcal{U}_U$, $A \in \mathcal{U}_U^m$, $\max\left\{\left\|\Phi(t,x,u,A)\right\|, \left\|\operatorname{grad}_x \Phi(t,x,u,A)\right\|\right\} \leq \lambda(t,x,U)$.*

($\boldsymbol{\vartheta}$) $\vartheta \in \mathbf{C_c^2}(\mathbb{R}^N; \mathbb{R}^{m \times n})$.

Then, for any $\bar{\mathcal{C}} > 0$ there exists a $T_* \in \widehat{I}$ and positive $\mathcal{L}, \mathcal{C}$ such that for any datum

$$\bar{u} \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R}^n) \quad \textit{with} \quad \begin{array}{rcl} \|\bar{u}_i\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)} & \leq & \bar{\mathcal{C}}, \\ \|\bar{u}_i\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^n)} & \leq & \bar{\mathcal{C}}, \\ \operatorname{TV}(\bar{u}_i) & \leq & \bar{\mathcal{C}}, \end{array} \tag{2}$$

problem (1) *admits a unique solution $u \in \mathbf{C^0}\left([0,T_*]; \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)\right)$ in the sense of Definition 2.1, satisfying for all $t \in [0,T_*]$ the bounds $\left\|u(t)\right\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)} \leq \mathcal{C}$, $\left\|u(t)\right\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^n)} \leq \mathcal{C}$ and $\operatorname{TV}(u(t)) \leq \mathcal{C}$. Moreover, if also $\bar{w}$ satisfies* (2) *and w is the corresponding solution to* (1)*, the following Lipschitz estimate holds: $\left\|u(t) - w(t)\right\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)} \leq \mathcal{L}\, \|\bar{u} - \bar{w}\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)}$.*

We complete the above statement proving the stability of solutions with respect to Φ_i and φ_i.

Theorem 2.3. *Let φ^1, φ^2 and Φ^1, Φ^2 satisfy satisfy **($\boldsymbol{\varphi}$)** and **($\boldsymbol{\Phi}$)** in Theorem 2.2, with the same function λ. Let ϑ^1, ϑ^2 satisfy **($\boldsymbol{\vartheta}$)**. Assume moreover that*

$$\int_{\widehat{I}} \int_{\mathbb{R}^N} \sup_{u \in \mathcal{U}_U} \lambda(t,x,u) \, \mathrm{d}x \, \mathrm{d}t < +\infty. \tag{3}$$

Then, the solutions $u^\ell \equiv (u_1^\ell, \ldots, u_n^\ell)$ to $\begin{cases} \partial_t u_i^\ell + \operatorname{div}_x \varphi_i^\ell(t,x,u_i,\vartheta^\ell * u) = \Phi_i^\ell(t,x,u_i,\vartheta^\ell * u) \\ u_i(0,x) = \bar{u}_i(x) \end{cases}$, *for $\ell = 1,2$, satisfy*

$$\begin{aligned} \left\|u^1(t) - u^2(t)\right\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)} \leq\ & C^* \left(\left\|\partial_u(\varphi^1 - \varphi^2)\right\|_{\mathbf{L^\infty}(I \times \mathbb{R}^N \times \mathcal{U}_U \times \mathcal{U}_U^m;\mathbb{R}^n)} \right. \\ & + \left\|\operatorname{grad}_{xA}(\varphi^1 - \varphi^2)\right\|_{\mathbf{L^1}(I \times \mathbb{R}^N; \mathbf{L^\infty}(\mathcal{U}_U \times \mathcal{U}_U^m; \mathbb{R}^{(N+m\times n)\times n}))} \\ & \left. + \left\|\Phi^1 - \Phi^2\right\|_{\mathbf{L^1}(I \times \mathbb{R}^N; \mathbf{L^\infty}(\mathcal{U}_U \times \mathcal{U}_U^m;\mathbb{R}^n))} + \left\|\vartheta^1 - \vartheta^2\right\|_{\mathbf{L^1}(I \times \mathbb{R}^N;\mathbb{R}^{m\times n})} \right). \end{aligned} \tag{4}$$

