Solutions to exercise set 13

$\boxed{1}$ Find the global maxima and minima for the function

$$f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 + 1$$

on the disk

$$D = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 4.$$

Solution.
The function f is continous and defined on a closed and bounded set D, we thus know that f will have global maxima and minima on D. We start by finding any extremas in the inner of D by setting the gradient of f equal to zero. By finding the partial derivatives we see that

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 - 2 \\ 2x_2 \end{pmatrix}.$$

Setting the gradient equal to zero we obtain the system of equations

$$\begin{aligned} 2x_1 - 2 &= 0 \\ 2x_2 &= 0, \end{aligned}$$

which have the solution $x_1 = 1$ og $x_2 = 0$. We calculate that $f(1, 0) = 0$. We now find the value of the function f on the boundary of D, that is the circle $S = \{(x, y) : x_1^2 + x_2^2 = 2^2\}$. We see that whenever $x_1^2 + x_2^2 = 4$ we have $f(x_1, x_2) = 4 - 2x_1 + 1 = 5 - 2x_1$, where $-2 \leq x_1 \leq 2$. From this we obtain that $f(x_1, x_2)$ is maximized on the boundary when $x_1 = -2$ and minimal when $x_1 = 2$. In both cases we have $x_2 = 0$. We calculate $f(2, 0) = 1$ and $f(-2, 0) = 9$. Thus it follows that f have a global maximum for $f(-2, 0) = 9$ and global minimum for $f(1, 0) = 0$. An alternative solution is to find the extremas on the boundary by parameterizing the circle with $x_1 = 2\cos\theta$ and $x_2 = 2\sin\theta$ for $0 \leq \theta < 2\pi$.

$\boxed{2}$ A rectangular prism consists of six rectangles. Find the largest possible volume of a rectangular prism when the surface area is 200m^2.

Løsning. We let the length, the width and the hight be denoted by l, b and h respectively. The surface area is

$$O = 2 \cdot (lb + lh + bh) = 200,$$

thus we can express the length as

$$l = \frac{100 - bh}{b + h}.$$

Thus the volume V is given by

$$V(b,h) = l \cdot b \cdot h = bh\frac{100-bh}{b+h}.$$

We need to the global maximum of the function $V(b,h)$ on the set

$$D = \{(b,h) \in \mathbb{R}^2 : b \geq 0, h \geq 0, bh \leq 100\}.$$

The inequality $bh \leq 100$ is due to the fact that $l \geq 0$ and that $l = \frac{100-bh}{b+h}$. We see that $V(b,h)$ can not have any global miximum on the boundary of D since whenever $bh = 100$ it follows that $V(b,h) = 0$ and we know $V \geq 0$. Thus, to find the maximum of V we need to find the extremal points of the function V on the interior of D. We start by finding the gradient of V. By applying the product rule and the quotient rule we see that

$$\nabla V(b,h) = \begin{pmatrix} \frac{-h^2(b^2+2bh-100}{(b+h)^2} \\ \frac{-b^2(h^2+2bh-100}{(b+h)^2} \end{pmatrix}$$

We are interested in whenever $\nabla V(b,h) = (0,0)^T$ but we can omit the cases where $b = 0$ or $h = 0$ since the volume then will be 0. Thus we need to solve the following equations

$$\begin{aligned} b^2 + 2bh &= 100 \\ h^2 + 2bh &= 100 \end{aligned}$$

In particular this means that

$$b^2 + 2bh = h^2 + 2bh$$

which gives $b = h$ since we know that $b \geq 0$ and $h \geq 0$. Thus we can now solve the equation

$$b^2 + 2b^2 = 100$$

which gives $h = b = 10/\sqrt{3}$. Further we calculate that when $h = b = 10/\sqrt{3}$ then $l = 10/\sqrt{3}$. This means that the volume is as large as possible when $l = b = h = \frac{10\sqrt{3}}{3}$. The volume is then $1000\sqrt{3}/9$.

$\boxed{3}$ Use Lagrange multipliers to find the maxima and minima of the functions under the given constraints.

