MA0002 Mathematical Methods B
Spring 2024

Exercise set 6

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1 Let

$$A = \begin{pmatrix} 2 & -1 & 9 \\ 0 & 3 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 9 & -7 & 3 \\ -1 & -2 & 6 \end{pmatrix}.$$

(a) Calculate $A + B$.

(c) Find the matrix C such that $A + B + C = \mathbf{0}$.

Solution. (a) We see that

$$A + B = \begin{pmatrix} 2 & -1 & 9 \\ 0 & 3 & -1 \end{pmatrix} + \begin{pmatrix} 9 & -7 & 3 \\ -1 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 11 & -8 & 12 \\ -1 & 1 & 5 \end{pmatrix}.$$

(b) For $A + B + C = \mathbf{0}$ to hold we must have $C = -(A + B)$, that is:

$$C = \begin{pmatrix} -11 & 8 & -12 \\ 1 & -1 & -5 \end{pmatrix}.$$

2 Let

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & -2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 7 & -1 \\ 0 & -2. \end{pmatrix}$$

(a) Calculate AB.

(b) Calculate BA.

Solution.

(a)

$$AB = \begin{pmatrix} 3 & 0 & 1 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 7 & -1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3*0+0*7+1*0 & 3*1+0*(-1)+1*(-2) \\ (-1)*0+(-2)*7+0*0 & (-1)*1+(-2)*(-1)+0*(-2) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -14 & 1 \end{pmatrix}$$

(b)

$$\begin{aligned}
BA &= \begin{pmatrix} 0 & 1 \\ 7 & -1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ -1 & -2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0*3+1*(-1) & 0*0+1*(-2) & 0*1+1*0 \\ 7*3+(-1)*(-1) & 7*0+(-1)*(-2) & 0*1+(-2)*0 \\ 0*3+(-2)*(-1) & 0*0+(-2)*(-2) & 0*1+(-2)*0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -2 & 0 \\ 22 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix}
\end{aligned}$$

$\boxed{3}$ Let

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) Calculate $3A$.

(b) Calculate A^3.

(c) What is the inverse matrix of A?

Solution.

(a)

$$3A = A + A + A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 3 \end{pmatrix}.$$

(b)

$$A^2 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3*3+0*0 & 3*0+0*1 \\ 0*3+1*0 & 0*0+1*1 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$A^3 = A^2 * A = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 9*3+0*0 & 9*0+0*1 \\ 0*3+1*0 & 0*0+1*1 \end{pmatrix} = \begin{pmatrix} 27 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) We use the theorem from the book at page 454. We see that $3*0-0*1 = 3 \neq 0$, so A is invertible and has inverse

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}.$$

$\boxed{4}$ Let

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \\ -1 & -2 & 1 \end{pmatrix}.$$

For each of the matrices A, B and C; find the inverse matrix or explain why it does not exist.

Solution.
We use the theorem from the book at page 454. We see that $1*(-1)-(-1)*0=-1\neq 0$, so A is invertible and has inverse

$$A^{-1}=-\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}.$$

(b) Vi calculate the determinate of the matrix B and see that $4*1-2*2=0$, so the matrix B is not invertible.
(c) We use Gaussian elimination on the matrix (AI_3):

$$\begin{aligned}
\begin{pmatrix}1 & 3 & 0 & 1 & 0 & 0\\ 0 & -1 & 2 & 0 & 1 & 0\\ -1 & -2 & 1 & 0 & 0 & 1\end{pmatrix}\ \cdot(-1) \sim &\begin{pmatrix}1 & 3 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 0 & -1 & 0\\ -1 & -2 & 1 & 0 & 0 & 1\end{pmatrix}\ 1\\
\sim &\begin{pmatrix}1 & 3 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 0 & -1 & 0\\ 0 & 1 & 1 & 1 & 0 & 1\end{pmatrix}\ -1\\
\sim &\begin{pmatrix}1 & 3 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 3 & 1 & 1 & 1\end{pmatrix}\ \cdot(1/3)\\
\sim &\begin{pmatrix}1 & 3 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3\end{pmatrix}\ 2\\
\sim &\begin{pmatrix}1 & 3 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 2/3 & -1/3 & 2/3\\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3\end{pmatrix}\ -3\\
\sim &\begin{pmatrix}1 & 0 & 0 & -1 & 1 & -2\\ 0 & 1 & 0 & 2/3 & -1/3 & 2/3\\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3\end{pmatrix}
\end{aligned}$$

We thus obtain that

$$A^{-1}=\begin{pmatrix}-1 & 1 & -2\\ 2/3 & -1/3 & 2/3\\ 1/3 & 1/3 & 1/3\end{pmatrix}.$$

$\boxed{5}$ Let

$$A=\begin{pmatrix}1 & 2\\ -1 & a\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix},\quad \text{and}\quad \mathbf{b}=\begin{pmatrix}b_1\\ b_2\end{pmatrix}$$

(a) Show that for $a\neq -2$ the equation $A\mathbf{x}=\mathbf{b}$ has exactly one solution.