*for a constant C^* dependent on the assumptions **($\boldsymbol{\varphi}$)**, **($\boldsymbol{\Phi}$)** and **($\boldsymbol{\vartheta}$)**, whose value is estimated in* (9)*,* (10) *and* (11)*.*

Proof of Theorem 2.3. Below, we often use the standard bound

$$\|\vartheta * u\|_{\mathbf{L^\infty}(I \times \mathbb{R}^N;\mathbb{R}^n)} \leq \|\vartheta\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^{m\times n})} \|u\|_{\mathbf{C^0}(I;\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}))}, \tag{5}$$

that holds for ϑ satisfying **(ϑ)** and $u \in \mathbf{C^0}(I; \mathbf{L^1}(\mathbb{R}^N; \mathbb{R}^n))$. By **($\vartheta$)**, we may assume that $\left\|\vartheta_{ji}\right\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \leq 1/n$ for all $j = 1, \ldots, m$ and $i = 1, \ldots, n$. This requirement simplifies several estimates below, since it ensures that, for $U \in \mathbb{R}^+$,

$$u_i(x) \in \mathcal{U}_U \text{ for all } i = 1, \ldots, n \text{ and } x \in \mathbb{R}^N \quad \Rightarrow \quad (\vartheta * u)(x) \in \mathcal{U}_U^m \text{ for all } x \in \mathbb{R}^N .$$

As in [3, Formula (5.3)], define

$$\Lambda(t, U) = \left\|\lambda(\cdot, \cdot, U)\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^N;\mathbb{R})} . \tag{6}$$

Fix positive U and R with

$$\|\bar{u}\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)} < R, \quad \|\bar{u}\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^n)} < U \quad \text{and} \quad \mathrm{TV}(\bar{u}) < R .$$

Introduce the $\mathbf{L^1}$ closed sphere centered at the initial datum $\bar{u}$ with radius R

$$B_{\mathbf{L^1}}(\bar{u}, R, U) = \left\{ u \in \mathbf{L^1}(\mathbb{R}^N; \mathbb{R}^n) \colon \|u - \bar{u}\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^n)} \leq R \text{ and } u(x) \in \mathcal{U}_U^n \right\} .$$

Throughout, we denote by C a quantity dependent only on λ and on the assumptions **(φ)**, **(Φ)** and **(ϑ)**, but independent of T, R and U. Similarly, C_U is a constant depending only on $\|\varphi\|_{\mathbf{W^{2,\infty}}(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^{n\times m})}$ and on $\|\Phi\|_{\mathbf{W^{1,\infty}}(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^n)}$.

For any positive $T \in \widehat{I}$, denote $I = [0, T]$ and define the map

$$\begin{array}{rccccc} \mathcal{T} \,:\, & \mathbf{C^0}\left(I; B_{\mathbf{L^1}}(\bar{u}, R, U)\right) & \times & \mathcal{P} & \to & \mathbf{C^0}\left(I; B_{\mathbf{L^1}}(\bar{u}, R, U)\right) \\ & w & , & (\varphi, \Phi, \vartheta) & \to & u \end{array} \tag{7}$$

where the parameter space is $\mathcal{P} = (\mathbf{C^2} \cap \mathbf{W^{2,\infty}})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_{\bar{U}} \times \mathcal{U}_{\bar{U}}^m; \mathbb{R}^{n\times N}) \times (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\widehat{I} \times \mathbb{R}^N \times \mathcal{U}_{\bar{U}} \times \mathcal{U}_{\bar{U}}^m; \mathbb{R}^n) \times \mathbf{C_c^2}(\mathbb{R}^N; \mathbb{R}^{m\times n})$.

