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Solutions to exercise set 14

$\boxed{1}$ In this exercise we are working with the matrices

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -7 & 2 \\ 1 & 4 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Calculate AB

(b) Is $AB = BA$? Justify your answer.

(c) Determine A^{-1}.

(d) Solve the equation $A\mathbf{x} = (1, 1, 1)^T$.

Solution (a)

$$AB = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 2 \\ 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 2 & -7 & 2 \\ 1 & 4 & 3 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 19 & 5 \\ -2 & -23 & -10 \end{pmatrix}$$

Solution (b)
We have $AB \neq BA$. We can show this by calculating the element in the first row and the first column of the matrix BA, which we see is $9 \neq 1$. Since AB and BA has at least one element that are different they cannot be the same matrix.

Solution(c). We use Gaussian elimination on the matrix (AI_3):

$$
\begin{aligned}
\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 1 & -4 & 0 & 0 & 0 & 1 \end{pmatrix}
&\sim \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 1 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix} && 1 \\
&\sim \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix} && \cdot(-1) \\
&\sim \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix} && 1 \\
&\sim \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 4 & 1 & 1 & 1 \end{pmatrix} && \cdot(1/4) \\
&\sim \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/4 \end{pmatrix} && 2 \\
&\sim \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/4 \end{pmatrix} && 4 \\
&\sim \begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/4 \end{pmatrix}
\end{aligned}
$$

Thus we have

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1/2 & -1/2 & -1/2 \\ 1/4 & 1/4 & 1/4 \end{pmatrix}$$

Solution (d) We solve the equation $A\mathbf{x} = (1,1,1)^T$ by using the inverse of A. We have

$$\mathbf{x} = A^{-1}(1,1,1)^T = \begin{pmatrix} 2 & -2 & -1 \\ 1/2 & -1/2 & -1/2 \\ 1/4 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \\ 3/4 \end{pmatrix}$$

Note: you may also solve the system of equations $A\mathbf{x} = (1,1,1)^T$ by using Gaussian elimination.

$\boxed{2}$ Let $f(x_1, x_2) = \sin(x_1^3 - 3x_2) + e^{-x_1} + x_2^3$. Find the gradient of f.

Solution. We start by findning the partial derivatives

$$\frac{\partial f}{\partial x_1} = 3x_1^2 \cos\left(x_1^3 - 3x_2\right) - e^{-x_1},$$

and

$$\frac{\partial f}{\partial x_2} = -3\cos\left(x_1^3 - 3x_2\right) + 3x_2^2.$$

Thus the gradient is given by

$$\nabla f(x_1, x_2) = \begin{pmatrix} 3x_1^2 \cos\left(x_1^3 - 3x_2\right) - e^{-x_1} \\ -3\cos\left(x_1^3 - 3x_2\right) + 3x_2^2 \end{pmatrix}$$

$\boxed{3}$ Let

$$L = \begin{bmatrix} 0 & 5 \\ 0.9 & 0 \end{bmatrix}$$

be the Leslie matrix for a population consisting of two age groups.

a) Find both the eigenvalues for the matrix L.

b) Give a biological interpretation of the largest eigenvalue.

c) Find the stable age distribution.

Solution(a).
We find the eigenvalues by solving the characteristic polynomial $\det(A - \lambda I) = 0$, that is

$$\det(A - \lambda I) = (-\lambda) \cdot (-\lambda) - 5 \cdot 0.9 = \lambda^2 - 4.5 = 0,$$

which has the solutions $\lambda_1 = (3\sqrt{2})/2$ and $\lambda_2 = -(3\sqrt{2})/2$.

Solution(b). The largest eigenvalue λ_1, describes the growth rate of the population.

