MA0002 Mathematical Methods B
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Solutions to exercise set 3

$\boxed{1}$ Solve the pure-time differential equation:

$$\frac{dx}{dt} = \sin(2\pi(t+1)), \text{ where } x(2) = 1.$$

We start by integrating with respect to t on both sides, and get

$$x(t) = \int \sin(2\pi(t+1))\,dt = -\frac{1}{2\pi}\cos(2\pi(t+1)) + C.$$

We compute C by evaluating at $t = 2$

$$1 = -\frac{1}{2\pi}\cos(2\pi(2+1)) + C = -\frac{1}{2\pi} + C.$$

then we get that $C = 1 + \frac{1}{2\pi}$, and

$$x(t) = -\frac{1}{2\pi}\cos(2\pi(t+1)) + 1 + \frac{1}{2\pi}.$$

$\boxed{2}$ Suppose that the amount of phosphorus in a lake at time t, denoted by $P(t)$, follows the equation

$$\frac{dP}{dt} = 2t + 3 \quad \text{with } P(0) = 0.$$

Find the amount of phosphorus at time $t = 5$.

We start by solving the differential equation by integrating on bouth sides

$$P(t) = \int (2t+3)\,dt = t^2 + 3t + C$$

Since $P(0) = 0$ we get that $C = 0$. We then compute the amount og phosphorus at time $t = 5$

$$P(5) = 25 + 15 = 40.$$

$\boxed{3}$ Solve the autonomous differential equations:

$$\frac{dx}{dt} = 2 - x, \text{ where } x(-2) = -1.$$

We first start by rearrange on the equation to get

$$\frac{dx}{2-x} = dt$$

Then we can integrate on bouth sides to get

$$\begin{aligned}\int \frac{dx}{2-x} &= \int dt \\ -\ln(2-x) &= t + C_0 \\ \ln(2-x) &= -t + C_0\end{aligned}$$

Then we take bouth sides as the power of e and get

$$\begin{aligned}2-x &= e^{-t}e^{C_0} \\ x(t) &= 2 - e^{C_0}e^{-t}\end{aligned}$$

We the put all the constants in to one

$$x(t) = 2 - Ce^{-t}$$

Then we compute C by evaluating at $t = -2$

$$\begin{aligned}-1 &= 2 - Ce^{2} \\ 3 &= Ce^{2} \\ C &= 3e^{-2}\end{aligned}$$

So the solution we get is

$$x(t) = 2 - 3e^{-t-2}.$$

$\boxed{4}$ Assume that $W(t)$ denotes the amount of radioactive material in a substance at time t. Radioactive decay is then described by the differential equation

$$\frac{dW}{dt} = -\lambda W(t), \qquad \text{with } W(0) = W_0.$$

(a) Find $W(t)$ by solving the differential equation.

We first start by rearrange on the equation to get

$$\frac{dW}{W} = -\lambda dt$$

Then we integrate on both sides to get

$$\begin{aligned}\int \frac{dW}{W} &= -\int \lambda dt \\ \ln W &= -\lambda t + C_0 \\ W(t) = e^{-\lambda t + C_0} &= Ce^{-\lambda t}\end{aligned}$$

We then sett in for $t = 0$ and get

$$W_0 = Ce^{-\lambda \cdot 0} = C$$

Then we have solved the differential equation

$$W(t) = W_0 e^{-\lambda t}$$

(b) Assume that $W(0) = 123$gr, $W(5) = 20$gr and that time is measured in minutes. Find the decay constant λ and determine the half-life of the radioactive substance.

We first sett inn for $W(0) = 123gr$ and get $W_0 = 123gr$, then we set inn for $W(5) = 20gr$ and get

$$20 = 123e^{-5\lambda}$$

Then we take the natural logorithm on both sides to get

$$\begin{aligned} \ln 20 &= \ln 123 - 5\lambda \\ \lambda &= -\frac{\ln 20 - \ln 123}{5} \approx 0.36. \end{aligned}$$

To determine the half life, we solve $W(t) = 123/2$gr for t

$$\begin{aligned} 123/2 &= 123e^{-0.36t} \\ t &\approx -\frac{\ln 1/2}{0.36} \approx 1.92. \end{aligned}$$

Then we have found the deca constant $\lambda \approx 0.36$ and the half-life of the substance to be about 1.92 minutes.

5 Use the partial fraction decomposition to solve

$$\frac{dy}{dx} = y(y-1), \text{ where } y_0 = 0 \text{ if } x_0 = 0.$$

First when we rearrange the equation we get

$$\frac{dy}{y(y-1)} = dx \tag{1}$$

Therefor in the partial-fraction method we want to find A and B such that

$$\frac{A}{y} + \frac{B}{y-1} = \frac{1}{y(y-1)}.$$

Then we have that

$$Ay - A + By = 1.$$

Which means that

$$\begin{aligned} A + B &= 0 \\ -A &= 1 \end{aligned}$$

We solve the system of equations to get that $A = -1$ and $B = 1$. If we now integrate on bouth sides of (1) we get

$$\int \left(-\frac{1}{y} + \frac{1}{y-1}\right) dy = \int dx$$
$$-\ln|y| + \ln|y-1| = x + C_0$$
$$\ln\left|\frac{y-1}{y}\right| = x + C_0$$
$$\left|\frac{y-1}{y}\right| = e^{x+C_0}$$
$$\frac{y-1}{y} = \pm e^{C_0} e^x$$
$$\frac{y-1}{y} = e^{C_0} e^x = Ce^x$$
$$y - 1 = yCe^x$$
$$y(1 - Ce^x) = 1$$
$$y = \frac{1}{1 - Ce^x}.$$

Then we compute C by evaluating with $y = 1$ and $x = 0$.

$$1 = \frac{1}{1-C}$$

which means that $C = 0$. Then we have the solution

$$y = 1.$$

$\boxed{6}$ Solve the differential equation:

$$\frac{dy}{dx} = x^2 e^y, \text{ where } y_0 = 1 \text{ if } x_0 = 1.$$

We first start by rearrange on the equation to get

$$e^{-y} dy = x^2 dx$$

Then we integrate on both sides to get

$$\int e^{-y} dy = \int x^2 dx$$
$$-e^{-y} = \frac{x^3}{3} + C_0$$
$$y = -\ln(C - \frac{x^3}{3})$$

Now we compute C by seting inn for $y = 1$ and $x = 1$

$$1 = -\ln(C - \frac{1}{3})$$
$$C = \frac{1}{3} + e^{-1}$$

Then we have solved the differential equation

$$y = -\ln(\frac{1}{3} + e^{-1} - \frac{x^3}{3}).$$