The map $\mathcal{T}$ is proved to be a contraction for T small in [3, Theorem 2.2]. Hence, the present proof consists in showing that $\mathcal{T}$ is Lipschitz continuous in φ, Φ, ϑ. We consider the three variables φ, Φ and ϑ separately and apply repeatedly [5, Theorem 2.6], as refined in [10, Theorem 2.5]. The assumptions **(H1*)** and **(H2*)** are verified in [3, Section 5]. We now check **(H3*)**.

In the estimates below we set for simplicity $\bar{u} = 0$.

*Step 1: **(H3*)** holds:.* Fix an index $i \in \{1, \ldots, n\}$, define $\Omega_T^U = I \times \mathbb{R}^n \times \mathcal{U}_U$ and for a fixed $w \in \mathbf{C^0}(I; B_{\mathbf{L^1}}(\bar{u}, R, U))$ denote

$$\begin{array}{rclcrcl} f(t,x,u) & = & \varphi_i^1\left(t, x, u, \vartheta^1 * w(t,x)\right) & \qquad & F(t,x,u) & = & \Phi_i^1\left(t, x, u, \vartheta^1 * w(t,x)\right) \\ g(t,x,u) & = & \varphi_i^2\left(t, x, u, \vartheta^2 * w(t,x)\right) & & G(t,x,u) & = & \Phi_i^2\left(t, x, u, \vartheta^2 * w(t,x)\right) \end{array} \tag{8}$$

Then, we directly have:

$\partial_u(f - g) \in \mathbf{L^\infty}(\Omega_T^U; \mathbb{R}^N)$ holds by **(φ)**. since $\partial_u(f - g) = \partial_u(\varphi_i^1 - \varphi_i^2)$.

$\partial_u(F - G) \in \mathbf{L^\infty}(\Omega_T^U; \mathbb{R}^N)$: holds by **($\varphi$)**, since $\partial_u(F - G) = \partial_u(\Phi_i^1 - \Phi_i^2)$.

$\int_0^T \int_{\mathbb{R}^N} \left\| \left((F-G)(t,x,\cdot)\right) - \left(\mathrm{div}_x(f-g)(t,x,\cdot)\right) \right\|_{\mathbf{L^\infty}([-U,U];\mathbb{R})} \mathrm{d}x\, \mathrm{d}t < +\infty$ holds, due to the inequality $\left| \left((F-G)(t,x,U)\right) - \left(\mathrm{div}_x(f-g)(t,x,U)\right) \right| \leq 4\lambda(t,x,U)$ and (3) applies.

We use below A as a dummy variable for the fourth argument in $\varphi^1, \varphi^2, \Phi^1$ and Φ^2.

Step 2: Dependence on φ. Assume that $\Phi^1 = \Phi^2$ and $\vartheta^1 = \vartheta^2$ in (8). Then, with reference to the notation in [10, Theorem 2.2] and using also [9, theorem 1], we have

$$\begin{array}{rcl} \kappa_0^* & \leq & C\, C_U\,(1 + RT) \qquad \text{(as in [3, Formula (5.6)])} \\ \kappa^* & \leq & \|\partial_u F\|_{\mathbf{L^\infty}(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} + \left\|\partial_u\, \mathrm{div}_x(f-g)\right\|_{\mathbf{L^\infty}(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} \\ & \leq & \left\|\Phi^1\right\|_{\mathbf{W^{1,\infty}}(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})} + \left\|\partial_u\, \mathrm{div}_x(\varphi^1 - \varphi^2)\right\|_{\mathbf{L^\infty}(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})} \end{array}$$

$$
\begin{aligned}
&\quad + \left\| \partial_u \operatorname{grad}_A(\varphi^1 - \varphi^2) \right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^m)} \|\operatorname{div}_x \vartheta\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^{m\times n})} \|w\|_{\mathbf{L}^1(I\times\mathbb{R}^N;\mathbb{R}^n)} \\
&\leq C\, C_U\,(1+R\,T) \\
\frac{e^{\kappa_0^* t} - e^{\kappa^* t}}{\kappa_0^* - \kappa^*} &\leq t\, e^{\max\{\kappa_0^*, \kappa^*\} t} \qquad \text{(by [10, Remark 2.8] and [3, Formula (5.16)])} \\
&\leq t\, e^{C C_U (1+RT) t}
\end{aligned}
$$