(b) Let $a = 1$. Solve the equation

$$A\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

by using the inverse matrix A^{-1}.

(c) Let $a = -2$. Determine conditions on b_1 and b_2 such that the equations $A\mathbf{x} = \mathbf{b}$ has

(i) infinitely many solutions,

(ii) no solutions.

(d) Explain your results from (a), (b) and (c) graphically. Tip: think about what the slope of the equations tells you.

Solution. (a) We see that the determinate $\det A = 1a - 2(-1) = 0$ if and only if $a = -2$. Thus we know that for $a \neq -2$ the matrix A is invertible. The equation will have the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ since $Ax = AA^{-1}\mathbf{b} = I_2\mathbf{b} = \mathbf{b}$.
(b) We start by finding the inverse matrix,

$$A^{-1} = \frac{1}{1+2}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

We can now calculate the solution of the system of equations:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(c) We start by row-reducing the matrix

$$\begin{pmatrix} 1 & 2 & b_1 \\ -1 & -2 & b_2 \end{pmatrix} \xrightarrow{R2*(-1)} A = \begin{pmatrix} 1 & 2 & b_1 \\ 1 & 2 & -b_2 \end{pmatrix} \xrightarrow{R2=R2-R1} A = \begin{pmatrix} 1 & 2 & b_1 \\ 0 & 0 & -b_2 - b_1 \end{pmatrix},$$

For the system of equations to be solvable er must thus have $x + 2y = b_1$ og $0 = -b_2 - b_1$. Thus if $b_2 \neq -b_1$ we have no solutions. However if $b_2 = -b_1$ we have infinitely many solutions, and they are given by

$$\{(x, y) : x = b_1 - 2y\}$$

(d) Vi study the system of equations

$$\begin{aligned} x_1 + 2x_2 &= b_1 \\ -x_1 + ax_2 &= b_2. \end{aligned}$$

We can express this as the two lines

$$\begin{aligned} x_2 &= -\frac{x_1}{2} + \frac{b_1}{2} \\ x_2 &= \frac{x_1}{a} + \frac{b_2}{a} \end{aligned}$$

There are three different cases for these lines:

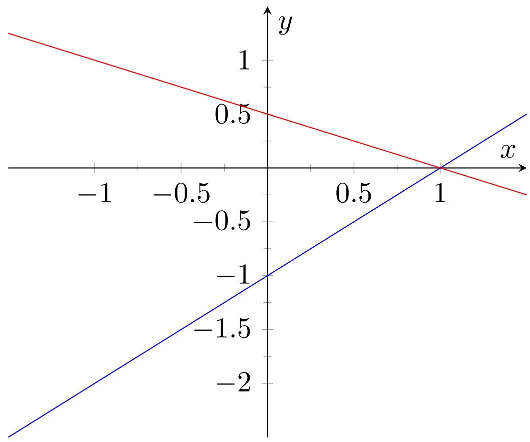

Figure 1: Figure for 5(b).

- The lines are the same line. (The system of equations have infinitely many solutions
- They are parallel but not the exact same line. (The system of equation has no solutions)
- The cross each other exactly one time. (The system of equation has exactly one solution).

In exercise (a) we shoved that the system had exactly one solution. This is the same as the lines crossing one time, which happens if and only if the lines has different slope, which happens exactly when $a \neq -2$.

In exercise (b) we found the point where the lines

$$x_2 = -\frac{x_1}{2} + \frac{1}{2}$$
$$x_2 = x_1 - 1$$

meets, as you can see on the figure. In exercise (c) we found conditions on b_1 and b_2. Asking when the system has no solutions is the same as asking when the lines are parallel, i.e when the slope is the same but the constant term is different. Asking when there are infinitely many solutions is the same as asking when the lines are equal, i.e have the same slope and constant term.