Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 5

$\boxed{1}$ a) Solve the linear system, then graph it to verify your solution.

$$\begin{aligned} 4x - y &= 8 \\ 3x + 2y &= -2 \end{aligned}$$

We start by noticing that

$$\begin{aligned} 4x - y = 8 &\iff y = 4x - 8 \\ 3x + 2y = -2 &\iff y = -1 - \frac{3x}{2} \end{aligned}$$

Then we can set in for y and get

$$\begin{aligned} 4x - 8 &= -1 - \frac{3x}{2} \\ \frac{11x}{2} &= 7 \\ x &= \frac{14}{11} \end{aligned}$$

Then we set in for x in one of the equations and get

$$y = -\frac{32}{11} \text{ and } x = \frac{14}{11}.$$

We graph the linear system as we seen in Figure 1.

b) Now consider the linear system

$$\begin{aligned} x - 2y &= 3 \\ ax - 4y &= 1 \end{aligned}$$

Solve the system to get a expression with a. For which values of a does the system have: only one solution, infinitely many solutions, no solutions?

We start the same way as in a), by noticing that

$$\begin{aligned} x - 2y = 3 &\iff y = \frac{x}{2} - \frac{3}{2} \\ ax - 4y = 1 &\iff y = a\frac{x}{4} - \frac{1}{4} \end{aligned}$$

We then set in for y and get

$$\begin{aligned} \frac{x}{2} - \frac{3}{2} &= a\frac{x}{4} - \frac{1}{4} \\ \frac{2-a}{4}x &= \frac{5}{4} \\ x &= \frac{5}{2-a} \end{aligned}$$

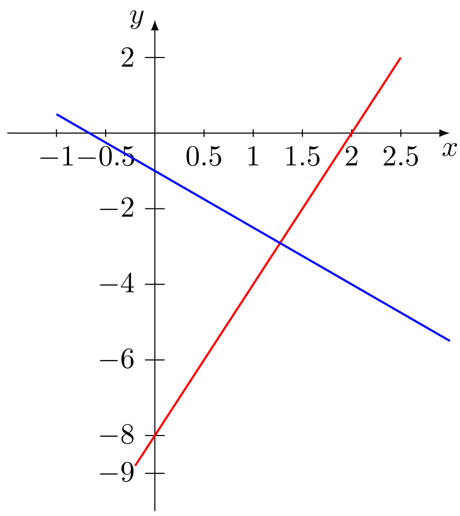

Figure 1: The graph of the system in task 1 a

Then we set in for y in one of the equations and get that

$$y = \frac{3a-2}{2(2-a)} \text{ and } x = \frac{5}{2-a}.$$

We see that this is a valid solution for all $a \neq 2$. So now we try setting in for $a = 2$ in the original system and get that

$$y = \frac{x}{2} - \frac{3}{2}$$
$$y = \frac{x}{2} - \frac{1}{4}$$

We see that these are two parallel lines that don't coincide. Which means that if $a = 2$ the system will have no solutions, but for all other values of a, the linear system will have exactly one solution.

c) Given the functions

$$f_1(x, y) = 2x - y \text{ and } f_2(x, y) = -x + y,$$

we define the vectors

$$\mathbf{f} = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find a matrix A, such that

$$\mathbf{f} = A \cdot \mathbf{u}.$$

If we set the matrix to be

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

we see that the equality holds.

$\boxed{2}$ Find all solutions (if there are any) to the linear systems a)

$$\begin{aligned} 2x + 5y &= 1 \\ x + 2y &= -1 \end{aligned}$$

We first see that

$$\begin{aligned} 2x + 5y = 1 &\iff y = \frac{1}{5} - \frac{2}{5}x \\ x + 2y = -1 &\iff y = -\frac{1}{2} - \frac{1}{2}x \end{aligned}$$

Then we set in for y and get

$$\begin{aligned} \frac{1}{5} - \frac{2}{5}x &= -\frac{1}{2} - \frac{1}{2}x \\ \frac{1}{10}x &= -\frac{7}{10} \\ x &= -7 \end{aligned}$$

We then set in for x and get

$$y = 3 \text{ and } x = -7.$$

b)

$$\begin{aligned} 3x - y &= 2 \\ -9x + 3y &= 1 \end{aligned}$$

We first see that

$$\begin{aligned} 3x - y = 2 &\iff y = 3x - 2 \\ -9x + 3y = 1 &\iff y = 3x + \frac{1}{3} \end{aligned}$$

Then we notice that these two are parallel lines that does not coincide, and the system has therefor no solutions.

c)

$$\begin{aligned} x - 2y &= -2 \\ -2x + 4y &= 4 \end{aligned}$$

We first see that

$$\begin{aligned} x - 2y = -2 &\iff y = \frac{1}{2}x + 1 \\ -2x + 4y = 4 &\iff y = \frac{1}{2}x + 1 \end{aligned}$$

We notice that these two are parallel lines that do coincide, which means that the system have infinitely many solutions. So if $x = a$ and $y = y = \frac{1}{2}a + 1$ for all real numbers a, this will be a solution for the linear system.