and clearly $\left\|\partial_u (f-g)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} = \left\|\partial_u(\varphi^1-\varphi^2)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})}$. Moreover,

$$
\int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{grad}_x (F - \operatorname{div}_x f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U;\mathbb{R}^N)} \mathrm{d}x\,\mathrm{d}t \leq C_U\, \Lambda(T,U)\,(1+R\,T+R^2\,T^2)
$$

the latter expression above is computed in [3, Formula (5.5)]. We also need to estimate

$$
\begin{aligned}
& \int_0^T \int_{\mathbb{R}^N} \left\| \big((F-G) - \operatorname{div}_x (f-g)\big)(t,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U;\mathbb{R})} \mathrm{d}x\,\mathrm{d}t \\
= & \int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{div}_x (f-g)(t,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U;\mathbb{R})} \mathrm{d}x\,\mathrm{d}t \\
\leq & \int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{div}_x (\varphi^1-\varphi^2)\big(t,x,\cdot,(\vartheta * w)(t,x)\big)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U;\mathbb{R})} \mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{grad}_A (\varphi^1-\varphi^2)\big(t,x,\cdot,(\vartheta * w)(t,x)\big)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U;\mathbb{R}^{m\times n})} \left\|\operatorname{div}_x (\vartheta * w)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^m)} \mathrm{d}x\,\mathrm{d}t \\
\leq & \int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{div}_x (\varphi^1-\varphi^2)(t,x,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})} \mathrm{d}x\,\mathrm{d}t \\
& + C_U \int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{grad}_A (\varphi^1-\varphi^2)(t,x,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^{m\times n})} \mathrm{d}x\,\mathrm{d}t \\
\leq & \; C_U \left\|\operatorname{grad}_{xA} (\varphi^1-\varphi^2)\right\|_{\mathbf{L}^1(I\times\mathbb{R}^N;\mathbf{L}^\infty(\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^{N+m\times n}))}.
\end{aligned}
$$

Inserting the estimates above in the one provided by [10, Theorem 2.5], we have:

$$
\begin{aligned}
& \left\|u^1(t) - u^2(t)\right\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}^n)} \\
\leq & \Big(1 + C C_U \Lambda(T,U)(1+RT+R^2T^2)\Big)\, T\, \mathrm{TV}(\bar u) e^{C C_U (1+RT)T} \left\|\partial_u(\varphi^1-\varphi^2)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})} \\
& + C_U\, e^{C\, C_U (1+R\,T)T} \left\|\operatorname{grad}_{xA}(\varphi^1-\varphi^2)\right\|_{\mathbf{L}^1(I\times\mathbb{R}^N;\mathbf{L}^\infty(\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^{N+m\times n}))}.
\end{aligned}
\tag{9}
$$

Step 3: Dependence on Φ. Assume $\varphi^1 = \varphi^2$ and $\vartheta^1 = \vartheta^2$, so that $f = g$ in (8). Then, again with reference to the notation in [10, Theorem 2.5], we have $\kappa^* = \|\partial_u F\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} \leq C_U$ so that

$$
\left\|u^1(t) - u^2(t)\right\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \leq C_U\, T \left\|\Phi^1 - \Phi^2\right\|_{\mathbf{L}^1(I\times\mathbb{R}^N;\mathbf{L}^\infty(\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}))}. \tag{10}
$$