Solution(c). To find the stable growth rate we must fine the eigenvector of the largest eigenvalue, that is an eigenvector belonging to λ_1. We do this by solving $(A - \lambda_1 I)\mathbf{x} = 0$:

$$\begin{aligned} A - \lambda_1 I | 0 &= \begin{pmatrix} -3\sqrt{2}/2 & 5 & 0 \\ 0.9 & -3\sqrt{2}/2 & 0 \end{pmatrix} \quad \cdot(-\sqrt{2}/3) \\ &\sim \begin{pmatrix} 1 & -5\sqrt{2}/3 & 0 \\ 0.9 & -3\sqrt{2}/2 & 0 \end{pmatrix} \quad -0.9 \\ &\sim \begin{pmatrix} 1 & -5\sqrt{2}/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Thus we obtain the equation $x_1 - 5\sqrt{2}/3x_2 = 0$, that is $x_1 = 5\sqrt{2}/3$. Thus an eigenvector is given by$x = (5\sqrt{2}/3, 1)^T$, this means that the stable age distribution is $\begin{pmatrix} 5\sqrt{2}/3 \\ 1 \end{pmatrix}$, this means that after a long time there will be $5\sqrt{2}/3 \approx 2.36$ zero-year-olds in the population for each one-year-old there is in the population.

$\boxed{4}$ Find the global maximum and minimum of $f(x_1, x_2) = 2x_1^2 + 3x_2^2 + x_1 - x_2$ if they exist.

We find the gradient of f by calculating the partial derivatives of f.

$$\nabla f(x_1, x_2) = \begin{pmatrix} 4x_1 + 1 \\ 6x_2 - 1 \end{pmatrix}$$

Since f is defined on all of $\mathbb{R}^2$ we know that all the critical points must satisfy the equation $\nabla f(x_1, x_2) = 0$. We see that

$$\nabla f(x_1, x_2) = 0$$

if and only if $4x_1 + 1 = 0$ and $6x_2 - 1 = 0$ that is $(x_1, x_2) = (-\frac{1}{4}, \frac{1}{6})$. Further we find the Hessian matrix of f f to determine which kind of critical pont $(-\frac{1}{4}, \frac{1}{6})$ is.

$$\text{Hess} f(x_1, x_2) = \begin{pmatrix} \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} \\ \\ \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$$

We see that both the determinante and the trace of the Hessian matrix is $4 > 0$ Thus it follows that $(-\frac{1}{4}, \frac{1}{6})$ is a global minimum and that f does not have any global maximum.

$\boxed{5}$ Solve the following initial-value problem.

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix},$$

where $y_1(0) = 1$ and $y_2(0) = 2$.

Solution.
We start by finding the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix}$ We find the eigenvalues by solving $\det(A - \lambda I) = 0$. We see that

$$\det(A - \lambda I) = (-3 - \lambda)(2 - \lambda) + 4 = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0$$

has the solution $\lambda_1 = -2$ and $\lambda_2 = 1$. We then find a corresponding eigenvector to each of the eigenvalues. We start by solving $(A - \lambda_1 I)\mathbf{u} = (0, 0)^T$, that is

$$\begin{pmatrix} -1 & 4 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus we obtain the equation $-y_1 + 4y_2 = 0$, that is $y_1 = 4y_2$. Thus an eigenvector for the eigenvalue λ_1 is $\mathbf{u} = (4, 1)^T$.

We now find a corresponding eigenvector for the eigenvalue $\lambda_2 = 1$ by solving $(A - \lambda_2 I)\mathbf{v} = (0, 0)^T$, that is

$$\begin{pmatrix} -4 & 4 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the equations $-4y_1 + 4y_2 = 0$ and $-y_1 + y_2 = 0$. That is $y_1 = y_2$. Thus an eigenvector is $\mathbf{v} = (1,1)^T$. Now we know that the solution of the differential equation is

$$\mathbf{y}(\mathbf{t}) = c_1 e^{\lambda_1 t}\mathbf{u} + c_2 e^{\lambda_2 t}\mathbf{v} = c_1 e^{-2t}\begin{pmatrix}4\\1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}1\\1\end{pmatrix}$$

Finally, we us the initial-value conditions to dertermine c_1 and c_2. We have

$$\mathbf{y}(0) = c_1(4,1)^T + c_2(1,1)^T = (4c_1 + c_2, c_1 + c_2)^T = (1,2)^T,$$

meaning that $4c_1 + c_2 = 1$ and $c_1 + c_2 = 2$, thus $c_1 = -1/3$ and $c_2 = 7/3$. Thus the solution is

$$\mathbf{y}(t) = -\frac{1}{3}e^{-2t}\begin{pmatrix}4\\1\end{pmatrix} + \frac{7}{3}e^{t}\begin{pmatrix}1\\1\end{pmatrix}$$