Norwegian University of Science and Technology
Department of Mathematical Sciences

MA0002 Mathematical Methods B
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Solutions to exercise set 1

$\boxed{1}$ Evaluate the indefinite integrals by making the given substitutions.

1. $\int 2x^3\sqrt{x^4+2}\,dx$ with $u = x^4+2$
   With $u = x^4+2$ we get $\frac{du}{dx} = 4x^3$and $du = 4x^3dx$. Then when we compute the integral we get
   $$\int 2x^3\sqrt{x^4+2}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\frac{2}{3}u^{3/2} + C = \frac{1}{3}(x^4+2)^{3/2} + C.$$
2. $\int x^2\cos(x^3-1)\,dx$, with $u = x^3-1$
   With $u = x^3-1$ we get $\frac{du}{dx} = 3x^2$ and $du = 3x^2dx$. Then when we compute the integral we get
   $$\int x^2\cos(x^3-1)\,dx = \int \frac{1}{3}\cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(x^3-1) + C.$$
3. $\int 3e^{2-x}\,dx$, with $u = 2-x$
   With $u = 2-x$ we get $\frac{du}{dx} = -1$ and $du = -dx$. Then when we compute the integral we get
   $$\int 3e^{2-x}\,dx = \int -3e^u\,du = -3e^u + C = -3e^{2-x} + C.$$
4. $\int \frac{x}{4-x}\,dx$, with $u = 4-x$
   With $u = 4-x$ we get $\frac{du}{dx} = -1$ and $du = -dx$. Then when we compute the integral we get
   $$\int \frac{x}{4-x}\,dx = \int -\frac{4-u}{u}\,du = -\int \frac{4}{u}\,du + \int du = -4\ln u + C_1 + u + C_2$$
   $$= -4\ln(4-x) + C_1 + 4 - x + C_2 = -x - 4\ln(4-x) + C.$$
   In the last equality we have joined the constants C_1, C_2 and 4 to one constant C.

$\boxed{2}$ Use substitution to evaluate the indefinite integrals.

1. $\int(4-x)^{2/3}\,dx$ We chose $u = 4-x$, then we get $\frac{du}{dx} = -1$ and $du = -dx$. We then calculate the integral
   $$\int(4-x)^{2/3}\,dx = -\int u^{2/3}\,du = -\frac{3}{5}u^{5/3} + C = -\frac{3}{5}(4-x)^{5/3} + C.$$

2. $\int (x^2 - 2x)(x^3 - 3x^2 + 3)^{2/3}\, dx$
We chose $u = x^3 - 3x^2 + 3$, then we get $\frac{du}{dx} = 3x^2 + 6x$ and $du = 3(x^2 - 2x)dx$. We then calculate the integral

$$\int (x^2-2x)(x^3-3x^2+3)^{2/3}\, dx = \int \frac{1}{3} u^{2/3}\, du = \frac{3}{15} u^{5/3} + C = \frac{1}{5}(x^3-3x^2+3)^{5/3} + C.$$

3. $\int \sin x\ e^{\cos x}\, dx$
We chose $u = \cos x$, then we get $\frac{du}{dx} = -\sin x$ and $du = -\sin(x)dx$. We then calculate the integral

$$\int \sin x\ e^{\cos x}\, dx = -\int e^u\, du = -e^u + C = -e^{\cos x} + C.$$

$\boxed{3}$ Let $g(x)$ be a continuous function whose derivative $g'(x)$ is also continuous. Use substitution to evaluate the indefinite integral

$$\int g'(x) e^{g(x)} dx.$$

We chose $u = g(x)$ so we get $\frac{du}{dx} = g'(x)$ and $du = g'(x)dx$. We then calculate the integral

$$\int g'(x) e^{g(x)}\, dx = \int e^u\, du = e^u + C = e^{g(x)} + C.$$

$\boxed{4}$ Use substitution to evaluate the definite integral

$$\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\, dx.$$

We chose $u = \cos x$ such that $\frac{du}{dx} = -\sin x$ and $du = -\sin(x)dx$
We then have two ways to continue:

1. We first compute the indefinite integral and get

$$\int \frac{\sin x}{\cos^2 x}\, dx = -\int \frac{1}{u^2}\, du = \frac{1}{u} + C = \frac{1}{\cos x} + C$$

and the compute

$$\left[\frac{1}{\cos x}\right]_0^{\pi/3} = \frac{1}{\cos(\pi/3)} - \frac{1}{\cos 0} = 2 - 1 = 1.$$

2. Or we can first compute the substitution values on thebounds of the integral $u(0) = 1$ and $u(\pi/3) = 1/2$. Then we compute the integral

$$\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\, dx = -\int_1^{1/2} \frac{1}{u^2}\, du = \int_{1/2}^{1} \frac{1}{u^2}\, du = \left[-\frac{1}{u}\right]_{1/2}^{1} = -1 - (-2) = 1.$$

We will use the second method onward.

$\boxed{5}$ Use integration by parts to evaluate the indefinite integrals.

1. $\int 3x \sin x \, dx$
   We chose $u(x) = 3x$ and $v'(x) = \sin x$ such that $u'(x) = 3$ and $v(x) = -\cos x$. Then we compute the integral
$$\int 3x \sin x \, dx = -3x \cos x - \int -3\cos(x) \, dx = -3x\cos x + 3\sin x + C.$$

2. $\int 2x^2 e^{-x} \, dx$
   We first chose $u_1(x) = 2x^2$ and $v_1'(x) = e^{-x}$ such that $u_1'(x) = 4x$ and $v_1(x) = -e^{-x}$ and by computation we get
$$\int 2x^2 e^{-x} \, dx = -2x^2 e^{-x} - \int (-4xe^{-x}) \, dx = -2x^2 e^{-x} + \int 4xe^{-x} \, dx.$$
   We then use integration by parts one more time with $u_2(x) = 4x$ and $v_2'(x) = e^{-x}$ such that $u_2'(x) = 4$ and $v_2(x) = -e^{-x}$. Then we we compute the integral to get
$$-2x^2e^{-x} + \int 4xe^{-x} \, dx = -2x^2e^{-x} - 4xe^{-x} + \int 4e^{-x} \, dx = -2e^{-x}(x^2 + 2x + 2).$$

$\boxed{6}$ Use integration by parts to evaluate the definite integrals.

1. $\int_0^{\pi/4} 2x \sin x \, dx$
   We start by choosing $u(x) = 2x$ and $v'(x) = \sin x$ such that $u'(x) = 2$ and $v(x) = -\cos x$. Then we get that
$$\int_0^{\pi/4} 2x \cos x \, dx = [-2x\cos x]_0^{\pi/4} - \int_0^{\pi/4} -2\cos(x) \, dx = [-2x\cos x + 2\sin(x)]_0^{\pi/4}$$
$$= -2\frac{\pi}{4}\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} - 0 - 0 = \sqrt{2}\left(1 - \frac{\pi}{4}\right) \approx 0.3034.$$

2. $\int_1^e \ln x^3 \, dx$
   We first notice that $\ln x^3 = 3\ln x$, then we chose $u(x) = \ln x$ and $v'(x) = 3$ such that $u'(x) = \frac{1}{x}$ and $v(x) = 3x$. Then we compute the intagral
$$\int_1^e \ln x^3 \, dx = \int_1^e 3\ln x \, dx = [3x\ln x]_1^e - \int_1^e 3 \, dx = [3x\ln x - 3x]_1^e$$
$$=3e - 3e - (0 - 3) = 3.$$

$\boxed{7}$ Use an appropriate substitution followed by integration by parts to evaluate
$$\int x^5 e^{-x^3/3} \, dx.$$

We start by using the substitution $y = -\frac{x^3}{3}$ such that $\frac{dy}{dx} = -x^2$ and $dy = -x^2 dx$. Then by computation we get
$$\int x^4 e^{-x^3/3} \, dx = \int -x^2 e^{-x^3/3} \, dy = \int 3ye^y \, dy$$

Then we use integration by parts to get $u(y) = 3y$ and $v'(y) = e^y$ such that $u'(y) = 3$ and $v(y) = e^y$. Then we get the integral to be

$$\int 3ye^y \, dy = 3ye^y - \int 3e^y \, dy = 3ye^y - 3e^y = -e^{-x^3/3}(x^3 + 3).$$