Step 4: Dependence on ϑ. We are left with the case $\varphi^1 = \varphi^2 = \varphi$ and $\Phi^1 = \Phi^2 = \Phi$. We fix for simplicity the notations $\varphi^{\vartheta^1} = \varphi^1\left(t,x,u,\vartheta^1 * w(t,x)\right)$, $\varphi^{\vartheta^2} = \varphi^1\left(t,x,u,\vartheta^2 * w(t,x)\right)$ and $\Phi^{\vartheta^1} = \Phi^1\left(t,x,u,\vartheta^1 * w(t,x)\right)$, $\Phi^{\vartheta^2} = \Phi^1\left(t,x,u,\vartheta^2 * w(t,x)\right)$. Then, always with reference to [10],

$$
\begin{aligned}
\kappa^* &\leq \|\partial_u F\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} + \left\|\partial_u \operatorname{div}_x (f-g)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} \\
&\leq \left\|\Phi^1\right\|_{\mathbf{W}^{1,\infty}(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})} + \left\|\partial_u \operatorname{div}_x \left(\varphi^{\vartheta^1} - \varphi^{\vartheta^2}\right)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\partial_u \operatorname{grad}_A \varphi^{\vartheta^1}\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^{m\times n})} \left\|\operatorname{div}_x(\vartheta^1 * w)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^m)}\\
&+\left\|\partial_u \operatorname{grad}_A \varphi^{\vartheta^2}\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R}^{m\times n})} \left\|\operatorname{div}_x(\vartheta^2 * w)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^m)}\\
\leq\ & C\, C_U\,(1+RT)\,,
\end{aligned}
$$

and $\kappa_0^* \leq C\, C_U\,(1+RT)$. Similarly to the previous case,

$$
\left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} \leq R\,\|\varphi\|_{\mathbf{W}^{\mathbf{1},\infty}(I\times\mathbb{R}^n\times\mathcal{U}_U\times\mathcal{U}_U^m;\mathbb{R})} \left\|\vartheta^1-\vartheta^2\right\|_{\mathbf{L}^\mathbf{1}(I\times\mathbb{R}^N;\mathbb{R}^{m\times n})}.
$$

We also need to estimate

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N} \left\|\left((F-G)-\operatorname{div}_x(f-g)\right)\right\|_{\mathbf{L}^\infty(\mathcal{U}_U;\mathbb{R})} \mathrm{d}x\,\mathrm{d}t\\
\leq\ & C_U\, R\left(\|\Phi\|_{\mathbf{W}^{\mathbf{1},\infty}(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})} + \|\varphi\|_{\mathbf{W}^{\mathbf{2},\infty}(I\times\mathbb{R}^N\times\mathcal{U}_U;\mathbb{R})}\right) \left\|\vartheta^1-\vartheta^2\right\|_{\mathbf{L}^\mathbf{1}(I\times\mathbb{R}^N;\mathbb{R}^{m\times n})},
\end{aligned}
$$

so that, following the same procedure used in Step 2,

$$
\begin{aligned}
\left\|u^1(t)-u^2(t)\right\|_{\mathbf{L}^\mathbf{1}(\mathbb{R}^N;\mathbb{R})} \leq\ & \left(1+C\Lambda(T,U)(1+RT+R^2T^2)\right) C_U\, R\,T\ \mathrm{TV}(\bar u)e^{C C_U(1+RT)T}\\
&\times\left\|\vartheta^1-\vartheta^2\right\|_{\mathbf{L}^\mathbf{1}(I\times\mathbb{R}^N;\mathbb{R}^{m\times n})}\\
&+C_U\, e^{C\,C_U(1+R\,T)T}\, R\left\|\vartheta^1-\vartheta^2\right\|_{\mathbf{L}^\mathbf{1}(I\times\mathbb{R}^N;\mathbb{R}^{m\times n})}.
\end{aligned} \tag{11}
$$

Summing up the estimates (9), (10) and (11) we obtain the Lipschitz continuous dependence of $\mathcal{T}$ defined in (7) on the parameter $p \equiv (\varphi, \Phi, \vartheta)$. A straightforward argument allows to conclude that also the fixed point of $\mathcal{T}$ is Lipschitz continuous in p. □