$\boxed{3}$ Find the augmented matrix and use it to solve the linear system.

$$
\begin{aligned}
2x - y + z &= -3 \\
3x + 2y - z &= 4 \\
x - y + 2z &= -5
\end{aligned}
$$

First we sett up the augmented matrix

$$
\left[\begin{array}{ccc|c}
2 & -1 & 1 & -3 \\
3 & 2 & -1 & 4 \\
1 & -1 & 2 & -5
\end{array}\right]
$$

Then we do the row reductions

$$
\begin{aligned}
&\left[\begin{array}{ccc|c}
2 & -1 & 1 & -3 \\
3 & 2 & -1 & 4 \\
1 & -1 & 2 & -5
\end{array}\right] \quad -1 \quad \sim \left[\begin{array}{ccc|c}
1 & 0 & -1 & 2 \\
3 & 2 & -1 & 4 \\
1 & -1 & 2 & -5
\end{array}\right] \quad -3 \quad -1 \\
\sim &\left[\begin{array}{ccc|c}
1 & 0 & -1 & 2 \\
0 & 2 & 2 & -2 \\
0 & -1 & 3 & -7
\end{array}\right] \quad \sim \left[\begin{array}{ccc|c}
1 & 0 & -1 & 2 \\
0 & -1 & 3 & -7 \\
0 & 2 & 2 & -2
\end{array}\right] \cdot(-1) \\
\sim &\left[\begin{array}{ccc|c}
1 & 0 & -1 & 2 \\
0 & 1 & -3 & 7 \\
0 & 2 & 2 & -2
\end{array}\right] \quad -2 \quad \sim \left[\begin{array}{ccc|c}
1 & 0 & -1 & 2 \\
0 & 1 & -3 & 7 \\
0 & 0 & 8 & -16
\end{array}\right] \cdot\left(\tfrac{1}{8}\right) \\
\sim &\left[\begin{array}{ccc|c}
1 & 0 & -1 & 2 \\
0 & 1 & -3 & 7 \\
0 & 0 & 1 & -2
\end{array}\right] \quad 3 \quad 1 \quad \sim \left[\begin{array}{ccc|c}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -2
\end{array}\right]
\end{aligned}
$$

Then we end up with the solution for the linear system

$$x = 0,\ y = 1,\ z = -2.$$

$\boxed{4}$ Three different species of insects are reared together in a laboratory cage. They are supplied with two different types of food each day. Each day each individual of species 1 consumes 3 units of food A and 5 units of food B, each individual of species 2 consumes 2 units of food A and 3 units of food B and each individual of species 3 consumes 1 units of food A and 2 units of food B. Each day, 500 units of food A and 900 units of food B are supplied. How many individuals of each species can be reared together? Is there more than one solution?
We first set x to be the number of insects of species 1, y insects of species 2 and z insects of species 3. Then we can set up a linear system of how much food of type A and B that gets eaten in the form of two equations.

$$
\begin{aligned}
3x + 2y + z &= 500 \\
5x + 3y + 2z &= 900
\end{aligned}
$$

We then set up the augmented matrix for this system and do the row reduction.

$$
\begin{aligned}
&\left[\begin{array}{ccc|c} 3 & 2 & 1 & 500 \\ 5 & 3 & 2 & 900 \end{array}\right] \; -2 \quad \sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 500 \\ -1 & -1 & 0 & -100 \end{array}\right] \; \cdot(-1) \\
\sim &\left[\begin{array}{ccc|c} 3 & 2 & 1 & 500 \\ 1 & 1 & 0 & 100 \end{array}\right] \quad \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 100 \\ 3 & 2 & 1 & 500 \end{array}\right] \; -3 \\
\sim &\left[\begin{array}{ccc|c} 1 & 1 & 0 & 100 \\ 0 & -1 & 1 & 200 \end{array}\right] \; \cdot(-1) \quad \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 100 \\ 0 & 1 & -1 & -200 \end{array}\right] \; -1 \\
\sim &\left[\begin{array}{ccc|c} 1 & 0 & 1 & 300 \\ 0 & 1 & -1 & -200 \end{array}\right]
\end{aligned}
$$

From this we get the linear system

$$
\begin{aligned}
x + z &= 300 \\
y - z &= -200
\end{aligned}
$$

which we can rewrite to the form

$$
\begin{aligned}
z + x &= 300 \\
z &= y + 200
\end{aligned}
$$

From this we can see that there should be a total of 300 insects of species 1 and 2, and there should be 200 more of species 3 than species 2. Then we have that all positive integer solutions that satisfy those two requirements will be valid solutions. One example is 50 of species 1, 50 of species 2 and 250 of species 3.