(a) $f(x_1, x_2) = x_1 \cdot x_2$ when $x^2 + y^2 = 1$

(b) $f(x_1, x_2) = x_1^2 - x_2^2$ when $2x_1 + x_2 = 1$

Solution (a)

We let $g(x_1, x_2) = x_1^2 + x_2^2 - 1$. We see that $\nabla f(x_1, x_2) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ and $\nabla g(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$. We need to find (x, y) such that

$$\nabla f(x_1, x_2) = \lambda \nabla g(x_1, x_2).$$

This we can rewrite into the system

$$\begin{aligned} x_2 &= \lambda 2x_1 \\ x_1 &= \lambda 2x_2, \end{aligned}$$

which is equivalent to

$$\lambda = x_2/(2x_1)$$
$$\lambda = x_1/(2x_2).$$

By solving for λ we obtain $x_1^2 = x_2^2$. Further we use that $g(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$. This means we have four solutions; $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. We see that $f(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}) = -\frac{1}{2}$ and $f(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}) = \frac{1}{2}$. That is $(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2})$ are maximum points and $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$ are minimum points since we are working on a closed curve $g(x, y)$.

Løsning (b)

Let $g(x_1, x_2) = 2x_1 + x_2 - 1$. We see that $\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 \\ x_2 \end{pmatrix}$ and $\nabla g(x_1, x_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We need to find (x, y) such that

$$\nabla f(x_1, x_2) = \lambda \nabla g(x_1, x_2).$$

We rewrite this into the following system of equations

$$2x_1 = \lambda$$
$$-2x_2 = \lambda$$

this gives $x_1 = -2x_2$. Using that $g(x_1, x_2) = 0$ we see that the only solution is $(2/3, -1/3)$. Thus $(2/3, -1/3)$ is a global maximum.

$\boxed{4}$ Solve the following initial-value problem

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix},$$

where $y_1(0) = 2$ and $y_2(0) = -1$.

Solution.

We start by finding the eigenvalues and the corresponding eigenvalues for the matrix

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}.$$

We find the eigenvalues by solving $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0,$$

has the solution $\lambda_1 = 1$ og $\lambda_2 = 2$. We now find a corresponding eigenvector for each of the eigenvalues by solving $(A - \lambda_1 I)\mathbf{u} = (0, 0)^T$:

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

this gives the equations $y_1 = y_1$ and $y_2 = 0$. Thus a corresponding eigenvector is $\mathbf{u} = (1,0)^T$. Similarly we find an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$ by solving $(A - \lambda_1 I)\mathbf{v} = (0,0)^T$:

$$A - \lambda_2 I = \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which have the solution $y_1 = -1$ and $y_1 = 3y_2$. Thus, $\mathbf{v} = (3,1)^T$ is a corresponding eigenvector. Thus the solutions of the differential equation is

$$\mathbf{x(t)} = c_1 e^{\lambda_1 t}\mathbf{u} + c_2 e^{\lambda_2 t}\mathbf{v} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Finally, we use the initial conditions to determine c_1 and c_2. We see that

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

this implies $c_2 = -1$ and $c_1 = 5$. Thus the solution is

$$\mathbf{x(t)} = 5e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$\boxed{5}$ Solve the following initial-value problem

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix},$$

where $y_1(0) = -1$ and $y_2(0) = -2$.

Solution
We start by finding the eigenvalues and the corresponding eigenvalues for the matrix $A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$ We find the eigenvalues by solving $\det(A - \lambda I) = 0$. We see that

$$\det(A - \lambda I) = (-1-\lambda)(-2-\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

has the solutions $\lambda_1 = -1$ and $\lambda_2 = -2$. We now find a corresponding eigenvector for each of the eigenvalues by solving $(A - \lambda_1 I)\mathbf{u} = (0,0)^T$, that is

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

That is $y_1 - y_2 = 0$, which again means $y_1 = y_2$. Thus an eigenvector is $\mathbf{u} = (1,1)^T$.

We now find an eigenvalue for the eigenvector $\lambda_2 = -2$ by solving $(A - \lambda_2 I)\mathbf{v} = (0,0)^T$, that is

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus we obtain the equation $x_1 = 0$. Thus $\mathbf{v} = (0,1)^T$ is an eigenvector. Thus the solution of the differential equation is

$$\mathbf{y}(\mathbf{t}) = c_1 e^{\lambda_1 t}\mathbf{u} + c_2 e^{\lambda_2 t}\mathbf{v} = c_1 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$$

We now use the initial conditions to determine c_1 and c_2. We see that

$$\mathbf{y}(0) = c_1(1,1)^T + c_2(0,1)^T = (c_1, c_1 + c_2)^T = (-1,-2)^T,$$

implying that $c_1 = -1$ and $c_2 = -2 - c_1 = -1$. Thus the solution of the initial value problem is

$$\mathbf{y}(t) = -e^{-t}\begin{pmatrix}1\\1\end{pmatrix} - e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$$