3. Application to the Laser Beam

The cutting of metal plates by means of a laser beam can be described through the following equations, introduced in [3, Section 3]:

$$
\left\{
\begin{array}{l}
\partial_t h_m + \operatorname{div}_x(h_m\, V) = \mathcal{L}\\
\partial_t h_s = -\mathcal{L}\,.
\end{array}
\right.
\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2 \tag{12}
$$

In this $3D$ framework, the laser beam is parallel to the vertical z axis and its trajectory is prescribed by the map $x_L = x_L(t)$. Above, h_m is the height of the melted metal and h_s is the height of solid part, both measured along the z axis. The vector V is the projection of the melted material velocity on the horizontal (x,y)-plane. The source $\mathcal{L}$ describes the laser position and intensity: it describes the net rate at which the solid part turns into melted. In particular,

$$
\begin{array}{rcl@{\qquad}rcl}
V &=& \left(w(t,x)-\mathcal{T}_g(t,x)h_m\right)\dfrac{-\operatorname{grad}_x(\eta * h_s)}{\sqrt{1+\left\|\operatorname{grad}_x(\eta*h_s)\right\|^2}}\,, & w(t,x) &=& \mathcal{W}\left(\left\|x-x_L(t)\right\|\right),\\
 & & & \mathcal{T}_g(t,x) &=& \tau_g\left(\left\|x-x_L(t)\right\|\right),\\
\mathcal{L} &=& \dfrac{i(t,x)}{1+\left\|\operatorname{grad}_x\left(\eta*(h_s+h_m)\right)\right\|^2}\,, & i(t,x) &=& \mathcal{I}\left(\left\|x-x_L(t)\right\|\right),
\end{array} \tag{13}
$$

where $\mathcal{T}_g$ is related to the shear stress, see [3, Formula (3.7)]; $\mathcal{W} = \mathcal{W}(x)$ describes the effect on the horizontal (x,y)-plane of the vertical wind that pushes the melted material and is produced around the laser beam at $x = x_L(t)$; the laser intensity is $\mathcal{I} = \mathcal{I}(x)$, again centered at the moving laser position $x = x_L(t)$. The system in (12) fits into (1) as shown by the following proposition.

Proposition 3.1 ([3, Proposition 3.1]). *Model* (12)–(13) *fits into* (1) *setting* $u \equiv (h_m, h_s)$ *and*

$$
\begin{aligned}
N &= 2\\
n &= 2\\
m &= 4\\
u_1 &= h_m\\
u_2 &= h_s
\end{aligned}
\qquad
\vartheta(x) = \begin{bmatrix} \partial_{x_1}\eta(x) & 0\\ 0 & \partial_{x_1}\eta(x)\\ \partial_{x_2}\eta(x) & 0\\ 0 & \partial_{x_2}\eta(x) \end{bmatrix}
\qquad
\begin{aligned}
\varphi_1(t,x,u_1,A) &= \frac{-\left(w(t,x) - \mathcal{T}_g(t,x)u_1\right)u_1}{\sqrt{1+(A_3)^2+(A_4)^2}} \begin{bmatrix} A_3\\ A_4 \end{bmatrix}\\
\varphi_2(t,x,u_2,A) &= 0\\
\Phi_1(t,x,u,A) &= \frac{i(t,x)}{\sqrt{1+(A_1+A_3)^2+(A_2+A_4)^2}}\\
\Phi_2(t,x,u,A) &= -\frac{i(t,x)}{\sqrt{1+(A_1+A_3)^2+(A_2+A_4)^2}}\,,
\end{aligned}
$$

Moreover, if $x_L \in (\mathbf{C^2} \cap \mathbf{W^{2,\infty}})([0,\widehat{T}];\mathbb{R}^2)$*,* $\mathcal{W}, \mathcal{I}, \tau_g \in \mathbf{C^2_c}(\mathbb{R};\mathbb{R})$ *and* $\eta \in \mathbf{C^3_c}(\mathbb{R}^2;\mathbb{R})$ *for a positive* $\widehat{T}$*, then, assumptions* **(**$\boldsymbol{\varphi}$**)**, **(**$\boldsymbol{\Phi}$**)**, **(**$\boldsymbol{\vartheta}$**)** *and* (3) *hold.*

(The verification of (3) is immediate and hence omitted).

