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Solutions to exercise set 10

$\boxed{1}$ Find the equation for the line that goes through the point $(1, 2)$ and is perpendicular to the vector $(-2, 1)^T$.

Solution. We know that a line through the point (x_0, y_0) orthogonal to the vector $(a, b)^T$ have the equation $a(x - x_0) + b(y - y_0) = 0$. (You might find page 496 in the book useful here). An equation for the line is $-2(x - 1) + (y - 2) = 0$, which we simplify to $y = 2x$.

$\boxed{2}$ Find the equation for the plane that goes through the point $(2, 0, -1)$ and is perpendicular to the vector $(1, 1, -2)^T$.

Solution The point $\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the plane if and only if $\mathbf{n}^T \cdot (\mathbf{r} - \mathbf{r}_0) = 0$, so the equation for the plane is given by

$$\begin{aligned} 0 &= \mathbf{n}^T \cdot (\mathbf{r} - \mathbf{r}_0) \\ &= (1, 1, -2) \begin{bmatrix} x - 2 \\ y \\ z + 1 \end{bmatrix} \\ &= x - 2 + y - 2(z + 1) \\ &= x + y - 2z - 4. \end{aligned}$$

Bonus We can parameterise the plane by letting $z = t$ and $y = s$ be free variables. Then we must have $x = 2t - s + 4$ so the plane is given by the range of the function $\mathbf{x} \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}(s, t) = \begin{pmatrix} 2t - s + 4 \\ s \\ t \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

$\boxed{3}$ Find a parametric equation for the line that goes through the points $(0, 2)$ and $(2, -1)$. Then find the equation for the line on standard form.

Solution. The direction of the line is $\mathbf{u} = \mathbf{x}_1 - \mathbf{x}_0 = (2, -3)^T$ and a parameterization of the line is

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_0 + t\mathbf{u} \\ &= \begin{pmatrix}0\\2\end{pmatrix} + t\begin{pmatrix}2\\-3\end{pmatrix}.\end{aligned}$$

or component-wise we have

$$\begin{aligned}x(t) &= 2t \\ y(t) &= 2 - 3t.\end{aligned}$$

Now by eliminating t we have the following connection between x and y:

$$3x + 2y = 3(2t) + 2(2 - 3t) = 4.$$

The line is given by: $3x + 2y = 4$.

$\boxed{4}$ We have

(1) A plane that passes through the point $(2, 0, -1)$ and has normal vector $(-1, 1, 3)^T$.
(2) A line passing through the points $(1, 0, -2)$ and $(-1, -1, 1)$.

Where does the plane and the line meet?

Solution. We know the plane is given by the equation

$$-(x - 2) + y + 3(z + 1) = 0,$$

that is

$$-x + y + 3z = -5.$$

Further we parameterize the line going through the points $(1, 0, -2)$ and $(-1, -1, 1)$: Letting $P_0 = (-1, -1, 1)$ and $\mathbf{u} = (-1, -1, 1)^T - (1, 0, -2)^T = (-2, -1, 3)^T$. Thus a parameterization is given by

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix} + \begin{pmatrix}-2\\-1\\3\end{pmatrix} t.$$

That is we may parameterize the line by $x = -1 - 2t$, $y = -1 - t$, $z = 1 + 3t$. Note that this is not unique, we could for instance have chosen $P_0 = (1, 0, -2)$ which would give another parameterization for the line. We know that all the points one the line satisfy $x = -1 - 2t$, $y = -1 - t$, $z = 1 + 3t$ for some $t \in \mathbb{R}$ and that all the point in the plane satisfy the equation $-x + y + 3z = -5$. TO find the point where the plane and the line meets we thus solve the equation

$$\begin{aligned}-(-1 - 2t) + (-1 - t) + 3(1 + 3t) &= -5 \\ 1 + 2t - 1 - t + 3 + 9t &= -5 \\ 10t &= -5 \\ t &= -\frac{4}{5}.\end{aligned}$$

Now inserting $t = -4/5$ into the parameterization of the line we see that the plane and the line meets in the point

$$(x, y, z) = \big(-1 - 2 * (-4/5), -1 - (-4/5), 1 + 3 * (-4/5)\big) = (3/5, -1/5, -7/5).$$

$\boxed{5}$ Evaluate the function

$$f(x_1, x_2) = \frac{2x_1 - x_2}{x_1^2 + x_2^2}$$

in the point $(1, 4)$.

Solution We see that

$$f(1,4) = \frac{2-4}{1^2+4^2} = -\frac{2}{17}$$

$\boxed{6}$ Find the largest possible domain and corresponding range of the function

$$f(x_1, x_2) = \sqrt{9 - x_1^2 - x_2^2}\,.$$

Then find the equation for the level curves $f(x_1, x_2) = c$ for the possible values of c.

Solution. The function is only defined when $9 - x_1^2 - x_2^2 \geq 0$. Thus the domain is

$$D = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 9,$$

we observe that this is a disc with center $(0, 0)$ and radius 3. Further we see that the range is $V = [0, 3]$ since $f(x_1, x_2)$ is a continuous function reaching its maximum when $(x_1, x_2) = 0$ and the obtaining the value $f(0, 0) = 3$, further the function reaches its minimum when $x_1^2 + x_2^2 = 9$ taking the value 0. The level curves of f is all points (x_1, x_2) where the function takes the constant fixed value c in our case that means

$$f(x_1, x_2) = 9 - x_1^2 - x_2^2 = c,$$

which we rewrite into

$$x^2 + y^2 = 9 - c.$$

Thus the level curves are all curves on the form $x_1^2 + x_2^2 = 9 - c$ for $0 \leq c \leq 9$. Observe that this is circles with center 0 and radius $\sqrt{9 - c}$.