Theorem 2.3 ensures the stability of solutions to (12)–(13) with respect to variations in the functions defining the model, namely: $x_L, \tau_g, \mathcal{W}, \mathcal{I}$ and η. A numerical investigation of the role of $\dot{x}_L$ is presented in [4]. Here, on the basis of Theorem 2.3, we can specify how the solutions to (12)–(13) depend on w and $\mathcal{T}_g$. Indeed, the bound (4) ensures that, calling (h^ℓ_m, h^ℓ_s) for $\ell = 1,2$ solutions to (12)–(13) corresponding to functions $w^\ell, \mathcal{T}^\ell_g$

$$
\begin{aligned}
& \left\|(h^1_m - h^2_m)(t)\right\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} + \left\|(h^1_s - h^2_s)(t)\right\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})}\\
\leq\ & C^* \Bigg(\left\|w^1 - w^2\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^2;\mathbb{R})} + \left\|\mathcal{T}^1_g - \mathcal{T}^2_g\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^2;\mathbb{R})}\\
& + \left\|\mathrm{grad}_x(w^1 - w^2)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^2;\mathbb{R}^2)} + \left\|\mathrm{grad}_x(\mathcal{T}^1_g - \mathcal{T}^2_g)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^2;\mathbb{R}^2)} \Bigg)\,.
\end{aligned}
$$

Acknowledgment: The author thanks R.M. Colombo (University of Brescia) and M. Herty (RWTH) for having introduced her to the problem. The present work was supported by the PRIN 2012 project *Nonlinear Hyperbolic Partial Differential Equations, Dispersive and Transport Equations* and by the GNAMPA 2015 project *Balance Laws: Theory and Applications.*

References

[1] R. M. Colombo, M. Garavello, and M. Lécureux-Mercier. A class of nonlocal models for pedestrian traffic. *Math. Models Methods Appl. Sci.*, 22(4):1150023, 34, 2012.

[2] R. M. Colombo, G. Guerra, M. Herty, and F. Marcellini. A hyperbolic model for the laser cutting process. *Appl. Math. Model.*, 37(14-15):7810–7821, 2013.

[3] R. M. Colombo and F. Marcellini. Nonlocal systems of balance laws in several space dimensions with applications to laser technology. *J. Differential Equations*, 259(11):6749–6773, 2015.

[4] R. M. Colombo, F. Marcellini, and E. Rossi. Biological and industrial models motivating nonlocal conservation laws: a review of analytic and numerical results. *Networks and Hetergeneous Media*, (11):49–67, 2016.

[5] R. M. Colombo, M. Mercier, and M. D. Rosini. Stability and total variation estimates on general scalar balance laws. *Commun. Math. Sci.*, 7(1):37–65, 2009.

[6] M. Di Francesco, P. A. Markowich, J.-F. Pietschmann, and M.-T. Wolfram. On the Hughes' model for pedestrian flow: the one-dimensional case. *J. Differential Equations*, 250(3):1334–1362, 2011.

[7] S. Göttlich, S. Hoher, P. Schindler, V. Schleper, and A. Verl. Modeling, simulation and validation of material flow on conveyor belts. *Applied Mathematical Modelling*, 38(13):3295–3313, 2014.

[8] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research Part B*, 36:507–535, 2002.

[9] S. N. Kružhkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[10] M. Lécureux-Mercier. Improved stability estimates on general scalar balance laws. *ArXiv e-prints*, Oct. 2010.